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The L^2–Alexander torsion is symmetric

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We show that the L^2–Alexander torsion of a 3–manifold is a symmetric function. This can be viewed as a generalization of the symmetry of the Alexander polynomial of a knot.

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1 Introduction

An *admissible triple* (N,ϕ,γ) consists of an irreducible, orientable, compact 3–manifold $N \neq S^1 \times D^2$ with empty or toroidal boundary, a class $\phi \neq 0 \in H^1(N;\mathbb{Z}) = \operatorname{Hom}(\pi_1(N),\mathbb{Z})$ and a homomorphism $\gamma\colon \pi_1(N) \to G$ such that ϕ factors through γ. In [4; 5] we used the L^2–torsion (see for example Lück [14]) to associate to an admissible triple (N,ϕ,γ) the L^2–Alexander torsion $\tau^{(2)}(N,\phi,\gamma)$ which is a function

$$\tau^{(2)}(N,\phi,\gamma)\colon \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$$

that is well defined up to multiplication by a function of the type $t \mapsto t^m$ for some $m \in \mathbb{Z}$. We recall the definition in Section 6.

The goal of this paper is to show that the L^2–Alexander torsion is symmetric. In order to state the precise symmetry result we need to recall that given a 3–manifold N the *Thurston norm* [16] of some $\phi \in H^1(N;\mathbb{Z}) = \operatorname{Hom}(\pi_1(N),\mathbb{Z})$ is defined as

$$x_N(\phi) := \min\{\chi_-(S) \mid S \subset N \text{ properly embedded surface dual to } \phi\}.$$

Here, given a surface S we define its complexity as $\chi_-(S) := -\chi(S')$, where S' is the result of deleting all components from S that are disks or spheres. Thurston [16] showed that x_N is a (possibly degenerate) norm on $H^1(N;\mathbb{Z})$. Now we can formulate the main result of this paper.

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Theorem 1.1 *Let (N,ϕ,γ) be an admissible triple. Then for any representative τ of $\tau^{(2)}(N,\phi,\gamma)$ there exists an $n \in \mathbb{Z}$ with $n \equiv x_N(\phi) \bmod 2$ such that*

$$\tau(t^{-1}) = t^n \cdot \tau(t) \text{ for any } t \in \mathbb{R}_{>0}.$$

It is worth looking at the case that $N = S^3 \setminus \nu K$ is the complement of a tubular neighborhood νK of an oriented knot $K \subset S^3$. We denote by $\phi_K\colon \pi_1(N) \to \mathbb{Z}$ the epimorphism sending the oriented meridian to 1. Let $\gamma\colon \pi_1(N) \to G$ be a homomorphism such that ϕ_K factors through γ. We define

$$\tau^{(2)}(K,\gamma) := \tau^{(2)}(S^3 \setminus \nu K, \phi_K, \gamma).$$

If we take $\gamma = \mathrm{id}$ to be the identity, then we showed in [4] that

$$\tau^{(2)}(K,\mathrm{id}) = \Delta_K^{(2)}(t) \cdot \max\{1,t\},$$

where $\Delta_K^{(2)}(t)\colon \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ denotes the L^2–Alexander invariant of Li and Zhang [12; 13], which was also studied by Dubois and Wegner [6; 7] and Aribi [1; 2].

If we take $\gamma = \phi_K$, then we showed in [4] that the L^2–Alexander torsion $\tau^{(2)}(K,\phi_K)$ is fully determined by the Alexander polynomial $\Delta_K(t)$ of K and that in turn $\tau^{(2)}(K,\phi_K)$ almost determines the Alexander polynomial $\Delta_K(t)$. In this sense the L^2–Alexander torsion can be viewed as a "twisted" version of the Alexander polynomial, and at least morally it is related to the twisted Alexander polynomial of Wada [20] and to the higher-order Alexander polynomials of Cochran [3] and Harvey [10]. We refer to [5] for more on the relationship and similarities between the various twisted invariants.

If K is a knot, then any Seifert surface is dual to ϕ_K and it immediately follows that $x(\phi_K) \leq \max\{2\cdot \mathrm{genus(K)} - 1, 0\}$. In fact an elementary argument shows that for any *non-trivial* knot we have the equality $x(\phi_K) = 2\cdot\mathrm{genus(K)} - 1$. In particular the Thurston norm of ϕ_K is odd. We thus obtain the following corollary to Theorem 1.1.

Theorem 1.2 *Let $K \subset S^3$ be an oriented non-trivial knot and let $\gamma\colon \pi_1(N) \to G$ be a homomorphism such that ϕ_K factors through γ. Then there exists an odd n with*

$$\tau^{(2)}(K,\gamma)(t^{-1}) = t^n \cdot \tau^{(2)}(K,\gamma)(t) \quad \text{for any } t \in \mathbb{R}_{>0}.$$

The proof of Theorem 1.1 has many similarities with the proof of the main theorem in Friedl, Kim and Kitayama [9], which in turn builds on the ideas of Turaev [17; 18; 19]. In an attempt to keep the proof as short as possible we will on several occasions refer to [9] and [17] for definitions and results.

Conventions All manifolds are assumed to be connected, orientable and compact. All CW–complexes are assumed to be finite and connected. If G is a group then we equip $\mathbb{C}[G]$ with the involution given by complex conjugation and by $\bar{g} := g^{-1}$ for $g \in G$. We extend this involution to matrices over $\mathbb{C}[G]$ by applying the involution to each entry. Given a ring R we will view all modules as left R–modules, unless we say explicitly otherwise. Furthermore, given a matrix $A \in M_{m,n}(R)$ we denote by $A\colon R^m \to R^n$ the R–homomorphism of left R–modules obtained by right multiplication with A and thinking of elements in R^m as the only row in a $(1,m)$–matrix.

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2 Euler structures

In this section we recall the notion of an Euler structure of a pair of CW–complexes and manifolds which is due to Turaev. We refer to [17; 18; 9] for full details. Throughout this paper, given a space X, we denote by $\mathcal{H}_1(X)$ the first integral homology group viewed as a multiplicative group.

2A Euler structures on CW–complexes

Let X be a CW–complex of dimension m and let Y be a proper subcomplex. We denote by $p\colon \widetilde{X} \to X$ the universal covering of X and we write $\widetilde{Y} := p^{-1}(Y)$. An *Euler lift* is a set of cells in $\widetilde{X}$ such that each cell of $X \setminus Y$ is covered by precisely one of the cells in the Euler lift.

Using the canonical left action of $\pi = \pi_1(X)$ on $\widetilde{X}$ we obtain a free and transitive action of π on the set of cells of $\widetilde{X} \setminus \widetilde{Y}$ lying over a fixed cell in $X \setminus Y$. If c and c' are two Euler lifts, then we can order the cells such that $c = \{c_{ij}\}$ and $c' = \{c'_{ij}\}$ and such that for each i and j the cells c_{ij} and c'_{ij} lie over the same i–cell in $X \setminus Y$. In particular there exist unique $g_{ij} \in \pi$ such that $c'_{ij} = g_{ij} \cdot c_{ij}$. We denote the projection map $\pi \to \mathcal{H}_1(X)$ by Ψ. We define

$$c'/c := \prod_{i=0}^{m} \prod_{j} \Psi(g_{ij})^{(-1)^i} \in \mathcal{H}_1(X).$$

We say that c and c' are *equivalent* if $c'/c \in \mathcal{H}_1(X)$ is trivial. An equivalence class of Euler lifts will be referred to as an *Euler structure*. We denote by $\mathrm{Eul}(X,Y)$ the set of Euler structures. If $Y=\varnothing$ then we will also write $\mathrm{Eul}(X)=\mathrm{Eul}(X,Y)$.

Given $g \in \mathcal{H}_1(X)$ and $e \in \mathrm{Eul}(X,Y)$ we define $g\cdot e \in \mathrm{Eul}(X,Y)$ as follows: pick representatives c for e and $\widetilde{g} \in \pi_1(X)$ for g, then act on one i–cell of c by $g^{(-1)^i}$. The resulting Euler lift represents an element in $\mathrm{Eul}(X,Y)$ which is independent of the choice of the cell. We denote by $g\cdot e$ the Euler structure represented by this new Euler lift. This defines a free and transitive $\mathcal{H}_1(X)$–action on $\mathrm{Eul}(X,Y)$, with $(g\cdot e)/e=g$.

If (X',Y') is a cellular subdivision of (X,Y), then there exists a canonical $\mathcal{H}_1(X)$–equivariant bijection $\sigma\colon \mathrm{Eul}(X,Y)\to \mathrm{Eul}(X',Y')$ which is defined as follows. Let $e\in \mathrm{Eul}(X,Y)$ and pick an Euler lift for (X,Y) which represents e. There exists a unique Euler lift for (X',Y') such that the cells in the Euler lift of (X',Y') are contained in the cells of the Euler lift of (X,Y). We denote by $\sigma(e)$ the Euler structure represented by this Euler lift. This map equals the map of Turaev [17, Section 1.2].

2B Euler structures of smooth manifolds

Let N be a manifold and let $\partial_0 N \subset \partial N$ be a union of components of ∂N such that $\chi(N,\partial_0 N)=0$. A *triangulation* of N is a pair (X,t) where X is a simplicial complex and $t\colon |X|\to N$ is a homeomorphism. Throughout this section we write $Y:=t^{-1}(\partial_0 N)$. For the most part we will suppress t from the notation. Following [17, Section I.4.1] we consider the projective system of sets $\{\mathrm{Eul}(X,Y)\}_{(X,t)}$, where (X,t) runs over all C^1–triangulations of N and where the maps are the $\mathcal{H}_1(N)$–equivariant bijections between these sets induced either by C^1–subdivisions or by smooth isotopies in N. We define $\mathrm{Eul}(N,\partial_0 N)$ by identifying the sets $\{\mathrm{Eul}(X,Y)\}_{(X,t)}$ via these bijections. We refer to $\mathrm{Eul}(N,\partial_0 N)$ as the set of *Euler structures on* $(N,\partial_0 N)$. For a C^1–triangulation X of N we get a canonical $\mathcal{H}_1(N)$–equivariant bijection $\mathrm{Eul}(X,Y)\to \mathrm{Eul}(N,\partial_0 N)$.

3 The L^2–torsion of a manifold

3A The L^2–torsion of a chain complex

First we recall some key properties of the Fuglede–Kadison determinant and the definition of the L^2–torsion of a chain complex of free based left $\mathbb{C}[G]$–modules. Throughout the section we refer to [14] and to [4] for details and proofs.

We fix a group G. Let A be a $k\times l$–matrix over $\mathbb{C}[G]$. There exists the notion of A being of *determinant class*. (To be more precise, we view the $k\times l$–matrix A as a map

$\mathcal{N}(G)^l \to \mathcal{N}(G)^k$, where $\mathcal{N}(G)$ is the von Neumann algebra of G, and then there is the notion of being of *determinant class*.) We treat this entirely as a black box, but we note that if G is residually amenable, eg a 3–manifold group [11] or solvable, then by [8] any matrix over $\mathbb{Q}[G]$ is of determinant class. If the matrix A is not of determinant class then for the purpose of this paper we define $\det_{\mathcal{N}(G)}(A) = 0$. On the other hand, if A is of determinant class, then we define

$$\det_{\mathcal{N}(G)}(A) := \text{Fuglede–Kadison determinant of } A \in \mathbb{R}_{>0}.$$

Here we do not assume that A is a square matrix. In an attempt to keep the paper as short as possible we will not provide the (somewhat lengthy) definition of the Fuglede–Kadison determinant. Instead we summarize a few key properties in the following theorem which is a consequence of [14, Example 3.12] and [14, Theorem 3.14].

Theorem 3.1 (1) *If A is a square matrix with complex entries such that the usual determinant* $\det(A) \in \mathbb{C}$ *is non-zero, then* $\det_{\mathcal{N}(G)}(A) = |\det(A)|$.

(2) *The determinant does not change if we swap two rows or two columns.*

(3) *Right multiplication of a column by $\pm g$, $g \in G$ does not change the determinant.*

(4) *For any matrix A over $\mathbb{C}[G]$ we have* $\det_{\mathcal{N}(G)}(A) = \det_{\mathcal{N}(G)}(\bar{A}^t)$.

Note that (2) implies that when we study Fuglede–Kadison determinants of homomorphisms we can work with unordered bases. Now let

$$C_* = (0 \to C_l \xrightarrow{\partial_l} C_{l-1} \xrightarrow{\partial_{l-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0)$$

be a chain complex of free left $\mathbb{C}[G]$–modules. We refer to [14] for the definition of the L^2–Betti numbers $b_i^{(2)}(C_*) \in \mathbb{R}_{\geq 0}$. Now suppose that the chain complex is equipped with bases $B_i \subset C_i$, $i = 0, \dots, l$. If one of the L^2–Betti numbers $b_i^{(2)}(C_*)$ is non-zero or if one the boundary maps is not of determinant class, then we define the L^2–torsion $\tau^{(2)}(C_*, B_*) := 0$. Otherwise we define the L^2*–torsion* to be

$$\tau^{(2)}(C_*, B_*) := \prod_{i=1}^{l} \det_{\mathcal{N}(G)}(A_i)^{(-1)^i} \in \mathbb{R}_{>0},$$

where the A_i denote the boundary matrices corresponding to the given bases. This definition is the multiplicative inverse of the exponential of the L^2–torsion as defined in [14, Definition 3.29].

3B The twisted L^2–torsion of CW–complexes and manifolds

Let (X,Y) be a pair of CW–complexes and let $e\in\mathrm{Eul}(X,Y)$. We denote by $p\colon \widetilde{X}\to X$ the universal covering of X and we write $\widetilde{Y}:=p^{-1}(Y)$. The deck transformation turns $C_*(\widetilde{X},\widetilde{Y})$ naturally into a chain complex of left $\mathbb{Z}[\pi_1(X)]$–modules.

Now let G be a group and let $\varphi\colon \pi(X)\to\mathrm{GL}(d,\mathbb{C}[G])$ be a representation. We view elements of $\mathbb{C}[G]^d$ as row vectors. Right multiplication via $\varphi(g)$ thus turns $\mathbb{C}[G]^d$ into a right $\mathbb{Z}[\pi_1(X)]$–module. We consider the chain complex

$$C_*^{\varphi}(X,Y;\mathbb{C}[G]^d):=\mathbb{C}[G]^d\otimes_{\mathbb{Z}[\pi_1(X)]}C_*(\widetilde{X},\widetilde{Y})$$

of left $\mathbb{C}[G]$–modules. Let $e\in\mathrm{Eul}(X,Y)$. We pick an Euler lift $\{c_{ij}\}$ that represents e. Throughout this paper we denote by $v_1,\dots,v_d$ the standard basis for $\mathbb{C}[G]^d$. We equip the chain complex $C_*^{\varphi}(X,Y;\mathbb{C}[G]^d)$ with the basis provided by the $v_k\otimes c_{ij}$. Therefore we can define

$$\tau^{(2)}(X,Y,\varphi,e):=\tau^{(2)}\big(C_*^{\varphi}(X,Y;\mathbb{C}[G]^d),\{v_k\otimes c_{ij}\}\big)\in\mathbb{R}_{\geq 0}.$$

Lemma 3.2 (1) *The number $\tau^{(2)}(X,Y,\varphi,e)$ is well defined.*

(2) *If $g\in\mathcal{H}_1(X)$, then*

$$\tau^{(2)}(X,Y,\varphi,ge)=\det\nolimits_{\mathcal{N}(G)}(\varphi(g^{-1}))\cdot\tau^{(2)}(X,Y,\varphi,e).$$

(3) *If (X',Y') is a cellular subdivision of (X,Y) and if $e'\in\mathrm{Eul}(X',Y')$ is the Euler structure corresponding to e, then*

$$\tau^{(2)}(X',Y',\varphi,e')=\tau^{(2)}(X,Y,\varphi,e).$$

The proofs are completely analogous to the proofs for ordinary Reidemeister torsion as given in [18; 9]. In the interest of space we will not provide the proofs.

Finally let N be a manifold and let $\partial_0 N\subset\partial N$ be a union of components of ∂N with $\chi(N,\partial_0 N)=0$. Let G be a group and let $\varphi\colon \pi(N)\to\mathrm{GL}(d,\mathbb{C}[G])$ be a representation. Let $e\in\mathrm{Eul}(N,\partial_0 N)$. Recall that for any C^1–triangulation $f\colon X\to N$ we get a bijection $\mathrm{Eul}(X,Y)\xrightarrow{f_*}\mathrm{Eul}(N,\partial_0 N)$. We define

$$\tau^{(2)}(N,\partial_0 N,\varphi,e):=\tau^{(2)}(X,Y,\varphi\circ f_*,f_*^{-1}(e)).$$

By Lemma 3.2(3) and the discussion in [17] the invariant $\tau^{(2)}(N,\partial_0 N,\varphi,e)\in\mathbb{R}_{\geq 0}$ is well defined, ie independent of the choice of the triangulation.

4 Duality for torsion of manifolds equipped with Euler structures

4A The algebraic duality theorem for L^2–torsion

Let G be a group and let V be a right $\mathbb{C}[G]$–module. We denote by $\overline{V}$ the left $\mathbb{C}[G]$–module with the same underlying abelian group but with the module structure given by $p \cdot_{\overline{V}} v := v \cdot_V \bar{p}$ for $p \in \mathbb{C}[G]$ and $v \in V$. If V is a left $\mathbb{C}[G]$–module then we can consider $\operatorname{Hom}_{\mathbb{C}[G]}(V, \mathbb{C}[G])$, the set of all left $\mathbb{C}[G]$–module homomorphisms. Since the range $\mathbb{C}[G]$ is a $\mathbb{C}[G]$–bimodule we can naturally view $\operatorname{Hom}_{\mathbb{C}[G]}(V, \mathbb{C}[G])$ as a right $\mathbb{C}[G]$–module.

In the following let C_* be a chain complex of length m of left $\mathbb{C}[G]$–modules with boundary operators ∂_i. Suppose that C_* is equipped with a basis B_i for each C_i. We denote by $C^\#$ the *dual chain complex* whose chain groups are the $\mathbb{C}[G]$–left modules $C_i^\# := \overline{\operatorname{Hom}_{\mathbb{C}[G]}(C_{m-i}, \mathbb{C}[G])}$ and where the boundary map $\partial_i^\#\colon C_{i+1}^\# \to C_i^\#$ is given by $(-1)^{m-i}\partial^*_{m-i-1}$. This means that for any $c \in C_{m-i}$ and $d \in C_{i+1}^\#$ we have $\partial_i^\#(d)(c) = (-1)^{m-i} d(\partial_{m-i}(c))$. We denote by $B_*^\#$ the bases of $C^\#$ dual to the bases B_*.

Lemma 4.1 *If $\tau^{(2)}(C_*, B_*) = 0$, then $\tau^{(2)}(C_*^\#, B_*^\#) = 0$, otherwise we have*

$$\tau^{(2)}(C_*, B_*) = \tau^{(2)}(C_*^\#, B_*^\#)^{(-1)^{m+1}}.$$

Proof By the proof of [14, Theorem 1.35(3)] the L^2–Betti numbers of C_* vanish if and only if the L^2–Betti numbers of $C_*^\#$ vanish. In particular, if either L^2–Betti number does not vanish, then both torsions are zero.

Now we suppose that the L^2–Betti numbers of C_* vanish. We denote by A_i the corresponding matrices of the boundary maps of C_*. The boundary matrices of the chain complex $C_*^\#$ with respect to the basis $B_*^\#$ are given by $(-1)^{m-i}\overline{A}_i^t$. Now the lemma is an immediate consequence of the definitions and of Theorem 3.1(4). □

4B The duality theorem for manifolds

Before we state our main technical duality theorem we need to introduce two more definitions.

(1) Let G be a group and let $\varphi\colon \pi \to \mathrm{GL}(d, \mathbb{C}[G])$ be a representation. We denote by $\varphi^\dagger$ the representation which is given by $g \mapsto \overline{\varphi(g^{-1})}^t$.

(2) Let N be an m–manifold and let $e \in \mathrm{Eul}(N, \partial N)$. Pick a triangulation X for N. We denote by Y the subcomplex corresponding to ∂N. Let $X^\dagger$ be the CW–complex that is given by the cellular decomposition of N dual to X. Pick

an Euler lift $\{c_{ij}\}$ that represents $e \in \operatorname{Eul}(X,Y) = \operatorname{Eul}(N,\partial N)$. For any i–cell c in $\widetilde{X} \setminus \widetilde{Y}$ we denote by $c^\dagger$ the unique oriented $(m-i)$–cell in $\widetilde{X}^\dagger$ which has intersection number $+1$ with c. The Euler lift $\{c_{ij}^\dagger\}$ defines an element in $\operatorname{Eul}(X^\dagger) = \operatorname{Eul}(N)$ that we denote by $e^\dagger$. This map is an $\mathcal{H}_1(N)$–equivariant bijection and we denote the inverse map $\operatorname{Eul}(N,\partial N) \to \operatorname{Eul}(N)$ again by $e \mapsto e^\dagger$. We refer to [15, Chapter 70], [18, Section 14] and [9, Section 4] for details.

Theorem 4.2 *Let N be an m–manifold. Let G be a group and let $\varphi\colon \pi(N) \to \mathrm{GL}(d,\mathbb{C}[G])$ be a representation. Let $e \in \operatorname{Eul}(N,\partial N)$. Then either $\tau^{(2)}(N,\partial N,\varphi,e)$ and $\tau^{(2)}(N,\varphi^\dagger,e^\dagger)$ are both zero, or the following equality holds:*

$$\tau^{(2)}(N,\partial N,\varphi,e) = \tau^{(2)}(N,\varphi^\dagger,e^\dagger)^{(-1)^{m+1}}.$$

Proof Pick a triangulation X for N and denote by Y the subcomplex corresponding to ∂N. Let $X^\dagger$ be the CW–complex which is given by the cellular decomposition of N dual to X. We identify $\pi = \pi_1(X) = \pi_1(N) = \pi_1(X^\dagger)$. We pick an Euler lift $\{c_{ij}\}$ which represents $e \in \operatorname{Eul}(N,\partial N) = \operatorname{Eul}(X,Y)$. We denote by $c_{ij}^\dagger$ the corresponding dual cells. The theorem follows from the definitions and the following claim.

Claim *Either both $\tau^{(2)}(C_*^\varphi(X,Y;\mathbb{C}[G]^d),\{v_k \otimes c_{ij}\})$ and $\tau^{(2)}(C_*^{\varphi^\dagger}(X^\dagger;\mathbb{C}[G]^d),\{v_k \otimes c_{ij}^\dagger\})$ are zero, or the following equality holds:*

$$\tau^{(2)}\big(C_*^\varphi(X,Y;\mathbb{C}[G]^d),\{v_k \otimes c_{ij}\}\big) = \tau^{(2)}\big(C_*^{\varphi^\dagger}(X^\dagger;\mathbb{C}[G]^d),\{v_k \otimes c_{ij}^\dagger\}\big)^{(-1)^{m+1}}.$$

In order to prove the claim we first note that there is a unique, sesquilinear paring

$$\begin{aligned} C_{m-i}(\widetilde{X},\widetilde{Y}) \times C_i(\widetilde{X}^\dagger) &\to \mathbb{Z}[\pi], \\ (a,b) &\mapsto \langle a,b\rangle := \sum_{g\in\pi} (a\cdot gb)g^{-1} \end{aligned}$$

such that $a \cdot b^\dagger = \delta_{ab}$ for any two cells a and b of $\widetilde{X} \setminus \widetilde{Y}$. Here *sesquilinear* means that for any $a \in C_{m-i}(\widetilde{X},\widetilde{Y})$, $b \in C_i(\widetilde{X}^\dagger)$ and $p,q \in \mathbb{Z}[\pi]$ we have $\langle pa,qb\rangle = q\langle a,b\rangle\bar{p}$. It is straightforward to see that the pairing is non-singular. It follows immediately from [18, Claim 14.4]) that these maps give rise to well-defined maps

$$\begin{aligned} C_i(\widetilde{X},\widetilde{Y}) &\to \overline{\operatorname{Hom}_{\mathbb{Z}[\pi]}(C_{m-i}(\widetilde{X}^\dagger),\mathbb{Z}[\pi])}, \\ a &\mapsto (b \mapsto \langle a,b\rangle) \end{aligned}$$

that define an isomorphism of based chain complexes of right $\mathbb{Z}[\pi]$–modules. In fact it follows easily from the definitions that the maps define an isomorphism

$$(C_*(\widetilde{X},\widetilde{Y}),\{c_{ij}\}) \to \big(\overline{\operatorname{Hom}_{\mathbb{Z}[\pi]}(C_{m-*}(\widetilde{X}^\dagger),\mathbb{Z}[\pi])},\{(c_{ij}^\dagger)^*\}\big)$$

of based chain complexes of left $\mathbb{Z}[\pi]$–modules. Tensoring these chain complexes with $\mathbb{C}[G]^d$ we obtain an isomorphism

$$\big(\mathbb{C}[G]^d \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X}, \widetilde{Y}), \{v_k \otimes c_{ij}\}\big) \\ \to \big(\mathbb{C}[G]^d \otimes_{\mathbb{Z}[\pi]} \overline{\mathrm{Hom}_{\mathbb{Z}[\pi]}(C_{m-*}(\widetilde{X}^\dagger), \mathbb{Z}[\pi])}, \{v_k \otimes (c_{ij}^\dagger)^*\}\big)$$

of based chain complexes of $\mathbb{C}[G]$–modules. Furthermore the maps

$$\mathbb{C}[G]^d \otimes_{\mathbb{Z}[\pi]} \overline{\mathrm{Hom}_{\mathbb{Z}[\pi]}(C_i(\widetilde{X}^\dagger), \mathbb{Z}[\pi])} \to \overline{\mathrm{Hom}_{\mathbb{C}[G]}\big(C_i^{\varphi^\dagger}(X^\dagger; \mathbb{C}[G]^d), \mathbb{C}[G]\big)}, \\ v \otimes f \mapsto \begin{pmatrix} C_i^{\varphi^\dagger}(X^\dagger; \mathbb{C}[G]^d) \to \mathbb{C}[G], \\ \qquad w \otimes \sigma \mapsto v\varphi(\overline{f(\sigma)})\overline{w}^t \end{pmatrix}$$

induce an isomorphism

$$\big(C_*^{\varphi}(X, Y; \mathbb{C}[G]^d), \{v_k \otimes c_{ij}\}\big) \to \big(C_*^{\varphi^\dagger}(X^\dagger; \mathbb{C}[G]^d)^\#, \{(v_k \otimes c_{ij}^\dagger)^\#\}\big)$$

of based chain complexes of $\mathbb{C}[G]$–modules. The claim follows from Lemma 4.1. □

5 Twisted L^2–torsion of 3–manifolds

5A Canonical structures on tori

Let T be a torus. We equip T with a CW–structure with one 0–cell p, two 1–cells x and y and one 2–cell s. We write $\pi = \pi_1(T, p)$ and by a slight abuse of notation we denote by x and y the elements in π represented by x and y. We denote by $\widetilde{T}$ the universal cover of T. There exist lifts of the cells such that the chain complex $C_*(\widetilde{T})$ of left $\mathbb{Z}[\pi]$–modules with respect to the bases given by these lifts is of the form

(1) $$0 \to \mathbb{Z}[\pi] \xrightarrow{(y-1 \;\; 1-x)} \mathbb{Z}[\pi]^2 \xrightarrow{\begin{pmatrix} 1-x \\ 1-y \end{pmatrix}} \mathbb{Z}[\pi] \to 0.$$

We refer to the corresponding Euler structure of T as the *canonical Euler structure on* T. Given a group G we say that a representation $\varphi\colon \pi \to \mathrm{GL}(1, \mathbb{C}[G])$ is *monomial* if for any $x \in \pi$ we have $\varphi(x) = zg$ for some $z \in \mathbb{C}$ and $g \in G$. The following is [4, Lemma 5.6].

Lemma 5.1 *Let $\varphi\colon \pi_1(T) \to \mathrm{GL}(1, \mathbb{C}[G])$ be a monomial representation such that $b_*^{(2)}(T; \mathbb{C}[G]) = 0$ and e be the canonical Euler structure on T. Then $\tau^{(2)}(T, \varphi, e) = 1$.*

5B Chern classes on 3–manifolds with toroidal boundary

Let N be a 3–manifold with toroidal incompressible boundary and let $e \in \mathrm{Eul}(N, \partial N)$. Let X be a triangulation for N. We denote the subcomplexes corresponding to the boundary components of N by $S_1 \cup \cdots \cup S_b$. We denote by $p\colon \widetilde{X} \to X$ and $p_i\colon \widetilde{S}_i \to S_i, i = 1, \dots, b$ the universal covering maps of X and $S_i, i = 1, \dots, b$. For each i we identify a component of $p^{-1}(S_i)$ with $\widetilde{S}_i$

Pick an Euler lift c that represents e. For each boundary torus S_i pick an Euler lift $\tilde{s}_i$ to $\widetilde{S}_i \subset p^{-1}(S_i) \subset \widetilde{X}$ that represents the canonical Euler structure. The set of cells $\{\tilde{s}_1, \dots, \tilde{s}_b, c\}$ defines an Euler structure $K(e)$ for N, which only depends on e. Put differently, we defined a map $K\colon \mathrm{Eul}(N, \partial N) \to \mathrm{Eul}(N)$ which is easily seen to be $\mathcal{H}_1(N)$–equivariant. Given $e \in \mathrm{Eul}(N)$ there exists a unique element $g \in \mathcal{H}_1(N)$ such that $e = g \cdot K(e^\dagger)$. Following Turaev [19, page 11] and [9, Section 6.3] we define $c_1(e) := g \in H_1(N; \mathbb{Z})$ and we refer to $c_1(e)$ as the *Chern class of e*.

5C Torsions of 3–manifolds

Let π and G be groups and let $\varphi\colon \pi \to \mathrm{GL}(1, \mathbb{C}[G])$ be a monomial representation. By the multiplicativity of the Fuglede–Kadison determinant, see [14, Theorem 3.14], given $g \in \pi$ the invariant $\det_{\mathcal{N}(G)}(\varphi(g))$ only depends on the homology class of g. Put differently, $\det_{\mathcal{N}(G)} \circ \varphi\colon \pi \to \mathbb{R}_{\geq 0}$ descends to a map $\det_{\mathcal{N}(G)} \circ \varphi\colon H_1(\pi; \mathbb{Z}) \to \mathbb{R}_{\geq 0}$.

Theorem 5.2 *Let N be a 3–manifold which is either closed or which has toroidal, incompressible boundary. Let G be a group and let $\varphi\colon \pi(N) \to \mathrm{GL}(1, \mathbb{C}[G])$ be a monomial representation such that $b_*^{(2)}(\partial N; \mathbb{C}[G]) = 0$. For any $e \in \mathrm{Eul}(N)$ we have*

$$\tau^{(2)}(N, \partial N, \varphi, e^\dagger) = \det_{\mathcal{N}(G)}(\varphi(c_1(e))) \cdot \tau^{(2)}(N, \varphi, e).$$

Proof The assumption that $b_*^{(2)}(\partial N; \mathbb{C}[G]) = 0$ together with the proof of [14, Theorem 1.35(2)] implies that $b_*^{(2)}(N; \mathbb{C}[G]) = 0$ if and only if $b_*^{(2)}(N, \partial N; \mathbb{C}[G]) = 0$. If both are non-zero, then both torsions $\tau^{(2)}(N, \partial N, \varphi, e^\dagger)$ and $\tau^{(2)}(N, \varphi, e)$ are zero. For the remainder of this proof we assume that $b_*^{(2)}(N; \mathbb{C}[G]) = 0$.

Pick a triangulation X for N. As usual denote by Y the subcomplex corresponding to ∂N. Let $e \in \mathrm{Eul}(N)$. Pick an Euler lift c_* which represents $e^\dagger \in \mathrm{Eul}(N, \partial N) = \mathrm{Eul}(X, Y)$. Denote the components of Y by $Y_1 \cup \cdots \cup Y_b$ and pick $\tilde{s}_*^1, \dots, \tilde{s}_*^b$ as in the previous section. We write $\tilde{s}_* = \tilde{s}_*^1 \cup \cdots \cup \tilde{s}_*^b$. Denote by $\{\tilde{s}_* \cup c_*\}$ the resulting Euler lift for X. Recall that this Euler lift represents $K(e^\dagger)$.

Claim $\tau^{(2)}\big(C_*^\varphi(X, Y; \mathbb{C}[G]), \{c_*\}\big) = \tau^{(2)}\big(C_*^\varphi(X; \mathbb{C}[G]), \{\tilde{s}_* \cup c_*\}\big).$

We consider the following short exact sequence of chain complexes

$$0 \to \bigoplus_{i=1}^{b} C_*^{\varphi}(Y_i;\mathbb{C}[G]) \to C_*^{\varphi}(X;\mathbb{C}[G]) \to C_*^{\varphi}(X,Y;\mathbb{C}[G]) \to 0,$$

with the bases $\{s_*^i\}_{i=1,\dots,b}$, $\{\tilde{s}_* \cup c_*\}$ and $\{c_*\}$. These bases are in fact compatible, in the sense that the middle basis is the image of the left basis together with a lift of the right basis. By Lemma 5.1 we have $\tau^{(2)}(C_*^{\varphi}(Y_i;\mathbb{C}[G]),\{\widetilde{s}_*^i\}) = 1$ for $i = 1,\dots,b$. Now it follows from the multiplicativity of torsion, see [14, Theorem 3.35], that

$$\tau^{(2)}\big(C_*^{\varphi}(X,Y;\mathbb{C}[G]),\{c_*\}\big) = \tau^{(2)}\big(C_*^{\varphi}(X;\mathbb{C}[G]),\{c_* \cup \tilde{s}_*\}\big).$$

Here we used that the complexes are acyclic. This concludes the proof of the claim.

Finally it follows from this claim, the definitions and Lemma 3.2 that

$$\begin{aligned}\tau^{(2)}(N,\partial N,\varphi,e^{\dagger}) &= \tau^{(2)}\big(C_*^{\varphi}(X,Y;\mathbb{C}[G]),\{c_*\}\big) = \tau^{(2)}\big(C_*^{\varphi}(X;\mathbb{C}[G]),\{\tilde{s}_* \cup c_*\}\big)\\ &= \tau^{(2)}(N,\varphi,K(e^{\dagger})) = \tau^{(2)}(N,\varphi,c_1(e)^{-1}e)\\ &= \det_{\mathcal{N}(G)}(\varphi(c_1(e)))\cdot\tau^{(2)}(N,\varphi,e).\end{aligned}$$ □

6 The symmetry of the L^2–Alexander torsion

Let $(N,\phi,\gamma\colon \pi_1(N)\to G)$ be an admissible triple and let $e \in \mathrm{Eul}(N)$. Given $t \in \mathbb{R}_{>0}$ we consider the representation $\gamma_t\colon \pi_1(N) \to \mathrm{GL}(1,\mathbb{C}[G])$ that is given by $\gamma_t(g) := (t^{\phi(g)}\gamma(g))$. We denote by $\tau^{(2)}(N,\phi,\gamma,e)$ the function

$$\begin{aligned}\tau^{(2)}(N,\phi,\gamma,e)\colon \mathbb{R}_{>0} &\to \mathbb{R}_{\geq 0},\\ t &\mapsto \tau^{(2)}(N,\gamma_t,e).\end{aligned}$$

For another $e' \in \mathrm{Eul}(N)$ we have $e' = ge$ for some $g \in \mathcal{H}_1(N)$. By Lemma 3.2

$$\tau^{(2)}(N,\phi,\gamma,ge)(t) = t^{-\phi(g)}\tau^{(2)}(N,\phi,g,e)(t) \text{ for all } t \in \mathbb{R}_{>0}.$$

Put differently, the functions $\tau^{(2)}(N,\phi,\gamma,e)$ and $\tau^{(2)}(N,\phi,\gamma,ge)$ are equivalent. We denote by $\tau^{(2)}(N,\phi,\gamma)$ the equivalence class of the functions $\tau^{(2)}(N,\phi,\gamma,e)$ and we refer to $\tau^{(2)}(N,\phi,\gamma)$ as the *L^2–Alexander torsion of* (N,ϕ,γ).

Proof of Theorem 1.1 Let $e \in \mathrm{Eul}(N)$ and $t \in \mathbb{R}_{>0}$. We write $\tau = \tau^{(2)}(N, \gamma, \phi, e)$. Note that $(\gamma_t)^\dagger = \gamma_{t^{-1}}$. It follows from Theorems 4.2 and 5.2 that

$$\begin{aligned} \tau(t) = \tau^{(2)}(N, \gamma, \phi, e) &= \tau^{(2)}(N, \gamma_t, e) \\ &= \tau^{(2)}(N, \partial N, (\gamma_t)^\dagger, e^\dagger) = \tau^{(2)}(N, \partial N, \gamma_{t^{-1}}, e^\dagger) \\ &= \det_{\mathcal{N}(G)}(\gamma_{t^{-1}}(c_1(e))) \cdot \tau^{(2)}(N, \gamma_{t^{-1}}, e) \\ &= \det_{\mathcal{N}(G)}(t^{-\phi(c_1(e))} c_1(e)) \cdot \tau^{(2)}(N, \gamma_{t^{-1}}, e) \\ &= t^{-\phi(c_1(e))} \cdot \tau^{(2)}(N, \gamma_{t^{-1}}, e) = t^{-\phi(c_1(e))} \cdot \tau(t^{-1}). \end{aligned}$$

Now it suffices to show that for any $\phi \in H^1(N; \mathbb{Z})$ we have $\phi(c_1(e)) = x_N(\phi)$ mod 2.

So let S be a Thurston norm minimizing surface which is dual to some $\phi \in H^1(N; \mathbb{Z})$. Since N is irreducible and since $N \neq S^1 \times D^2$ we can arrange that S has no disk components. Therefore we have

$$x_N(\phi) \equiv \chi_-(S) \equiv b_0(\partial S) \mod 2\mathbb{Z}.$$

On the other hand, by [19, Lemma VI.1.2] and [19, Section XI.1] we have that $b_0(\partial S) \equiv c_1(e) \cdot S \mod 2\mathbb{Z}$ where $c_1(e) \cdot S$ is the intersection number of $c_1(e) \in H_1(N) = \mathcal{H}_1(N)$ with S. Since S is dual to ϕ, we obtain the desired equality

$$\phi(c_1(e)) \equiv c_1(e) \cdot S \equiv b_0(\partial S) \equiv \chi_-(S) \equiv x_N(\phi) \mod 2\mathbb{Z}. \qquad \square$$

Finally, a *real admissible triple* (N, ϕ, γ) is defined like an admissible triple, except that now we also allow ϕ to lie in $H^1(N; \mathbb{R}) = \mathrm{Hom}(\pi_1(N), \mathbb{R})$. The same definition as in Section 6 associates to (N, ϕ, e) a function $\tau^{(2)}(N, \phi, e)\colon \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ that is well defined up to multiplication by a function of the form $t \mapsto t^r$ for some $r \in \mathbb{R}$. The same argument as in the proof of Theorem 1.1 gives us the following result.

Theorem 6.1 *Let (N, ϕ, γ) be a real admissible triple. Then for any representative τ of $\tau^{(2)}(N, \phi, \gamma)$ there exists an $r \in \mathbb{R}$ such that $\tau(t^{-1}) = t^r \cdot \tau(t)$ for any $t \in \mathbb{R}_{>0}$.*

The only difference to Theorem 1.1 is that for real cohomology classes $\phi \in H^1(N; \mathbb{R})$ we cannot relate the exponent r to the Thurston norm of ϕ.

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