The Kumaraswamy Generalized Power Weibull Distribution

Mahmoud Ali Selim * and Abdullah Mahmoud Badr

[1] Department of Statistics, Faculty of Commerce, Al-Azher University, Egypt
& King Khalid University, Saudi Arabia

Abstract

A new family of distributions called Kumaraswamy-generalized power Weibull (Kgpw) distribution is proposed and studied. This family has a number of well known sub-models such as Weibull, exponentiated Weibull, Kumaraswamy Weibull, generalized power Weibull and new sub-models, namely, exponentiated generalized power Weibull, Kumaraswamy generalized power exponential distributions. Some statistical properties of the new distribution include its moments, moment generating function, quantile function and hazard function are derived. In addition, maximum likelihood estimates of the model parameters are obtained. An application as well as comparisons of the Kgpw and its sub-distributions is given.

Keywords: Generalized power Weibull distribution, Kumaraswamy distribution, Maximum likelihood estimation, Moment generating function, Hazard rate function.

1. Introduction

In reliability models, the probability distributions are most often used as time to failure distributions. In same context, the reliability model quality significantly depends on the success in selecting appropriate probability distribution of the phenomenon under discussion. During the past decades, a specific group of the classical distributions such as, exponential, Weibull and Rayleigh distributions were used for modeling lifetime data. However, in practice, we find that most of these distributions are not flexible enough to accommodate different phenomena. For this reason, the statisticians have worked on development and extend of these distributions to become more flexible and more suited for modeling data in practice. The traditional Weibull distribution by Waloddi Weibull (1951) is one of the most used lifetime distributions for modeling lifetime data. However, the Weibull distribution does not provide non-monotone failure rates that are common in reliability and survival analysis. Many versions of generalized Weibull distribution have arisen out of the need to improve its properties. The first generalization of Weibull distribution provides bathtub shaped hazard rate is the exponentiated Weibull distribution due to Mudholkar et al. (1995). The exponentiated Weibull distribution can be used quite effectively to analyze the lifetime data in place of Weibull distribution. Also, Nikulin and Haghighi (2006) proposed a new generalization of the Weibull distribution by introducing an additional shape parameter, which they called the generalized power Weibull distribution. The random variable X has the generalized power Weibull (gpw) distribution if its cdf and pdf are

$$G_{gpw}(x) = 1 - \exp\left\{1 - \left(1 + \left(\frac{x}{\lambda}\right)^{\alpha}\right)^{\theta}\right\}, \qquad \alpha, \lambda, \theta > 0, \ \ x > 0 \tag{1}$$

and

$$g_{gpw}(x) = \frac{\alpha\theta}{\lambda^{\alpha}} x^{\alpha-1}\left(1 + \left(\frac{x}{\lambda}\right)^{\alpha}\right)^{\theta-1} \exp\left\{1 - \left(1 + \left(\frac{x}{\lambda}\right)^{\alpha}\right)^{\theta}\right\}, \ \ \alpha, \lambda, \theta > 0, \ \ x > 0 \tag{2}$$

where λ is a scale parameter and α and θ are two shape parameters. It is reduced to the standard Weibull distribution when, $\theta = 1$. Nikulin and Haghighi (2006) showed that the hazard rate function of the generalized power Weibull distribution has nice and flexible properties and can be constant, monotone and non-monotone shaped. This distribution is often used for constructing accelerated failures times models that describe dependence of the lifetime distribution on explanatory variables. They also illustrated that the gpw provides a good fit to the well-known randomly censored survival times data for patients at arm A of the head-and-neck cancer clinical trial by using a chi-squared goodness-of-fit test.

Eugene et al. (2002) introduced the beta-generated family to generalize the continuous probability distribution as follows

$$F_B(x) = \frac{1}{B(a,b)} \int_0^{G(x)} u^{a-1}(1-u)^{b-1} du \quad (3)$$

the corresponding pdf of the beta generated distribution is

$$f(x) = \frac{1}{B(a,b)} g(x) G(x)^{a-1} [1-G(x)]^{b-1} \quad (4)$$

where $B(\alpha, \beta)$ is the beta function, g(x) and G(x) are the pdf and cdf for parent probability distribution. The formula given in (4) has been used first by Eugene et al. (2002) to generate the beta–normal distribution. After that, number of authors proposed a new beta-generated family of distributions include Famoye et al. (2005) introduced the beta-Weibull distribution, Nadarajah and Kotz (2004) introduced the beta Gumbel distribution Nadarajah and Kotz (2006) introduced the beta exponential distribution, Kong et al. (2007) introduced the beta-gamma distribution, Cordeiro and Lemonte (2011) introduced beta Laplace distribution. Mameli and Musio (2013) introduced the beta skew-normal distribution. Recently, Merovci and Sharma (2014) introduced beta-Lindley distribution, Jafari et al. (2014) introduced beta-Gompertz distribution, Chukwu and Ogunde (2015) introduced Beta Mekaham distribution and MirMostafaee et al. (2015) introduced beta Lindley distribution. For a good review of beta- generated distributions, one may refer to Lee et al. (2013).

Kumaraswamy (1980) proposed a two-parameter distribution on (0, 1), so-called Kumaraswamy distribution, and denoted by *Kum(a, b)*. Its cumulative distribution function (cdf) is

$$F_{kum}(x) = 1 - (1-x^a)^b \ , \quad 0 < x < 1 \quad (5)$$

and its density function is

$$f_{kum}(x) = abx^{a-1}(1-x^a)^{b-1} \ , \quad 0 < x < 1 \quad (6)$$

Kum(a, b) distribution, according to Jones (2009) like the beta distribution, can be unimodal, uniantimodal, increasing, decreasing or constant and has advantage over the beta distribution that, Kum(a, b) distribution does not involve any special function such as the beta function and its cumulative distribution function has a simple closed form. For this reasons, Cordeiro and de Castro (2011) developed the beta-generated family by employing the Kumaraswamy distribution instead of beta distribution. For an arbitrary baseline cdf G(x), Cordeiro and de Castro (2011) defined the cdf and pdf of Kumaraswamy generalized distributions (Kum-G), respectively, as follow

$$F(x) = 1 - \{1 - G(x)^a\}^b \quad (7)$$

and

$$f(x) = abg(x)G(x)^{b-1}\{1 - G(x)^a\}^{b-1} \quad (8)$$

where $g(x) = dG(x)/dx$ and $a > 0$ and $b > 0$ are two additional shape parameters of the G(x) distribution which role are to govern skewness and tail weights of the generated distribution. This type of generalizations contains distributions with unimodal and bathtub shaped hazard rate functions and have some desirable structural properties compared with beta-generated family of distributions, for detail see Cordeiro and de Castro (2011) and Jones (2009). Several generalized distributions from (5) have been studied in the literature including, the Kw-Weibull distribution by Cordeiro et al. (2010), Kw-Gumbel distribution by Cordeiro et al. (2011), Kw-generalized gamma distribution by Pascoa et al. (2011), Kw-log-logistic distribution by Tiago et. al. (2012), Kw-modified Weibull distribution by Cordeiro et. al. (2012), Recently, Gosh (2014) introduced Kw-half-Cauchy distribution, Antonio at. el. (2014) introduced Kw-generalized Rayleigh distribution and (Rocha et al. 2015) introduced Kw- Gompertz distribution.

In this paper, we apply the works of Kumaraswamy (1980), Cordeiro and de Castro (2011), Nikulin and Haghighi (2006) in order to study the mathematical properties of a new distribution referred to as the Kumaraswamy generalized power Weibull (Kgpw) distribution. The rest of the article is organized as follows. Section 2 introduces the Kumaraswamy generalized power Weibull distribution. Some statistical properties of the Kumaraswamy generalized power Weibull distribution are discussed in Section 3. The pdf of order statistics of the Kgpw model is introduced in Section 4. Maximum likelihood estimation is investigated in Section 5. In Section 6, real data set are used to illustrate the usefulness of the Kgpw model. Concluding comments are given in Section 7.

2. The Kumaraswamy Generalized Power Weibull Distribution

In this section, we introduce the pdf and the cdf of Kgpw distribution by setting the gpw baseline functions (1) and (2) in Equations (5) and (6), then the cdf and pdf of the Kgpw distribution are obtained as follow

$$F_{kgpw}(x) = 1 - \left[1 - \left(1 - e^{1-\left(1+\left(\frac{x}{\lambda}\right)^{\alpha}\right)^{\theta}}\right)^{a}\right]^{b}, \qquad a, b, \alpha, \lambda, \theta > 0, \;\; x > 0 \tag{9}$$

and

$$f_{kgpw}(x) = \frac{ab\alpha\theta}{\lambda^{\alpha}} x^{\alpha-1}\left(1 + \left(\frac{x}{\lambda}\right)^{\alpha}\right)^{\theta-1} e^{1-\left(1+\left(\frac{x}{\lambda}\right)^{\alpha}\right)^{\theta}}\left[1 - e^{1-\left(1+\left(\frac{x}{\lambda}\right)^{\alpha}\right)^{\theta}}\right]^{a-1}$$
$$\times\left\{1 - \left[1 - e^{1-\left(1+\left(\frac{x}{\lambda}\right)^{\alpha}\right)^{\theta}}\right]^{a}\right\}^{b-1} \tag{10}$$

where λ is a scale parameter and the others parameters a, b, α and θ are shape parameters. The possible shapes of the pdf and cdf of Kgpw distribution are provided for five combinations of the parameters in Figure 1 and Figure 2, respectively. The shapes in Figure 1, show that the pdfs of Kgpw distribution can be monotonically decreasing or positively skewed.

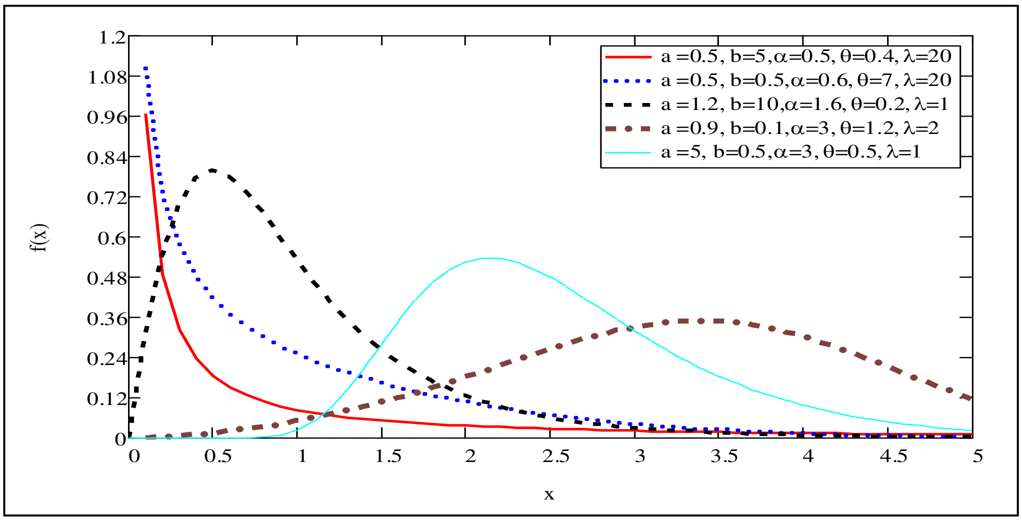


Figure (1): Some Possible Shapes of the Kgpw Density Function

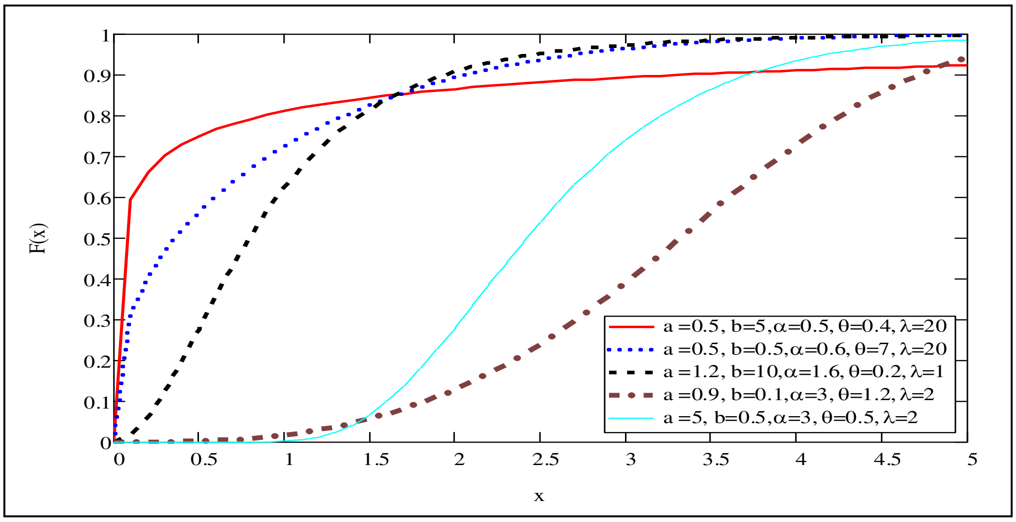


Figure (2): Some Possible Shapes of the Kgpw Cumulative Density Function

2.1 Some sub-models of the Kgpw

The Kgpw distribution is very flexible seeing as this distribution includes several well-known distributions as sub-models based on special values of the parameters $(a, b, \theta$ and $\alpha)$. These sub-models are

1) Setting $a = b = 1$, we obtain generalized power Weibull distribution (Nikulin and Haghighi 2006) with cdf:

$$F(x) = 1 - e^{1-\left(1+\left(\frac{x}{\lambda}\right)^{\alpha}\right)^{\theta}},$$

2) Setting $b = 1$, we obtain exponentiated generalized power Weibull distribution (new) with cdf:

$$F(x) = \left(1 - e^{1-\left(1+\left(\frac{x}{\lambda}\right)^{\alpha}\right)^{\theta}}\right)^{a},$$

3) Setting $\theta = 1$, we obtain Kumaraswamy Weibull distribution distribution (cordiro et. al. 2010) with cdf:

$$F(x) = 1 - \left(1 - \left(1 - e^{-\left(\frac{x}{\lambda}\right)^{\alpha}}\right)^{a}\right)^{b},$$

4) Setting $\alpha = 1$ we obtain Kumaraswamy generalized power exponential distribution (new) with cdf:

$$F(x) = 1 - \left(1 - \left(1 - e^{1-\left(1+\frac{x}{\lambda}\right)^{\theta}}\right)^{a}\right)^{b},$$

5) Setting $b = \theta = 1$, we obtain exponentiated Weibull distribution (Mudholkar et al.(1995)) with cdf:

$$F(x) = \left(1 - e^{-\left(\frac{x}{\lambda}\right)^{\alpha}}\right)^{a},$$

6) Setting $\theta = a = b = 1$ we obtain Weibull distribution with cdf:

$$F(x) = 1 - e^{-\left(\frac{x}{\lambda}\right)^{\alpha}},$$

7) Setting $\alpha = 1, b = 1$ we obtain exponentiated generalized power exponential distribution (new) with cdf:

$$F(x) = \left(1 - e^{1-\left(1+\frac{x}{\lambda}\right)^{\theta}}\right)^{a},$$

8) Setting $\alpha = \theta = 1$ we obtain Kumaraswamy exponential distribution with cdf:

$$F(x) = 1 - \left(1 - \left(1 - e^{-\frac{x}{\lambda}}\right)^{a}\right)^{b},$$

9) Setting $b = \theta = \alpha = 1$, we obtain exponentiated exponential distribution (Gupta and Kundu (2001)) with cdf:

$$F(x) = \left(1 - e^{-\frac{x}{\lambda}}\right)^{a},$$

10) Setting $\alpha = a = b = 1$, we obtain the exponential extension distribution (Nadarajah and Haghighi (2011)) with cdf:

$$F(x) = 1 - e^{1-\left(1+\frac{x}{\lambda}\right)^{\theta}},$$

11) Setting $\alpha = \theta = a = b = 1$, we obtain exponential distribution with cdf:

$$F(x) = 1 - e^{-\frac{x}{\lambda}},$$

12)Setting $\theta = 1, \alpha = 2$, we obtain Kumaraswamy Burr type X distribution (NEW) with cdf:

$$F(x) = 1 - \left(1 - \left(1 - e^{-\left(\frac{x}{\lambda}\right)^2}\right)^a\right)^b,$$

13) Setting $b = \theta = 1, \alpha = 2$ we obtain Burr type X distribution with cdf:

$$F(x) = \left(1 - e^{-\left(\frac{x}{\lambda}\right)^2}\right)^a.$$

2.2 Expansions for the cumulative and density functions

The expansion for the cumulative distribution function of Kgpw can be derived by using the generalized binomial theorem. For any real number $r > 0$ and $|z| < 1$ the binomial expansion is

$$(1-z)^r = \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} z^i \tag{11}$$

where $\binom{r}{i} = \frac{r(r-1)\ldots(r-i+1)}{i!}$.

Using the binomial expansion (11) in equation (9), we get the cdf as a power series expansion as follows

$$F(x) = 1 - \sum_{i=0}^{\infty} p_i \, G_{gpw}(x)^{ai} \tag{12}$$

where $p_i = (-1)^i \binom{b}{i}$ and $G_{gpw}(x)$ denotes the gpw cumulative distribution with parameters α, λ and θ . Which means that, $G_{gpw}(x)^{ai}$ denotes the cdf of exponentiated generalized power Weibull (egpw) with parameters α, λ, θ and ai. Using the binomial expansion (11), again in the last term of (12), we get

$$F(x) = 1 - \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j} \binom{b}{i} \binom{ai}{j} e^{j\left(1-\left(1+\left(\frac{x}{\lambda}\right)^{\alpha}\right)^{\theta}\right)} \tag{13}$$

Differentiating (13) with respect to x gives the expansion of pdf as follow

$$f_{kgpw}(x) = \frac{\alpha\theta}{\lambda^{\alpha}} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j} \binom{b}{i} \binom{ai}{j} j \, x^{\alpha-1} \left(1 + \left(\frac{x}{\lambda}\right)^{\alpha}\right)^{\theta-1} e^{j\left(1-\left(1+\left(\frac{x}{\lambda}\right)^{\alpha}\right)^{\theta}\right)} \tag{14}$$

2.3 The hazard and survival functions

Failure rates or hazard rates are important subject in the industry, engineered system, finance and fundamental to the plan of social security, medical insurance and safe systems in a wide variety of applications. The hazard rate function (hrf) of the random variable *T* that has the Kgpw is given by

$$h(t) = \frac{ab\theta\alpha t^{\alpha-1} \left(1 + \left(\frac{t}{\lambda}\right)^{\alpha}\right)^{\theta-1} e^{1-\left(1+\left(\frac{t}{\lambda}\right)^{\alpha}\right)^{\theta}} \left[1 - e^{1-\left(1+\left(\frac{t}{\lambda}\right)^{\alpha}\right)^{\theta}}\right]^{a-1}}{\lambda^{\alpha} \left(1 - \left(1 - e^{1-\left(1+\left(\frac{t}{\lambda}\right)^{\alpha}\right)^{\theta}}\right)^{a}\right)}, \quad t > 0 \tag{15}$$

Using the expansions in (13) and (14), the hrf of the Kgpw distribution in (15) can be expressed in the mixture form as follows

$$h(t) = \frac{\alpha\theta \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j} \binom{b}{i} \binom{ai}{j} j \, t^{\alpha-1} \left(1 + \left(\frac{t}{\lambda}\right)^{\alpha}\right)^{\theta-1} e^{j\left(1-\left(1+\left(\frac{t}{\lambda}\right)^{\alpha}\right)^{\theta}\right)}}{\lambda^{\alpha} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j} \binom{b}{i} \binom{ai}{j} e^{j\left(1-\left(1+\left(\frac{t}{\lambda}\right)^{\alpha}\right)^{\theta}\right)}}, \quad t > 0$$

Note that for all b, θ, λ we have

$$h(0)=\begin{cases}0 & for\ a,\alpha>1\\ \dfrac{b\theta}{\lambda} & for\ a=\alpha=1\\ \infty & for\ a,\alpha<1\end{cases}$$

The hazard rate function of Kgpw distribution can be have various shapes, including constant, monotonically decreasing or increasing, bathtub and upside down bathtub. More specifically, the hazard rate curve is

(a) monotone increasing if either $a\alpha>1$ and $\alpha\theta>1$ or $a\alpha=1$ and $\theta>1$,
(b) monotone decreasing if either $0<a\alpha<1$ and $\alpha\theta<1$ or $0<a\alpha<1$ and $\alpha\theta=1$,
(c) unimodal (inverted bathtub shaped) if $a\alpha>1$ and $0<\alpha\theta<1$,
(d) bathtub shaped if $0<a\alpha<1$ and $\alpha\theta>1$,
(e) constant, $h(t)=\frac{b}{\lambda}$ if $a=\alpha=\theta=1$.

Figure 3, provides plots of the hazard function of Kgpw distribution for some selected parameters values. These plots show flexibility of hazard rate function that makes the Kgpw hazard rate function useful and suitable for non-monotone hazard behaviors that are more likely to be observed in real life situations.

In reliability theory there are several important functions such as the survival function *s(t)*, reverse hazard function *r(t)* and the cumulative hazard rate function *H(t)*. These functions corresponding of the Kgpw distribution, take the following forms:

$$s(t)=1-F(x)=\left(1-\left(1-e^{1-\left(1+\left(\frac{t}{\lambda}\right)^{\alpha}\right)^{\theta}}\right)^{a}\right)^{b},\quad t>0, \tag{16}$$

$$r(t)=\frac{ab\theta\alpha t^{\alpha-1}\left(1+\left(\frac{t}{\lambda}\right)^{\alpha}\right)^{\theta-1}e^{1-\left(1+\left(\frac{t}{\lambda}\right)^{\alpha}\right)^{\theta}}\left[1-e^{1-\left(1+\left(\frac{t}{\lambda}\right)^{\alpha}\right)^{\theta}}\right]^{a-1}\left\{1-\left[1-e^{1-\left(1+\left(\frac{t}{\lambda}\right)^{\alpha}\right)^{\theta}}\right]^{a}\right\}^{b-1}}{\lambda^{\alpha}\left[1-\left(1-\left(1-e^{1-\left(1+\left(\frac{t}{\lambda}\right)^{\alpha}\right)^{\theta}}\right)^{a}\right)^{b}\right]}\quad t>0 \tag{17}$$

and

$$H(t)=-b\ln\left(1-\left(1-e^{1-\left(1+\left(\frac{t}{\lambda}\right)^{\alpha}\right)^{\theta}}\right)^{a}\right),\quad t>0 \tag{18}$$

respectively.

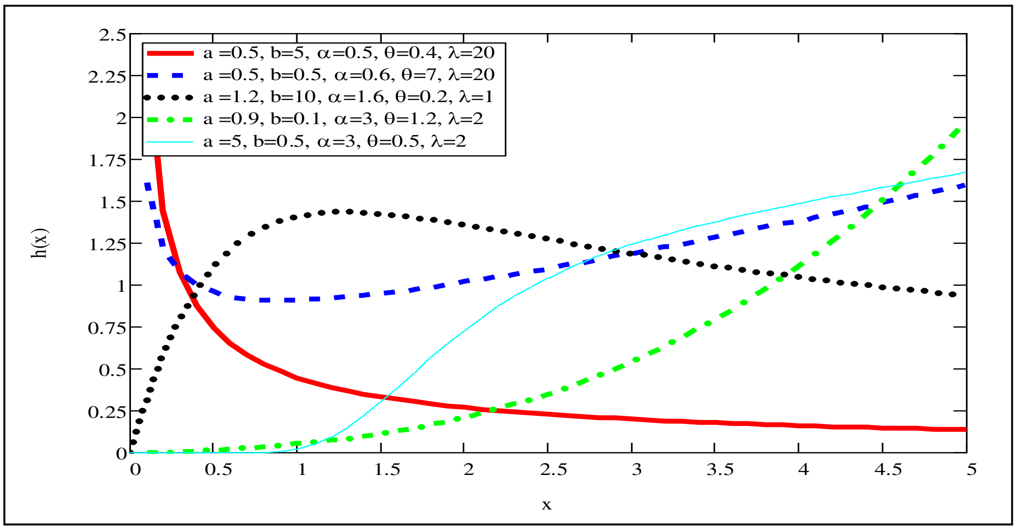


Figure (3): Some Possible Shapes of the Kgpw Hazard Rate Function

3. The Statistical Properties of Kgpw Distribution

In this section, we present the quantile function, the moments and the moment generating function, skewness, kurtosis and random variables generation for the Kgpw distribution.

3.1 Quantile function

There are several measures for location and dispersion such as median, the interquartile range, the quartiles, the skewness and the kurtosis can be obtained by using the quantile function. The definition of the q-th quantile is the real solution of the following equation

$F(x_q) = q,$ where $0 \le q \le 1$

Thus, the quantile function *Q(q)* corresponding of the Kgpw distribution is

$$Q(q) = \lambda\left\{\left(1 - ln\left[1 - \left(1 - (1-q)^{1/b}\right)^{1/a}\right]\right)^{1/\theta} - 1\right\}^{1/\alpha} \tag{19}$$

The median *M(x)* of Kgpw distribution can be obtained from previous function, by setting $q = 0.5$, as follows

$$M(X) = \lambda\left\{\left(1 - ln\left[1 - \left(1 - (0.5)^{1/b}\right)^{1/a}\right]\right)^{1/\theta} - 1\right\}^{1/\alpha} \tag{20}$$

Also, the quartiles of the Kgpw distribution can be obtained by putting $q = 0.25$ and $q = 0.75$ in (19).

3.2 Skewness and kurtosis

The statistical measures of skewness and kurtosis play important role in describing shape characteristics of the probability distributions. The Bowley's skewness measure based on quartiles (Kenney and Keeping, (1962)) is given by

$$Sk = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)} \tag{21}$$

and the Moors' kurtosis measure based on octiles (Moors (1988)) is given by

$$Ku = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)} \tag{22}$$

The previous measures Sk and Ku have a number of advantages compared to the classical measures of skewness and kurtosis, e.g. they are less sensitive to outliers and they exist for the distributions even without defined the moments.

3.3 Random variables generation

The quantile function of the Kgpw has a closed form, which makes the simulation from this distribution easier. When the parameters a, b, α, λ and θ are known, we can generate Kgpw random variables from the quantile function (19) as follows

$$X = \lambda\left\{\left(1 - ln\left[1 - \left(1 - (1-u)^{1/b}\right)^{1/a}\right]\right)^{1/\theta} - 1\right\}^{1/\alpha} \tag{23}$$

where, u is generated number from the Uniform distribution (0, 1).

3.4 Moments and moment generating function

If X has the Kgpw distribution, the moments and moment generating function are given by the following theorems

Theorem 1. If X is a random variable having the pdf (2), For integer value of r/α, the r-th moment about zero can be determined as

$$E(X^r) = \lambda^r \varphi_{ijk} \frac{e^j}{j^{\frac{k}{\theta}}} \Gamma\left(\frac{k}{\theta}+1, j\right) \tag{24}$$

where, $\varphi_{ijk} = \sum_{i=1}^{b} \sum_{j=1}^{ai} \sum_{k=0}^{r/\alpha} (-1)^{i+j+\frac{r}{\alpha}-k} \binom{b}{i}\binom{ai}{j}\binom{r/\alpha}{k}$.

Proof. From definition, the r-th moment of Kgpw distribution is

$$E(X^r) = \int_0^\infty x^r \, f_{kgpw}(x) dx \tag{25}$$

By setting $f_{Kgpw}(\mathrm{x})$ from (14) in previous equation yields

$$E(X^r) = \frac{\alpha\theta}{\lambda^\alpha} \sum_{i=0}^{b-1} \sum_{j=0}^{a(i+1)-1} (-1)^{i+j} \binom{b}{i}\binom{ai}{j} j \int_0^\infty x^{r+\alpha-1} \left(1+\left(\frac{x}{\lambda}\right)^\alpha\right)^{\theta-1} e^{j\left(1-\left(1+\left(\frac{x}{\lambda}\right)^\alpha\right)^\theta\right)} dx \tag{26}$$

Substitution $v^{\frac{1}{\theta}} = 1 + \left(\frac{x}{\lambda}\right)^\alpha$ in the previous equation and after that using the binomial expansion, we get

$$E(X^r) = \lambda^r \sum_{i=1}^{b} \sum_{j=1}^{ai} \sum_{k=0}^{r/\alpha} (-1)^{i+j+\frac{r}{\alpha}-k} \binom{b}{i}\binom{ai}{j}\binom{r/\alpha}{k} j e^j \int_1^\infty v^{\frac{k}{\theta}} e^{-jv} \, dv \tag{27}$$

The last term is $\Gamma\left(\frac{k}{\theta}+1, j\right)/j^{\frac{k}{\theta}+1}$. Therefore, (27) can be reduced to (24).

Theorem 2. If X is a random variable having the pdf (2), for integer value of r/α the moment generating function is

$$M_x(t) = \lambda^r \sum_{r=0}^{\infty} \frac{t^r}{r!} \varphi_{ijk} \frac{e^j \Gamma\left(\frac{k}{\theta}+1, j\right)}{j^{\frac{k}{\theta}}} \tag{28}$$

Proof. The moment generating function of Kgpw distribution is given by

$$M_x(t) = \int_0^\infty e^{tx} \, f_{kgpw}(x) dx$$

Using the fact that $e^{tx} = \sum_{r=0}^{\infty} \frac{(tx)^r}{r!}$, we get

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r) \tag{29}$$

Inserting (24) in equation (29) yields the mgf of Kgpw in (28).

The numerical values of mean and variance for various choices of parameters are given in Table 1and Table 2, respectively.

Table 1: Mean of the Kgpw Distribution for Some Values of a, b, θ, α and $\lambda = 3$

		θ=1.5			θ=2.5			θ=3.5		
a	b	α=1.5	α=2.5	α=3.5	α=1.5	α=2.5	α=3.5	α=1.5	α=2.5	α=3.5
1	1	1.85	2.135	2.312	1.213	1.666	1.94	0.937	1.43	1.741
	2	1.218	1.657	1.927	0.824	1.315	1.636	0.644	1.137	1.475
	3	0.947	1.422	1.727	0.649	1.137	1.474	0.51	0.986	1.332
	4	0.789	1.275	1.597	0.545	1.023	1.366	0.43	0.889	1.236
	5	0.685	1.17	1.502	0.475	0.942	1.287	0.376	0.819	1.165
2	1	2.483	2.614	2.697	1.602	2.017	2.245	1.23	1.724	2.007
	2	1.874	2.212	2.395	1.244	1.734	2.015	0.966	1.492	1.81
	3	1.599	2.012	2.239	1.075	1.589	1.893	0.839	1.371	1.704
	4	1.432	1.884	2.137	0.97	1.494	1.812	0.759	1.291	1.633
	5	1.315	1.791	2.062	0.896	1.425	1.752	0.703	1.233	1.58
3	1	2.845	2.858	2.883	1.816	2.191	2.386	1.388	1.867	2.13
	2	2.264	2.498	2.62	1.484	1.943	2.192	1.147	1.666	1.964
	3	1.998	2.32	2.486	1.326	1.817	2.089	1.029	1.562	1.876
	4	1.835	2.205	2.398	1.227	1.735	2.021	0.955	1.494	1.817
	5	1.72	2.122	2.334	1.156	1.674	1.971	0.902	1.444	1.773
4	1	3.092	3.016	3	1.96	2.3	2.474	1.493	1.956	2.204
	2	2.533	2.682	2.76	1.646	2.075	2.299	1.267	1.775	2.057
	3	2.277	2.518	2.639	1.497	1.961	2.209	1.157	1.682	1.979
	4	2.118	2.413	2.561	1.402	1.886	2.149	1.088	1.62	1.928
	5	2.007	2.337	2.503	1.335	1.832	2.104	1.037	1.575	1.89
5	1	3.278	3.13	3.082	2.065	2.378	2.535	1.57	2.02	2.256
	2	2.736	2.815	2.859	1.766	2.168	2.374	1.356	1.851	2.121
	3	2.488	2.661	2.747	1.623	2.063	2.291	1.252	1.765	2.051
	4	2.334	2.562	2.675	1.533	1.994	2.237	1.185	1.709	2.004
	5	2.225	2.491	2.621	1.469	1.944	2.196	1.138	1.668	1.969

Table 1 indicates that, the mean of Kgpw distribution is decreasing when b and θ increasing with fixed the others parameters. While, the mean of Kgpw distribution is increasing when increasing a or α with fixed the others parameters.

Table 2: Variance of the Kgpw Distribution for Some Values of *a*, *b*, *θ*, *α* and *λ*= 3

		θ=1.5			θ=2.5			θ=3.5		
a	b	α=1.5	α=2.5	α=3.5	α=1.5	α=2.5	α=3.5	α=1.5	α=2.5	α=3.5
1	1	1.287	0.707	0.46	0.474	0.379	0.288	0.266	0.265	0.221
	2	0.597	0.451	0.338	0.246	0.262	0.226	0.145	0.189	0.178
	3	0.372	0.342	0.278	0.162	0.205	0.191	0.097	0.15	0.153
	4	0.264	0.279	0.241	0.118	0.171	0.169	0.072	0.126	0.135
	5	0.201	0.237	0.215	0.092	0.147	0.152	0.056	0.109	0.123
2	1	1.177	0.504	0.286	0.399	0.249	0.166	0.216	0.168	0.123
	2	0.619	0.335	0.21	0.235	0.18	0.13	0.133	0.125	0.099
	3	0.432	0.267	0.176	0.172	0.149	0.113	0.1	0.105	0.087
	4	0.338	0.228	0.157	0.139	0.13	0.103	0.081	0.093	0.08
	5	0.28	0.203	0.143	0.118	0.117	0.095	0.069	0.085	0.075
3	1	1.071	0.407	0.217	0.345	0.192	0.12	0.182	0.126	0.087
	2	0.581	0.268	0.155	0.208	0.137	0.092	0.115	0.093	0.068
	3	0.418	0.215	0.13	0.157	0.113	0.079	0.089	0.079	0.06
	4	0.335	0.185	0.115	0.13	0.1	0.072	0.074	0.07	0.055
	5	0.284	0.166	0.106	0.113	0.091	0.067	0.065	0.064	0.051
4	1	0.993	0.35	0.18	0.308	0.159	0.096	0.16	0.104	0.069
	2	0.543	0.228	0.126	0.186	0.112	0.072	0.101	0.075	0.053
	3	0.396	0.182	0.104	0.142	0.092	0.061	0.079	0.063	0.046
	4	0.321	0.157	0.092	0.119	0.082	0.055	0.067	0.056	0.042
	5	0.275	0.141	0.084	0.104	0.074	0.051	0.059	0.051	0.039
5	1	0.934	0.312	0.157	0.281	0.139	0.082	0.144	0.089	0.058
	2	0.512	0.201	0.107	0.17	0.096	0.06	0.091	0.063	0.043
	3	0.375	0.16	0.088	0.13	0.079	0.051	0.071	0.053	0.037
	4	0.305	0.138	0.078	0.109	0.069	0.046	0.06	0.047	0.034
	5	0.263	0.123	0.071	0.096	0.063	0.042	0.053	0.043	0.031

Table 2 indicates that, the variance of Kgpw distribution will decrease with increase the parameters values.

4. Order Statistics

Suppose $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ denote the order statistics of a random sample of size n drawn from a continuous distribution with cdf $F(x)$ and pdf $f(x)$, then the pdf of $X_{(l)}$ is given by

$$f_{l:n}(x)=\frac{n!}{(n-l)!\,(r-l)!}f(x)[F(x)]^{l-1}[1-F(x)]^{n-l} \tag{30}$$

New, If X is a random variable following Kgpw distribution then, by substituting F(x) and f(x) in eqs (9) and (10) in to eq (30), we get the Kgpw density of the l-th order statistics as follows

$$f_{l:n}(x)=\frac{n!}{(n-l)!\,(l-1)!}\frac{ab\alpha\theta}{\lambda^{\alpha}}x^{\alpha-1}\left(1+\left(\frac{x}{\lambda}\right)^{\alpha}\right)^{\theta-1}e^{1-\left(1+\left(\frac{x}{\lambda}\right)^{\alpha}\right)^{\theta}}\left[1-e^{1-\left(1+\left(\frac{x}{\lambda}\right)^{\alpha}\right)^{\theta}}\right]^{a-1}$$
$$\times\left\{1-\left[1-\left[1-e^{1-\left(1+\left(\frac{x}{\lambda}\right)^{\alpha}\right)^{\theta}}\right]^{a}\right]^{b}\right\}^{l-1}\left\{1-\left[1-e^{1-\left(1+\left(\frac{x}{\lambda}\right)^{\alpha}\right)^{\theta}}\right]^{a}\right\}^{b(n-l+1)-1} \tag{31}$$

when $l=1$ and when $l=n$, the pdf of order statistics become

$$f_{1:n}(x)=n\frac{ab\alpha\theta}{\lambda^{\alpha}}x^{\alpha-1}\left(1+\left(\frac{x}{\lambda}\right)^{\alpha}\right)^{\theta-1}e^{1-\left(1+\left(\frac{x}{\lambda}\right)^{\alpha}\right)^{\theta}}\left[1-e^{1-\left(1+\left(\frac{x}{\lambda}\right)^{\alpha}\right)^{\theta}}\right]^{a-1}$$
$$\times\left\{1-\left[1-e^{1-\left(1+\left(\frac{x}{\lambda}\right)^{\alpha}\right)^{\theta}}\right]^{a}\right\}^{nb-1} \tag{32}$$

and

$$f_{n:n}(x)=n\frac{ab\alpha\theta}{\lambda^{\alpha}}x^{\alpha-1}\left(1+\left(\frac{x}{\lambda}\right)^{\alpha}\right)^{\theta-1}e^{1-\left(1+\left(\frac{x}{\lambda}\right)^{\alpha}\right)^{\theta}}\left[1-e^{1-\left(1+\left(\frac{x}{\lambda}\right)^{\alpha}\right)^{\theta}}\right]^{a-1}$$
$$\times\left\{1-\left[1-\left[1-e^{1-\left(1+\left(\frac{x}{\lambda}\right)^{\alpha}\right)^{\theta}}\right]^{a}\right]^{b}\right\}^{n-1}\left\{1-\left[1-e^{1-\left(1+\left(\frac{x}{\lambda}\right)^{\alpha}\right)^{\theta}}\right]^{a}\right\}^{b-1} \tag{33}$$

respectively.

5. Maximum Likelihood Estimation

In this section, we determine the maximum likelihood estimates ($MLEs$) for the parameters $\sigma=(a,b,\alpha,\lambda,\theta)$ of the Kgpw distribution. Let $x_1,x_1,\ldots,x_n$ be a complete random sample of size n from the Kgpw distribution. The likelihood function (LF) is given by

$$L(\sigma|x)=\left(\frac{ab\theta\alpha}{\lambda^{\alpha}}\right)^{n}\prod_{i=1}^{n}{x_i}^{\alpha-1}\left(1+\left(\frac{x_i}{\lambda}\right)^{\alpha}\right)^{\theta-1}e^{1-\left(1+\left(\frac{x_i}{\lambda}\right)^{\alpha}\right)^{\theta}}\left[1-e^{1-\left(1+\left(\frac{x_i}{\lambda}\right)^{\alpha}\right)^{\theta}}\right]^{a-1}$$
$$\times\left\{1-\left[1-e^{1-\left(1+\left(\frac{x_i}{\lambda}\right)^{\alpha}\right)^{\theta}}\right]^{a}\right\}^{b-1} \tag{34}$$

and the log-likelihood function ($logL$) is given by

$$logL=n\ln\left(\frac{ab\theta\alpha}{\lambda^{\alpha}}\right)+n+(\alpha-1)\sum_{i=1}^{n}\ln x_i+(\theta-1)\sum_{i=1}^{n}\ln\left(1+\left(\frac{x_i}{\lambda}\right)^{\alpha}\right)-\sum_{i=1}^{n}\left(1+\left(\frac{x_i}{\lambda}\right)^{\alpha}\right)^{\theta}$$
$$+(a-1)\sum_{i=1}^{n}\ln\left(1-e^{1-\left(1+\left(\frac{x_i}{\lambda}\right)^{\alpha}\right)^{\theta}}\right)+(b-1)\sum_{i=1}^{n}\ln\left[1-\left(1-e^{1-\left(1+\left(\frac{x_i}{\lambda}\right)^{\alpha}\right)^{\theta}}\right)^{a}\right] \tag{35}$$

The derivatives of the $logL$ with respect to the unknown parameters a,b,α,λ and θ are

$$\frac{\partial\ln L}{\partial a}=\frac{n}{a}+\sum_{i=1}^{n}\ln\left(1-\omega_{\theta,\alpha,\lambda}\right)-(b-1)\sum_{i=1}^{n}\frac{\ln\left(1-\omega_{\theta,\alpha,\lambda}\right)\left(1-\omega_{\theta,\alpha,\lambda}\right)^{a}}{1-\left(1-\omega_{\theta,\alpha,\lambda}\right)^{a}} \tag{36}$$

$$\frac{\partial \ln L}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} \ln\left(1 - \left(1 - \omega_{\theta,\alpha,\lambda}\right)^{a}\right) \tag{37}$$

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} - n\ln\lambda + \sum_{i=1}^{n} \ln x_i + (\theta - 1)\sum_{i=1}^{n} \frac{\ln\left(\frac{x_i}{\lambda}\right)\left(\frac{x_i}{\lambda}\right)^{\alpha}}{\varphi_{\alpha,\lambda}} - \theta\sum_{i=1}^{n} \ln\left(\frac{x_i}{\lambda}\right)\left(\frac{x_i}{\lambda}\right)^{\alpha} {\varphi_{\alpha,\lambda}}^{\theta-1}$$
$$\times\left\{1 - \frac{(a-1)\omega_{\theta,\alpha,\lambda}}{1-\omega_{\theta,\alpha,\lambda}} + \frac{a(b-1)\left(1-\omega_{\theta,\alpha,\lambda}\right)^{a-1}\omega_{\theta,\alpha,\lambda}}{1-\left(1-\omega_{\theta,\alpha,\lambda}\right)^{a}}\right\} \tag{38}$$

$$\frac{\partial \ln L}{\partial \lambda} = -\frac{\alpha n}{\lambda} + \frac{\alpha\theta}{\lambda^{\alpha+1}}\sum_{i=1}^{n}\left[{x_i}^{\alpha}{\varphi_{\alpha,\lambda}}^{\theta-1}\right] - \frac{\alpha(\theta-1)}{\lambda^{\alpha+1}}\sum_{i=1}^{n}\left[\frac{{x_i}^{\alpha}}{\varphi_{\alpha,\lambda}}\right] + \frac{\alpha\theta}{\lambda^{\alpha+1}}\sum_{i=1}^{n} {x_i}^{\alpha}{\varphi_{\alpha,\lambda}}^{\theta-1}$$
$$\times\,\omega_{\theta,\alpha,\lambda}\left\{\frac{(a-1)}{\omega_{\theta,\alpha,\lambda}-1} + \frac{a(b-1)\left(1-\omega_{\theta,\alpha,\lambda}\right)^{a-1}}{1-\left(1-\omega_{\theta,\alpha,\lambda}\right)^{a}}\right\} \tag{39}$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\ln\left(\varphi_{\alpha,\lambda}\right) - \sum_{i=1}^{n}\ln\left(\varphi_{\alpha,\lambda}\right){\varphi_{\alpha,\lambda}}^{\theta} + (a-1)\sum_{i=1}^{n}\frac{\ln\left(\varphi_{\alpha,\lambda}\right)\omega_{\theta,\alpha,\lambda}{\varphi_{\alpha,\lambda}}^{\theta}}{1-\omega_{\theta,\alpha,\lambda}}$$
$$-a(b-1)\sum_{i=1}^{n}\frac{\ln\left(\varphi_{\alpha,\lambda}\right)\omega_{\theta,\alpha,\lambda}{\varphi_{\alpha,\lambda}}^{\theta}\left(1-\omega_{\theta,\alpha,\lambda}\right)^{a-1}}{1-\left(1-\omega_{\theta,\alpha,\lambda}\right)^{a}} \tag{40}$$

where, $\omega_{\theta,\alpha,\lambda} = e^{1-\left(1+\left(\frac{x_i}{\lambda}\right)^{\alpha}\right)^{\theta}}$, $\varphi_{\alpha,\lambda} = \left(1+\left(\frac{x_i}{\lambda}\right)^{\alpha}\right)$.

The maximum likelihood estimates of a, b, α, λ and θ are the simultaneous solutions of the equations $\frac{\partial \ln L}{\partial a} = 0$, $\frac{\partial \ln L}{\partial b} = 0$, $\frac{\partial \ln L}{\partial \alpha} = 0$, $\frac{\partial \ln L}{\partial \lambda} = 0$, $\frac{\partial \ln L}{\partial \theta} = 0$. These equations cannot be solved analytically and statistical software can be used to solve them numerically by using iterative techniques like the Newton-Raphson algorithm.

6. Application

In this section, we have given an application of Kgpw distribution using real data set to illustrate that Kgpw distribution provides significant improvements over its sub-models Weibull (W) and generalized power Weibull (gpw). The real data set is taken from Badar and Priest (1982). The data represent the strength data measured in GPA, for single carbon fibers were tested under tension at gauge lengths of 1, 10, 20 and 50 mm. For illustrative purpose, we consider only the data set consisting the single fibers of 20 mm, with a sample of size 63. The data are: 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

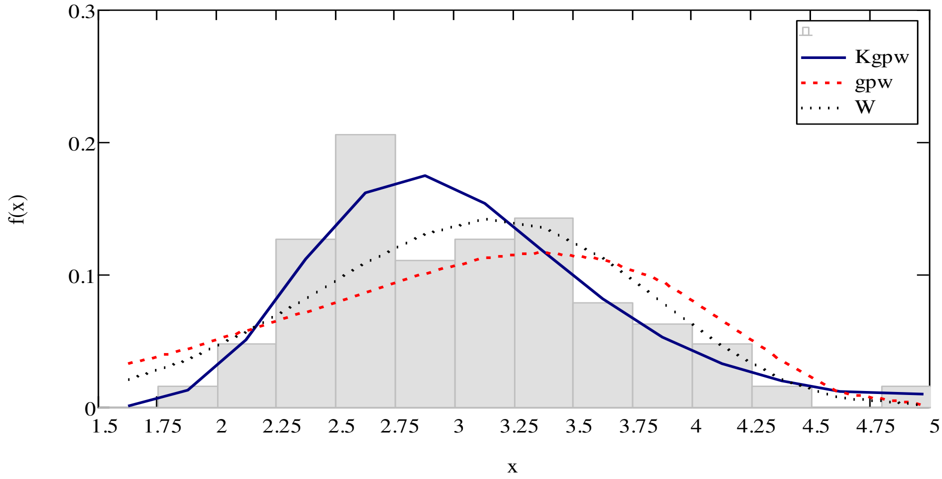
In Table 3. the maximum likelihood estimates of the unknown parameters of the Kgpw, gpw and Weibull distributions are given, along with the criteria likelihood $(-\ell_n(k))$, Akaike information criterion (*AIC*), corrected Akaike information criterion (*CAIC*) and Hannan-Quinn criterion (HQC) (see Hannan and Quinn (1978)). These criteria take the following forms, $AIC = -2ln\ell_n(k) + 2k$, $AICC = -2\,AIC + \frac{2k(k+1)}{n-k-1}$ and $HQC = -2ln\ell_n(k) + 2kln(ln(n))$ where, $\ell_n(k)$ be the maximum likelihood of a model with number of parameters k based on a sample of size n. Also, the plots of the empirical and estimated cdf's of these distributions are given in Figure (4) as a graphical illustration of the goodness of fit for these data.

Table 3: The estimated parameters and statistics $-\ell_n(k)$, *AIC*, *AICC* and *HQC* for fitted models

	Estimates					Statistics			
Model	a	b	α	λ	θ	$-\ell_n(k)$	AIC	AICC	HQC
Weibull	-	-	5.049	3.315	-	61.957	127.914	128.114	129.6
gpw	-	-	3.151	19.824	179.363	69.271	144.542	144.949	147.071
Kgpw	40.071	1.407	2.215	0.67	0.467	56.346	122.692	123.745	126.907

It is observed from Table 3 that $-\ell_n(k)$, AIC, $AICc$ and HQC are lowest in case of Kgpw distribution. Therefore, we can conclude that Kgpw distribution performs better than Wibull and gpw distributions. The figures (4a) and (4b) also confirm a good fit of the Kgpw model for the data set.

(a)



(b)

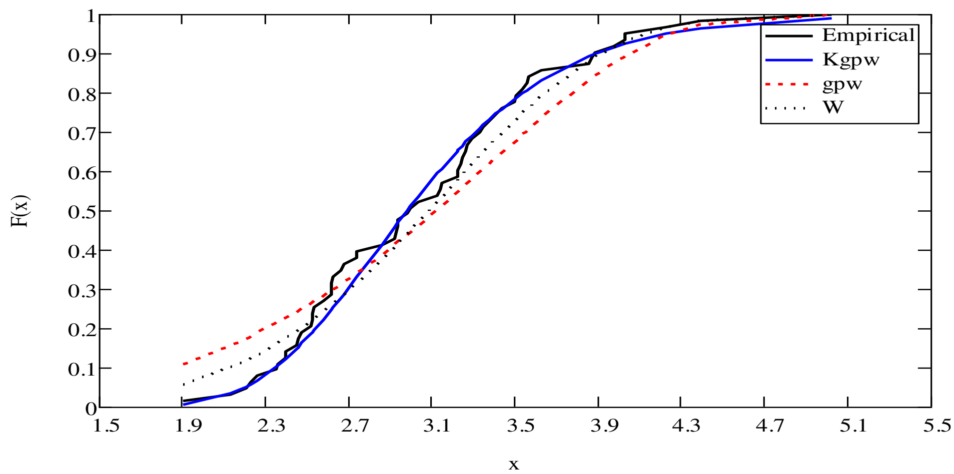


Figure 4: (a) Estimated pdfs of the Kgpw Distribution and its Sub-Models for the Strength Data from Badar and Priest (1982). (b) Empirical and Estimated cdfs of the Kgpw Distribution and its Sub-Models for the Strength Data from Badar and Priest (1982).

7. Concluding Remarks

In this article, we define a new model, which is called the Kumaraswamy generalized power Weibull distribution. The new distribution generalizes the generalized power Weibull distribution defined by Nikulin and Haghighi (2006). Some mathematical properties are derived and plots of the pdf, cdf and hazard function are presented to show the flexibility of the new distribution. The maximum likelihood estimation for the model parameters is discussed. Finally, an application of the proposed model to a real data set is given to illustrate that Kgpw distribution can be used quite effectively to provide better fits than other available models.

References

1. Antonio E. Gomes, Cibele Q. da-Silva, Gauss M. Cordeiro and Edwin M.M. Ortega (2014). A new lifetime model: the Kumaraswamy generalized Rayleigh distribution, Journal of Statistical Computation and Simulation, 84(2), 290-309.
2. Bader, M. G., and Priest, A. M. (1982). Statistical aspects of fibre and bundle strength in hybrid composites, Progress in science and engineering of composites, 1129-1136.
3. Chukwu, A., and A. Ogunde. (2015). On the beta Mekaham distribution and Its applications. American Journal of Mathematics and Statistics 5 (3):137-143.
4. Cordeiro, G.M. and Lemonte, A.J. (2011). The beta Laplace distribution, Statistics and Probability Letters, 81, 973–982.
5. Cordeiro, G.M., de Castro, M. (2011). A new family of generalized distributions, Journal of Statistical Computation and Simulation, 81, 883-898.
6. Cordeiro, G.M., Ortega, E.M.M. and Nadarajah, S. (2010). The Kumaraswamy Weibull distribution with application to failure data, Journal of the Franklin Institute, 347, 1399–1429.
7. Cordeiro, G.M., Ortega, E.M.M. and Silva, G.O. (2012). The Kumaraswamy modified Weibull distribution: theory and applications, Journal of Statistical Computation and Simulation, DOI:10.1080/00949655.2012.745125, (print).
8. Cordeiro, GM, Nadarajah, S and Ortega, E.M.M. (2011). The Kumaraswamy Gumbel distribution, Statistical Methods and Applications, 21(2), 139–168.
9. Eugene, N., Lee, C., Famoye, F. (2002). Beta-normal distribution and its applications, Communications in Statistics - Theory and methods, 31(4), 497–512.
10. Famoye, F., Lee, C. and Olumolade, O. (2005). The beta-Weibull distribution, Journal of Statistical Theory and Applications, 4(2), 121-136.
11. Ghosh, I. (2014). The Kumaraswamy-half-Cauchy distribution: properties and applications, Journal of Statistical Theory and Applications, 13(2), 122-134.
12. Gupta, R. D. and Kundu, D. (2001). Exponentiated exponential family; an alternative to gamma and Weibull, Biometrical Journal, 33(1), 117-130.
13. Hannan, E. J., and B. G. Quinn (1978),The determination of the order of an autoregression, Journal of the Royal Statistical Society, B, 41, 190-195.
14. Jafari, A. A., S. Tahmasebi, and M. Alizadeh. (2014). The beta-Gompertz distribution. Revista Colombiana de Estadística 37 (1):141-158.
15. Jones, M. C. (2009). Kumaraswamy's distribution: a beta-type distribution with some tractability advantages, Statistical Methodology, 6(1), 70–81.
16. Kenney, J. F. and Keeping, E. S. (1962). *Mathematics of Statistics*, Part One. Third Edition, Princeton, New Jersey: Van Nostrand.
17. Kong, L., Lee, C. and Sepanski, J.H. (2007). On the properties of beta-gamma distribution, Journal of Modern Applied Statistical Methods, 6, 187–211.
18. Kumaraswamy, P. (1980). Generalized probability density function for double-bounded random processes, Journal of Hydrology, 462, 79–88.
19. Lee, C., Famoye, F., Alzaatreh. (2013). A Methods for generating families of univariate continuous distributions in the recent decades, WIREs Comput. Stat., 5, 219–238.
20. Mameli, V., Musio, M. (2013). A generalization of the skew-normal distribution: the beta skew-normal, Communications in Statistics-Theory and Methods, 42(12), 2229-2244.
21. Merovci, F., and V. K. Sharma. (2014). The beta-Lindley distribution: Properties and Applications. Journal of Applied Mathematics, 2014, 1-10 .
22. MirMostafaee, S., M. Mahdizadeh, and S. Nadarajah. (2015). The beta Lindley distribution. Journal of Data Science 13 (3).
23. Mudholkar, G. S., Srivastava, D. K. and Freimer, M. (1995). The exponentiated Weibull family: A reanalysis of the bus-motor-failure data, Technometrics, 37, 436-445.

24. Nadarajah S., Kotz, S. (2004). The beta Gumbel distribution, Mathematical Problems in Engineering, 4, 323–332.
25. Nadarajah, S. and Kotz, S. (2006). The beta exponential distribution, Reliability Engineering and System Safety, 91, 689-697.
26. Nikulin, M., Haghighi, F. (2006). A chi-squared test for the generalized power Weibull family for the head-and-neck cancer censored data, Journal of Mathematical Sciences, 133, 1333–1341.
27. Pascoa, A.R.M., Ortega, E.M.M. and Cordeiro, G.M. (2011). The Kumaraswamy generalized gamma distribution with application in survival analysis, Statistical Methodology, 8, 411–433.
28. Rocha, R., S. Nadarajah, V. Tomazella, F. Louzada, and A. Eudes. (2015). New defective models based on the Kumaraswamy family of distributions with application to cancer data sets. Statistical methods in medical research:0962280215587976.
29. Tiago Viana Flor de Santana, Edwin M.M. Ortega, Gauss M. Cordeiro and Giovana O. Silva, (2012). The Kumaraswamy-log-logistic distribution, J. Statist. Theor. and Applica., 11(3), 265-291.
30. Weibull, W. (1951). A statistical distribution function of wide applicability, J. Appl. Mech., 18, 293-297.

Mahmoud Ali Selim is an assistant Professor of Statistics in the Department of Statistics at Al-Azhar University. He received Ph.D. (2010) in Statistics from Al-Azhar University, Cairo, Egypt. He is now on leave at King Khalid University, Saudi Arabia. His research interests include probability distribution, reliability, statistical inferences involving order statistics and record values.