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THE KRULL INTERSECTION THEOREM II

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Let R be a commutative ring, I an ideal in R and A an R-module. We always have $0 \subseteq \{a \in A \mid (1-i)a = 0 \exists i \in I\} \subseteq I \bigcap_{n=1}^{\infty} I^n A \subseteq \bigcap_{n=1}^{\infty} I^n A$. In this paper we inyestigate conditions under which certain of these containments may or may not be replaced by equality.

1. Introduction. This paper is a continuation of **[1]**. In §2 we show that for a nonminimal principal prime (p), $J = \bigcap_{n=1}^{\infty} (p)^n$ is a prime ideal and $pJ = J$. An example is given to show that the condition that (p) be nonminimal is necessary. We also consider the question of when a prime ideal minimal over a principal ideal has rank one. Of particular interest is the example of a domain D with a doubly generated ideal I such that $\bigcap_{n=1}^{\infty} I^n \neq I \bigcap_{n=1}^{\infty} I^n$. In §3 we prove that $\bigcap_{n=1}^{\infty} I^n A = I \bigcap_{n=1}^{\infty} I^n A$ for any finitely generated module A over a valuation ring. In §4 we consider certain converses to the usual Krull Intersection Theorem for Noetherian rings. It is shown that for (R, M) a quasi-local ring whose maximal ideal M is finitely generated, many classical results for local rings are actually equivalent to the ring R being Noetherian.

2. Some examples and counterexamples. In **[1]** we remarked that for a ring R the following statements are equivalent: (1) $\dim R = 0$, (2) $\bigcap_{n=1}^{\infty} I^n A = I \bigcap_{n=1}^{\infty} I^n A$ for all finitely generated ideals I and all R-modules A, (3) $\bigcap_{n=1}^{\infty} x^n A = x \bigcap_{n=1}^{\infty} x^n A$ for $x \in R$ and all R-modules A. This raises the question: For which ideals I in a ring R do we have $I \bigcap_{n=1}^{\infty} I^n A = \bigcap_{n=1}^{\infty} I^n A$ for all R-modules A? A modification of the example on page 11 of **[1]** yields

THEOREM 2.1. *For a quasi-local ring (R, M) and an ideal I the following statements are equivalent:*

(1) *$I^n = I^{n+1}$ for some n,*

(2) *for every R-module A, $I \bigcap_{n=1}^{\infty} I^n A = \bigcap_{n=1}^{\infty} I^n A$.*

Proof. The implication (1) $\Rightarrow$ (2) is clear. Suppose that (2) holds but $I^n \supsetneq I^{n+1}$ for all $n > 0$. Choose $i_n \in I^n - I^{n+1}$. Let $F = Rx \oplus (\oplus \sum_{i=1}^{\infty} Ry_i)$ be the free R-module on $\{x, y_1, y_2, \cdots\}$ and let G be the sub-module of F generated by the set $\{x - i_1 y_1, x - i_2 y_2, \cdots\}$ and let $A = F/G$. One can then verify that $I \bigcap_{n=1}^{\infty} I^n A \neq \bigcap_{n=1}^{\infty} I^n A$.

It is well-known [7, page 74] that if P is an invertible prime ideal in a domain, then $J = \bigcap_{n=1}^{\infty} P^n$ is a prime ideal, $J = PJ$ and any prime ideal properly contained in P is actually contained in J. We generalize this result. Recall that an ideal I is finitely generated and locally principal if and only if it is a multiplication ideal (i.e., any ideal contained in I is a multiple of I) and a weak-cancellation ideal (for two ideals A and B, $AI \subseteq BI$ implies $A \subseteq B + (0\colon I)$). (For example, see **[2]** or **[8]**.)

THEOREM 2.2. *Let R be a ring and P a nonminimal finitely generated locally-principal prime ideal of R and set $J = \bigcap_{n=1}^{\infty} P^n$. Then*

(1) *J is prime,*

(2) *$PJ = J$, and*

(3) *any prime ideal properly contained in P is contained in J.*

Proof. Let $a, b \in R \ni a, b \not\in J$. We show that $ab \not\in J$. Choose n, m such that $a \in P^n - P^{n+1}$ and $b \in P^m - P^{m+1}$. Then since P^n and P^m are multiplication ideals, we get (a) $= P^n A_1$ and (b) $= P^m B_1$ where $A_1 \not\subseteq P$ and $B_1 \not\subseteq P$. Now (a)(b) $\subseteq P^{n+m+1}$ implies $A_1 B_1 P^{n+m} \subseteq P^{n+m+1}$. Since P^{n+m} is a weak-cancellation ideal, $A_1 B_1 \subseteq P + (0\colon P^{n+m})$. Let $Q \subsetneq P$ be a prime ideal, then $(0\colon P^{n+m})P^{n+m} = 0 \subsetneq Q$ gives $(0\colon P^{n+m}) \subseteq Q \subsetneq P$ and hence $A_1 B_1 \subseteq P$. Thus A_1 or $B_1 \subseteq P$, a contradiction. Hence J is prime. Let $j \in J$, then $j \in P$ so (j) $= PA$. Since P is a nonminimal prime, $P \not\subseteq J$, hence $A \subseteq J$, so $j \in PJ$. For (3), let Q be a prime ideal properly contained in P and let $q \in Q$. Then $(q) = PQ_1 \subseteq Q$ and $P \not\subseteq Q$ implies $Q_1 \subseteq Q \subset P$. Continuing we get $(q) \subseteq J$.

COROLLARY 2.3. *Let (p) be a nonminimal principal prime ideal. Then $J = \bigcap_{n=1}^{\infty} (p)^n$ is prime, $pJ = J$ and prime ideal $Q \subsetneq (p)$ is contained in J.*

The above corollary is false if (p) is a minimal prime ideal. For example, in $Z/(4)$ $\bigcap_{n=1}^{\infty} (\bar{2})^n$ is not prime. However, in this example condition (2) still holds. In the following example we show that condition (2) may also fail.

EXAMPLE 2.4. Let k be a field and let $R = k[X, Z, Y_1, Y_2, \cdots]$ be the polynomial ring over k in indeterminants $X, Z, Y_1, Y_2, \cdots$. Let $A = (X - ZY_1, X - Z^2Y_2, X - Z^3Y_3, \cdots)$ and put $\bar{R} = R/A$. Then (X, Z) is a prime ideal of R minimal over A and hence $\overline{(X, Z)}$ is a minimal prime ideal of $\bar{R}$ ($-$ denotes passage to $\bar{R}$). Moreover, $\overline{(X, Z)} = (\bar{Z})$, so $(\bar{Z})$ is a minimal principal prime ideal of $\bar{R}$. However, $\bigcap_{n=1}^{\infty} (\bar{Z})^n \neq (\bar{Z}) \bigcap_{n=1}^{\infty} (\bar{Z})$ because $\bar{X} \in \bigcap_{n=1}^{\infty} (\bar{Z})^n$ but $\bar{X} \not\in (\bar{Z}) \bigcap_{n=1}^{\infty} (\bar{Z})^n$.

The Principal Ideal Theorem states that a prime ideal in a Noetherian domain minimal over a principal ideal has rank one. In general a prime ideal minimal over a principal ideal need not have rank one. In fact, a principal prime (p) has rank one if and only if $\bigcap_{n=1}^{\infty}(p)^n = 0$. More generally, if P is a rank one prime, any $a \in P$ must satisfy $\bigcap_{n=1}^{\infty}(a)^n = 0$ (see Corollary 1.4 **[9]** or Theorem 1 **[1]**). This raises the question: In a domain, does a prime P minimal over a principal ideal (a) with $\bigcap_{n=1}^{\infty}(a)^n = 0$ imply that rank $P = 1$? This question is answered in the negative by Example 5.2 **[9]**. Finally we ask the question: In a domain, does a finitely generated prime P satisfying $\bigcap_{n=1}^{\infty} P^n = 0$, minimal over a principal ideal, have rank 1? While we are not able to answer this question, we do show that there can not be "too many" primes below P.

THEOREM 2.5. *Let R be a domain and let P be a finitely generated prime ideal minimal over a principal ideal Rx. Then* rank $P = 1$ *if and only if* $\cap\{Q \in \operatorname{Spec}(R) \mid Q$ *is directly below* $P\} = 0$.

Proof. The implication ($\Rightarrow$) is clear. Conversely, let $\{Q_\alpha\}$ be the set of prime ideals directly below P (this set is nonempty by Zorn's Lemma). The hypothesis of the theorem is preserved by passage to R_p, so we may assume that R is quasi-local. Thus (R, P) is quasi-local, P is finitely generated, and Rx is P-primary. By Theorem 1 **[1]**, $\bigcap_{n=1}^{\infty} P^n \subseteq \cap\{Q \mid Q$ directly below $P\} = 0$. Let $(\hat{R}, \hat{P})$ be the P-adic completion of R. Then $(\hat{R}, \hat{P})$ is a complete (Noetherian) local ring. Now $\hat{R}x$ is still $\hat{P}$-primary, so by the Principal Ideal Theorem, $\dim \hat{R} \leqq 1$. If $\dim \hat{R} = 0$, then $\hat{P}^n = 0$ for some n and hence $P^n = 0$. This contradiction shows that $\dim \hat{R} = 1$. Let $P_1, \cdots, P_n$ be the minimal primes of $\hat{R}$ and let $Q_i = P_i \cap R$. Now $\cap\{Q \mid Q$ directly below $P\} = 0$ implies that there exist infinitely many primes directly below P. Hence $\exists y \in Q_0 - \bigcup_{i=1}^{n} Q_i$ where Q_0 is a prime directly below P. Now $\hat{R}y \not\subseteq \bigcup_{i=1}^{n} P_i$, so $\hat{R}y$ is $\hat{P}$-primary. Hence $\hat{R}y \cap R$ is P-primary. But by Theorem 1**[1]** we see that Q_0 is closed in the P-adic topology, and hence $\hat{R}y \cap R \subseteq Q_0$. This is a contradiction because $\hat{R}y \cap R$ is P-primary.

The proof of Theorem 2.5 does yield the following result. Let P be a finitely generated prime ideal in a domain minimal over a principal ideal. Then rank $P = 1$ if and only if $\bigcap_{n=1}^{\infty} P_P^n = 0$ (or equivalently, if $\bigcap_{n=1}^{\infty} P^{(n)} = 0$ where $P^{(n)}$ is the n-symbolic power of P).

We end this section with an example of a domain D and a doubly generated ideal I in D satisfying $I \bigcap_{n=1}^{\infty} I^n \neq \bigcap_{n=1}^{\infty} I^n$. This is the best possible counterexample as $\bigcap_{n=1}^{\infty}(x)^n = (x)\bigcap_{n=1}^{\infty}(x)^n$ for all principal ideals in a domain.

EXAMPLE 2.6. Let k be a field, $S = k[W, W^{\frac{1}{2}}, W^{\frac{1}{3}}, W^{\frac{1}{5}}, \cdots]$, and $R_0 = S[X, U_2, U_3, U_5, U_7, \cdots]$. Then $R_0[Y, 1/Y]$ is a graded domain, with degree $R_0 = 0$, degree $Y = 1$ and degree $1/Y = -1$. Let R be the graded subdomain $R_0[Y, (W^{\frac{1}{2}} - XU_2)/Y, (W^{\frac{1}{3}} - XU_3)/Y, \cdots]$. Then $I = (X, Y)$ is a homogeneous ideal of R. Put $J = \bigcap_{n=1}^{\infty} I^n$ so that J is also a homogeneous ideal. We show that $J \neq IJ$.

Write $Z_p = W^{1/p} - XU_p$. Then $R_0 = k[W, Z_2, Z_3, Z_5, \cdots, X, U_2, U_3, U_5, \cdots]$. We have the relation $(Z_p + XU_p)^p = W$ and hence

$$Z_p^p = W - X^pU_p^p - \binom{p}{1}Z_pX^{p-1}U_p^{p-1} - \cdots - \binom{p}{p-1}Z_p^{p-1}XU_p.$$

Note that R_0 is spanned as a k-vector space by the monomials $Z_{p_1}^{e_1}\cdots Z_{p_r}^{e_r}W^{n_0}X^{n_1}U_{q_1}^{f_1}\cdots U_{q_s}^{f_s}$, where $0 < e_i < p_i$. We show that these monomials are k-independent, and thus form a k-basis. To see this, define the degree of the monomial $W^{e_1/p_1+\cdots+e_r/p_r+n_0}X^{n_1}U_{q_1}^{f_1}\cdots U_{q_s}^{f_s}$ $(0 < e_i < p_i)$ to be $(e_1/p_1 + \cdots + e_r/p_r + n_0, n_1, 0, \cdots, 0, f_1, 0, \cdots, 0, f_s, 0\cdots)$ where f_i appears in the s_i-th position after n_1 if q_i is the s_i-th prime. Order the degrees lexicographically. Then define the degree of a polynomial to be the degree of the largest term. We find that the degree of $Z_{p_1}^{e_1}\cdots Z_{p_r}^{e_r}\cdots Z_{p_r}^{e_r}W^{n_0}X^{n_1}U_{q_1}^{f_1}\cdots U_{q_s}^{f_s}$ $(0 < e_i < p_i)$ to be $(e_1/p_1 + \cdots + e_r/p_r + n_0, n_1, 0, \cdots, f_1, \cdots, f_s, 0, \cdots)$ as above. Each such monomial has a different degree, and hence these monomials are k-independent. Let us write $T = k[X, W, U_2, U_3, Y_5, \cdots]$. We see that $R_0 = T \oplus R_{0z}$ as a T-module, where R_{0z} is generated as a T-module by the $Z_{p_1}^{e_1}\cdots Z_{p_r}^{e_r}$, $0 < e_i < p_i$, $r \geqq 1$. Let H be the ideal of R_0 generated by the Z_p's. Since $H \supset R_{0z}$, we have $H = (H \cap T) \oplus R_{0z}$ as a T-module. Now

$$\begin{aligned}[I^m]_0 = [(X, Y)^m]_0 &= X^mR_0 + X^{m-1}YR_{-1} + \cdots + Y^mR_{-m} \\ &= X^mR_0 + X^{m-1}H + \cdots XH^m = (X, H)^m\end{aligned}$$

as an ideal of R_0. Notice that since $W = (Z_p + XU_p)^p$, we have $W \in (X, H)^m$ for all m. Now $H \cap T$ is generated as a T-module by the $W - X^pU_p^p$. Thus (X, H) is generated by X, W, and the $Z_{p_1}^{e_1}\cdots Z_{p_r}^{e_r}$ $(r \geqq 1)$ and $(X, H)^m$ is generated by X^m, W, the $Z_{p_1}^{e_1}\cdots Z_p^{e_r}W$ $(r \geqq 1)$, and the $Z_{p_1}^{e_1}\cdots Z_{p_r}^{e_r}X^{n_0}$ with $e_1 + \cdots + e_r + n_0 \geqq m$. It follows that $J_0 = \bigcap_{m=1}^{\infty}(X, H)^m = WR_0$.

We claim that $W \notin [IJ]_0 = XJ_0 + YJ_{-1}$. In fact, we claim that $W \notin XJ_0 + YR_{-1} = XWR_0 + H$. Since $H \supset R_{0z}$, the ideal

$$(XWR_0 + H) \cap T = (XW, W - X^2U_2^2, W - X^3U_3^3, W - X^5U_5^5, \cdots).$$

Suppose that $W \in XJ_0 + YR_{-1}$, then

$$W = aXW + b_2(W - X^2U_2^2) + \cdots + b_p(W - X^pU_p^p),$$

$a, b_i \in T$. Write $b_i = c_i + \lambda_i$ where $\lambda_i \in k$ and $c_i \in T$ with no constant term. Cancelling W, we get

$$\lambda_2 X^2U_2^2 + \cdots + \lambda_p X^pU_p^p = aXW - c_2X^2U_2^2 - \cdots - c_pX^pU_p^p.$$

But this is a contradiction since none of the terms on the left appear on the right.

3. Valuation rings. We call a ring R a valuation ring if any two ideals of R are comparable. In Theorem 2 **[1]** we proved that for R a Prüfer domain, I an ideal in R and A a torsion-free R-module, $I\bigcap_{n=1}^{\infty} I^nA = \bigcap_{n=1}^{\infty} I^nA$. In this section we prove that for R a valuation ring, I an ideal in R and A a finitely generated R-module, $I\bigcap_{n=1}^{\infty} I^nA = \bigcap_{n=1}^{\infty} I^nA$. We begin with the ring case.

THEOREM 3.1. *Let V be a valuation ring and I a nonzero ideal in V. Then exactly one of the following occurs*:

(1) *$I = I^2$ is prime,*

(2) *$I^n \supsetneq I^{n+1}$ for all n, $\bigcap_{n=1}^{\infty} I^n$ is a prime ideal in V, and $\bigcap_{n=1}^{\infty} I^n = \bigcap_{n=1}^{\infty} (i)^n$ for any $i \in I - I^2$. In particular, $\bigcap_{n=1}^{\infty} I^n = I\bigcap_{n=1}^{\infty} I^n$.*

(3) *$I^n = 0$ for some n.*

Proof. First suppose that $I = I^2$ and let $ab \in I$. Suppose that $a, b \notin I$, so that $I \subsetneq (a)$ and $I \subsetneq (b)$. Hence $I = I^2 \subseteq (a)(b) \subseteq I$ so $I = (ab)$. Thus $I = I^2$ implies $I = 0$, a contradiction. Next suppose that $I \neq I^2$, but $I^n \supsetneq I^{n+1} = I^{n+2}$. Let $i \in I^n - I^{n+1}$. Then for $m > 1$, $I^{n+1} = I^{mn} \supseteq (i)^m \supseteq I^{m(n+1)} = I^{n+1}$, in particular $(i)^2 = (i)^3$, so $(i)^2 = 0$. Hence $0 = (i)^2 \supseteq I^{2(n+1)} = I^{n+1}$. Finally, suppose that $I^n \supsetneq I^{n+1}$ for all n. For $i \in I - I^2$, $I \supseteq (i) \supseteq I^2$, so that $I^n \supseteq (i)^n \supseteq I^{2n}$ and hence $\bigcap_{n=1}^{\infty} I^n = \bigcap_{n=1}^{\infty} (i)^n$. Suppose that $xy \in \bigcap_{n=1}^{\infty} I^n$. If $x, y \notin \bigcap_{n=1}^{\infty} I^n$, then there exist integers s and t such that $I^s \subsetneq (x)$ and $I^t \subsetneq (y)$. Hence $I^{s+t} \subseteq (xy) \subseteq \bigcap_{n=1}^{\infty} I^n$ so $I^{s+t} = I^{s+t+1}$. This contradiction shows that $\bigcap_{n=1}^{\infty} I^n$ must be prime. Suppose that $x \in \bigcap_{n=1}^{\infty} I^n$. Then $x = si^2$ for some $s \in V$ and $i \in I$. Hence si or $i \in \bigcap_{n=1}^{\infty} I^n$ because $\bigcap_{n=1}^{\infty} I^n$ is prime. Thus $\bigcap_{n=1}^{\infty} I^n = I\bigcap_{n=1}^{\infty} I^n$.

THEOREM 3.2. *Let V be a valuation ring, I an ideal in V and A a finitely generated V-module. Then $\bigcap_{n=1}^{\infty} I^nA = I\bigcap_{n=1}^{\infty} I^nA$.*

Proof. By the previous theorem we are reduced to the case where $I = (i)$ is a principal ideal and $\bigcap_{n=1}^{\infty} (i)^n$ is prime. Put $B = (\bigcap_{n=1}^{\infty} (i)^n)A$, so that $B \subseteq \bigcap_{n=1}^{\infty} (i)^nA$. It suffices to show that $\bigcap_{n=1}^{\infty} (i)^n(A/B) =$

$(i)\bigcap_{n=1}^{\infty}(i)^n(A/B)$. But as $\operatorname{ann}(A/B)\supseteq\bigcap_{n=1}^{\infty}(i)^n$, we may assume that $\bigcap_{n=0}^{\infty}(i)^n=0$, so that V is a valuation domain. Let $A=Va_1+\cdots+Va_s$ and assume that $\operatorname{ann}(a_1)\supseteq\cdots\supseteq\operatorname{ann}(a_s)$. We may assume that $(i)^n\supseteq\operatorname{ann}(a_1)$ (for otherwise $i^na_1=0$ for large n and hence we may assume that $A=Va_2+\cdots+Va_s$). Thus $0=\bigcap_{n=1}^{\infty}(i)^n\supseteq\operatorname{ann}(a_1)$, so that A is actually torsion-free. The result now follows from Lemma 1 **[1]**.

4. "Almost" Noetherian rings. Let R be a Noetherian ring, I an ideal in R, and A a finitely generated R-module. One version of the Krull Intersection Theorem states that $\bigcap_{n=1}^{\infty}I^nA=\{x\in A\mid(1-i)x=0\ \exists i\in I\}$. In fact, by Theorem 3 **[1]** this holds for R locally Noetherian and A locally finitely generated. In this section we consider to what extent the converse is true. We begin with the quasi-local case.

THEOREM 4.1. *Let (R,M) be a quasi-local ring whose maximal ideal M is finitely generated. Then the following statements are equivalent*:

(1) *R is Noetherian,*

(2) *$\bigcap_{n=1}^{\infty}M^nN=0$ for all finitely generated R-modules N,*

(3) *every finitely generated ideal of R has a primary decomposition,*

(4) *for finitely generated ideals A and B of R, there exists an integer n such that $(A+B^l)\cap(A:B^l)=A$ for $l\geqq n$,*

(5) *$\bigcap_{n=1}^{\infty}(M^n+A)=A$ for all finitely generated ideals A of R,*

(6) *$B=A+MB$ with A a finitely generated ideal of R implies $A=B$.*

Proof. The implications $(1)\Rightarrow(2)$ and $(1)\Rightarrow(3)$ are well known. Assume that (3) holds and let A and B be finitely generated ideals. Suppose that $A=Q_1\cap\cdots\cap Q_m$ where Q_i is P_i-primary. Assume that $B\subseteq P_i$ precisely for $i>k$. For $i\leqq k$, $(Q_i:B^n)B^n\subseteq Q_i$ and $B^n\not\subseteq P_i$ implies $(Q_i:B^n)=Q_i$ for all n. For $i>k$, there exists an integer n_i such that $B^{n_i}\subseteq Q_i$ because B is finitely generated. Set $n=\max\{n_i\}$. Then for $l\geqq n$, $A:B^l=Q_1\cap\cdots\cap Q_k$ and $A+B^l\subseteq Q_{k+1}\cap\cdots\cap Q_m$. Hence $A\subseteq(A:B^l)\cap(A+B^l)\subseteq Q_1\cap\cdots\cap Q_m=A$. Next we show that (4) implies (5). Let A be a finitely generated ideal of R. Clearly $A\subseteq\bigcap_{n=1}^{\infty}(M^n+A)$. Suppose that $x\in\bigcap_{n=1}^{\infty}(M^n+A)$. Then by (4) $A+(x)M=(A+(x)M+M^k)\cap((A+(x)M):M^k)$ for large k. But $x\in A+M^k$ so $A+(x)M=A+(x)$. Thus $x\in A$ by Nakayama's Lemma. Setting $N=R/A$ we see that (2) implies (5). As (6) holds in any (Noetherian) local ring, it remains to prove $(5)\Rightarrow(1)$ and $(6)\Rightarrow(1)$. Suppose that R is not Noetherian. Then there exists an ideal $P\neq M$ maximal with respect to not being finitely generated and P is necessarily prime. Let $z\in M-P$.

Then $P+(z)$ is finitely generated, say by $p_1+r_1z, \cdots, p_n+r_nz$ where $p_1, \cdots, p_n \in P$. We claim that $P=(p_1, \cdots, p_n)$. Let $p \in P \subseteq P+(z)$, so that

$$p = a_1(p_1+r_1z)+ \cdots + a_n(p_n+r_nz) =$$
$$= a_1p_1 + \cdots + a_np_n + (a_1r_1 + \cdots + a_nr_n)z.$$

Since P is a prime ideal and $z \notin P$, $a_1r_1+\cdots+a_nr_n \in P$. Hence $P = (p_1, \cdots, p_n)+Pz = (p_1, \cdots, p_n)+P^nZ^n$ for $n \geqq 1$. Thus either (5) or (6) implies that $P=(p_1, \cdots, p_n)$.

It is necessary to assume that M is finitely generated as is seen by the example $R = k[\{X_i\}_{i=1}^{\infty}]/(\{x_i\}_{i=1}^{\infty})^2$ where $k[\{x_i\}_{i=1}^{\infty}]$ is the polynomial ring over the field k in countably-many indeterminates. If we replace the quasi-local ring (R, M) with a quasi-semilocal ring $(R, M_1, \cdots, M_n)$ where $M_1, \cdots, M_n$ are finitely generated and replace M with $J = M_1 \cap \cdots \cap M_n$, then Theorem 4.1 remains true. The equivalence of (1) and (5) is a slight generalization of Exercise 4 [**5**, page 246]. Condition (4) has been studied in [**4**].

COROLLARY 4.2. *For a ring R the following statements are equivalent*:

(1) *R is locally Noetherian,*

(2) $\bigcap_{n=1}^{\infty}(M^n+A) = \{r \in R \mid (1-m)r \in A \ \exists m \in M\}$ *for all finitely generated ideals A of R and all maximal ideals M of R, and for every maximal ideal M of R, M_M is a finitely generated ideal in R_M.*

Proof. (1) $\Rightarrow$ (2). The first statement follows from Theorem 3 [**1**] applied to the ring R/A which is locally Noetherian. The second statement is obvious. (2) $\Rightarrow$ (1). Follows from the previous theorem.

THEOREM 4.3. *For a ring R the following conditions are equivalent*:

(1) *R is Noetherian,*

(2) *the maximal ideals of R are finitely generated and every finitely generated ideal of R has a primary decomposition.*

Proof. That (1) $\Rightarrow$ (2) is well-known. Therefore we may assume that R satisfies (2). It follows from Theorem 4.1 that R is locally Noetherian. Theorem 1.4 [**3**] gives that R is Noetherian.

The results of this section raise the question: Is a locally Noetherian ring whose maximal ideals are finitely generated necessarily Noetherian? The answer is no.

EXAMPLE 4.4. The ring $R = Z[\{x/p \mid p \text{ a prime}\}]$ is two dimensional, integrally closed, locally Noetherian with all maximal ideals finitely generated, but R is not Noetherian. In fact, R is not even a Krull domain.

This ring is given in **[6]** as an example of a locally polynomial ring over Z which is not a polynomial ring over Z. We wish to thank Professor R. Gilmer for pointing out this example to us.[1]

First, the ring R is not Noetherian because the ideal $(\{x/p \mid p$ a prime$\})$ is not finitely generated. The maximal ideals of R have the form $(p, f(x/p))$ where $p \in Z$ is prime and $f(x/p)$ is an irreducible polynomial (in x/p) mod p. The remaining statements follow from the fact that R localized at a maximal ideal M (with $M \cap Z = (p)$) is a localization of the polynomial ring $Z_{(p)}[x/p]$ at $M_{Z-(p)}$.

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[1] This example is due to P. Eakin, R. Gilmer, and W. Heinzer.

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