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THE KURZWEIL-HENSTOCK THEORY OF STOCHASTIC INTEGRATION

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Abstract. The Kurzweil-Henstock approach has been successful in giving an alternative definition to the classical Itô integral, and a simpler and more direct proof of the Itô Formula. The main advantage of this approach lies in its explicitness in defining the integral, thereby reducing the technicalities of the classical stochastic calculus. In this note, we give a unified theory of stochastic integration using the Kurzweil-Henstock approach, using the more general martingale as the integrator. We derive Henstock's Lemmas, absolute continuity property of the primitive process, integrability of stochastic processes and convergence theorems for the Kurzweil-Henstock stochastic integrals. These properties are well-known in the classical (non-stochastic) integration theory but have not been explicitly derived in the classical stochastic integration.

Keywords: stochastic integral, Kurzweil-Henstock, convergence theorem

MSC 2010: 26A39, 60H05

1. Introduction

Stochastic calculus has been well developed in the study of stochastic integrals, see [3], [8], [14], [18], [19], [20], [21], [22], [32]. In the classical theory of integration the Riemann integral with uniform mesh is found to be deficient. In the 1950s, J. Kurzweil and R. Henstock independently modified the Riemann integral by using non-uniform meshes, that is, meshes that vary from point to point. It turns out that this integral is more general than the classical Riemann integral and the Lebesgue integral, see [4], [5], [6], [9], [10], [11].

Along this line of thought, the Henstock approach, also known as the generalized Riemann aproach, has been used to study stochastic integrals, see [2], [7], [10], [12], [13], [15], [16], [17], [23], [24], [25], [26], [27], [31], [33]. The advantage of the generalized Riemann approach is that it gives an explicit and intuitive definition of the stochastic integral using L^2-convergence. Even for the stochastic integrals, it turns out that Henstock's definition encompasses the classical stochastic integrals, see [23], [24], [25], [26], [27]. The Henstock approach was also used to characterize stochastic integrable processes in [29] and an integration-by-part formula is also derived for stochastic integrals, see [28]. The Henstock approach has also been shown to be able to give an easier and more direct proof of the Itô Formula, see [30].

In this note, we shall establish the results of the theory of stochastic integration using the Kurzweil-Henstock approach. We shall use an L^2-martingale as the integrator in most of our discussion. The Kurzweil-Henstock approach is well-known for its explicitness for the classical integration theory. In addition, we also establish the convergence theorems for the stochastic integrals. As Brownian motions are special cases of L^2-martingales, the results of this paper encompass Itô's stochastic integration as well (which consider Brownian motion as the integrator).

2. Setting and definition of the integral

Let $(\Omega, \mathscr{F}, P)$ be a probability space, and $\{\mathscr{F}_t\}$ an increasing family of σ-subfields of $\mathscr{F}$ for $t \in [a,b]$, that is, $\mathscr{F}_r \subset \mathscr{F}_s$ for $a \leqslant r < s \leqslant b$ with $\mathscr{F}_b = \mathscr{F}$. The probability space together with its family of increasing σ-subfields is called a *standard filtering space* and denoted by $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$.

A process $\{\varphi(t,\omega)\colon\ t \in [a,b]\}$ on $(\Omega, \mathscr{F}, P)$ is a family of $\mathscr{F}$-measurable functions (which are called random variables) on $(\Omega, \mathscr{F}, P)$. We also denote the process $\varphi(t,\omega)$ by $\varphi_t(\omega)$.

The process $\{\varphi_t(\omega)\colon\ t \in [a,b]\}$ is said to be adapted to the filtering $\{\mathscr{F}_t\}$ if for each $t \in [a,b]$, φ_t is $\mathscr{F}_t$-measurable. In this paper, we shall fix the standard filtering space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$ and simplify it to the process φ being adapted.

Definition 1. A measurable process $X = \{X(t,\omega),\ t \in [a,b]\}$ defined on $[a,b]$ is called a *martingale* if

(i) X is adapted to the filtration, that is, X_t is $\mathscr{F}_t$-measurable for each $t \in [a,b]$;
(ii) $\int_\Omega |X_t| \, \mathrm{d}P < \infty$ for almost all $t \in [a,b]$; and
(iii) $E(X_t | \mathscr{F}_s) = X_s$ for all $t \geqslant s$.

For a given random variable φ on $(\Omega, \mathscr{F}, P)$, let $E(\varphi)$ denote its expectation in the probability space, that is,

$$E(\varphi) = \int_\Omega \varphi \, \mathrm{d}P,$$

and let us use the notation $E(X_t|\mathscr{F}_s)$ to denote the conditional expectation of X_t given $\mathscr{F}_s$, which is defined as the $\mathscr{F}_s$-measurable random variable such that

$$\int_A E(X_t|\mathscr{F}_s)\,\mathrm{d}P = \int_A X_t\,\mathrm{d}P$$

for each $A \in \mathscr{F}_s$. By the Radon-Nikodym Theorem, $E(X_t|\mathscr{F}_s)$ exists and is well-defined.

Remark 1a. As an easy consequence of (iii) of Definition 1 and the definition of conditional expectation, for all $t \neq s$ we have

(a) $E(X_t) = E(X_s)$;

(b) $E(X_t - X_s|\mathscr{F}_s) = 0$; and

(c) if θ is a random variable that is $\mathscr{F}_s$-measurable, then

$$E(\theta(X_t - X_s)|\mathscr{F}_s) = \theta E(X_t - X_s|\mathscr{F}_s).$$

It is also clear that for any random variable φ,

(d) $E(\varphi) = E(E(\varphi|\mathscr{F}_s))$.

A martingale X is said to be an L^2-martingale if, in addition to satisfying the conditions (i), (ii) and (iii) above, we also have

$$\sup_{t\in[a,b]} \int_\Omega |X_t|^2\,\mathrm{d}P < \infty.$$

In this note, the integrator we shall be considering is an L^2-martingale. We shall also assume that the integrator is cadlag, that is, it has sample paths which are right-continuous with left limits. Throughout our discussion, we fix an L^2-martingale as the integrator and denote it by X. All stochastic integrals are taken with respect to the integrator.

Definition 2. For any L^2-martingale X, there exists an increasing adapted stochastic process $\langle X\rangle$ such that $X^2 - \langle X\rangle$ is also an L^2-martingale. This process $\langle X\rangle$ is called the *quadratic variation process* associated with the martingale X.

The proof of existence of the quadratic variation process for an L^2-martingale can be found in classical textbooks, for example, see [35, pp. 212–213, Proposition 11.20]. It is also shown in [3] that the process $\langle X\rangle$ is an adapted cadlag process provided the L^2-martingale is cadlag.

For each $(c,d] \subset [a,b]$, we denote the measure induced by the quadratic variation process by

$$\mu_X[c,d] = E(\langle X\rangle_d - \langle X\rangle_c).$$

Definition 3. Let δ be a positive function on $[a,b]$. A finite collection D of interval-point pairs $\{((\xi_i, v_i], \xi_i)\colon\ i = 1,2,3,\ldots,n\}$ is a δ-fine belated *partial* division of $[a,b]$ if

(i) $(\xi_i, v_i]$, $i = 1,2,3,\ldots n$, are disjoint left-open subintervals of $[a,b]$; and

(ii) each $[\xi_i, v_i]$ is δ-fine belated, that is, $[\xi_i, v_i] \subset [\xi_i, \xi_i + \delta(\xi_i))$.

Given $\eta > 0$, the partial division D is said to fail to cover $[a,b]$ by at most μ_X-measure η if

$$\left|\mu_X[a,b] - \sum_{i=1}^{n} \mu_X[\xi_i, v_i]\right| \leqslant \eta.$$

In this note, for each positive function δ on $[a,b]$ and each positive constant η we shall always assume that there exists a δ-fine belated partial division of $[a,b]$ which fails to cover $[a,b]$ by at most μ_X-measure η (see for example [12, p. 52]). As a special case, if X is the classical Brownian motion, then $\langle X\rangle_t \equiv t$ (see for example [34, p. 288], and [1]), hence μ_X is the Lebesgue measure.

Definition 4. An adapted process $f = \{f_t\colon\ t \in [a,b]\}$ is said to be Kurzweil-Henstock stochastic integrable on $[a,b]$ (with respect to X) if there exists an $A \in L^2(\Omega)$ such that for any $\varepsilon > 0$ there exists a positive function δ on $[a,b]$ and a positive number $\eta > 0$ such that for any δ-fine belated partial division $D = \{((\xi_i, v_i], \xi_i)\colon\ i = 1,2,3,\ldots,n\}$ of $[a,b]$ which fails to cover $[a,b]$ by at most η, we have

$$E\left(\sum_{i=1}^{n} f_{\xi_i}[X_{v_i} - X_{\xi_i}] - A\right)^2 \leqslant \varepsilon.$$

Definition 4a. In Definition 4, if X is replaced by a classical Brownian motion, then the process f is said to be Itô-Henstock integrable to A (see [2]).

Proposition 5. *If an adapted process f is Kurzweil-Henstock stochastic integrable, then the integral of f is unique up to a set of P-measure zero.*

The proof of Proposition 5 follows easily from definition, hence it is omitted.

Subsequently, in view of the above uniqueness proposition, we shall denote the integral of the adapted process f with respect to the L^2-martingale X by the symbol

$$\int_a^b f_t \, \mathrm{d}X_t.$$

Example 6. Let $h\colon \Omega \to \mathbb{R}$ be a bounded random variable on $(\Omega, \mathscr{F}, P)$, that is, there exists $M > 0$ such that $|h(\omega)| \leqslant M$ for all $\omega \in \Omega$. Let $s \in [a,b]$ be fixed and let h be $\mathscr{F}_t$-measurable. Suppose $t = s$ implies $f_t(\omega) = h(\omega)$ for all

$\omega \in \Omega$; and suppose $t \neq s$ implies $f_t(\omega) = 0$ for all $\omega \in \Omega$. Then f is Kurzweil-Henstock stochastic integrable to zero on $[a, b]$. We remark that this function f is also stochastic integrable to 0 under the classical setting (see for example [2]).

P r o o f. For any $\varepsilon > 0$ there exists a positive function δ on $[a, b]$ such that whenever $((\xi, v], \xi)$ is δ-fine, we have

$$0 \leqslant E(\langle X\rangle_v - \langle X\rangle_\xi) \leqslant \frac{\varepsilon}{M^2}.$$

This is possible since $\langle X\rangle_t$ is right-continuous, hence $E\langle X\rangle_t$ is right-continuous by Dominated Convergence Theorem for integration with respect to measures. Consider any δ-fine belated partial division $D = \{((\xi_j, v_j], \xi_j)\}_{j=1}^m$ of $[a, b]$. Assume that $s = \xi_j$ for some j; otherwise, it is trivial since $f_{\xi_j} \equiv 0$ for all $j = 1, 2, 3, \ldots, m$. Thus,

$$\begin{aligned} E\Big(\sum_j (f_{\xi_i}[X_{v_j} - X_{\xi_j}] - 0)\Big)^2 &= E(f_s[X_{v_j} - X_{\xi_j}])^2 \\ &\leqslant M^2 E(X_{v_j} - X_s)^2 = M^2 E(\langle X\rangle_{v_j} - \langle X\rangle_s) \\ &< M^2\Big(\frac{\varepsilon}{M^2}\Big) = \varepsilon. \end{aligned}$$

The above inequality holds for any δ-fine belated partial division of $[a, b]$. Hence, f is Kurzweil-Henstock stochastic integrable to zero on $[a, b]$. □

Notation. We shall use the symbol $S(f, D, \delta, \eta)$ to denote the Riemann sum

$$\sum_{i=1}^n f_{\xi_i}(X_{v_i} - X_{\xi_i})$$

where $D = \{((\xi_i, v_i], \xi_i)\colon\ i = 1, 2, 3, \ldots, n\}$ is a δ-fine belated partial division which covers $[a, b]$ except for a set of μ_X-measure not exceeding η.

Proposition 7 (Cauchy Criterion). *Let f be an adapted process on $[a, b]$. Then f is Kurzweil-Henstock stochastic integrable on $[a, b]$ if and only if for each $\varepsilon > 0$ there exist a positive function δ on $[a, b]$ and a positive constant η such that whenever $D_1 = \{((\xi_i, v_i], \xi_i)\}_{i=1}^n$ and $D_2 = \{((\eta_j, t_j], \eta_j)\}_{j=1}^m$ are two δ-fine belated partial divisions of $[a, b]$ failing to cover $[a, b]$ by at most η, we have*

$$E\bigg(\bigg|\sum_{i=1}^n f_{\xi_i}(X_{v_i} - X_{\xi_i}) - \sum_{j=1}^m f_{\eta_j}(X_{t_j} - X_{\eta_j})\bigg|^2\bigg) \leqslant \varepsilon.$$

The proof of Cauchy Criterion is analogous to the classical case for Henstock (nonstochastic) integral (see [9]), hence it is omitted here.

Theorem 8. *Let f be an adapted process on $[a,b]$. Then f is Kurzweil-Henstock stochastic integrable on $[a,b]$ if and only if there exist $A \in L^2(\Omega)$, a decreasing sequence of $\{\delta_n(\xi)\}$ of positive functions defined on $[a,b]$, and a decreasing sequence of positive numbers $\{\eta_n\}$ such that whenever D_n is a δ_n-fine belated partial division of $[a,b]$ that fails to cover it by at most μ_X-measure η_n, we have*

$$\lim_{n\to\infty} E(|S(f, D_n, \delta_n, \eta_n) - A|^2) = 0.$$

Furthermore,

$$A = \int_a^b f_t \,\mathrm{d}X_t.$$

Theorem 8 provides an alternative definition of the Kurzweil-Henstock stochastic integral using limits of sequences of Riemann sums. This was first used in [30] for Henstock's Itô integral and [25] for Multiple Wiener integral, where the integrators are Brownian motion. The proof is similar to the two cases with obvious modifications, hence we omit the proof here.

2. Non-stochastic properties

We shall use the term "non-stochastic properties" to refer to the properties of the integral which can be established using the standard approach of Kurzweil-Henstock (non-stochastic) integration, without using the results on probability. The term "stochastic properties" will be used to refer to those properties that can only be established using the properties of probability, and which are not generally shared with classical Kurzweil-Henstock (non-stochastic) integration theory. In this section we shall state without proving some basic non-stochastic properties of the integral. These non-stochastic properties can be derived easily using the non-uniform Riemann approach (see, for example, [9] for the details of the various proofs).

Proposition 9. *Let f and g be adapted processes on $[a,b]$ which are Kurzweil-Henstock stochastic integrable on $[a,b]$, and let $\alpha \in \mathbb{R}$. Then both $f+g$ and αf are Kurzweil-Henstock stochastic integrable on $[a,b]$, and furthermore,*

(i) $\int_a^b (f_t + g_t)\,\mathrm{d}X_t = \int_a^b f_t\,\mathrm{d}X_t + \int_a^b g_t\,\mathrm{d}X_t$,
(ii) $\int_a^b \alpha f_t\,\mathrm{d}X_t = \alpha \int_a^b f_t\,\mathrm{d}X_t$.

Proposition 10. *Let f be Kurzweil-Henstock stochastic integrable on $[a,c]$ and $[c,b]$. Then f is Kurzweil-Henstock stochastic integrable on $[a,b]$; further,*

$$\int_a^b f_t\,\mathrm{d}X_t = \int_a^c f_t\,\mathrm{d}X_t + \int_c^b f_t\,\mathrm{d}X_t.$$

Proposition 11. *If f is Kurzweil-Henstock stochastic integrable on $[a,b]$, then f is integrable on any subinterval $[c,d]$ of $[a,b]$.*

Henstock's Lemma is a crucial result of classical (nonstochastic) integration theory. The proof of Lemma 12 is parallel to that for classical integrals, hence the proof is omitted.

Lemma 12 (Henstock's Lemma). *Let f be Kurzweil-Henstock stochastic integrable on $[a,b]$ and let $F(u,v) = (\mathscr{K}HS)\int_u^v f_t\,\mathrm{d}X_t$ for any $(u,v] \subset [a,b]$. Then for each $\varepsilon > 0$ there exists a positive function δ on $[a,b]$ such that whenever $D = \{((\xi_i, v_i], \xi_i)\}_{i=1}^n$ is a δ-fine belated* partial *division of $[a,b]$, we have*

$$E\Bigg(\Bigg|\sum_{i=1}^{n}(f_{\xi_i}(X_{v_i} - X_{\xi_i}) - F(\xi_i, v_i))\Bigg|^2\Bigg) \leqslant \varepsilon.$$

In Definition 4, the Riemann sum $\sum f_{\xi_i}(X_{v_i} - X_{u_i})$ over almost the entire interval $[a,b]$ was used as an "approximation" of A with a small "accumulated error" from the classical integration theory. In Lemma 12 (Henstock's Lemma), the "absolute error" $f_{\xi_i}(X_{v_i} - X_{u_i}) - F(\xi_i, v_i)$ is small over *any* subinterval of $[a,b]$. Henstock's Lemma asserts that these two are equivalent.

3. Stochastic properties and isometric equality

In this section we derive the stochastic properties of the Kurzweil-Henstock stochastic integral and the isometric equality. Throughout this section, we let the integrator X be an L^2-martingale.

For a stochastic process Y on $[a,b]$ and for a given subinterval $J = (u,v] \subset [a,b]$ we let $Y(J)$ denote the random variable $Y_v - Y_u$ throughout our discussion from this section onwards.

Lemma 13. *Let f be Kurzweil-Henstock stochastic integrable on $[a,b]$ and let $F(I) = \int_I f_t\,\mathrm{d}X_t$, where I is a left-open subinterval of $[a,b]$. Let J and K be two disjoint left-open subintervals of $[a,b]$. Then*

(i) *$E(F(J)) = 0$;*
(ii) *F has the orthogonal increment property, that is, $E(F(J)F(K)) = 0$;*
(iii) *$E(X(J)F(K)) = 0$;*
(iv) *$E(F(a,t)|\mathscr{F}_s) = F(a,s)$ if $s \leqslant t$; and*
(v) *$E\big((f_{\xi_i}X(J) - F(J))(f_{\xi_j}X(K) - F(K))\big) = 0$, where ξ_i is the left-end point of J and ξ_j the left-end point of K.*

P r o o f. By using the properties listed in Remark 1a, we have

$$\begin{aligned}
E(f_\xi(X_v - X_\xi) &= E(E(f_\xi(X_v - X_\xi)|\mathscr{F}_\xi)) \\
&= E(f_\xi E(X_v - X_\xi|\mathscr{F}_\xi)) \\
&= E(f_\xi \cdot 0) = 0.
\end{aligned}$$

Hence, $E(S(f, D, \delta, \eta)) = 0$. By Theorem 8, we obtain $E(F(J)) = 0$, completing the proof of (i).

To prove (ii) and (iii), suppose that $(\xi_i, v_i]$ and $(\xi_j, v_j]$ are disjoint. Without loss of genereality, we may assume that $v_i \leqslant \xi_j$. Then

$$\begin{aligned}
E(f_{\xi_i}[X_{v_i} - X_{\xi_i}]f_{\xi_j}[X_{v_j} - X_{\xi_j}]) &= E(E((f_{\xi_i}[X_{v_i} - X_{\xi_i}]f_{\xi_j}[X_{v_j} - X_{\xi_j}])|\mathscr{F}_{\xi_j})) \\
&= E(f_{\xi_i}[X_{v_i} - X_{\xi_i}]f_{\xi_j}E([X_{v_j} - X_{\xi_j}]|\mathscr{F}_{\xi_j})) = 0,
\end{aligned}$$

noting that the last step follows since $f_{\xi_i}(X_{v_i} - X_{\xi_i})$ is $\mathscr{F}_{\xi_j}$-measurable. Let $D(J)$ and $D(K)$ denote the partial divisions of J and K respectively. Then, for any δ_n and η_n and any two disjoint intervals J and K of $[a, b]$,

$$E(S(f, D(J), \delta_n, \eta_n)S(f, D(K), \delta_n, \eta_n)) = 0.$$

Further, by Theorem 8, choose $\{\delta_n\}$ and $\{\eta_n\}$ such that

$$\lim_{n\to\infty} E(|S(f, D(J), \delta_n, \eta_n) - F(J)|^2) = 0$$

and

$$\lim_{n\to\infty} E(|S(f, D(K), \delta_n, \eta_n) - F(K)|^2) = 0;$$

then

$$E(F(J)F(K)) = \lim_{n\to\infty} E(S(f, D(J), \delta_n, \eta_n)S(f, D(K), \delta_n, \eta_n)) = 0.$$

Similarly,

$$E(X(J)F(K)) = \lim_{n\to\infty} E(X(J)S(f, D(K), \delta_n, \eta_n)) = 0,$$

thereby completing the proofs of (ii) and (iii).

To prove (iv), let $s \leqslant t$. First note that

$$E(G_t|\mathscr{F}_s) = G_s$$

where

$$G_t = \sum_{i=1}^{n} \eta_i(X_{v_i\wedge t} - X_{u_i\wedge t})$$

and each η_i is $\mathscr{F}_{u_i}$-measurable while $u \wedge t$ is the minimum of u and t. Thus

$$E(S(f, D, \delta, \eta)|\mathscr{F}_s) = (D)\sum f_\xi(X_{v\wedge s} - X_{u\wedge s}).$$

Hence we have

$$E(F(a,b)|\mathscr{F}_s) = F(a,s).$$

Similarly,

$$E(F(a,t)|\mathscr{F}_s) = F(a,s).$$

By (iii) and (iv), we get (v), completing the proof. □

Lemma 14. *Let f be Kurzweil-Henstock stochastic integrable on $[a,b]$ and let $F(u,v) = (\mathscr{K}HS)\int_u^v f_t\,\mathrm{d}X_t$. Let $D = \{((\xi_i, v_i], \xi_i)\}_{i=1}^n$ be a δ-fine belated partial division of $[a,b]$. Then*

$$\text{(i)}\quad E\biggl(\biggl|\sum_{i=1}^n f_{\xi_i}(X_{v_i} - X_{\xi_i})\biggr|^2\biggr) = E\biggl[\sum_{i=1}^n f_{\xi_i}^2(\langle X\rangle_{v_i} - \langle X\rangle_{\xi_i})\biggr],$$

$$\text{(ii)}\quad E\biggl(\biggl|\sum_{i=1}^n f_{\xi_i}(X_{v_i} - X_{\xi_i}) - F(\xi_i, v_i)\biggr|^2\biggr) = E\biggl(\sum_{i=1}^n |f_{\xi_i}(X_{v_i} - X_{\xi_i}) - F(\xi_i, v_i)|^2\biggr).$$

P r o o f. To prove (i), note that

$$\begin{aligned} &E\Bigl(\sum_i f_{\xi_i}[X_{v_i} - X_{\xi_i}]\Bigr)^2 \\ &\quad= E\Bigl\{\sum_i (f_{\xi_i}[X_{v_i} - X_{\xi_i}])^2 + \sum_{i\neq j}(f_{\xi_i} f_{\xi_j}[X_{v_i} - X_{\xi_i}][X_{v_j} - X_{\xi_j}])\Bigr\} \\ &\quad= \sum_i \{E([f_{\xi_i}]^2[X_{v_i} - X_{\xi_i}]^2)\} = E\Bigl(\sum_i f_{\xi_i}^2(\langle X\rangle_{v_i} - \langle X\rangle_{\xi_i})\Bigr) \end{aligned}$$

thereby completing the proof of (i).

The proof of (ii) follows immediately from Part (v) of Lemma 13. □

Lemma 15 (Henstock's Lemma—Strong Version). *Let f be Kurzweil-Henstock stochastic integrable on $[a,b]$ and let $F(u,v) = \int_u^v f_t\,\mathrm{d}X_t$ for any $(u,v] \subset [a,b]$. Then for every $\varepsilon > 0$ there exists a positive function δ on $[a,b]$ such that whenever $D = \{((\xi_i, v_i], \xi_i)\}_{i=1}^n$ is a δ-fine belated* partial *division of $[a,b]$ we have*

$$E\biggl(\sum_{i=1}^n |f_{\xi_i}(X_{v_i} - X_{\xi_i}) - F(\xi_i, v_i)|^2\biggr) \leqslant \varepsilon.$$

P r o o f. This is an immediate consequence of Part (v) of Lemma 13 and Lemma 14. □

Note that Lemmas 12 and 15 are different as in the former, the expression involves the expected value of the square of the sum while in the latter it is the expected value of the sum of squares. We are able to switch the order of squaring and summation by virtue of the stochastic properties; in the general integration theory, the switching of order of the two operations is not true in general.

Definition 16. Let f be an adapted process on $[a, b]$. Then f is said to be $\langle X\rangle$-integrable to $A \in L^2(\Omega)$ on $[a, b]$ if for each $\varepsilon > 0$ there exist a positive function δ on $[a, b]$ and a positive constant η such that whenever $D = \{((\xi, v], \xi)\}$ is a δ-fine belated partial division of $[a, b]$ failing to cover $[a, b]$ by at most μ_X-measure η, we have

$$E\Big(\Big|(D)\sum f_\xi(\langle X\rangle_V - \langle X\rangle_\xi) - A\Big|\Big) \leqslant \varepsilon.$$

We denote A by $\int_a^b f_t \, \mathrm{d}\langle X\rangle_t$. We shall also denote the above Riemann-Stieltjes sum $\sum(D)f_\xi(\langle X\rangle_v - \langle X\rangle_\xi)$ by $\tilde{S}(f, D, \delta, \eta)$.

Note that if X is a classical Brownian motion, the above Riemann-Stieltjes sum is the usual Riemann sum, which defines the McShane integral (which is equivalent to the Lebesgue integral).

Remark. The corresponding result of Theorem 8 also holds for $f \in \mathscr{L}^2$, that is, an adapted process f on $[a, b]$ is μ_X-integrable to $A \in L^2$ on $[a, b]$ if and only if there exist a positive sequence of functions δ_n on $[a, b]$ and a positive sequence of constants η_n such that whenever D_n is a δ_n-fine belated partial division of $[a, b]$ that fails to cover $[a, b]$ by at most μ_X-measure η_n, we have

$$\lim_{n\to\infty} E(|\tilde{S}(f^2, D, \delta_n, \eta_n) - A|) = 0.$$

The technique of the proof of the above is similar to that for Theorem 8, which was quoted from [30]. Hence we omit the proof here.

Theorem 17 (Isometric Equality)**.** *Let f be Kurzweil-Henstock stochastic integrable on $[a, b]$ and $f \in \mathscr{L}^2$. Then*

$$E\bigg(\Big(\int_a^b f_t \, \mathrm{d}X_t\Big)^2\bigg) = E\Big(\int_a^b f_t^2 \, \mathrm{d}\langle X\rangle_t\Big).$$

We remark that the integral on the left in the above equation is the Kurzweil-Henstock stochastic integral while that on the right in the above equation is the usual Henstock-Stieltjes integral.

P r o o f. By Theorem 8 and the remark preceding Theorem 17, there exist a decreasing sequence $\{\delta_n\}$ of positive functions on $[a,b]$ and a decreasing sequence η_n of positive numbers such that for any δ_n-fine belated partial division $D_n = \{((\xi_i^{(n)}, v_i^{(n)}], \xi_i^{(n)})\}_{i=1}^{p(n)}$ that fails to cover $[a,b]$ by at most μ_X-measure η_n, we have

$$\lim_{n\to\infty} E\biggl(\biggl|S(f, D_n, \delta_n, \eta_n) - \int_a^b f_t \,\mathrm{d}X_t\biggr|^2\biggr) = 0$$

and

$$\lim_{n\to\infty} E\biggl(\biggl|\tilde{S}(f^2, D_n, \delta_n, \eta_n) - \int_a^b f_t^2 \,\mathrm{d}\langle X\rangle_t\biggr|\biggr) = 0.$$

Then

$$\begin{aligned}\lim_{n\to\infty} E\biggl(\biggl(\int_a^b f_t \,\mathrm{d}X_t\biggr)^2\biggr) &= \lim_{n\to\infty} E(|S(f, D_n, \delta_n, \eta_n)|^2) \\ &= \lim_{n\to\infty} E\biggl(\sum_{i=1}^{p(n)} f_{\xi_i^{(n)}}(X_{v_i^{(n)}} - X_{\xi_i^{(n)}})\biggr)^2 \\ &= \lim_{n\to\infty} E(\tilde{S}(f^2, D_n, \delta_n, \eta_n)) = E\biggl(\int_a^b f_t^2 \,\mathrm{d}\langle X\rangle_t\biggr),\end{aligned}$$

which completes the proof. □

Theorem 18. *Let f and g be Kurzweil-Henstock stochastic integrable on $[a,b]$. Then*

(i) $E\bigl(\int_a^b f_t \,\mathrm{d}X_t\bigr) = 0$;

(ii) $E\bigl(\sum\limits_{i=1}^{n} \int_{\xi_i}^{v_i} f_t \,\mathrm{d}X_t\bigr)^2 = \sum\limits_{i=1}^{n} E\bigl(\int_{\xi_i}^{v_i} f_t \,\mathrm{d}X_t\bigr)^2$ *for any finite collection* $\{(\xi_i, v_i]\}_{i=1}^n$ *of disjoint subintervals of* $[a,b]$; *and*

(iii) $E\bigl(\bigl(\int_a^b f_t \,\mathrm{d}X_t\bigr)\bigl(\int_a^b g_t \,\mathrm{d}X_t\bigr)\bigr) = E\bigl(\int_a^b f_t g_t \,\mathrm{d}\langle X\rangle_t\bigr)$ *if f and g are also in* $\mathscr{L}^2$.

P r o o f. The results (i) and (ii) follow from Lemma 13 by taking limits. For a proof of (iii), it is instructional for the reader to go through the proof of Theorem 17 with the obvious modifications. □

Theorem 19. *Let f be Kurzweil-Henstock stochastic integrable on any subinterval $[a,b]$ of $[0,\infty)$ and let $F_s = \int_0^s f_t \,\mathrm{d}X_t$. Then the stochastic process $\{F_s\colon s \geqslant 0\}$ is an L^2-martingale with respect to the natural filtration.*

P r o o f. This result follows from (iv) of Lemma 13. □

4. Absolute continuity property

In this section we shall discuss the relation between the stochastic analogue of absolute continuity and the Kurzweil-Henstock stochastic integral. As in the previous section, we fix the integrator X as an L^2-martingale.

Definition 20. Let F be a stochastic process on $[a,b]$. Then F is said to have the AC_X-property if for each $\varepsilon > 0$ there exists $\eta > 0$ such that whenever $D = \{[u,v]\}$ is a partial division of $[a,b]$ with $\mu_X\Big(\bigcup\limits_D [u_i, v_i]\Big) \leqslant \eta$, we have

$$E\Big((D)\Big|\sum(F_v - F_u)\Big|^2\Big) \leqslant \varepsilon.$$

It follows from the martingale property that

$$\mu_X\Big(\bigcup_D [u_i, v_i]\Big) = E\Big((D)\sum_i(\langle X\rangle_{v_i} - \langle X\rangle_{u_i})\Big).$$

Lemma 21. *Let f be Kurzweil-Henstock stochastic integrable on $[a,b]$. Given $\varepsilon > 0$, there exist a positive function δ on $[a,b]$ and a positive number η such that*

$$E\Big(\Big|(D)\sum f_\xi[X_v - X_\xi]\Big|\Big)^2 \leqslant \varepsilon$$

for any δ-fine belated partial division $D = \{((\xi, v], \xi)\}$ which fails to cover $[a,b]$ by μ_X-measure at most η.

Proof. Given $\varepsilon > 0$, there exist a positive function δ on $[a,b]$ and a positive number η such that for any δ-fine belated division D_0 that fails to cover $[a,b]$ by at most η, we have

$$E\Big(\Big|(D_0)\sum f_\xi[X_v - X_\xi] - \int_a^b f_t \,\mathrm{d}X_t\Big|\Big)^2 \leqslant \frac{\varepsilon}{4}.$$

Let $D = \{((\xi, v], \xi)\}$ be a δ-fine belated partial division of $[a,b]$ such that

$$E\Big((D)\sum(\langle X\rangle_v - \langle X\rangle_\xi)\Big) \leqslant \eta.$$

Construct a δ-fine belated partial division D_1 of $[a,b]$ that fails to cover $[a,b]$ by at most η and is disjoint from D, such that $D \cup D_1$ is also a δ-fine belated partial division of $[a,b]$ failing to cover $[a,b]$ by at most η. Hence

$$E\Big(\Big|(D \cup D_1)\sum f_\xi[X_v - X_\xi] - \int_a^b f_t \,\mathrm{d}X_t\Big|\Big)^2 \leqslant \frac{\varepsilon}{4}.$$

Consequently,

$$\begin{aligned}E\Big(\Big|(D)\sum f_\xi[X_v-X_\xi]\Big|^2\Big) &= E\Big|(D\cup D_1)\sum f_\xi[X_v-X_\xi]-\int_a^b f_t\,\mathrm{d}X_t\\ &\quad+\int_a^b f_t\,\mathrm{d}X_t-(D_1)\sum f_\xi[X_v-X_\xi]\Big|^2\\ &\leqslant 2E\Big(\Big|(D\cup D_1)\sum f_\xi[X_v-X_\xi]-\int_a^b f_t\,\mathrm{d}X_t\Big|^2\Big)\\ &\quad+2E\Big(\Big|\int_a^b f_t\,\mathrm{d}X_t-(D_1)\sum f_\xi[X_v-X_\xi]\Big|^2\Big)\\ &\leqslant 2\frac{\varepsilon}{4}+2\frac{\varepsilon}{4}=\varepsilon,\end{aligned}$$

thereby completing the proof. □

Theorem 22. *Let f be Kurzweil-Henstock stochastic integrable on $[a,b]$. Let $\Phi(u)=\int_a^u f_t\,\mathrm{d}X_t$. Then Φ has the AC_X property.*

P r o o f. Let $\varepsilon>0$ be given. By Lemma 21 there exist η and a positive function δ on $[a,b]$ such that whenever $D_1=\{((\xi,v],\xi)\}$ is a δ-fine belated partial division of $[a,b]$ with $E\big((D_1)\sum(\langle X\rangle_v-\langle X\rangle_\xi)\big)\leqslant\eta$ we have

$$E\Big(\Big|(D_1)\sum f_\xi[X_v-X_\xi]\Big|^2\Big)\leqslant\varepsilon.$$

Let $\{(a_i,b_i]\}_{i=1}^N$ be a finite collection of disjoint subintervals from $[a,b]$, where

$$E\Big(\sum_{i=1}^N(\langle X\rangle_{b_i}-\langle X\rangle_{a_i})\Big)\leqslant\eta.$$

Then f is Kurzweil-Henstock stochastic integrable on each $[a_i,b_i]$, $i=1,2,3,\ldots,N$. On each $[a_i,b_i]$ there exist a positive function δ_i and a positive number η_i such that

$$E\Big(\Big|(D_i)\sum f_\xi[X_v-X_\xi]-\int_{a_i}^{b_i} f_t\,\mathrm{d}X_t\Big|^2\Big)\leqslant\frac{\varepsilon}{2^{2i}}$$

whenever $D_i=\{((\xi,v],\xi)\}$ is a δ_i-fine belated partial division of $[a_i,b_i]$ which fails to cover $[a_i,b_i]$ by at most η_i, and it is clear that $\sum\eta_i\leqslant\eta$. We may assume that $\delta_i(\xi)<\delta(\xi)$ for each $i=1,2,\ldots,N$. Now $D=\bigcup_{i=1}^N D_i$ is a δ-fine belated partial division of $[a,b]$ with

$$E\Big(\Big(\bigcup_i D_i\Big)\sum(\langle X\rangle_v-\langle X\rangle_\xi)\Big)\leqslant E\Big(\sum(\langle X\rangle_{b_i}-\langle X\rangle_{a_i})\Big)\leqslant\eta,$$

so that we have

$$E\biggl(\biggl|\biggl(\bigcup_{i=1}^{N} D_i\biggr)\sum f_\xi[X_v - X_\xi]\biggr|\biggr)^2 \leqslant \varepsilon.$$

Consequently,

$$\begin{aligned} E\biggl(\biggl|\sum_i \int_{a_i}^{b_i} f_t \,\mathrm{d}X_t\biggr|^2\biggr) &\leqslant 2E\biggl(\biggl|\sum_i \biggl\{\int_{a_i}^{b_i} f_t \,\mathrm{d}X_t - (D_i)\sum f_\xi[X_v - X_\xi]\biggr\}\biggr|^2\biggr) \\ &\quad + 2E\biggl(\biggl|\sum_i (D_i)\sum f_\xi[X_v - X_\xi]\biggr|^2\biggr) \\ &\leqslant 2\Biggl\{\sum_i \sqrt{E\biggl(\biggl|\int_{a_i}^{b_i} f_t \,\mathrm{d}X_t - (D_i)\sum f_\xi[X_v - X_\xi]\biggr|^2\biggr)}\Biggr\}^2 + 2\varepsilon \\ &\leqslant 2\biggl\{\sum_{i=1}^{\infty} \frac{\sqrt{\varepsilon}}{2^i}\biggr\}^2 + 2\varepsilon \leqslant 4\varepsilon, \end{aligned}$$

showing that Φ possesses the AC_X property, thereby completing our proof. □

Theorem 23. *Let f and F be stochastic processes on $[a,b]$. Then f is Kurzweil-Henstock stochastic integrable on $[a,b]$ with $F(u,v) = \int_u^v f_t \,\mathrm{d}X_t$ if and only if F has the AC_X property and for every $\varepsilon > 0$ there exists a positive function δ on $[a,b]$ such that whenever $D = \{((\xi_i, v_i], \xi_i)\}_{i=1}^n$ is a δ-fine belated partial division of $[a,b]$, we have*

$$E\biggl(\sum_{i=1}^{n}(f_{\xi_i}(X_{v_i} - X_{\xi_i}) - F(\xi_i, v_i))\biggr)^2 \leqslant \varepsilon.$$

Proof. Sufficiency is guaranteed by Theorem 22 and Lemma 15. We just need to prove the converse. Given $\varepsilon > 0$, choose a positive η such that whenever $\{(u_i, v_i]\}_{i=1}^m$ is a finite collection of subintervals of $[a,b]$ with

$$E\biggl(\sum_{i=1}^{m}(\langle X\rangle_{v_i} - \langle X\rangle_{u_i})\biggr) \leqslant \eta$$

we have

$$E\biggl(\biggl|\sum_{i=1}^{m} F(u_i, v_i)\biggr|^2\biggr) \leqslant \varepsilon.$$

Choose a δ-fine partial division $D = \{((\xi, v], \xi)\}$ of $[a,b]$ such that it fails to cover $[a,b]$ by at most $\eta > 0$. Then the measure of the part of $[a,b]$ not covered by D is at most η. Denote this finite collection of subintervals as $\{(s_i, t_i]\}_{i=1}^N$. Hence

$$E\Big(\Big|(D)\sum f_\xi(X_v - X_\xi) - F(a,b)\Big|^2\Big) \leqslant 2E\Big(\Big|(D)\sum (f_\xi(X_v - X_\xi) - F(\xi,v))\Big|^2\Big) + 2E\Big(\Big|\sum_{i=1}^N F(s_i,t_i)\Big|^2\Big) \leqslant 2\varepsilon + 2\varepsilon = 4\varepsilon,$$

showing that f is Kurzweil-Henstock stochastic integrable to $F(a,b)$ on $[a,b]$. Similarly, we can show that f is Kurzweil-Henstock stochastic integrable to $F(a,v)$ on $[a,v]$ for any $v \in [a,b]$, hence completing the proof. □

5. Integrable Functions

In this section we shall prove that if $f \in \mathscr{L}^2$ is μ_X-measurable, then f is Kurzweil-Henstock stochastic integrable.

We use the same setting throughout: fix $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$ to be the standard filtering space as in the previous sections. An adapted process f on $[a,b]$ is said to be a *simple step process* on $[a,b]$ if f can be written as

$$f_t(\omega) = \alpha_0(\omega)1_{\{a\}}(t) + \sum_{i=1}^n \alpha_i(\omega)1_{(u_i,v_i]}(t),$$

where each α_i is a $\mathscr{F}_{u_i}$-measurable bounded random variable for each $i = 1,2,\ldots,n$; the random variable α_0 is $\mathscr{F}_a$-measurable. Let $\{(u_i,v_i]\}_{i=1}^n$ be a finite collection of disjoint left-open subintervals of $[a,b]$ with $\bigcup(u_i,v_i] = (a,b]$.

Lemma 24. *Let f be a simple step process as given above. Then f is integrable and we have*

$$\int_a^b f_t \,\mathrm{d}X_t = \sum_{i=1}^n \alpha_i(X_{v_i} - X_{u_i})$$

and

$$E\Big(\int_a^b f_t \,\mathrm{d}X_t\Big)^2 = E\Big(\int_a^b f_t^2 \,\mathrm{d}\langle X\rangle_t\Big).$$

P r o o f. We only need to consider the adapted simple step process

$$f_t = \alpha_0 1_{\{a\}}(t) + \alpha_1 1_{(a,b]}(t),$$

where $|\alpha_0(\omega)| \leqslant M$ and $|\alpha_1(\omega)| \leqslant M$ for all $\omega \in \Omega$. By Example 6, we may assume that $\alpha_0 \equiv 0$, i.e. $f_t = \alpha_1 1_{(a,b]}(t)$.

Let δ be a positive function on $[a,b]$. For any subinterval $(u,v]$, define

$$F(u,v] = \alpha_1(X_{b\wedge v} - X_{a\wedge u}).$$

Let $D = \{((\xi,v],\xi)\}$ be a δ-fine belated partial division of $[a,b]$. Then

$$(D)\sum(f_\xi(X_v - X_\xi) - \alpha_1(X_v - X_\xi)) = 0.$$

Further, it is clear that $\alpha_1(X_v - X_\xi)$ possesses the AC_X property. By Theorem 23, we get the required result. □

Theorem 25. *Let $f \in \mathscr{L}^2$ be μ_X-measurable. Then f is Kurzweil-Henstock stochastic integrable on $[a,b]$.*

Proof. According to the classical theory of integration with respect to measures, there exists a sequence $\{f^{(n)}\}$ of simple step processes such that

$$E\biggl(\int_a^b (f_t^{(n)} - f_t)^2 \,\mathrm{d}\langle X\rangle_t\biggr) \to 0$$

as $n \to \infty$. Hence

$$E\biggl(\int_a^b (f_t^{(n)} - f_t^{(m)})^2 \,\mathrm{d}\langle X\rangle_t\biggr) \to 0$$

as $n,m \to \infty$.

On the other hand,

$$E\biggl(\int_a^b (f_t^{(n)} - f_t^{(m)}) \,\mathrm{d}X_t\biggr)^2 = E\biggl(\int_a^b (f_t^{(n)} - f_t^{(m)})^2 \,\mathrm{d}\langle X\rangle_t\biggr).$$

Thus there exists $A \in L^2(\Omega)$ such that

$$E\biggl(\int_a^b f_t^{(n)} \,\mathrm{d}X_t - A\biggr)^2 \to 0$$

as $n \to \infty$. We claim that f is Kurzweil-Henstock stochastic integrable to A.

Let $\varepsilon > 0$. Then there exists a positive integer n such that

$$E\biggl(\int_a^b f_t^{(n)} \,\mathrm{d}X_t - A\biggr)^2 \leqslant \varepsilon$$

and

$$E\biggl(\int_a^b (f_t^{(n)} - f)^2 \,\mathrm{d}\langle X\rangle_t\biggr) \leqslant \varepsilon.$$

Let n be fixed and let $f^{(n)}$ be denoted by g in the following formulas:

$$\begin{aligned}E(|S(f,D,\delta,\eta)-A|^2) \leqslant 4E(|S(f,D,\delta,\eta)-S(g,D,\delta,\eta)|^2)\\ + 4E\Bigl(\Bigl|S(g,D,\delta,\eta)-\int_a^b g_t\,\mathrm{d}X_t\Bigr|^2\Bigr)\\ + 4E\Bigl(\Bigl|\int_a^b g_t\,\mathrm{d}X_t - A\Bigr|^2\Bigr);\end{aligned}$$

and

$$E(|S(f,D,\delta,\eta)-S(g,D,\delta,\eta)|^2) = E(|\tilde{S}((f-g)^2,D,\delta,\eta)|).$$

Therefore, f is Kurzweil-Henstock stochastic integrable to A on $[a,b]$. □

6. Convergence theorems

It was shown in [2], [23], [24], [25] that the Kurzweil-Henstock stochastic integral encompasses the classical Itô integral. The techniques of proofs use implicitly some form of convergence theorems.

In this section we shall establish three convergence theorems for the Kurzweil-Henstock stochastic integrals. The integrator X in this section is restricted to the classical Brownian motion.

From the previous section, any process A on $[a,b]$ can be treated as a random variable defined on all left-open intervals by letting $A[u,v]$ denote $A_v - A_u$ for any subinterval $[u,v]$ of $[a,b]$.

Definition 26 (Mean convergence). Let A and $A^{(n)}$, $n=1,2,3,\ldots$, be stochastic processes on $[a,b]$. Then $A^{(n)}$ is said to *converge in mean* to A if given $\varepsilon>0$ there exists a positive integer N such that for any finite collection of disjoint intervals $\{[u_i,v_i]\colon\ i=1,2,3,\ldots,q\}$ we have

$$E\Bigl(\sum_{i=1}^{q}\{A^{(n)}(u_i,v_i)-A(u_i,v_i)\}\Bigr)^2 \leqslant \varepsilon$$

for all $n \geqslant N$.

Theorem 27 (Mean Convergence Theorem). *Let $f^{(n)}$, $n=1,2,\ldots$, be a sequence of Itô-Henstock integrable processes (see Definition* 4a*) on $[a,b]$ and let f be an adapted process such that*

1. *for almost all $t\in[a,b]$, $E(f_t^{(n)}-f_t)^2 \to 0$ as $n\to\infty$;*
2. *$E\bigl(\int_a^b (f^{(n)}-f^{(m)})_t\,\mathrm{d}X_t\bigr)^2 \to 0$ as $n,m\to\infty$.*

Then f is Itô-Henstock integrable on $[a,b]$ and

$$E\left(\int_a^b (f^{(n)} - f)_t \, \mathrm{d}X_t\right)^2 \to 0$$

as $n \to \infty$.

We shall skip the proof of Theorem 27 which was extensively used (and proved) in [2], [23], [24], [25] to show the scope of Henstock's approach. As a natural corollary of the Mean Convergence Theorem, we have the Uniform Convergence Theorem for the Kurzweil-Henstock stochastic integral:

Corollary 28 (Uniform Convergence Theorem). *Let $f^{(n)}$, $n = 1, 2, \ldots$, be a sequence of Itô-Henstock stochastic integrable processes on $[a,b]$ and let f be a stochastic process on $[a,b]$ adapted to the standard filtering space such that*

1. *$E(f_t^{(n)} - f_t)^2 \to 0$ uniformly as $n \to \infty$;*
2. *$E\left(\int_a^b (f^{(n)} - f^{(m)})_t \, \mathrm{d}X_t\right)^2 \to 0$ as $n, m \to \infty$.*

Then f is Itô-Henstock integrable on $[a,b]$ with

$$E\left(\int_a^b (f^{(n)} - f)_t \, \mathrm{d}X_t\right)^2 \to 0$$

as $n \to \infty$.

Note that the Uniform Convergence Theorem is a direct consequence of the Mean Convergence Theorem, where the choice of n is independent of the value of $t \in [a,b]$ as in the proof of the Mean Convergence Theorem (see [2], [23], [24], [25]). Finally, we state and prove the Dominated Convergence Theorem for the Kurzweil-Henstock stochastic integral.

Theorem 29 (Dominated Convergence Theorem). *Let $f^{(n)}$, $n = 1, 2, 3, \ldots$, be a sequence of Itô-Henstock integrable processes (see Definition* 4a*) on $[a,b]$. Let f and g be stochastic processes on $[a,b]$ having the following properties:*

(i) *$E\left(f_t^{(n)} - f_t\right)^2 \to 0$ as $n \to \infty$ for almost all $t \in [a,b]$;*

(ii) *$|f_t(\omega)| \leqslant g_t(\omega)$ for almost all $\omega \in \Omega$ and almost all $t \in [a,b]$; and $E[g_t^2]$ is Lebesgue integrable over $[a,b]$.*

Then f is Itô-Henstock integrable on $[a,b]$. Furthermore,

$$E\left(\int_a^b (f^{(n)} - f)_t^2 \, \mathrm{d}Xt\right) \to 0$$

as $n \to \infty$.

P r o o f. By the Isometric Equality (Theorem 17), $E[{f_t^{(n)}}^2]$ is Lebesgue integrable on $[a,b]$. By the Dominated Convergence Theorem for the classical Lebesgue integral, $E[f_t^2]$ is integrable on $[a,b]$ and $(L)\int_a^b E[f_t^{(n)} - f_t]^2 \,\mathrm{d}t \to 0$ as $n \to \infty$. Hence, we have that $(L)\int_a^b E[f_t^{(n)} - f_t^{(m)}]^2 \,\mathrm{d}t \to 0$ as $m,n \to \infty$. Applying Theorem 17 again,

$$E\biggl(\int_a^b (f^{(n)} - f^{(m)})_t \,\mathrm{d}X_t\biggr)^2 \to 0$$

as $n,m \to \infty$. Finally, the result is obtained by applying the Mean Convergence Theorem to $\{f^{(n)}\}$. □

7. Conclusion

This note is an extension of our earlier papers where we attempted to show that the Kurzweil-Henstock approach could be used to study stochastic integrals. In this note, we have used the Kurzweil-Henstock approach to provide a unified theory of stochastic integration theory. Using this approach, some useful results of classical (non-stochastic) integration theory can be shown to be satisfied by the stochastic integral. These results include Henstock's Lemmas, the absolute continuity of the primitive and convergence theorems.

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