THE $\lambda(\phi^4)_2$ QUANTUM FIELD THEORY WITHOUT CUTOFFS

III. *The physical vacuum*

BY

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1. Introduction

In this series of papers we construct a quantum field theory model. This model describes a spin-zero boson field ϕ with a nonlinear ϕ^4 selfinteraction in two dimensional space time. The corresponding classical field equation is

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + m_0^2 \phi + 4\lambda\phi^3 = 0. \tag{1.1}$$

The classical field ϕ is by definition a real valued function of x and t which is a solution to (1.1). The quantum field ϕ is also a function of x and t, but its values $\phi(x, t)$ are densely defined bilinear forms on some Hilbert space. The quantum field ϕ is a solution to (1.1),

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provided that ϕ^3 is suitably interpreted. The quantum field $\phi(x, t)$ depends continuously on x and t and the derivatives in (1.1) are taken in the sense of distributions.

Certain averages of the quantum field

$$\phi(f) = \int \phi(x, t)\, f(x, t)\, dx\, dt \tag{1.2}$$

and

$$\phi(f, t) = \int \phi(x, t)\, f(x)\, dx \tag{1.3}$$

are self adjoint operators on the Hilbert space. In addition, the quantum field ϕ satisfies the canonical equal time commutation relations

$$\left[\phi(f, t), \frac{\partial}{\partial t}\phi(f, t)\right] = i \int f(x)^2\, dx. \tag{1.4}$$

The construction of a quantum field ϕ satisfying (1.1)–(1.4) was one of the main results of the first papers in this series [10, 12] (denoted hereafter as I, II). General background material can be found in [9, 17, 19].

The field ϕ constructed in I and II is a bilinear form on the Fock space $\mathcal{F}$, the Hilbert space for noninteracting bosons. The Cauchy data for the quantum field

$$\phi(x, 0) \quad \text{and} \quad \left(\frac{\partial}{\partial t}\phi\right)(x,0) \tag{1.5}$$

are the same Cauchy data that give a solution to the free field equation

$$\frac{\partial^2}{\partial t^2}\phi_{\text{free}} - \frac{\partial^2}{\partial x^2}\phi_{\text{free}} + m_0^2\phi_{\text{free}} = 0, \tag{1.6}$$

and that satisfy (1.4). In other words,

$$\phi(x, 0) = \phi_{\text{free}}(x, 0) \tag{1.7}$$

and

$$\left(\frac{\partial\phi}{\partial t}\right)(x, 0) = \left(\frac{\partial}{\partial t}\phi_{\text{free}}\right)(x, 0). \tag{1.8}$$

It is expected that this quantum field theory of Iand II, with ϕ realized on the Fock space $\mathcal{F}$, will not possess all the properties conventionally assumed in quantum field theory. For example, it seems that there will not exist a Hamiltonian operator H to generate time translations

$$\phi(x, t + \tau) = e^{itH\tau}\phi(x, t)\, e^{-itH\tau}.$$

Also there will not exist a vacuum vector (a translation invariant ground state of H). The cause of these difficulties is made clear by the circle of ideas and results known as Haag's theorem [16, 24]. Haag's theorem states that if the field $\phi \neq \phi_{\text{free}}$ satisfies (1.7)–(1.8) and if there is a Hamiltonian H for ϕ, then H will not possess a unique Euclidean invariant vacuum vector.

In order to understand and deal with these difficulties—the lack of a Hamiltonian operator H on $\mathfrak{F}$ and of a vacuum vector Ω in $\mathfrak{F}$—we first consider the cutoff field equation

$$\frac{\partial^2}{\partial t^2}\phi - \frac{\partial^2}{\partial x^2}\phi + m_0^2\phi + 4\lambda g\phi^3 = 0. \tag{1.9}$$

Here the spatial cutoff $g(x)$ is a smooth, positive function that equals one on some bounded set and that vanishes off some larger bounded set. Corresponding to the cutoff field equation (1.9), there is a Hamiltonian operator

$$H(g) = H_0 + \lambda \int :\phi^4(x):g(x)\,dx - E_g = H_0 + H_{I,g} - E_g. \tag{1.10}$$

The operator H_0 is the free field Hamiltonian and it corresponds to the free field equation (1.6). The interaction energy operator $H_{I,g}$ contains the spatial cutoff g. The operator $H_0 + H_{I,g}$ is a self adjoint operator [10, 11] and is bounded from below [22, 8]. The constant E_g is chosen so that

$$0 = \inf \{\text{spectrum } H(g)\}.$$

The constant E_g is one of the standard renormalization counterterms of quantum field theory. E_g is called the self energy of the vacuum. E_g is finite because of the spatial cutoff and because of the limitation to only one space dimension [13]. Both g and the restriction to a single space variable serve to reduce the singularity of the perturbing operator $H_{I,g}$.

In I, II we constructed the Heisenberg picture dynamics for (1.1). We started from the locally correct equation (1.9) for which the solution is

$$\phi_g(x,t) = e^{iH(g)t}\,\phi(x,0)\,e^{-iH(g)t}. \tag{1.11}$$

The solution ϕ_g is independent of $g(\cdot)$ provided $g(y)=1$ for $|y| \leq |x| + |t|$. Thus by patching together different local solutions (1.11), corresponding to $g=1$ on different sets, we constructed a solution ϕ to (1.1). This solution is associated with a time translation automorphism σ_τ implemented locally by $H(g)$ as in (1.11).

$$\sigma_\tau(\phi(x,t)) = \phi(x, t+\tau). \tag{1.12}$$

While a vacuum vector Ω does not exist in $\mathfrak{F}$, we showed in II that the approximate

Hamiltonian $H(g)$ does have a vacuum. In fact, zero is an isolated eigenvalue of $H(g)$ with multiplicity one. Thus there is a vector Ω_g in $\mathcal{F}$ such that

$$H(g)\Omega_g = 0, \quad \|\Omega_g\| = 1, \tag{1.13}$$

and Ω_g is unique up to a scalar multiple.

We wish to obtain the Hamiltonian H and the vacuum Ω corresponding to (1.1) by taking the limit $g(\cdot)\to 1$. We cannot use the most conventional types of limits because as $g(\cdot)\to 1$,

$$E_g \to -\infty \tag{1.14}$$

[13]. Furthermore, perturbation theory predicts that

$$\Omega_g \to 0, \tag{1.15}$$

in the sense of weak convergence on $\mathcal{F}$. In fact (1.15) can be compared to the simpler weak limit in $L_2(R)$,

$$g(\cdot)/\|g\|_2 \to 0.$$

Instead of studying these quantities, we define the expectation values

$$\omega_g(A) = (\Omega_g, A\Omega_g), \tag{1.16}$$

where A is a bounded function of the field operators (1.2) or (1.3). The set of all such A form a C^* algebra $\mathfrak{A}$ of bounded operators on $\mathcal{F}$, and $\omega_g \in \mathfrak{A}^*$, the dual space of $\mathfrak{A}$. Furthermore, ω_g is positive and has norm one, and thus ω_g is a *state* in the sense of C^* algebras. We use a limiting process as $g(\cdot)\to 1$ to obtain a limiting expectation value $\omega(A)$, and then $\omega \in \mathfrak{A}^*$ is necessarily a state also.

According to the Gelfand–Segal construction, ω comes from a vector Ω in some new Hilbert space $\mathcal{F}_{\text{ren}}$ in the sense that

$$\omega(A) = (\Omega, \pi(A)\Omega), \quad \|\Omega\| = 1, \tag{1.17}$$

where

$$\pi\colon A \to \pi(A) = A_{\text{ren}}$$

is a representation of $\mathfrak{A}$ as operators on $\mathcal{F}_{\text{ren}}$. The Schrödinger picture dynamics exists on $\mathcal{F}_{\text{ren}}$. In the Schrödinger picture, the time translation is given by a one parameter family of unitary operators

$$U(t) = e^{-iHt}. \tag{1.18}$$

If

$$A \to A(t) = \sigma_t(A)$$

is the Heisenberg picture dynamics constructed in I, II, then

$$\pi(\sigma_t(A)) = e^{iHt}\,\pi(A)\,e^{-iHt}. \tag{1.19}$$

Here H is positive and Ω is a vacuum for H,

$$H\Omega = 0. \tag{1.20}$$

The operator H is the renormalized Hamiltonian, and it is the limit of the $H(g)$ in the sense that

$$(\pi(B)\,\Omega,\, e^{iHt}\,\pi(A)\,\Omega)$$

is obtained through a limit as $g(\cdot)\to 1$ of

$$(B\Omega_g,\, e^{iH(g)t}A\Omega_g).$$

We call Ω the physical vacuum and π the physical representation. The vectors in $\mathcal{F}_{\text{ren}}$ are called physical vectors.

The space translations are also given by a one parameter group, commuting with the $U(t)$ and leaving Ω fixed.

$$\pi(\sigma_{x,t}(A)) = e^{iHt-iPx}\,\pi(A)\,e^{-iHt+iPx}, \qquad e^{iPx}\,\Omega = \Omega. \tag{1.21}$$

It is well known that certain representations of $\mathfrak{A}$ (coming from states of $\mathfrak{A}$ via the Gelfand–Segal construction) cannot be extended to the unbounded field operators $\phi(f)$ of (1.2), (1.3). Such states and representations seem to be totally unsuited for use in physics. The deepest result of this paper is the fact that the physical representation π is not one of these pathological representations. We now formulate this result in a stronger and more precise form.

The C^* algebra $\mathfrak{A}$ is an inductive limit. Let $\mathcal{B}$ be a bounded region of space time (or a bounded region of space at time zero). Let $\mathfrak{A}(\mathcal{B})$ be the weakly closed (von Neumann) algebra generated by bounded functions of the operators (1.2) or (1.3), but with f restricted to have support in $\mathcal{B}$. Then $\mathfrak{A}$ is defined as the norm closure of

$$\bigcup_{\mathcal{B}} \mathfrak{A}(\mathcal{B}).$$

We prove that π is locally equivalent to the representation of $\mathfrak{A}$ as operators on the Fock space $\mathcal{F}$; in brief π is *locally Fock*. For each bounded region $\mathcal{B}$, there is a unitary operator $U_{\mathcal{B}}$,

$$U_{\mathcal{B}}\colon \mathcal{F}\to \mathcal{F}_{\text{ren}} \tag{1.22}$$

such that for all A in $\mathfrak{A}(\mathcal{B})$,

$$\pi(A) = A_{\text{ren}} = U_{\mathcal{B}}\,A\,U_{\mathcal{B}}^*. \tag{1.23}$$

We have remarked that there is no vacuum vector Ω in $\mathcal{F}$. It immediately follows from the locally Fock property that for each bounded region $\mathcal{B}$ there is a vector Ω_B in $\mathcal{F}$ which serves as a vacuum vector for the algebra $\mathfrak{A}(\mathcal{B})$. In other words, for all $A \in \mathfrak{A}(\mathcal{B})$,

$$\omega(A) = (\Omega, A_{\text{ren}}\Omega) = (\Omega_B, A\Omega_B). \tag{1.24}$$

The vector Ω_B could be called a local vacuum. The vector Ω_B is not unique, but one choice would be $U_B^*\Omega$. The unitary U_B is also not unique.

An important consequence of the locally Fock property of π is the fact that the field ϕ exists on the physical Hilbert space $\mathcal{F}_{\text{ren}}$. Let $\phi(f)$ be self adjoint on $\mathcal{F}$. This is the case if, for example, f is a real, twice differentiable function with compact support [II]. Then for supp. $f \subset \mathcal{B}$ and for real s,

$$s \to \pi(e^{is\phi(f)}) = U_B\, e^{is\phi(f)}\, U_B^*$$

is a weakly continuous one parameter group of unitary operators on $\mathcal{F}_{\text{ren}}$. We let

$$\pi(\phi(f)) = \phi_{\text{ren}}(f) = U_B\,\phi(f)\, U_B^*$$

be the self adjoint operator that is the infinitesimal generator of the group.(1) Furthermore,

$$\phi_{\text{ren}}(x, t) = U_B\phi(x, t)\, U_B^*, \quad (x, t) \in \mathcal{B},$$

is a densely defined bilinear form depending continuously on x and t, and ϕ_{ren} is a solution to (1.1)–(1.4). All the local properties of ϕ established in II go over immediately to ϕ_{ren} For example, ϕ_{ren} is local, so that $\phi_{\text{ren}}(f_1)$ and $\phi_{\text{ren}}(f_2)$ commute if the supports of f_1 and f_2 are spacelike separated. Also, $\phi_{\text{ren}}(x, t)$ transforms correctly under the space-time translation group.

$$\phi_{\text{ren}}(x+\alpha, t+\tau) = e^{iH\tau - iP\alpha}\phi_{\text{ren}}(x, t)\, e^{-iH\tau + iP\alpha}.$$

The Haag–Kastler axioms [15] are valid for $\mathfrak{A}$ [12, 27] and carry over to $\pi(\mathfrak{A})$ on $\mathcal{F}_{\text{ren}}$.

The main technical step in proving the locally Fock property of π is an estimate on E_g,

$$-MV \leqslant E_g.$$

Here M is a positive constant and V is essentially the length of the support of the spatial cutoff $g(\cdot)$. From this estimate it easily follows that

$$\omega_g\left(\frac{H_0}{V}\right) \leqslant \text{const.}$$

(1) Our definition of ϕ_{ren} is $Z_3^{\frac{1}{2}}$ times the usual renormalized field. The field strength renormalization constant Z_3 is chosen to normalize the one particle states produced by the field ϕ. Perturbation theory predicts that Z_3 is strictly positive in our model. Hence $Z_3^{-\frac{1}{2}}$ should be finite and neglecting Z_3 is merely a matter of convenience.

We use space translation averaging in the limiting process which defines ω, and therefore we are able to conclude that

$$\omega(H_0^{\mathrm{loc}}) \leqslant \text{const.},$$

where H_0^{loc} resembles a local energy operator for free particles. An estimate of this nature, valid for a sequence ω_n of Fock states implies that

$$\omega_n \restriction \mathfrak{A}(\mathcal{B})$$

lies in a *norm* compact subset of the dual $\mathfrak{A}(\mathcal{B})^*$. We note that the set of all states is w^* compact, and that any state $\varrho \in \mathfrak{A}^*$ can be reached as a w^* limit point of Fock states [5, 7]. In particular, the pathological states referred to above are w^* limits of Fock states. However, a norm convergent limit of Fock states is normal on $\mathfrak{A}(\mathcal{B})$, and this excludes the unwanted states. In order to complete the construction of U_B, we use the result of Araki [1] that $\mathfrak{A}(\mathcal{B})$ is a factor of type III, for any bounded open set $\mathcal{B}$ at time zero. We also use a result of Griffin [14] that isomorphisms between separable factors of type III are unitarily implementable.

There are several properties of $\Omega \in \mathcal{F}_{\mathrm{ren}}$ which we have not established. For example, the uniqueness of Ω is one of the Wightman axioms. Also Ω should be in the domain of any polynomial in the field operators $\phi_{\mathrm{ren}}(f)$.

The main phenomenon that we encounter in this paper is the necessity of changing Hilbert spaces. This is a common feature of quantum field theory. In all cases this phenomenon can be traced to the fact that the fundamental objects, such as the Hamiltonian or energy operator, are given in the form

$$(\text{Self adjoint operator}) + (\text{Perturbation})$$

with the perturbation exceedingly singular. (In the $(\phi^4)_2$ theory, the perturbation is a bilinear form, but not an operator on $\mathcal{F}$. In fact the perturbation is so singular that the total Hamiltonian, as a bilinear form on Fock space, is unbounded below. It does not yield an operator on Fock space, but on $\mathcal{F}_{\mathrm{ren}}$ the renormalized Hamiltonian is a positive self adjoint operator.) Two important reasons for this phenomenon are the translation invariance of the Hamiltonian and the fact that the interaction involves an infinite number of degrees of freedom.

The results of this paper extend to the interaction $P(\phi)_2$, where P is a semibounded polynomial. L. Rosen [28] has proved for this model that the cutoff Hamiltonian $H(g) = H_0 + H_{I,g}$ is essentially self adjoint and has a vacuum vector. We expect that the conclusions of section 2 of this work remain valid for certain models, even when the estimates on the vacuum self energy derived in section 5 fail to hold. In particular, we expect that if the vacuum energy in a finite volume has a logarithmic (but not a linear) ultraviolet divergence, the corresponding physical representation is still locally Fock. In two dimensional space-time, the interaction $P(\phi)+Q(\phi)\bar{\psi}\psi$ has this property. Here $P(\phi)$ is a polynomial bounded from below and $Q(\phi)$ is a polynomial of lower degree than P. The two dimensional Yukawa interaction $\bar{\psi}\psi\phi$ also has a logarithmic vacuum energy divergence so we expect that the theory is locally Fock.

The logical order of the following sections is 5, 3, 4, 2. We have rearranged this order to present results, first, followed by details of increasing complexity.

2. The physical representation and renormalization

In this section we construct the physical vacuum vector Ω and the corresponding Hilbert space $\mathfrak{H}_{\text{ren}}$. On $\mathfrak{H}_{\text{ren}}$ we construct the renormalized self adjoint Hamiltonian H, and the self adjoint momentum operator P. The space time translations are implemented by the unitary group

$$U(\alpha, \tau) = e^{iH\tau - iP\alpha} \tag{2.1}$$

and the vacuum satisfies

$$H\Omega = P\Omega = 0. \tag{2.2}$$

Furthermore, H and P commute and $H \geqslant 0$.

We construct Ω as a limit of the vacuum vectors Ω_g of the cutoff Hamiltonians $H(g)$. For concreteness, we take $g(x)$ to be a nonnegative C_0^∞ function equal to one on the interval $[-3, 3]$. Let

$$g_n(x) = g(x/n) \tag{2.3}$$

and for $A \in \mathfrak{A}$, let

$$\omega_n(A) = \frac{1}{n} \int (\Omega_{g_n}, \sigma_\alpha(A)\Omega_{g_n}) h(\alpha/n)\, d\alpha, \tag{2.4}$$

where σ_α is the space translation automorphism (see II, sections 3.6 and 4). The function $h(\cdot) \geqslant 0$ is C^∞, has support in $[-1, 1]$ and

$$\int h(x)\, dx = 1 = \frac{1}{n} \int h(\alpha/n)\, d\alpha. \tag{2.5}$$

By a general compactness principle for states on a C^* algebra, there is a w^* convergent subnet ω_{n_j} of the states $\{\omega_n\}$.

Thus for each $A \in \mathfrak{A}$,

$$\omega_{n_j}(A) \to \omega(A).$$

The limiting state ω can be used to construct the inner product of a new Hilbert space $\mathfrak{H}_{\text{ren}}$, and the operators $A \in \mathfrak{A}$ can be realized as operators on $\mathfrak{H}_{\text{ren}}$. The advantage of working on the Hilbert space $\mathfrak{H}_{\text{ren}}$ is that the space–time automorphisms σ_a are given by unitary operators (2.1). We state some results.

THEOREM 2.1. *Let ω be a w^* limit point of the sequence $\omega_n \in \mathfrak{A}^*$. There is a subsequence of the ω_n which converges w^* to ω. There is a separable Hilbert space $\mathcal{F}_{\text{ren}}$, a $*$ isomorphism π of $\mathfrak{A}$, a continuous unitary representation U of R^2, and a vector $\Omega \in \mathcal{F}_{\text{ren}}$ such that*

$$\omega(A) = (\Omega, \pi(A)\Omega) = (\Omega, A_{\text{ren}}\Omega) \tag{2.6}$$

$$U(a)\pi(A)U(-a) = \pi(\sigma_a(A)) \tag{2.7}$$

and

$$U(a)\Omega = \Omega. \tag{2.8}$$

The spectrum of the generator H of time translations is contained in the interval $[0, \infty)$.

The existence of $\mathcal{F}_{\text{ren}}$, π and Ω satisfying (2.6) is the Gelfand–Segal theorem [21]. We will prove Theorem 2.1 using the locally Fock property of π, which also leads to the properties of ϕ_{ren} explained in section 1.

THEOREM 2.2. *Let ω be a w^* limit point of the sequence $\{\omega_n\}$ and let π be the corresponding representation of $\mathfrak{A}$. Let $\mathcal{B}$ be a bounded region of space time or of space at $t=0$. Then there is a unitary operator $U_{\mathcal{B}}$: $\mathcal{F} \to \mathcal{F}_{\text{ren}}$ such that for $A \in \mathfrak{A}(\mathcal{B})$*

$$\pi(A) = U_{\mathcal{B}} A U_{\mathcal{B}}^*. \tag{2.9}$$

In short, we say that ω and π are *locally Fock.*

The locally Fock property of ω rests on the following theorem.

THEOREM 2.3. *Let $\mathcal{B}$ be a bounded region of space time or of space at $t=0$. Then the sequence*

$$\omega_n \upharpoonright \mathfrak{A}(\mathcal{B}) \in \mathfrak{A}(\mathcal{B})^* \tag{2.10}$$

lies in a norm compact subset of $\mathfrak{A}(\mathcal{B})^*$. Any limit point $\omega \upharpoonright \mathfrak{A}(\mathcal{B})$ is normal.

The proof of Theorem 2.3 will be given in the following sections. We introduce some von Neumann algebra terminology [2]. A state ϱ of a von Neumann algebra $\mathfrak{M}$ is called *normal* if

$$\sup \varrho(A_\alpha) = \varrho\,(\sup A_\alpha) \tag{2.11}$$

for each monotone increasing generalized sequence $\{A_\alpha\}$, $A_\alpha \in \mathfrak{M}$, provided the sequence A_α is bounded from above, so that $\sup A_\alpha$ exists.

Let us consider linear functionals on $\mathfrak{M}$ of the form $A \to l(A)$,

$$l(A) = \sum_{i=1}^{\infty} (A\theta_i, \psi_i), \tag{2.12}$$

where
$$\sum_{i=1}^{\infty} \|\theta_i\|^2 < \infty, \quad \sum_{i=1}^{\infty} \|\psi_i\|^2 < \infty. \tag{2.13}$$

Ultraweak convergence in $\mathfrak{M}$ is defined by convergence of all linear functionals of the form (2.12)–(2.13), and conversely each ultraweakly continuous linear functional on $\mathfrak{M}$ can be represented in the form (2.12)–(2.13) [2, page 40]. A state ϱ is ultraweakly continuous if and only if it is normal, and every ultraweakly continuous linear functional can be written as a linear combination of four normal states [2, page 54]. The norm of $l \in \mathfrak{M}^*$ defined by
$$\|l\| = \sup \{|l(A)|: A \in \mathfrak{M}, \|A\| \leqslant 1\}. \tag{2.14}$$

The A in (2.14) can be restricted to a subalgebra $\mathfrak{M}_0$ of $\mathfrak{M}$ which is weakly or ultraweakly dense in $\mathfrak{M}$, by Kaplansky's density theorem [2, page 46, page 43].

We apply these concepts to the von Neumann algebra $\mathfrak{A}(\mathcal{B})$ and the weakly dense subalgebra $\mathfrak{A}_0(\mathcal{B})$ generated (algebraically) by the operators
$$\{e^{i\phi(f)}, e^{i\dot{\phi}(f)}: \text{supp } f \subset \mathcal{B}, f \in C^\infty\}, \tag{2.15}$$

where $\mathcal{B}$ is a region of space time or of space at $t=0$.

Proof of Theorem 2.2. We first prove that $\pi \restriction \mathfrak{A}(\mathcal{B})$ is ultraweakly continuous. The right and left multiplications
$$A \to AC \quad \text{and} \quad A \to CA$$

are ultraweakly continuous on $\mathcal{B}(\mathcal{J})$. For C_1 and C_2 in some $\mathfrak{A}(\mathcal{B}_1)$, with $\mathcal{B}_1 \supset \mathcal{B}$,
$$A \to C_1^* A C_2 \in \mathfrak{A}(\mathcal{B}_1)$$

is ultraweakly continuous. By Theorem 2.3, $\omega \restriction \mathfrak{A}(\mathcal{B}_1)$ is normal, and hence ultraweakly continuous. Thus
$$A \to \omega(C_1^* A C_2) = (\pi(C_1)\,\Omega, \pi(A)\,\pi(C_2)\,\Omega) \tag{2.16}$$

is ultraweakly continuous. Since the vectors $\pi(C_i)\Omega$ run over a dense subset of $\mathcal{J}_{\text{ren}}$, and since $\pi(A)$ is bounded, the map
$$A \to \pi(A), \quad A \in \mathfrak{A}(\mathcal{B})$$

is ultraweakly–weakly continuous. (This map is also ultraweakly continuous.)

Since $\mathfrak{A}(\mathcal{B})$ is ultraweakly separable, $\pi(\mathfrak{A}(\mathcal{B}))$ has a countable, weakly dense subset.

Since the weak and strong closures of a linear subspace of $\mathcal{F}_{\text{ren}}$ coincide, $\pi(\mathfrak{A}(\mathcal{B}))$ has a countable, strongly dense subset. But

$$\bigcup_{\mathcal{B}} \pi(\mathfrak{A}(\mathcal{B}))\,\Omega$$

is dense in $\mathcal{F}_{\text{ren}}$, so $\mathcal{F}_{\text{ren}}$ is separable also.

In proving the theorem, we replace $\mathcal{B}$ by a bounded open set at time zero (the domain of dependence of $\mathcal{B}$), since this change merely enlarges $\mathfrak{A}(\mathcal{B})$. For such a $\mathcal{B}$ at time zero, Araki has shown that $\mathfrak{A}(\mathcal{B})$ is a factor of type III [1]. His proof requires a computation that a certain operator is not of trace class, and he shows that the operator in question is closed but not everywhere defined due to the conditions at the boundary of $\mathcal{B}$. Thus his proof which is given in three spatial dimensions is valid in one spatial dimension, and is somewhat easier in the latter case.

The kernel of $\pi \restriction \mathfrak{A}(\mathcal{B})$ is a weakly closed two sided ideal. Since there are no proper weakly closed two sided ideals in any factor, we see that ker $(\pi \restriction \mathfrak{A}(\mathcal{B}))$ equals zero and $\pi \restriction \mathfrak{A}(\mathcal{B})$ is an isomorphism. According to a theorem of E. Griffin [14], $\pi \restriction \mathfrak{A}(\mathcal{B})$, as an isomorphism between separable type III factors, is unitarily implemented. This completes the proof.

We remark that the argument of the above paragraph combined with the fact that there are no norm closed two sided ideals in a factor of type III shows that $\mathfrak{A}$ is simple, and thus π is an isomorphism of $\mathfrak{A}$.

Proof of Theorem 2.1. On the metric space $\mathfrak{A}^*(\mathcal{B})$, compactness implies sequential compactness. Thus using Theorem 2.3, we can find a subsequence ω_{n_j} with

$$\|(\omega - \omega_{n_j}) \restriction \mathfrak{A}(\mathcal{B})\| \to 0.$$

Using the diagonal process, the subsequence can be chosen independently of $\mathcal{B}$.

The existence of the unitary $U(a)$ satisfying (2.7)–(2.8) follows from the Gelfand–Segal construction if for each space–time translation σ_a,

$$\omega(\sigma_a(A)) = \omega(A). \tag{2.17}$$

Thus we must show that ω is a fixed point of σ_a^*. Since ω is norm continuous, it is sufficient to take A in the dense subalgebra

$$\bigcup_{\mathcal{B}} \mathfrak{A}(\mathcal{B}).$$

Thus we let A belong to $\mathfrak{A}(\mathcal{B})$ for some region $\mathcal{B}$, and we choose n sufficiently large so that for all $(x, t) \in \mathcal{B}$,

$$n > |x| + |t|.$$

We define $\mathcal{B}_{|\tau|}$ as the set of points within the distance $|\tau|$ of $\mathcal{B}$. With the above restrictions, $H(g_n)$ is a Hamiltonian for $\mathcal{B}_{|\tau|+|\alpha|}$, provided that $|\alpha| < n$ and $|\tau| < n$. The restriction on α is satisfied if

$$\alpha \in \operatorname{supp} h(\cdot/n).$$

Under these restrictions
$$\sigma_\alpha(\sigma_\tau(A)) \in \mathfrak{A}(\mathcal{B}_{|\tau|+|\alpha|})$$

and
$$\sigma_\alpha(\sigma_\tau(A)) = \sigma_\tau(\sigma_\alpha(A)) = e^{iH_n t}\sigma_\alpha(A)\, e^{-iH_n\tau},$$

see II. We have set
$$H_n = H(g_n) = H_0 + H_{I,g_n} - E_{g_n}. \tag{2.18}$$

Since
$$e^{-iH_n\tau}\Omega_{g_n} = \Omega_{g_n},$$

we see that for α in supp $h(\cdot/n)$ and $|\tau| < n$,

$$(\Omega_{g_n}, \sigma_\alpha(\sigma_\tau(A))\,\Omega_{g_n}) = (\Omega_{g_n}, \sigma_\alpha(A)\,\Omega_{g_n})$$

and
$$\omega_n(\sigma_\tau(A)) = \omega_n(A).$$

The formula (2.17) for time translations follows immediately.

In order to prove (2.17) for space translations, we substitute $\sigma_\beta(A)$ for A in (2.4) and perform a change of variables $\alpha \to \alpha - \beta$ in the integration. This leads to the estimate

$$|\omega_n(\sigma_\beta(A) - A)| = \frac{1}{n}\left|\int (\Omega_{g_n}, \sigma_\alpha(A)\,\Omega_{g_n})\{h((\alpha-\beta)/n) - h(\alpha/n)\}\,d\alpha\right| \leq \|A\| \int \left| h\left(\alpha - \frac{\beta}{n}\right) - h(\alpha)\right| d\alpha.$$

This vanishes as $n \to \infty$, so that for $A \in \mathfrak{A}(\mathcal{B})$,

$$\omega(\sigma_\beta(A)) = \omega(A),$$

(2.17) is valid, and $U(a)$ does exist.

We prove now the strong continuity of $U(a)$. The vectors $\pi(A)\Omega$ are dense in $\mathcal{J}_{\text{ren}}$ for A in $\bigcup_{\mathcal{B}} \mathfrak{A}(\mathcal{B})$ for bounded space–time regions $\mathcal{B}$. If $\sigma_a(A)$ is in some $\mathfrak{A}(\mathcal{B})$, we have

$$U(a)\,\pi(A)\,\Omega = \pi(\sigma_a(A))\,\Omega$$

by (2.7) and (2.8). By (2.9),

$$U(a)\,\pi(A)\,\Omega = U_{\mathcal{B}}\,\sigma_a(A)\,U_{\mathcal{B}}^*\,\Omega. \tag{2.19}$$

Since σ_a is implemented locally on $\mathcal{J}$ by unitary operators of the form

$$\exp\,(iH(g)\,\tau)\,\exp\,(-iP\alpha),$$

the right side of (2.19) is strongly continuous in a. This proves the strong continuity of $U(a)$ on a dense set of vectors, which is sufficient.

The remaining assertion of the theorem is that $H \geqslant 0$. Let

$$H = \int \mu dE(\mu)$$

be the spectral resolution of H. Then we define

$$F(\tau) = \omega(A^* \sigma_\tau(A)) = (\pi(A)\,\Omega, e^{iH\tau}\pi(A)\,\Omega) = \int e^{i\mu\tau} d\,\|E(\mu)\,\pi(A)\Omega\|^2, \tag{2.20}$$

In order to prove that H has nonnegative spectrum, we must show that only the half line $[0, \infty)$ contributes to the integral (2.20). For A in some local algebra $\mathfrak{A}(\mathcal{B})$, we infer from Theorem 2.3 that $F(\tau)$ is the limit through a subsequence of the similar functions

$$F_n(\tau) = \omega_n(A^* \sigma_\tau(A)) = \frac{1}{n}\int (\sigma_\alpha(A^*)\,\Omega_{g_n}, e^{iH_n\tau}\sigma_\alpha(A)\,\Omega_{g_n}) \times h(\alpha/n)\,d\alpha,$$

where H_n is defined in (2.18). Since $H_n \geqslant 0$, $F_n(\tau)$ has a Fourier transform $\tilde{F}_n(\mu)$ with support in the half line $[0, \infty)$. The subsequence $F_{n_j}(\tau)$ converges pointwise to F, and is uniformly bounded,

$$|F_{n_j}(\tau)| \leqslant \|A\|^2.$$

Hence the F_{n_j} converge to F as tempered distributions, and consequently the Fourier transforms $\tilde{F}_{n_j}$ converge as tempered distributions also. Their limit $\tilde{F}$ must therefore have support in $[0, \infty)$, and so only the interval $[0, \infty)$ contributes to (2.20). As the vectors $\pi(A)\Omega$, $A \in \mathfrak{A}(\mathcal{B})$, are dense in $\mathcal{J}_{\text{ren}}$, this shows that $H \geqslant 0$.

3. Local number operators

In this section we define local number operators $N_{\mathcal{B}}$, $N_{\tau,\mathcal{B}}$ and $N_{\tau,\zeta}$ and we investigate their properties. Each of these operators is the biquantization of an operator on the one particle space. We reduce the study of such quantized operators to the study of the corresponding one particle operators. We analyze the one particle operators in detail, and thereby arrive at results for the quantized operators. In Section 3.3 we assume the estimates derived in Section 5, and we show that expectation values of the above local number operators in the approximate vacuum states ω_n are bounded uniformly in n. We shall use such results in Section 4.

3.1. One particle operators. The one particle space consists of Lebesque square integrable functions $L_2(R^1)$. We study operators $\boldsymbol{c}$ on $L_2(R^1)$ that are defined on the dense domain

$S(R^1)$, the Schwartz space of rapidly decreasing C^∞ functions with rapidly decreasing derivatives. We require that c be a bounded transformation from $S(R^1)$ to $L_2(R^1)$,

$$\|cf\|_2 \leqslant \|f\|_s, \tag{3.1.1}$$

where $\|\cdot\|_s$ denotes some Schwartz space norm. Such an operator c can be represented by a tempered distribution kernel. The Fourier transform is an operator of this type.

$$f(x) = (2\pi)^{-\frac{1}{2}} \int \tilde{f}(p)\, e^{-ipx}\, dp. \tag{3.1.2}$$

Corresponding to the physical interpretation of position and momentum, we say that Fourier transformation connects the momentum representation with the configuration space (position) representation. This physical identification corresponds to the introduction of Newton–Wigner position coordinates.

To each bounded operator $c_p: S(R^1) \to L_2(R^1)$ in the momentum representation, there is a bounded operator c_x: $S(R^1) \to L_2(R^1)$ in the configuration space representation, where

$$(c_x f)(x) = (2\pi)^{-\frac{1}{2}} \int (c_p \tilde{f})(p)\, e^{-ipx}\, dp. \tag{3.1.3}$$

Conversely given c_x there is a corresponding c_p.

We now mention a few other examples. Let $\mu(p)$ be the energy of a particle of mass m_0. The operator $c_p = \mu(p)^\tau$ of multiplication by

$$\mu(p)^\tau = (p^2 + m_0^2)^{\tau/2} \tag{3.1.4}$$

maps $S(R^1)$ onto $S(R^1)$.

The configuration space operator μ_x^τ corresponding to $\mu(p)^\tau$ is convolution by a kernel k_τ. Here $k_\tau(x)$ is C^∞ except for $x=0$. If $\tau/2$ is a nonnegative integer, k_τ has support at $x=0$. Otherwise $k_\tau(x)$ decreases exponentially at infinity. Explicitly [25, page 185],

$$k_\tau(x) = \frac{2^{\tau/2+1}}{\Gamma\left(-\frac{\tau}{2}\right)} \left(\frac{m_0}{|x|}\right)^{(\tau+1)/2} \int_0^\infty e^{-m_0|x|\cosh t} \cosh\left(\frac{t}{2} + \tau t\right) dt, \tag{3.1.5}$$

from which we see that for $\tau \geqslant -1$,

$$\left|\frac{d^n}{dx^n} k_\tau(x)\right| \leqslant O(e^{-m_0|x|}), \quad \text{as } |x| \to \infty, \tag{3.1.6}$$

for $n = 0, 1, 2, \ldots$ For $\tau < -1$, estimates of the form (3.1.6) hold if m_0 is replaced by $m_0 - \varepsilon$, for any $\varepsilon > 0$.

A third example of an operator C_x which we will need is multiplication by the characteristic function $E(x)$ of an interval $\mathcal{B}=[a, b]$. The Fourier transformed operator c_p is convolution by

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}(e^{i(a+b)/2})\frac{\sin((a-b)p/2)}{p}=(2\pi)^{-\frac{1}{2}}\tilde{E}(p). \tag{3.1.7}$$

A fourth example of an operator c_x is multiplication by a C_0^∞ function $\zeta(x)$. The corresponding c_p is convolution by $(2\pi)^{-\frac{1}{2}}\tilde{\zeta}(p)\in S(R^1)$. Both c_x and c_p map $S(R^1)$ into itself.

Thus the operator

$$c_x=\zeta(x)\,\mu_x^\tau\zeta(x) \tag{3.1.8}$$

is a bounded transformation of $S(R^1)$ into itself, and it is a localization of the operator μ_x^τ. We also want a localized version of μ_x^τ with a sharp boundary. Thus we wish to replace $\zeta(x)$ in (3.1.8) by $E(x)$.

THEOREM 3.1.1. *Let $\tau<\frac{1}{4}$. Then $E(x)\mu_x^\tau E(x)$ is a bounded operator from $S(R^1)$ to $L_2(R^1)$. Let $\tau<\frac{1}{2}$. If $\zeta(x)$ is a positive C_0^∞ function equal to one in a neighborhood of the support of $E(x)$, and if $\varepsilon>0$, then as bilinear forms on $S(R^1)\times S(R^1)$,*

$$E(x)\,\mu_x^\tau E(x)\leqslant \text{const. }\zeta(x)\,\mu_x^{2\tau+\varepsilon}\zeta(x), \tag{3.1.9}$$

where the constant depends only on τ, ε and supp E.

Proof. It is sufficient to prove that for $\tau<\frac{1}{4}$, $\varepsilon>0$, and $f\in S(R^1)$,

$$(f, E\mu_x^{2\tau}Ef)\leqslant \text{const. }(f, \mu_x^{4\tau+\varepsilon}f). \tag{3.1.10}$$

In that case
$$\|E\mu_x^\tau Ef\|^2\leqslant\|\mu_x^\tau Ef\|^2=(f, E\mu_x^{2\tau}Ef)\leqslant \text{const. }\|\mu_x^{2\tau+\varepsilon/2}f\|^2, \tag{3.1.11}$$

which shows that $E(x)\,\mu_x^\tau E(x)$ is an operator and a bounded transformation from $S(R^1)$ to $L_2(R^1)$. Furthermore, $f=\zeta f+(1-\zeta)f$ and

$$Ef=E\zeta f.$$

Hence
$$(f, E\mu_x^\tau Ef)=(\zeta f, E\mu_x^\tau E\zeta f)\leqslant \text{const. }(f, \zeta\mu_x^{2\tau+\varepsilon}\zeta f),$$

which is (3.1.9).

In order to prove (3.1.10), we write

$$(f, E\mu_x^\tau Ef)=(f, E\mu_x^\tau f)+(f, E[\mu_x^\tau, E]f).$$

Since
$$|(f, E\mu_x^\tau f)|\leqslant\|f\|\,\|\mu_x^\tau f\|\leqslant \text{const. }\|\mu_x^\tau f\|^2,$$

we infer that
$$(f, E\mu_x^\tau Ef)\leqslant \text{const. }(f, \mu_x^{2\tau}f)+|(f, E[\mu_x^\tau, E]f)|. \tag{3.1.12}$$

We now estimate
$$(f, E[\mu_x^\tau, E]f) = (\tilde{f}, g), \tag{3.1.13}$$

where
$$g(p) = \frac{1}{(2\pi)} \int \tilde{E}(p-q)\,(\mu(q)^\tau - \mu(s)^\tau)\,\tilde{E}(q-s)\,\tilde{f}(s)\,dq\,ds.$$

Since for $\tau \leqslant 1$,
$$|\mu(q)^\tau - \mu(s)^\tau| \leqslant \text{const. } \mu(q-s)^\tau,$$

and since by (3.1.4),
$$|\tilde{E}(p)| \leqslant \text{const. } \mu(p)^{-1},$$

we have
$$|g(p)| \leqslant (h * |\tilde{f}|)(p), \tag{3.1.14}$$

where
$$h(p) = \text{const.} \int \mu(p-q)^{-1} \mu(q)^{-1+\tau} dq. \tag{3.1.15}$$

We now show that for any $\varepsilon > 0$, there is a constant such that $h(p) \leqslant \text{const. } \mu(p)^{-1+\tau+\varepsilon}$. We break up the integral in (3.1.15) into two regions:

$$\text{I}: |q| \leqslant \tfrac{1}{2}|p|$$

and
$$\text{II}: |q| > \tfrac{1}{2}|p|.$$

In region I, $\mu(p-q) \geqslant \text{const. } \mu(p)$, so that

$$\int_{\text{I}} \mu(p-q)^{-1} \mu(q)^{-1+\tau} dq \leqslant \text{const. } \mu(p)^{-1} \int_{\text{I}} \mu(q)^{-1+\tau} dq \leqslant \text{const. } \mu(p)^{-1+\tau},$$

since
$$\int_{\text{I}} \mu(q)^{-1+\tau} dq \leqslant \text{const.} \int_1^{|p|} |q|^{-1+\tau} dq \leqslant \text{const. } \mu(p)^\tau.$$

On the other hand, for $\varepsilon > 0$,

$$\mu(q)^{-\varepsilon/2} \leqslant \text{const. } \mu(p-q)^{-\varepsilon/2} \mu(p)^{\varepsilon/2},$$

so that
$$\int_{\text{II}} \mu(p-q)^{-1} \mu(q)^{-1+\tau} dq \leqslant \text{const. } \mu(p)^{\varepsilon/2} \int_{\text{II}} \mu(p-q)^{-1-\varepsilon/2} \mu(q)^{-1+\tau+\varepsilon/2}\, dq.$$

But in the region II, $\mu(q)^{-1} \leqslant 2\,\mu(p)^{-1}$, so

$$\int_{\text{II}} \mu(p-q)^{-1} \mu(q)^{-1+\tau} dq \leqslant \text{const. } \mu(p)^{(\varepsilon/2)-1+\tau+\varepsilon/2},$$

from which we conclude that
$$h(p) \leqslant \text{const. } \mu(p)^{-1+\tau+\varepsilon}. \tag{3.1.16}$$

We now bound (3.1.13) by writing for $\nu \geq 0$,

$$|(\tilde{f}, g)| \leq \|\tilde{f}\|_{2-\nu/(1+\nu)} \|g\|_{2+\nu} \leq \|\tilde{f}\|_{2-\nu/(1+\nu)} \|h * |\tilde{f}|\|_{2+\nu}, \tag{3.1.17}$$

where subscripts denote L_p norms. By the Hausdorff–Young inequality

$$\|h * |\tilde{f}|\|_{2+\nu} \leq \|\tilde{h}(x)|\tilde{f}|^{\sim}(x)\|_{2-\nu/(1+\nu)}$$

and by the Hölder inequality

$$\leq \|\tilde{h}(x)\|_{(4+2\nu)/\nu} \|f\|_2,$$

since

$$\||\tilde{f}|^{\sim}\|_2 = \||\tilde{f}|\|_2 = \|f\|_2.$$

Again applying the Hausdorff–Young inequality

$$\|h * |\tilde{f}|\|_{2+\nu} \leq \|h\|_{1+\nu/(4+\nu)} \|f\|_2, \tag{3.1.18}$$

and by (3.1.16), $h \in L_{1+\nu/(4+\nu)}$ if

$$(1-\tau)\left(1 + \frac{\nu}{4+\nu}\right) > 1.$$

This can be satisfied if

$$\tau < \tfrac{1}{2} \tag{3.1.19}$$

by choosing ν sufficiently large so that $\nu > (1-2\tau)/4\tau$, or in other words

$$\tau < \frac{\nu}{4+2\nu}. \tag{3.1.20}$$

For such a value of ν, we have by (3.1.17)

$$|(\tilde{f}, g)| \leq \text{const. } \|\tilde{f}\|_{2-\nu/(1+\nu)} \|f\|_2.$$

On the other hand, we shall prove that for certain ν satisfying (3.1.20) and $\varepsilon > 0$,

$$\|\tilde{f}\|_{2-\nu/(1+\nu)} \leq \text{const. } \|\mu_x^{\tau+\varepsilon/2} f\|_2, \tag{3.1.21}$$

so that

$$|(f, E[\mu_x^{\tau}, E]f)| = |(\tilde{f}, g)| \leq \text{const. } \|\mu_x^{\tau+\varepsilon/2} f\|_2 \|f\|_2 \leq \text{const. } (f, \mu_x^{2\tau+\varepsilon} f). \tag{3.1.22}$$

Since this τ is restricted to $\tau < \frac{1}{2}$, the inequalities (3.1.22) and (3.1.12) complete the proof of (3.1.10). In order to establish (3.1.21), we write

$$\|\tilde{f}\|_{2-\nu/(1+\nu)}^{2-\nu/(1+\nu)} = \int |\tilde{f}(p)|^{2-\nu/(1+\nu)} dp = \int |\mu(p)^{\sigma} \tilde{f}(p)|^{2-\nu/(1+\nu)} \mu(p)^{-\sigma(2-\nu/(1+\nu))} dp$$

$$\leq \|\mu(p)^{\sigma} \tilde{f}\|_2^{2-\nu/(1+\nu)} \|\mu(p)^{-2\sigma+\sigma\nu/(1+\nu)}\|_{2(1+\nu)/\nu}.$$

The function $\mu(p)^{-2\sigma+\sigma\nu/(1+\nu)}$ is in $L_{2(1+\nu)/\nu}$ if

$$\sigma\left(2-\frac{\nu}{1+\nu}\right)\left(2\cdot\frac{1+\nu}{\nu}\right)>1.$$

This is true for any σ satisfying

$$\sigma>\frac{\nu}{4+2\nu}.$$

Since ν may be any positive number satisfying (3.1.20), we can choose ν so that

$$\tau+\varepsilon/2>\frac{\nu}{4+2\nu}>\tau,$$

and then choose $\sigma=\tau+\varepsilon/2$. Hence

$$\|\tilde{f}\|_{2-\nu/(1+\nu)}\leqslant \text{const. } \|\mu(p)^{\tau+\varepsilon/2}\tilde{f}\|_2=\text{const. } \|\mu_x^{\tau+\varepsilon/2}f\|_2,$$

which is (3.1.21). This completes the proof of the theorem.

For E, ζ, τ as above, the operators

$$E\mu_x^{\tau}E \text{ and } \zeta\mu_x^{\tau}\zeta$$

are positive operators with domain $S(R^1)$. Let c_τ and s_τ denote their respective Friedrichs extensions. We note that for

$$f\in \mathcal{D}(s_{2\tau+\varepsilon}^{\frac{1}{2}}),$$

we have

$$f\in \mathcal{D}(c_\tau^{\frac{1}{2}})$$

and for a constant depending only on supp E, τ and ε,

$$\|c_\tau^{\frac{1}{2}}f\|^2\leqslant \text{const. } \|s_{2\tau+\varepsilon}^{\frac{1}{2}}f\|. \tag{3.1.23}$$

This is a consequence of the fact that inequality (3.1.9) extends by continuity to the closure of $S(R^1)$ in the norm

$$|||f|||_1=\|\mu_x^{\tau+\varepsilon/2}\zeta f\|,$$

which is the domain of $s_{2\tau+\varepsilon}^{\frac{1}{2}}$. Furthermore,

$$|||f|||_2=\|\mu_x^{\tau/2}Ef\|\leqslant \text{const. } |||f|||_1$$

so that the domain of $c_\tau^{\frac{1}{2}}$ includes the domain of $s_{2\tau+\varepsilon}^{\frac{1}{2}}$.

THEOREM 3.1.2. *Let $0<\tau<\frac{1}{4}$ and let c_τ be the Friedrichs extension of*

$$E(x)\,\mu_x^\tau E(x),$$

where $E(x)$ is the characteristic function for a bounded interval $\mathcal{B}$. Then c_τ commutes with the orthogonal projection of $L_2(R^1)$ onto $L_2(\mathcal{B})$, and

$$[c_\tau \restriction L_2(\mathcal{B})]^{-1}$$

is a compact operator.

Proof. The projection of $L_2(R^1)$ onto $L_2(\mathcal{B})$ is given by multiplication by $E(x)$. Clearly $E(x)\mu_x^\tau E(x)$ commutes with multiplication by $E(x)$ on $\mathcal{S}(R^1)$, since $E(x)^2=E(x)$. By continuity, this extends as a bilinear form to the domain $\mathcal{D}(c_\tau^{\frac{1}{2}})\times\mathcal{D}(c_\tau^{\frac{1}{2}})$, and so holds on $\mathcal{D}(c_\tau)$ as an operator equality.

In order to show that c_τ has a compact resolvent on $L_2(\mathcal{B})$, it is sufficient to show that the resolvent of $c_\tau^{\frac{1}{2}}$ is compact on $L_2(\mathcal{B})$. The latter is true if whenever $\mathcal{D}$ is a set of vectors such that

$$\mathcal{D}\subset\mathcal{D}(c_\tau^{\frac{1}{2}}\restriction L_2(\mathcal{B})) \tag{3.1.24}$$

and

$$\sup_{f\in\mathcal{D}}\|c_\tau^{\frac{1}{2}}f\|<\infty, \tag{3.1.25}$$

then $\mathcal{D}$ has a compact closure. It is sufficient to replace $\mathcal{D}(c_\tau^{\frac{1}{2}}\restriction L_2(\mathcal{B}))$ in (3.1.24) by any core of $c_\tau^{\frac{1}{2}}\restriction L_2(\mathcal{B})$. By definition of the Friedrichs extension, $\mathcal{S}(R^1)$ is a core for $c_\tau^{\frac{1}{2}}$ and $E(x)\,\mathcal{S}(R^1)$ is a core for $c_\tau^{\frac{1}{2}}\restriction L_2(\mathcal{B})$.

It is convenient to imbed $L_2(\mathcal{B})$ in $L_2(R^1)$ and to write a nonlocal expression for $c_\tau^{\frac{1}{2}}$. For $f\in E(x)\,\mathcal{S}(R^1)$, we have

$$\|c_\tau^{\frac{1}{2}}f\|^2=\|\mu_x^{\tau/2}E(x)f\|^2=\|\mu_x^{\tau/2}f\|^2, \tag{3.1.26}$$

where we call this nonlocal since $\mu_x^{\tau/2}f\in L_2(R^1)$. We remark that (3.1.26) implies a continuity condition on f. Let $f_a(x)=f(x+a)$, so that

$$\|f_a-f\|^2=\int|\tilde f(p)\,(e^{-ipa}-1)|^2\,dp.$$

Since for $0<\tau\leqslant 1$,
$$|(e^{-ipa}-1)|\leqslant 2\,|pa|^\tau\leqslant 2\,|a|^\tau\mu(p)^\tau,$$

we infer that
$$\|f_a-f\|\leqslant 2\,|a|^\tau\left(\int|\tilde f(p)|^2\mu(p)^{2\tau}dp\right)^{\frac{1}{2}}=2\,|a|^\tau\,\|\mu_x^\tau f\|.$$

Thus a uniform bound (3.1.25) for $f\in\mathcal{D}$, gives

$$\sup_{f\in\mathcal{D}}\|f_a-f\|\leqslant\text{const.}\,|a|^\tau,$$

or equicontinuity of the functions in $\mathcal{D}$. It is a classical result that an equicontinuous set $\mathcal{D}$ of L_2 functions with support in a bounded interval has a compact closure $\mathcal{D}^-$. This completes the proof of the theorem.

3.2. Quantized operators on Fock space. We first review [1] some standard notation. The Fock space $\mathcal{F}$ for noninteracting bosons is the Hilbert space completion of the direct sum of n-particle Fock spaces,

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}_n. \tag{3.2.1}$$

Here $\mathcal{F}_0$ is the complex numbers and $\mathcal{F}_n$ is the completion in $L_2(R^n)$ of the n-fold symmetric tensor product of $\mathcal{F}_1 = L_2(R_1)$. Thus a vector $f = \{f_n\} \in \mathcal{F}$ is a sequence of n particle vectors in $\mathcal{F}_n$, $\|f\|^2 = \sum_{n=0}^{\infty} \|f_n\|^2$. As for the one particle space, we represent n particle vectors in the momentum representation $f_n(k_1, \ldots, k_n)$ or in the configuration representation $f_n(x_1, \ldots, x_n)$.

$$f_n(x_1, \ldots, x_n) = (2\pi)^{-n/2} \int e^{-ikx} \tilde{f}(k_1, \ldots, k_n)\, dk, \tag{3.2.2}$$

where $kx = \sum_{i=1}^{n} k_i x_i$ and $dk = \prod_{i=1}^{n} dk_i$. Fourier transformation is an isomorphism of $\mathcal{F}_n$ and of $\mathcal{F}$. We use a domain $\mathcal{D}_0 \subset \mathcal{F}$ of well behaved vectors, The vectors in $\mathcal{D}_0$ have a finite number of particles with wave functions in the Schwartz space $\mathcal{S}(R^n)$,

$$\mathcal{D}_0 = \{f : f \in \mathcal{F},\ f_n = 0 \text{ for large } n, \text{ and } f_n \in \mathcal{S}(R_n)\} \tag{3.2.3}$$

Fourier transformation maps $\mathcal{D}_0$ onto itself.

We define the annihilation operator $a(k)$ on $\mathcal{D}_0$, which maps $\mathcal{F}_n$ onto $\mathcal{F}_{n-1}$. For $f \in \mathcal{D}_0$,

$$(a(k)\, f)_{n-1}(k_1, \ldots, k_{n-1}) = n^{\frac{1}{2}} f_n(k, k_1, \ldots, k_{n-1}), \tag{3.2.4}$$

so that $a(k)\,\mathcal{D}_0 \subset \mathcal{D}_0$. Clearly $\quad [a(k), a(k')]\,\mathcal{D}_0 = 0$

and any product $a(k_1) \ldots a(k_\beta)$ is defined on the domain $\mathcal{D}_0$.

The adjoint $a(k)^*$ of $a(k)$ has domain zero as an operator, but it is a densely defined bilinear form on $\mathcal{D}_0 \times \mathcal{D}_0$. Even though $a(k)^*$ is not a densely defined operator, we follow convention and call it the creation operator for a particle with momentum k.

Any monomial of creation and annihilation operators

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(1) Formulations of Fock space and operators on it may be found in many places. Convenient for our point of view is the systematic treatment of Kristensen, Mejlbo and Poulsen [20]. Creation and annihilation operators as bilinear forms were also considered by Galindo [6]. Our use in [10] of weak integrals of bilinear forms has been questioned [23], and for this reason we present the elementary details leading to formula (3.2.7). We note that every Wightman field $\phi(x)$ is a densely defined bilinear form with C^∞ dependence on x [3].

$$a^*(k_1) \dots a^*(k_\alpha) a(k_1) \dots a(k_\beta), \tag{3.2.5}$$

with the creation operators to the left of the annihilators, is a bilinear form on $\mathcal{D}_0 \times \mathcal{D}_0$ with values

$$(f, a^*(k_1) \dots a^*(k_\alpha) a(p_1) \dots a(p_\beta) g) \in \mathcal{S}(R^{\alpha+\beta}). \tag{3.2.6}$$

Thus if $c_{\alpha\beta}(k; p) \in \mathcal{S}'(R^{\alpha+\beta})$ is a tempered distribution,

$$C_{\alpha\beta} = \int c_{\alpha\beta}(k; p)\, a^*(k_1) \dots a^*(k_\alpha)\, a(p_1) \dots a(p_\beta)\, dk\, dp \tag{3.2.7}$$

is a bilinear form on $\mathcal{D}_0 \times \mathcal{D}_0$ that exists as a weak integral of bilinear forms. The $C_{\alpha\beta}$ of (3.2.7) is a *Wick ordered monomial* and is the quantization of the tempered distribution $c_{\alpha\beta}(k; p)$.

If $c_{\alpha\beta}(k;p)$ is the kernel of a bounded operator $c_{\alpha\beta}$ from $\mathcal{S}(R^\beta)$ to $L_2(R^\alpha)$, then the bilinear form $C_{\alpha\beta}$ on $\mathcal{D}_0 \times \mathcal{D}_0$ determines an unbounded operator on $\mathcal{F}$ with domain $\mathcal{D}_0$. In order to establish this, we note that if $i_{\gamma\gamma}$ is the identity operator on $L_2(R^\gamma)$, then $c_{\alpha\beta} \otimes i_{\gamma\gamma}$ is a bounded operator from $\mathcal{S}(R^\beta) \otimes \mathcal{S}(R^\gamma)$ into $L_2(R^{\alpha+\gamma})$, and it extends uniquely to a bounded operator from $\mathcal{S}(R^{\beta+\gamma})$ into $L_2(R^{\alpha+\gamma})$.

Symmetrization is a projection on L_2 and on $\mathcal{S}$. Thus the symmetrization of $c_{\alpha\beta} \otimes i_{\gamma\gamma}$ in the $\beta+\gamma$ initial variables and in the $\alpha+\gamma$ final variables yields a bounded operator from $\mathcal{S}(R^{\beta+\gamma})$ to $L_2(R^{\alpha+\gamma})$, and an unbounded operator from $\mathcal{F}_{\beta+\gamma}$ to $\mathcal{F}_{\alpha+\gamma}$ with domain $\mathcal{S}(R^{\beta+\gamma}) \cap \mathcal{F}_{\beta+\gamma}$. Finally, $C_{\alpha\beta}$ is the sum over γ of the symmetrizations of $((\alpha+\gamma)!/\gamma!)\,((\beta+\gamma)!/\gamma!)^{\frac{1}{2}}$ $c_{\alpha\beta} \otimes i_{\gamma\gamma}$ with domain $\mathcal{D}_0$.

If c_{11} is a bounded operator from $\mathcal{S}(R^1)$ to $L_2(R^1)$, then the operator C_{11} has a closure on $\mathcal{F}$ if the operator c_{11} has a closure as an unbounded operator from $L_2(R^1)$ to $L_2(R^1)$. If $c_{\alpha\beta}$ is bounded from $L_2(R^\beta)$ to $L_2(R^\alpha)$, then $C_{\alpha\beta}$ has a closure with domain containing $\mathcal{D}(N^{(\alpha+\beta)/2})$, where N is the total number of particles operator defined by

$$(Nf)_n = nf_n. \tag{3.2.8}$$

If $\|c_{\alpha\beta}\|$ is the norm of $c_{\alpha\beta}$ as a transformation from $L_2(R^\beta)$ to $L_2(R^\alpha)$ then

$$\|(N+I)^{-\alpha/2} C_{\alpha\beta} (N+I)^{-\beta/2}\| \leq \|c_{\alpha\beta}\|. \tag{3.2.9}$$

This we derive from the bound

$$\begin{aligned} |(f_{\alpha+\gamma}, C_{\alpha\beta} g_{\beta+\gamma})| &\leq \left(\frac{(\alpha+\gamma)!}{\gamma!} \frac{(\beta+\gamma)!}{\gamma!}\right)^{\frac{1}{2}} \|c_{\alpha\beta}\|\, \|f_{\alpha+\gamma}\|\, \|g_{\beta+\gamma}\| \\ &\leq \|c_{\alpha\beta}\|\, (\alpha+\gamma)^{\alpha/2} \|f_{\alpha+\gamma}\|\, (\beta+\gamma)^{\beta/2} \|g_{\beta+\gamma}\|. \end{aligned}$$

If $c_{\alpha\beta}$ maps $\mathcal{S}(R^\beta)$ continuously into $\mathcal{S}(R^\alpha)$, then $C_{\alpha\beta}$ maps $\mathcal{D}_0$ into $\mathcal{D}_0$.

By Fourier transformation we define the annihilation operator $A(x)$ for a particle at the space point x,

$$A(x) = (2\pi)^{-\frac{1}{2}} \int a(p)\, e^{-ipx}\, dp, \tag{3.2.10}$$

corresponding to the quantization of $c_{01}(p) = (2\pi)^{-1/2} \exp(-ipx)$. In the configuration space representation, for $f \in \mathcal{D}_0$,

$$(A(x)f)_{n-1}(x_1, \ldots, x_{n-1}) = n^{\frac{1}{2}} f_n(x, x_1, \ldots, x_{n-1}). \tag{3.2.11}$$

As above, we can quantize a tempered distribution $c_{\alpha\beta}(x; y)$ yielding

$$C_{\alpha\beta} = \int c_{\alpha\beta}(x; y)\, A^*(x_1) \ldots A^*(x_\alpha)\, A(y_1) \ldots A(y_\beta)\, dx\, dy, \tag{3.2.12}$$

a bilinear form on $\mathcal{D}_0 \times \mathcal{D}_0$. These $C_{\alpha\beta}$ have the same properties as the $C_{\alpha\beta}$ in the momentum representation discussed above.

We also use the Fock space $\mathcal{F}(\mathcal{B})$ defined over the one particle space $L_2(\mathcal{B})$ where $\mathcal{B}$ is an interval in R^1.

$$\mathcal{F}(\mathcal{B}) = \bigoplus_{n=0}^{\infty} \mathcal{F}_n(\mathcal{B}), \tag{3.2.13}$$

where $\mathcal{F}_n(\mathcal{B})$ is the Hilbert space completion of the n-fold symmetric tensor product of $L_2(\mathcal{B}) = \mathcal{F}_1(\mathcal{B})$. If $\mathcal{B} \subset \mathcal{B}_1$, then $\mathcal{F}(\mathcal{B}) \subset \mathcal{F}(\mathcal{B}_1)$. If $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ and $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$, then

$$\mathcal{F}(\mathcal{B}) = \mathcal{F}(\mathcal{B}_1) \otimes_s \mathcal{F}(\mathcal{B}_2), \tag{3.2.14}$$

where $\otimes_s$ denotes the completion of the symmetric tensor product.

We now specialize to study the biquantization C of operators c on the one particle space $\mathcal{F}_1$. For simplicity, we do not write c_{11} or C_{11}. We consider operators c studied in Section 3.1; they are defined on the domain $\mathcal{S}(R^1)$ and are continuous transformations of $\mathcal{S}(R^1)$ into $L_2(R^1)$. Then on the domain $\mathcal{D}_0$,

$$C = \int c(x, y)\, A^*(x)\, A(y)\, dx\, dy, \tag{3.2.15}$$

where $c(x, y)$ is the kernel of c.

We give some examples which we call local number operators:

$$N_B \text{ is the quantization of multiplication by } E(x). \tag{3.2.16}$$

$$N_\tau \text{ is the quantization of } \mu_x^\tau. \tag{3.2.17}$$

$$N_{\tau,\zeta} \text{ is the quantization of } \zeta(x)\mu_x^\tau\zeta(x). \tag{3.2.18}$$

$$N_{\tau,B} \text{ is the quantization of } E(x)\mu_x^\tau E(x), \text{ for } \tau < \tfrac{1}{4}. \tag{3.2.19}$$

In the case $\tau = 1$, the operator $N_{1,\zeta}$ provides a local energy operator H_0^{loc}, discussed in Section 1.

THEOREM 3.2.1. *The operator* $C \upharpoonright \mathcal{D}_0$ *of* (3.2.15) *has any of the following properties if and only if the operator* $c \upharpoonright \mathcal{S}(R^1)$ *has the same property.*

(a) *The operator is essentially self adjoint.*

(b) *The operator is positive.*

(c) *The operator is positive and the Friedrichs extension* (*or closure*) *has a compact resolvent.*

Proof. (a) If $C \upharpoonright \mathcal{D}_0$ has a self adjoint closure, the domain of the self adjoint closure contains a dense set of analytic vectors. Since C commutes with the projection of $\mathcal{J}$ onto $\mathcal{J}_1$, there is a dense set of one particle analytic vectors, obtained as limits of one particle states in $\mathcal{S}(R^1)$. Thus they are in the domain of the closure of c, and c is essentially self adjoint. Conversely, if $c \upharpoonright \mathcal{S}(R^1)$ has a self adjoint closure, the domain of the closure c^- contains a total set of analytic unit vectors $e_i(x)$, $i=1,2,\ldots$ The vectors $e_{i_1\ldots i_n}=e_{i_1}\otimes_s\ldots\otimes_s e_{i_n}$ are n-particle vectors in the domain of the closure of $C \upharpoonright \mathcal{D}_0$. Clearly they are C^∞ vectors for C^-, since

$$C^- e_{i_1\ldots i_n} = c^- e_{i_1}\otimes_s\ldots\otimes_s e_{i_n}+\ldots+e_{i_1}\otimes_s\ldots\otimes_s c^- e_{i_n}.$$

Furthermore if,
$$\|(c^-)^m e_i\| \leqslant \beta_i^m m!,$$

then
$$\|(C^-)^m e_{i_1\ldots i_n}\| \leqslant n^m\beta^m m!, \qquad \beta=\max\{\beta_{i_1},\ldots,\beta_{i_n}\}.$$

Thus the $e_{i_1\ldots i_n}$ are analytic vectors for C^- and $C \upharpoonright \mathcal{D}_0$ is essentially self adjoint.

(b) If C is positive, c is clearly positive. Conversely, suppose that c is positive and $f_n(x_1,\ldots,x_n)\in\mathcal{D}_0\cap\mathcal{J}_n$. Then for each $x_2,\ldots,x_n$, $f_n(x_1,x_2,\ldots,x_n)\in\mathcal{S}(R^1)$ and

$$\int \overline{f}_n(x_1,x_2,\ldots,x_n)\,c(x,y)\,f_n(y,x_2,\ldots,x_n)\,dx\,dy \geqslant 0$$

is a function in $\mathcal{S}(R^{n-1})$. Thus integrating over $x_2,\ldots,x_n$ we have shown that c is a positive operator on each n particle coordinate and hence C is positive on $\mathcal{D}_0\cap\mathcal{J}_n$, and so on $\mathcal{D}_0$.

(c) Let C_F and c_F be the Friedrichs extensions of $C \upharpoonright \mathcal{D}_0$ and $c \upharpoonright \mathcal{S}(R^1)$ respectively. It is sufficient to show that $c_F^{\frac{1}{2}}$ has a compact resolvent if and only if $C_F^{\frac{1}{2}}$ does. Note that $\mathcal{D}_0$ is a core for $C_F^{\frac{1}{2}}$ and $\mathcal{S}(R^1)$ is a core for $c_F^{\frac{1}{2}}$. The Friedrichs extension C_F commutes with the projection P_1 onto $\mathcal{J}_1$ as a bilinear form on $\mathcal{D}(C_F^{\frac{1}{2}})\times\mathcal{D}(C_F^{\frac{1}{2}})$, and hence as an operator on $\mathcal{D}(C_F)$. Thus $C_F^{\frac{1}{2}}$ commutes with P_1, and if $C_F^{\frac{1}{2}}$ has a compact resolvent, so does $c_F^{\frac{1}{2}}=P_1C_F^{\frac{1}{2}}P_1$. Conversely, if $c_F^{\frac{1}{2}}$ has a compact resolvent, we choose orthonormal eigenvectors e_i of $c_F^{\frac{1}{2}}$ corresponding to eigenvalues $0\leqslant\lambda_i$, $\lambda_i\to\infty$. Since $e_i=\lim_{j\to\infty}e_{ij}$ where $e_{ij}(x)\in\mathcal{S}(R^1)$ and

$$\lim_{j,k\to\infty}(e_{ij}-e_{ik}, ce_{ij}-e_{ik})=0, \tag{3.2.20}$$

it follows that $e_{i_1\ldots i_n}=e_{i_1}\otimes_s \ldots \otimes_s e_{i_n}$ is in the domain of $C_F^{\frac{1}{2}}$, and hence is an eigenvector with eigenvalue $\lambda_{i_1}+\ldots+\lambda_{i_n}$. These eigenvectors are total in $\mathcal{J}$, so that $C_F^{\frac{1}{2}}$ has been diagonalized. It has discrete spectrum and eigenvalues

$$\sum_{i=0}^{\infty} n_i \lambda_i \tag{3.2.21}$$

for nonnegative integers n_i, only a finite number being non-zero. Since $\lambda_i\to\infty$, the eigenvalues have no point of accumulation except ∞ and hence $C_F^{\frac{1}{2}}$ has a compact resolvent. This completes the proof.

COROLLARY 3.2.2. (a) *The operators $N_{\mathcal{B}}$ and N are essentially self adjoint on $\mathcal{D}_0$. The operators $N_{\tau,\zeta}$ for $\zeta(x)\geqslant 0$ and $N_{\tau,\mathcal{B}}$ for $\tau<\frac{1}{4}$ are positive.*

(b) *If $0<\tau<\frac{1}{4}$, the Friedrichs extension of $N_{\tau,\mathcal{B}}$ has a compact resolvent on $\mathcal{J}(\mathcal{B})$.*

(c) *Let $\mathcal{B}$ be a bounded interval and*

$$\mathcal{B}=\bigcup_{j=1}^{J}\mathcal{B}_j.$$

Then for $0<\tau<\frac{1}{4}$ the Friedrichs extension of

$$\sum_{j=1}^{J} N_{\tau,\mathcal{B}_j}$$

has a compact resolvent on $\mathcal{J}(\mathcal{B})$.

Proof. We need only prove (b) and (c). Since the Friedrichs extension of $E\mu_x^\tau E$ commutes with the projection of $L_2(R^1)$ onto $L_2(\mathcal{B})$ (see the proof of Theorem 3.1.2) a similar argument shows that the Friedrichs extension of $N_{\tau,\mathcal{B}}$ commutes with the orthogonal projection $P_{\mathcal{B}}$ of $\mathcal{J}$ onto $\mathcal{J}(\mathcal{B})$. The corollary now follows from part (c) of the theorem, which can be proved for operators $C \restriction P_{\mathcal{B}}\mathcal{D}_0$ and $c\restriction E\mathcal{S}(R^1)$ on $\mathcal{J}(\mathcal{B})$ and $L_2(\mathcal{B})$. The compactness of the resolvent of $(E\mu_x^\tau E)_F$ is proved in Theorem 3.1.2. This completes the proof of (b). The proof of (c) follows from the compactness on $L_2(\mathcal{B})$ of the resolvent of

$$\Big(\sum_{j=1}^{J} E_j\mu_x^\tau E_j\Big)_F, \tag{3.2.22}$$

where E_j is the characteristic function of $\mathcal{B}_j$. Noting that $\|f\|^2\leqslant\sum_{j=1}^{J}\|E_j f\|^2$, we follow the proof of Theorem 3.1.2 in order to establish the compactness of the resolvent of the operator (3.2.22).

3.3. Vacuum expectation values of local number operators. In this section we assume the results of Section 5, and give estimates on the vacuum expectation values of local number operators. We first recall some definitions. In terms of the $A(x)$ and $A(x)^*$, the local fields $\phi(x)$ and $\pi(x)$ are defined by

$$\phi(x)=\frac{1}{\sqrt{2}}\,\mu_x^{-\frac{1}{2}}(A^*(x)+A(x)) \tag{3.3.1}$$

$$\pi(x)=\frac{i}{\sqrt{2}}\,\mu_x^{\frac{1}{2}}(A^*(x)-A(x)), \tag{3.3.2}$$

and are $S(R^1)$ valued bilinear forms on $\mathcal{D}_0\times\mathcal{D}_0$.

In Section 2 we defined the approximate vacuum ω_n by

$$\omega_n(A)=\frac{1}{n}\int(\Omega_{g_n},\sigma_\alpha(A)\,\Omega_{g_n})\;h(\alpha/n)\,d\alpha. \tag{3.3.3}$$

THEOREM 3.3.1. (a) *Let $\mathcal{B}$ be a bounded interval in R_1. Then*

$$\omega_n(N_{\mathcal{B}})\leqslant M, \tag{3.3.4}$$

where M is a translation invariant constant, independent of n.

(b) *If $0\leqslant\zeta(x)\in C_0^\infty$, and $\tau\leqslant 1$, then*

$$\omega_n(N_{\tau,\zeta})\leqslant M, \tag{3.3.5}$$

where M is a translation invariant constant, independent of n.

(c) *If $f=\bar{f}\in S(R^1)$, then*

$$\omega_n(\phi(f)^2+\pi(f)^2)\leqslant|f|^2,$$

where $|f|$ is a Schwartz space norm of f, independent of n.

(d) *If $\tau<\frac{1}{4}$*

$$\omega_n(N_{\tau,\mathcal{B}})\leqslant M, \tag{3.3.6}$$

where the constant M is translation invariant and independent of n.

Proof. (a) Let $\mathcal{B}=[a,b]$ and

$$c_n(x,y)=\frac{1}{n}\int E(x-\alpha)\,E(y-\alpha)\,h(\alpha/n)\,d\alpha\delta(x-y),$$

so that

$$\omega_n(N_{\mathcal{B}})=(\Omega_{g_n},C_n\Omega_{g_n}). \tag{3.3.7}$$

Since
$$\left|\int E(x-\alpha)\,E(y-\alpha)\,h(\alpha/n)\,d\alpha\right| \leqslant \|h\|_\infty\,\|E\|_1 = \|h\|_\infty\,|b-a|,$$
$c_n(x, y)$ is the kernel of an operator on $\mathcal{J}_1$ with norm bounded by
$$\text{const.}\ \frac{1}{n}\,|b-a|.$$
Thus by (3.2.9),
$$\|(N+I)^{-\frac{1}{2}}\,C_n\,(N+I)^{-\frac{1}{2}}\| \leqslant \text{const.}\ \frac{1}{n}\,|b-a|$$
and by (3.3.7)
$$\omega_n(N_B) \leqslant \text{const.}\ \frac{1}{n}\,|b-a|\,\omega_n(N) \leqslant \text{const.}\ |b-a|$$
by Theorem 5.1.

(b) We follow the above proof with
$$c_n(x, y) = \frac{1}{n}\int \zeta(x-\alpha)\,k_\tau(x-y)\,\zeta(y-\alpha)\,h(\alpha/n)\,d\alpha.$$
Let $\zeta_\alpha = \zeta(\cdot - \alpha)$. We note that for $f \in S(R^1)$,
$$\begin{aligned}
(f, c_n f) &= \frac{1}{n}\int (f, \zeta_\alpha\mu_x^\tau\zeta_\alpha f)\,h(\alpha/n)\,d\alpha \leqslant \frac{1}{n}\,\|h\|_\infty\left\{\int (f, \mu_x^{\tau/2}\zeta_\alpha^2\mu_x^{\tau/2} f)\,d\alpha\right.\\
&\quad \left. + \int |(f, [\zeta_\alpha, \mu_x^{\tau/2}]\,[\mu_x^{\tau/2}, \zeta_\alpha] f)|\,d\alpha + 2\int |(f, \mu_x^{\tau/2}\zeta_\alpha[\zeta_\alpha, \mu_x^{\tau/2}] f)|\,d\alpha\right\}\\
&\leqslant \frac{1}{n}\,\|h\|_\infty\left\{\int \|\zeta_\alpha\mu_x^{\tau/2} f\|^2\,d\alpha + \int \|[\mu_x^{\tau/2}, \zeta_\alpha] f\|^2\,d\alpha + 2\int \|[\mu_x^{\tau/2}, \zeta_\alpha] f\|\,\|\zeta_\alpha\mu_x^{\tau/2} f\|\,d\alpha\right\}\\
&\leqslant \frac{2}{n}\,\|h\|_\infty\int (\|\zeta_\alpha\mu_x^{\tau/2} f\|^2 + \|[\mu_x^{\tau/2}, \zeta_\alpha] f\|^2)\,d\alpha\\
&= \frac{2}{n}\,\|h\|_\infty\left(\|\zeta\|_2^2\,\|\mu_x^{\tau/2} f\|^2 + \int \|[\mu_x^{\tau/2}, \zeta_\alpha] f\|^2\,d\alpha\right).
\end{aligned}$$
The commutator is a bounded operator, since
$$\begin{aligned}
|([\mu_x^{\tau/2}, \zeta] f)^\sim(p)| &\leqslant (2\pi)^{-\frac{1}{2}}\int |(\mu(p)^{\tau/2} - \mu(q)^{\tau/2})\,\tilde{\zeta}(p-q)\,\tilde{f}(q)|\,dq\\
&\leqslant \text{const.}\int (p-q)^{\tau/2}\,|\tilde{\zeta}(p-q)\,\tilde{f}(q)|\,dq
\end{aligned}$$
and therefore
$$\|[\mu_x^{\tau/2}, \zeta] f\|_2 \leqslant \text{const.}\ \|(\mu^{\tau/2}\,|\tilde{\zeta}|)^\sim\|_\infty\,\|f\|_2 \leqslant M\|f\|_2,$$
as $\tilde{\zeta}$ is rapidly decreasing.

Let $B=[a,b]$ be a neighborhood of the support of ζ, let E be the characteristic function of B, and let $E_\alpha = E(\cdot - \alpha)$. Then

$$\int \|[\mu_x^{\tau/2}, \zeta_\alpha] E_\alpha f\|^2 d\alpha \leqslant \text{const.} \int \|E_\alpha f\|^2 d\alpha \leqslant \text{const.}\ (b-a)\|f\|^2.$$

Furthermore, the operator

$$[\mu_x^{\tau/2}, \zeta_\alpha](I-E_\alpha) = \zeta_\alpha \mu_x^{\tau/2}(I-E_\alpha)$$

is an integral operator with a kernel dominated by const. $e^{-m_0(|x-\alpha|+|y-\alpha|)}$. (See (3.1.6).) Thus the operator

$$K_\alpha = [\mu_x^{\tau/2}, \zeta_\alpha](I-E_\alpha)(1+(x-\alpha)^2)$$

has an L_2 kernel and is consequently bounded with a bound M_1 independent of α. It follows that

$$\int \|[\mu_x^{\tau/2}, \zeta_\alpha](I-E_\alpha) f\|^2 d\alpha = \int \|K_\alpha (1+(x-\alpha)^2)^{-1} f\|^2 d\alpha$$
$$\leqslant M_1^2 \int \|(1+(x-\alpha)^2)^{-1} f\|^2 d\alpha = M_1^2 \int (1+t^2)^{-2} dt\, \|f\|^2.$$

Combining these results

$$(f, c_n f) \leqslant \frac{1}{n} \text{ const.}\ (\|\mu_x^{\tau/2} f\|^2 + \|f\|^2) \leqslant \frac{1}{n} \text{ const.}\ \|\mu_x^{\tau/2} f\|^2.$$

Therefore there is a constant M such that

$$\frac{M}{n}\mu_x^\tau - c_n$$

is a positive operator, and hence by Theorem 3.2.1 b we have after biquantization

$$\frac{M}{n} N_\tau - C_n \geqslant 0.$$

Thus for $\tau \leqslant 1$,
$$\omega_n(C_n) \leqslant \frac{M}{n}\omega_n(N_\tau) \leqslant \frac{M}{n} \text{ const. } \omega_n(H_0).$$

By Theorem 5.1 and Remark 2, equation (5.10),

$$\omega_n(C_n) \leqslant \text{const.}$$

(c) Since
$$\phi(f)^2 + \pi(f)^2 = :\phi(f)^2: + :\pi(f)^2: + \|\mu_x^{-\frac{1}{2}} f\|^2 + \|\mu_x^{\frac{1}{2}} f\|^2, \tag{3.3.8}$$

we need only consider the Wick ordered terms in (3.3.8). Each such term has j annihilators, $j=0, 1, 2,$ and $2-j$ creators. We now show that it is sufficient to consider the terms with $j=1$. Consider, for example, the pure creation term $(A^*(\mu_x^{\frac{1}{2}} f))^2$. We have

$$|\omega_n((A^*(\mu_x^{\frac{1}{2}} f))^2)| \leqslant \omega_n(A^*(\mu_x^{\frac{1}{2}} f)\, A(\mu_x^{\frac{1}{2}} f)) + \tfrac{1}{2} \|\mu_x^{\frac{1}{2}} f\|^2.$$

Similarly $$\omega_n(\phi(f)^2 + \pi(f)^2) \leqslant \text{const. } \omega_n(A^*(\mu_x^{\frac{1}{2}} f)\, A(\mu_x^{\frac{1}{2}} f)) + |f|^2 \tag{3.3.9}$$

for a suitable norm $|f|$. We estimate (3.3.9) as in the proof of parts (a) and (b).

(d) From Theorem 3.1.1, and Theorem 3.2.1 b, we conclude that on $\mathcal{D}_0 \times \mathcal{D}_0$,

$$N_{\tau, B} \leqslant \text{const. } N_{2\tau+\varepsilon, \zeta}, \tag{3.3.10}$$

where $\varepsilon > 0$. Hence (3.3.10) extends to the domain $\mathcal{D} \times \mathcal{D}$, where $\mathcal{D}$ is the domain of the square root of the Friedrichs extension of $N_{2\tau+\varepsilon, \zeta}$. The desired inequality now follows from part (b) of this theorem, by choosing ε sufficiently small so that $2\tau + \varepsilon \leqslant 1$.

4. Norm compactness of the approximate vacuums

In this section we use the bounds on the local number operators derived in Section 3. We give a proof of Theorem 2.3, and hence complete the proof of Theorems 2.1 and 2.2. We restate our result as Theorem 4.1, in a somewhat more general form.

THEOREM 4.1. *Let ω_n be a sequence of normal states defined on the algebra $\mathfrak{A}$ and suppose that for some $\tau > 0$ and for each $\zeta(x) \geqslant 0$ in C_0^∞,*

$$\omega_n(N_{\tau, \zeta}) \leqslant M, \tag{4.1}$$

where $M = M(\zeta)$ a translation invariant constant independent of n. Then the sequence $\omega_n \restriction \mathfrak{A}(\mathcal{B})$ lies in a norm compact subset of the dual $\mathfrak{A}(\mathcal{B})^*$, and any limit point $\omega \restriction \mathfrak{A}(\mathcal{B})$ is normal.

Remarks. 1. The state ω_n defined in (2.4) is a normal state on $\mathfrak{A}$, since the vector state $(\Omega_{q_n}, \cdot\, \Omega_{q_n})$ is normal and the space translation automorphism σ_α in (2.4) is implemented by a one parameter continuous unitary group of operators on $\mathcal{J}$. Hence σ_α extends to a strongly continuous automorphism of all bounded operators on $\mathcal{J}$.

2. In Theorem 3.3.1 we established the bound (4.1) for all $\tau \leqslant 1$ in the case that the ω_n are the approximate vacuum states defined in (2.4) for the $(\phi^4)_2$ interaction. Thus a proof of Theorem 4.1 provides a proof of Theorem 2.3.

3. If the inequality (4.1) holds for some $\tau > 0$, then for $\sigma \leqslant \tau$, a similar inequality holds.

$$\omega_n(N_{\sigma,\zeta}) \leqslant m_0^{\tau-\sigma}\,\omega_n(N_{\tau,\zeta}).$$

If $\sigma < \frac{1}{2}\tau$, and $\sigma < \frac{1}{4}$, then for a bounded interval $\mathcal{B}$ we conclude from (4.1) and the proof of Theorem 3.3.1 (d) that

$$\omega_n(N_{\sigma,\mathcal{B}}) \leqslant M, \tag{3.3.6a}$$

for a translation invariant constant $M = M(\mathcal{B})$ that is independent of n. The estimate (3.3.6(a)), for some $\sigma > 0$, is the crucial estimate in the proof of Theorem 4.1.

The method of proof is to use (3.3.6(a)) to replace the sequence ω_n by a regularized sequence ω_n^ε that is close to ω_n in norm.

$$\|\omega_n - \omega_n^\varepsilon\| \leqslant \varepsilon. \tag{4.2}$$

In the regularized state ω_n^ε, there will be only a finite number of particles in each bounded region of space, and that number will grow at most as a power of the diameter of the region. These bounds on ω_n^ε will be uniform in n.

The algebra $\mathfrak{A}_0(\mathcal{B})$ defined in (2.15) is weakly dense in $\mathfrak{A}(\mathcal{B})$. Since

$$\|(\omega_n^\varepsilon - \omega_m^\varepsilon) \restriction \mathfrak{A}(\mathcal{B})\| = \|(\omega_n^\varepsilon - \omega_m^\varepsilon) \restriction \mathfrak{A}_0(\mathcal{B})\|,$$

it is sufficient to prove that any sequence $\omega_n^\varepsilon \restriction \mathfrak{A}_0(\beta)$ has a norm convergent subsequence.

The next step in the proof is to approximate $C \in \mathfrak{A}_0(\mathcal{B})$ by an operator C^ε such that

$$|\omega_n^\varepsilon(C - C^\varepsilon)| \leqslant \varepsilon \|C\|, \tag{4.3}$$

where C^ε is independent of n. The approximation C^ε of C is obtained by expanding the exponentials in C in power series, each term of which is a product of unbounded field operators. Because of the regularity of ω_n^ε, the expectation value $\omega_n^\varepsilon(C)$ has a corresponding infinite series expansion which converges uniformly in n. To obtain C^ε, we neglect the tail of this expansion and we also localize each term in a finite interval of space. The localization also leads to a small error, uniform in n, and satisfying (4.3).

Finally it will be possible to apply the compactness of the resolvent of the localized number operator $N_{\tau,\mathcal{B}}$ proved in Corollary 3.2.2 (c), in order to prove that the twice regularized expectation values $\omega_n^\varepsilon(C^\varepsilon)$ have a subsequence such that

$$|(\omega_{n_i}^\varepsilon - \omega_{n_j}^\varepsilon)(C^\varepsilon)| \leqslant o(1)\,\|C\|. \tag{4.4}$$

From (4.2)–(4.4) we will conclude that Theorem 4.1 is valid.

We prove Theorem 4.1 in the case where $\mathcal{B}$ is an open interval of space whose closure is contained in (0, 1). This restriction on $\mathcal{B}$ is for convenience only. Let

$$X_j = [j, j+1).$$

The local number operators N_{X_j}, $j=0, \pm 1, \ldots$, are commuting self adjoint operators. Let $P_{l,j}$ denote the spectral projection of the self adjoint operator N_{X_j} for the spectral interval $[0, l]$.

If $0 \leqslant \zeta(x)$ is a C_0^∞ function equal to one on X_0, then $\zeta_j(x) = \zeta(x+j)$ equals one on X_j and

$$N_{X_j} \leqslant N_{0,\zeta_j}.$$

By (3.3.6 (a)) in the case $\tau = 0$,
$$\omega_n(N_{X_j}) \leqslant M_1. \tag{4.5}$$

The constant M_1 is independent of n and of j. In the following, we introduce constants $M_2, \ldots$, independent of n and of C in $\mathfrak{A}$.

From (4.5) we have
$$\omega_n(I - P_{l,j}) \leqslant M_1 l^{-1}. \tag{4.6}$$

We use this fact to construct the approximate vacuum ω_n^ε, by keeping a finite number of particles in each interval X_j. For some constant M_2 to be chosen later, we define

$$l = l(j) = M_2 \varepsilon^{-2}(1 + j^2). \tag{4.7}$$

We define the projection
$$P = \prod_{j=-\infty}^{\infty} P_{l(j),j}, \tag{4.8}$$

and we define the linear functional $\omega_n^\varepsilon(\cdot)$ on $\mathfrak{A}$ by

$$\omega_n^\varepsilon(C) = \omega_n(PCP). \tag{4.9}$$

By (4.8) and the fact that the $P_{l,j}$ are commuting projections, we have

$$(I-P) \leqslant \sum_{j=-\infty}^{\infty} (I - P_{l(j),j}). \tag{4.10}$$

Thus
$$\omega_n(I-P) \leqslant \sum_{j=-\infty}^{\infty} \omega_n(I - P_{l(j),j})$$

and by (4.6) − (4.7)
$$\leqslant \varepsilon^2 \left(\frac{M_1}{M_2}\right) \sum_{j=-\infty}^{\infty} (1+j^2)^{-1}.$$

We now choose M_2 large, so that

$$\omega_n(I-P) \leqslant (\varepsilon/3)^2. \tag{4.11}$$

In order to verify that ω_n and ω_n^ε are close in norm, we write

$$\omega_n(C) - \omega_n^\varepsilon(C) = \omega_n((I-P)\,C(I-P)) + \omega_n((I-P)\,CP) + \omega_n(PC(I-P)). \tag{4.12}$$

Note that
$$|\omega_n(A^*B)|^2 \leqslant \omega_n(A^*A)\,\omega_n(B^*B). \tag{4.13}$$

Since
$$\|P\| = 1 = \|I-P\|,$$

and
$$(I-P)^2 = (I-P),$$

we have
$$|\omega_n(C) - \omega_n^\varepsilon(C)| \leqslant 3\,\|C\|\,[\omega_n(I-P)]^{\frac{1}{2}}.$$

Thus from (4.11) we conclude that

$$|\omega_n(C) - \omega_n^\varepsilon(C)| \leqslant \varepsilon\,\|C\|.$$

In terms of the states,
$$\|\omega_n - \omega_n^\varepsilon\| \leqslant \varepsilon. \tag{4.14}$$

We now study an operator $C \in \mathfrak{A}_0(\mathcal{B})$. Thus

$$C = \sum_{j=1}^{J} \alpha_j e^{i\phi(f_j) + i\pi(h_j)}. \tag{4.15}$$

The test functions f_j and h_j are smooth and their support is contained in $\mathcal{B}$. For such a C, it is obvious what an expansion in terms of field operators means. First we expand C in terms of annihilation and creation operators A and A^*. We write

$$C = \sum_{\alpha,\beta=0}^{\infty} C_{\alpha\beta}, \tag{4.16}$$

where $C_{\alpha\beta}$ is a Wick ordered monomial of degree α in creation operators A^* and of degree β in annihilation operators A. Thus

$$C_{\alpha\beta} = \int c_{\alpha\beta}(x_1, \ldots, x_\alpha, x_{\alpha+1}, \ldots, x_{\alpha+\beta})\,A^*(x_1)\ldots A^*(x_\alpha)\,A(x_{\alpha+1})\ldots A(x_{\alpha+\beta})\,dx. \tag{4.17}$$

The kernel $c_{\alpha\beta}(x)$ is symmetric in the α creation variables $x_1, \ldots, x_\alpha$ and in the β annihilation variables $x_{\alpha+1}, \ldots, x_{\alpha+\beta}$. We note several properties of the operators $C_{\alpha\beta}$, the kernels $c_{\alpha\beta}(x)$, and the corresponding operator $c_{\alpha\beta}$ from $\mathcal{F}_\beta$ to $\mathcal{F}_\alpha$.

Suppose that the kernel $c_{\alpha\beta}(x)$ has support in the region

$$x_i \in \mathcal{B}_i, \qquad i = 1, \ldots, \alpha+\beta,$$

for intervals $\mathcal{B}_i \subset R^1$. Then if

$$N_+ = \prod_{i=1}^{\alpha} N_{B_i}, \qquad N_- = \prod_{i=\alpha+1}^{\alpha+\beta} N_{B_i}, \tag{4.18}$$

we have

$$\|(N_+ + I)^{-\frac{1}{2}} C_{\alpha\beta} (N_- + I)^{-\frac{1}{2}}\| \leqslant \|c_{\alpha\beta}\|. \tag{4.19}$$

The bound (4.19) is a localization of (3.2.9), and is a consequence of the local action of $A^*(x)$ and $A(x)$ on the Newton–Wigner wave functions $f_n(x_1, ..., x_n) \in \mathfrak{J}_n$. (See Section 3.2.) We also note that for $\psi_\alpha \in \mathfrak{J}_\alpha$, $\psi_\beta \in \mathfrak{J}_\beta$,

$$\|c_{\alpha\beta}\| = \sup_{\substack{\|\psi_\alpha\|=1 \\ \|\psi_\beta\|=1}} |(\alpha!\,\beta!)^{-\frac{1}{2}} (\psi_\alpha, C_{\alpha\beta}\psi_\beta)| \tag{4.20}$$

where $C_{\alpha\beta}$ is the quantization (4.17) of $c_{\alpha\beta}$ on Fock space. Thus for

$$0 \leqslant \gamma \leqslant \min(\alpha, \beta),$$

we infer from (4.20) that

$$|(\psi_\alpha, C_{\alpha-\gamma,\beta-\gamma}\psi_\beta)| \leqslant \|\psi_\alpha\|\,\|\psi_\beta\| \left\{\frac{\alpha!}{\gamma!}\frac{\beta!}{\gamma!}\right\}^{\frac{1}{2}} \|c_{\alpha-\gamma,\beta-\gamma}\| \tag{4.21}$$

LEMMA 4.2. *For a bounded operator C of the form* (4.16) – (4.17),

$$\|c_{\alpha\beta}\| \leqslant 2^{\alpha+\beta} \|C\|. \tag{4.22}$$

Proof. Note that

$$(\psi_\alpha, C_{\alpha\beta}\psi_\beta) = (\psi_\alpha, \{C - \sum_{\gamma=1}^{\min(\alpha,\beta)} C_{\alpha-\gamma,\beta-\gamma}\}\psi_\beta)$$

By (4.20) – (4.21)

$$\|c_{\alpha\beta}\| \leqslant (\alpha!\,\beta!)^{-\frac{1}{2}} \left\{\|C\| + \sum_{\gamma=1}^{\min(\alpha,\beta)} \frac{(\alpha!\,\beta!)^{\frac{1}{2}}}{\gamma!} \|c_{\alpha-\gamma,\beta-\gamma}\|\right\}$$

$$= \|C\| \left\{(\alpha!\,\beta!)^{-\frac{1}{2}} + \sum_{\gamma=1}^{\min(\alpha,\beta)} \frac{1}{\gamma!} \frac{\|c_{\alpha-\gamma,\beta-\gamma}\|}{\|C\|}\right\}. \tag{4.23}$$

We reason by induction on $\min(\alpha, \beta)$. If $\min(\alpha, \beta) = 0$, then the sum in (4.23) does not occur and

$$\|c_{\alpha\beta}\| \leqslant \|C\|.$$

If $\min(\alpha, \beta) \neq 0$, then by the induction hypothesis (4.22) is valid for each $\|c_{\alpha-\gamma,\beta-\gamma}\|$ in the sum on the right side of (4.23). Thus

$$\|c_{\alpha\beta}\| \leqslant \|C\| \left\{(\alpha!\,\beta!)^{-\frac{1}{2}} + \sum_{\gamma=1}^{n} \frac{1}{\gamma!} 2^{\alpha+\beta-2\gamma}\right\} \leqslant \|C\| \left\{(a!\,\beta!)^{-\frac{1}{2}} + \sum_{\gamma=1}^{\infty} 2^{\alpha+\beta} \frac{4^{-\gamma}}{\gamma!}\right\}$$

$$= \|C\| \{(\alpha!\,\beta!)^{-\frac{1}{2}} + 2^{\alpha+\beta}(e^{\frac{1}{4}} - 1)\} \leqslant 2^{\alpha+\beta}\|C\|,$$

and this completes the proof of Lemma 4.2.

We want to show that $c_{\alpha\beta}(x)$ is exponentially small at infinity in each of the variables x_i. For this purpose we define a localization multi-index $L=\{L_1, L_2, ..., L_{\alpha+\beta}\}$. We let L_k be an integer corresponding to the localization of the coordinate x_k in the interval X_{L_k}. We use the localization indices L to localize the operator $C_{\alpha\beta}$. We define

$$c^L_{\alpha\beta}(x) = \text{Sym.}\, c_{\alpha\beta}(x) \prod_{k=1}^{\alpha+\beta} E(X_{L_k}), \quad L=\{L_k\}, \tag{4.24}$$

where $E(X_j)$ is the characteristic function for the interval X_j. The symmetrization in (4.24) symmetrizes the kernel in the α creation variables $(x_1, ..., x_\alpha)$ and in the β annihilation variables $(x_{\alpha+1}, ..., x_{\alpha+\beta})$. The corresponding localized operator $c^L_{\alpha\beta}$ from $\mathfrak{F}_\beta$ to $\mathfrak{F}_\alpha$ has the kernel $c^L_{\alpha\beta}(x)$. We denote by $C^L_{\alpha\beta}$ the localized, quantized operator arising from $c^L_{\alpha\beta}$.

$$C^L_{\alpha\beta} = \int c^L_{\alpha\beta}(x)\, A^*(x_1) \dots A^*(x_\alpha)\, A(x_{\alpha+1}) \dots A(x_{\alpha+\beta})\, dx. \tag{4.25}$$

We define
$$N^L_+ = \prod_{k=1}^{\alpha} N_{X_{L_k}} \tag{4.26}$$

and
$$N^L_- = \prod_{k=\alpha+1}^{\alpha+\beta} N_{X_{L_k}}. \tag{4.27}$$

Then by (4.19)
$$\|(N^L_+ + I)^{-\frac{1}{2}}\, C^L_{\alpha\beta}\, (N^L_- + I)^{-\frac{1}{2}}\| \leqslant \|c^L_{\alpha\beta}\|. \tag{4.28}$$

We note that the kernels $c_{\alpha\beta}(x)$ from operators $C \in \mathfrak{A}_0(\mathcal{B})$ are functions in $\mathcal{S}(R^{\alpha+\beta})$. Thus each term in (4.24) before symmetrization is a bounded operator from $L_2(R^\beta)$ to $L_2(R^\alpha)$ with a norm bounded by $\|c_{\alpha\beta}\|$. Hence the same holds for $c^L_{\alpha\beta}(x)$ and

$$\|c^L_{\alpha\beta}\| \leqslant \|c_{\alpha\beta}\|.$$

Thus by Lemma 4.2,
$$\|c^L_{\alpha\beta}\| \leqslant 2^{\alpha+\beta}\, \|C\|. \tag{4.29}$$

We now define a measure $D = D(L)$ of the total distance from the origin of the localization L in $C^L_{\alpha\beta}$.

$$D = D(L) = \sum_{k=1}^{\alpha+\beta} |L_k|. \tag{4.30}$$

LEMMA 4.3. *There is a constant M_4 independent of C such that for $C \in \mathfrak{A}_0(\mathcal{B})$,*

$$\|c^L_{\alpha\beta}\| \leqslant M_4^{\alpha+\beta}\, e^{-m_0 D(L)}\, \|C\|. \tag{4.31}$$

This lemma improves the bound (4.29). We postpone the proof to the end of this section. We remark, however, that the basis for the localization (4.31) is the fact that when

C is expanded in terms of the fields $\phi(y)$ and $\pi(y)$, only points $y \in \mathcal{B}$ enter the expansion. The expansion (4.17) approximately preserves this localization, and (4.31) is a quantitative estimate of the loss of locality that results from using the A^*, A expansion.

We now study the operator $PC^L_{\alpha\beta}P$ where P is the projection defined in (4.8). By definition P projects onto a subspace of $\mathcal{F}$ with a finite number $l(j)$ of particles in each interval X_j. Since the localization L restricts the kernel $c^L_{\alpha\beta}(x)$ to have compact support, the operator $C^L_{\alpha\beta}$ creates or annihilates particles in a bounded region. Hence by (4.25),

$$PC^L_{\alpha\beta}P \tag{4.32}$$

is a bounded operator.

LEMMA 4.4. *Given $\varepsilon_1 > 0$, there exists a constant M_5 such that for all $C \in \mathfrak{A}_0(\mathcal{B})$,*

$$\|PC^L_{\alpha\beta}P\| \leq M_5^{\alpha+\beta} e^{-(m_0-\varepsilon_1)D(L)} \|C\|.$$

Proof. We first prove that given $\varepsilon_1 > 0$, there is a constant M_6 such that

$$\|P(N^L_+ + I)^{\frac{1}{2}} P\| \, \|P(N^L_- + I)^{\frac{1}{2}} P\| \leq M_6^{\alpha+\beta} e^{\varepsilon_1 D(L)}.$$

By the definition of P and the fact that each N_{X_j} commutes with P, we have

$$\|P(N^L_+ + I)^{\frac{1}{2}} P\| \, \|P(N^L_- + I)^{\frac{1}{2}} P\| \leq \prod_{k=1}^{\alpha+\beta} (l(L_k)+1)^{\frac{1}{2}}. \tag{4.33}$$

For any $\varepsilon_1 > 0$, there is a constant M_6 such that

$$(l(L_k)+1)^{\frac{1}{2}} = \{M_2 \varepsilon^{-2}(1+L_k^2)+1\}^{\frac{1}{2}} \leq M_6 e^{\varepsilon_1 |L_k|}.$$

Thus
$$\prod_{k=1}^{\alpha+\beta} (l(L_k)+1)^{\frac{1}{2}} \leq M_6^{\alpha+\beta} e^{\varepsilon_1 D(L)}.$$

Using (4.33),
$$\|P(N^L_+ + I)^{\frac{1}{2}} P\| \, \|P(N^L_- + I)^{\frac{1}{2}} P\| \leq M_6^{\alpha+\beta} e^{\varepsilon_1 D(L)}.$$

Using estimate (4.28), we find that

$$\begin{aligned}\|PC^L_{\alpha\beta}P\| &= \|P(N^L_+ + I)^{\frac{1}{2}} (N^L_+ + I)^{-\frac{1}{2}} C^L_{\alpha\beta} (N^L_- + I)^{-\frac{1}{2}} (N^L_- + I)^{\frac{1}{2}} P\| \\ &\leq \|P(N^L_+ + I) P\| \, \|P(N^L_- + I)^{\frac{1}{2}} P\| \, \|(N^L_+ + I)^{-\frac{1}{2}} C^L_{\alpha\beta} (N^L_- + I)^{-\frac{1}{2}}\| \\ &\leq M_6^{\alpha+\beta} e^{\varepsilon_1 D(L)} \|c^L_{\alpha\beta}\|\end{aligned}$$

and by Lemma 4.3,
$$\leq (M_4 M_6)^{\alpha+\beta} e^{-(m_0-\varepsilon_1)D(L)} \|C\|,$$

to complete the proof.

Our next estimate bounds $D(L)$ from below.

LEMMA 4.5. *If P and $C^L_{\alpha\beta}$ are defined as in* (4.8) *and* (4.25), *and if*

$$PC^L_{\alpha\beta}P \neq 0, \tag{4.34}$$

then there is a constant M_7 such that

$$D(L) + 1 \geqslant M_7(\alpha^{\frac{4}{3}} + \beta^{\frac{4}{3}}). \tag{4.35}$$

Proof. For fixed values of α and β, we can minimize (4.30) by choosing the $|L_k|$ as small as possible. However, (4.34) will hold only if $C^L_{\alpha\beta}$ both creates and annihilates no more than $l(j)$ particles in each region X_j. For this reason it is convenient to count the number of particles created and annihilated in the region X_j for the localization L. We define $L_+(j)$ and $L_-(j)$ to be these numbers:

$$L_+(j) = \text{the number of } L_k = j, \quad \text{for } k = 1, 2, \ldots, \alpha.$$

$$L_-(j) = \text{the number of } L_k = j, \quad \text{for } k = \alpha + 1, \ldots, \alpha + \beta.$$

These numbers must satisfy

$$\sum_{j=-\infty}^{\infty} L_+(j) = \alpha, \quad \text{and} \quad \sum_{j=-\infty}^{\infty} L_-(j) = \beta. \tag{4.36}$$

In addition, (4.34) will be valid only if

$$L_+(j) \leqslant l(j) = M_2 \varepsilon^{-2}(1 + j^2), \tag{4.37}$$

and

$$L_-(j) \leqslant l(j) = M_2 \varepsilon^{-2}(1 + j^2). \tag{4.38}$$

In order to minimize $D(L)$ consistent with (4.34), we choose

$$L_\pm(j) = \begin{cases} l(j) = M_2 \varepsilon^{-2}(1 + j^2), & \text{if } |j| < J_\pm \\ 0, & \text{if } |j| > J_\pm. \end{cases} \tag{4.39}$$

The maximum value J_+ or J_- of any localization index $|L_k|$ in L, and the number of indices with this maximum value, are fixed by the requirements (4.36). Using (4.39) we conclude that there is a constant M_8 (approximately equal to $M_2\varepsilon^{-2}/3$) such that (excluding the trivial case $J_+ = 0$)

$$\alpha = \sum_{|j| < J_+} M_2 \varepsilon^{-2}(1 + j^2) + L_+(J_+) \leqslant M_8 J_+^3, \tag{4.40}$$

and similarly (excluding the trivial case $J_- = 0$),

$$\beta \leqslant M_8 J_-^3. \tag{4.41}$$

For our choice of L, there is a constant M_9, independent of α and β, such that

$$D(L) = \sum_{k=1}^{\alpha+\beta} |L_k| = \sum_{|j| \leqslant J_+} |j| L_+(j) + \sum_{|j| \leqslant J_-} |j| L_-(j) \geqslant M_9 (J_+^4 + J_-^4),$$

where we have used (4.39). Thus by (4.40) – (4.41) which bound J_+ and J_-,

$$D(L) \geqslant M_9 M_8^{-\frac{4}{3}} (\alpha^{\frac{4}{3}} + \beta^{\frac{4}{3}}).$$

If $J_+ = 0$, then $\alpha \leqslant M_2 \varepsilon^{-2}$ and if $J_- = 0$, then $\beta \leqslant M_2 \varepsilon^{-2}$. The proof is thus complete.

These estimates are useful for constructing the operator C^ε and for analyzing the error $C - C^\varepsilon$. Let us define the set of localization indices

$$\mathfrak{L}_M = \{L: |L_k| \leqslant M,\ k = 1, 2, \ldots, \alpha+\beta\}. \tag{4.42}$$

If $L \in \mathfrak{L}_M$, then all the particles that are created or annihilated by $C_{\alpha\beta}^L$ are in the region of space

$$Z = Z_M = \bigcup_{|j| \leqslant M} X_j. \tag{4.43}$$

We define for $C \in \mathfrak{A}_0(\mathcal{B})$,

$$C^\varepsilon = \sum_{\alpha, \beta = 0}^{M} \sum_{L \in \mathfrak{L}_M} C_{\alpha\beta}^L, \tag{4.44}$$

where $M = M(\varepsilon)$ will be chosen later, independently of C.

LEMMA 4.6 (a). *There exists a constant M_{10} independent of M such that for $C \in \mathfrak{A}_0(\mathcal{B})$ and for C^ε defined in* (4.44),

$$\|PC^\varepsilon P\| \leqslant M_{10} \|C\|. \tag{4.45}$$

(b) *Given $\varepsilon > 0$, there exists $M = M(\varepsilon)$ sufficiently large such that for all $C \in \mathfrak{A}_0(\mathcal{B})$*

$$\|P(C - C^\varepsilon) P\| \leqslant \varepsilon \|C\|, \tag{4.46}$$

and

$$|\omega_n^\varepsilon (C - C^\varepsilon)| \leqslant \varepsilon \|C\|. \tag{4.47}$$

Proof. By Lemma 4.4 with $\varepsilon_1 = \varepsilon$,

$$\|PC^\varepsilon P\| \leqslant \sum_{\alpha, \beta = 0}^{M} \sum_{L \in \mathfrak{L}_M}' M_5^{\alpha+\beta} e^{-(m_0 - \varepsilon) D(L)} \|C\|,$$

where $\sum_L'$ denotes the sum over those L for which $PC_{\alpha\beta}^L P \neq 0$, namely the L which satisfy the hypotheses of Lemma 4.5. Thus

$$\|PC^{\varepsilon}P\| \leq M_3 \|C\| \sum_{\alpha,\beta=0}^{M} \sum_{L\in\mathfrak{L}_M} M_5^{\alpha+\beta} \exp\{-\tfrac{1}{2}m_0 D(L) - M_{11}(\alpha^{\frac{3}{2}}+\beta^{\frac{3}{2}})\}, \tag{4.48}$$

where $M_{11} = M_7(\frac{1}{2}m_0 - \varepsilon)$ and $M_3 = e^{m_0-\varepsilon}$. The sum over all L converges exponentially fast, for

$$\sum_L \exp\{-\tfrac{1}{2}m_0 D(L)\} = \left[\sum_{l=-\infty}^{\infty} e^{-m_0|l|/2}\right]^{\alpha+\beta} = (M_{12})^{\alpha+\beta}. \tag{4.49}$$

The sum over the tail, $L \notin \mathfrak{L}_M$, is characterized by the fact that for some k, $|L_k| > M$. Thus

$$\sum_{L\notin\mathfrak{L}_M} \exp\{-\tfrac{1}{2}m_0 D(L)\} \leq (\alpha+\beta)\, e^{-Mm_0/2}(M_{12})^{\alpha+\beta}. \tag{4.50}$$

Using the estimate (4.49) in (4.48) shows that

$$\|PC^{\varepsilon}P\| \leq M_3 \|C\| \sum_{\alpha,\beta=0}^{M} (M_5 M_{12})^{\alpha+\beta} \exp\{-M_{11}(\alpha^{\frac{3}{2}}+\beta^{\frac{3}{2}})\}, \tag{4.51}$$

which converges faster than exponentially as $M\to\infty$. Hence (4.48) is bounded independently of $M = M(\varepsilon)$.

To establish (4.46), we write

$$\|P(C-C^{\varepsilon})P\| \leq \sum_{\alpha,\beta=0}^{M} \sum_{L\notin\mathfrak{L}_M} + \sum_{\substack{\alpha>M\\ \beta\geq 0}} \sum_{L} + \sum_{\substack{0\leq\alpha\leq M\\ \beta>M}} \sum_{L} \|PC^{L}_{\alpha\beta}P\|. \tag{4.52}$$

From (4.50) we conclude that the first sum in (4.52) is bounded by const. $\exp(-m_0 M/2)$, which converges to zero exponentially fast as $M\to\infty$. The remaining two terms in (4.52) converge to zero faster than exponentially as $M\to\infty$, as a consequence of the estimate on the tail of (4.51). Hence by choosing $M = M(\varepsilon)$ sufficiently large, we can assure that

$$\|P(C-C^{\varepsilon})P\| \leq \varepsilon\|C\|.$$

Lastly, we remark that

$$|\omega_n^{\varepsilon}(C-C^{\varepsilon})| = |\omega_n(P(C-C^{\varepsilon})P)| \leq \|P(C-C^{\varepsilon})P\| \leq \varepsilon\|C\|, \tag{4.53}$$

to complete the proof.

We next study the convergence of the sequence $\{\omega_n^{\varepsilon}\}$ of approximate states applied to the approximate operators C^{ε}. In other words, we study the convergence of the states ω_n on the bounded operators $PC^{\varepsilon}P$.

As a first reduction, we restrict our attention to the Fock space of a bounded region. Let Z be the interval defined in (4.43). Then if Z_1 is the complement of Z in R^1,

$$R^1 = Z \cup Z_1 \tag{4.54}$$

and by (3.2.14)
$$\mathcal{F} = \mathcal{F}(Z) \otimes_s \mathcal{F}(Z_1) \tag{4.55}$$

With this decomposition we write

$$P = (P_0 \otimes I)\,(I \otimes P_1). \tag{4.56}$$

We now study the local number operator N_{τ, X_j} defined in (3.2.19).

Here we choose $\mathcal{B} = X_j$ and we restrict τ to be less than $\frac{1}{4}$. Since the Friedrichs extension $(N_{\tau, X_j})_F$ of $N_{\tau, X_j} \restriction \mathcal{D}_0$ commutes with the projection onto $\mathcal{F}(X_i)$ for all i, it follows that the various $(N_{\tau, X_j})_F$ commute. Also $(N_{\tau, X_j})_F$ commutes with each N_{X_i} and with the spectral projections $P_{l, i}$. Thus $(N_{\tau, X_j})_F$ commutes with $P_0 \otimes I$, with $I \otimes P_1$, and with P. We now let N_{τ, X_j} denote the Friedrichs extension of $N_{\tau, X_j} \restriction \mathcal{D}_0$.

As a consequence of the commutativity with P,

$$\omega_n^\varepsilon (N_{\tau, X_j}) = \omega_n (P N_{\tau, X_j} P) = \omega_n (N^{\frac{1}{2}}_{\tau, X_j} P N^{\frac{1}{2}}_{\tau, X_j}) \leqslant \omega_n (N_{\tau, X_j}).$$

We choose $\tau > 0$ sufficiently small so that (3.3.6 a) is valid. (See Remark 3 following Theorem 4.1.) Hence

$$\omega_n^\varepsilon (N_{\tau, X_j}) \leqslant \text{const.}$$

We now choose M as in (4.42) – (4.44) and define

$$N_{\tau, M} = \Big(\sum_{|j| \leqslant M} N_{\tau, X_j}\Big)_F. \tag{4.57}$$

Thus
$$\omega_n^\varepsilon (N_{\tau, M}) \leqslant \text{const.} \tag{4.58}$$

We note that $N_{\tau, M}$ commutes with each $P_0 \otimes I$ and with $I \otimes P_1$. Also $N_{\tau, M}$ leaves $\mathcal{F}(Z)$ invariant, and by Corollary 3.2.2 c,

$$N_{\tau, M} \restriction \mathcal{F}(Z)$$

has a compact resolvent.

We now study the sequence $\{\omega_n^\varepsilon\}$ of functionals restricted to $\mathfrak{B}(Z)$, the algebra of all bounded operators on the Hilbert space $\mathcal{F}(Z)$. Each functional ω_n^ε is normal, since ω_n is normal by assumption in Theorem 4.1, and P is a projection. The normal functional

$$\omega_n^\varepsilon \restriction \mathfrak{B}(Z)$$

has the form
$$\omega_n^\varepsilon (A) = \operatorname{Tr} (\Lambda_n A), \quad A \in \mathfrak{B}(Z), \tag{4.59}$$

where Λ_n is a positive trace class operator in $\mathfrak{B}(Z)$ with

$$\|\omega_n^\varepsilon \restriction \mathfrak{B}(Z)\| = \operatorname{Tr} (\Lambda_n) \leqslant 1. \tag{4.60}$$

(See [2].) The operator $N_{\tau, M}$ in (4.57) can be restricted to $\mathcal{F}(Z)$, and thus by (4.58)

$$\omega_n^\varepsilon(N_{\tau,M} \restriction \mathcal{F}(Z)) \leqslant M. \tag{4.61}$$

We wish to study the convergence of

$$\omega_n^\varepsilon(C^\varepsilon) = \omega_n(PC^\varepsilon P).$$

We note that $PC^\varepsilon P = P_0 C^\varepsilon P_0 \otimes P_1$ where we regard C^ε on the right side as an operator on $\mathcal{F}(Z)$. Thus

$$\omega_n^\varepsilon(C^\varepsilon) = \operatorname{Tr}(\Lambda_n P_0 C^\varepsilon P_0).$$

The desired convergence of the $\omega_n^\varepsilon(C^\varepsilon)$ now follows from

LEMMA 4.7. *Let $\mathfrak{B}$ be the algebra of all bounded operators on some Hilbert space $\mathcal{H}$, and let N be a positive (unbounded) operator with N^{-1} a compact element of $\mathfrak{B}$. Then the set*

$$\{\Lambda: \Lambda \in \mathfrak{B},\ 0 \leqslant \Lambda \leqslant I,\ \operatorname{Tr}(\Lambda N) \leqslant 1\} = \mathfrak{J}_N$$

is compact in the trace norm.

Proof. Choose an orthonormal basis $\{e_i\}$ for $\mathcal{H}$ consisting of eigenvectors of N and let b_i be the corresponding eigenvalues. For Λ in $\mathfrak{J}_N$, we have

$$\operatorname{Tr}(\Lambda N) = \operatorname{Tr}(N^{\frac{1}{2}} \Lambda N^{\frac{1}{2}}) = \sum_i \lambda_{ii} b_i$$

if (λ_{ij}) is the matrix representing the operator Λ. If $P = \Lambda^{\frac{1}{2}}$ and if ϱ_{ij} is the corresponding matrix, then

$$\operatorname{Tr}(\Lambda N) = \sum_{i<j} (b_i + b_j) |\varrho_{ij}|^2 + \sum_i b_i |\varrho_{ii}|^2.$$

Let Λ_n be a sequence in $\mathfrak{J}_N$ and let $P_n = \Lambda_n^{\frac{1}{2}}$. Since $b_i \to \infty$ as $i \to \infty$, we see that a subsequence P_n of the P's converge to a limit P in the Hilbert–Schmidt norm. For a trace class operator A, the trace norm $\|A\|_1$ is given by

$$\|A\|_1 = \sup_U \operatorname{Tr}(UA),$$

where the supremum runs over all unitary operators. Thus if $\Lambda = P^2$,

$$\|\Lambda_n - \Lambda\|_1 = \sup_U \operatorname{Tr}(U(\Lambda_n - \Lambda)) = \sup_U \{\operatorname{Tr}(U(P_n - P)P_n) + \operatorname{Tr}(UP(P_n - P))\}$$

$$\leqslant \sup_U \|U(P_n - P)\|_2 \|P_n\|_2 + \sup_U \|UP\|_2 \|P_n - P\|_2 = (\|P_n\|_2 + \|P\|_2)\|P_n - P\|_2 \to 0,$$

with

$$\|P\|_2^2 = \sum_{i,j} |\varrho_{ij}|^2$$

denoting the Hilbert–Schmidt norm, and n belonging to the subsequence.

We apply this lemma to the case $\mathfrak{J} = \mathfrak{J}(Z)$ and $\mathfrak{B} = \mathfrak{B}(Z)$. We use the operator $N_{\tau,M}$ of (4.57) to give an

$$N = (N_{\tau,M} + I)$$

for Lemma 4.7. The states $\omega_n^\varepsilon \restriction \mathfrak{B}(Z)$ satisfy the hypotheses of the lemma by (4.59) – (4.61). By (4.58) and Corollary 3.3.2 (c) the operator N has the desired properties. Thus there is a norm convergent subsequence ω_{n_j} satisfying

$$|(\omega_{n_i} - \omega_{n_j})(PC^\varepsilon P)| \leqslant o(1)\,\|PC^\varepsilon P\| \leqslant o(1)\,\|C\|$$

by (4.45). For this subsequence

$$|(\omega_{n_i}^\varepsilon - \omega_{n_j}^\varepsilon)(C^\varepsilon)| \leqslant o(1)\,\|C\|, \tag{4.62}$$

which is the announced bound (4.4).

In order to complete the proof of Theorem 4.1, we write for $C \in \mathfrak{A}_0(\boldsymbol{B})$.

$$\begin{aligned} |(\omega_{n_i} - \omega_{n_j})(C)| \leqslant |(\omega_{n_i} - \omega_{n_i}^\varepsilon)(C)| + |\omega_{n_i}^\varepsilon(C - C^\varepsilon)| + |(\omega_{n_i}^\varepsilon - \omega_{n_j}^\varepsilon)(C^\varepsilon)| + |\omega_{n_j}^\varepsilon(C^\varepsilon - C)| \\ + |(\omega_{n_j}^\varepsilon - \omega_{n_j})(C)| \leqslant 4\,\varepsilon\,\|C\| + o(1)\,\|C\|. \end{aligned} \tag{4.63}$$

Here we have chosen n_i, n_j to belong to the convergent subsequence in (4.62), and we used (4.14), (4.46) and (4.62) to dominate (4.63). Since

$$\|(\omega_{n_i} - \omega_{n_j}) \restriction \mathfrak{A}_0(\boldsymbol{B})\| = \|(\omega_{n_i} - \omega_{n_j}) \restriction \mathfrak{A}(\boldsymbol{B})\|,$$

we conclude that the subsequence in (4.63) is norm convergent on $\mathfrak{A}(\boldsymbol{B})$.

It is known and easy to prove that the norm limit of a sequence of normal states is normal. Thus any norm limit ω of the ω_n is normal on $\mathfrak{A}(\boldsymbol{B})$, and Theorem 4.1 is established.

We now return to prove Lemma 4.3. We analyze the expansion of $C \in \mathfrak{A}_0(\boldsymbol{B})$ in terms of the local, time zero fields. We start by proving a useful lemma.

Let α, β, δ, ε, μ, ν be nonnegative integers satisfying $\alpha + \beta = \delta + \varepsilon$, $\mu \leqslant \delta$, $\nu \leqslant \varepsilon$ and $\alpha - \delta = \nu - \mu$.

Lemma 4.8. *Let $c_{\delta\varepsilon}$ be a bounded operator from $L_2(R^\varepsilon)$ to $L_2(R^\delta)$ with bound $\|c_{\delta\varepsilon}\|$. Let k_C and k_A be Hilbert–Schmidt operators on $L_2(R^\mu)$ and on $L_2(R^\nu)$ respectively, with Hilbert–Schmidt norms*

$$\|k_C\|_2 \text{ and } \|k_A\|_2.$$

Then

$$\int k_A(x_1, \dots, x_\nu; w_1, \dots, w_\nu)\, c_{\delta\varepsilon}(x_{\nu+1}, \dots, x_\alpha, z_1, \dots, z_\mu; w_1, \dots, w_\nu, y_{\mu+1}, \dots, y_\beta) \times k_C(z_1, \dots, z_\mu; y_1, \dots, y_\mu)\, dw\, dz$$

is the kernel of a bounded operator $c_{\alpha\beta}$ *from* $L_2(R^\beta)$ *to* $L_2(R^\alpha)$ *with norm* $\|c_{\alpha\beta}\|$ *satisfying*

$$\|c_{\alpha\beta}\| \leq \|k_A\|_2 \, \|c_{\delta\varepsilon}\| \, \|k_C\|_2.$$

Proof. We introduce the variables

$$x_a = (x_1, \dots, x_\nu), \quad x_b = (x_{\nu+1}, \dots, x_\alpha), \quad x = (x_a, x_b),$$
$$y_a = (y_1, \dots, y_\mu), \quad y_b = (y_{\mu+1}, \dots, y_\beta), \quad y = (y_a, y_b)$$
$$w = (w_1, \dots, w_\nu), \quad z = (z_1, \dots, z_\mu).$$

Then for $f_\alpha \in \mathcal{J}_\alpha$, $g_\beta \in \mathcal{J}_\beta$,

$$|(f_\alpha, c_{\alpha\beta} g_\beta)| = \left| \int f_\alpha(x)\, k_A(x_a; w)\, c_{\delta\varepsilon}(x_b, z; w, y_b)\, k_C(z; y_a)\, g_\beta(y)\, dx\, dy\, dw\, dz \right|.$$

By the definition of $\|c_{\delta\varepsilon}\|$,

$$|(f_\alpha, c_{\alpha\beta} g_\beta)| \leq \|c_{\delta\varepsilon}\| \int \|f_\alpha(x_a, \cdot)\|_2 \, \|k_C(\cdot\,; y_a)\|_2 \, \|k_A(x_a; \cdot)\|_2 \, \|g_\beta(y_a, \cdot)\|_2 \, dx_a\, dy_a,$$

and by the Schwartz inequality

$$|(f_\alpha, c_{\alpha\beta} g_\beta)| \leq \|c_{\alpha\beta}\| \, \|k_C\|_2 \, \|k_A\|_2 \, \|f_\alpha\| \, \|g_\beta\|,$$

to complete the proof.

We now expand $C \in \mathfrak{A}_0(\mathcal{B})$ in terms of ϕ and π fields. Let

$$C_\gamma = \sum_{\alpha+\beta=\gamma} C_{\alpha\beta}.$$

Then

$$C_\gamma = \sum_{\mu+\nu=\gamma} B_{\mu\nu},$$

where

$$B_{\mu\nu} = \int b_{\mu\nu}(y) \colon \phi(y_1) \dots \phi(y_\mu)\, \pi(y_{\mu+1}) \dots \pi(y_{\mu+\nu}) \colon dy.$$

The locality of $C \in \mathfrak{A}_0(\mathcal{B})$ means that

$$\operatorname{supp} b_{\mu\nu}(y_1, \dots, y_{\mu+\nu}) \in \mathcal{B} \times \mathcal{B} \times \dots \times \mathcal{B}.$$

The kernels $b_{\mu\nu}(y)$ are symmetric in the μ variables $y_1, \dots, y_\mu$ corresponding to the ϕ's, and also in the ν variables $y_{\mu+1}, \ldots, y_{\mu+\nu}$ corresponding to the π's. For fixed $\mu+\nu=\gamma$, there are exactly $\gamma+1$ different $B_{\mu\nu}$'s and for fixed $\alpha+\beta=\gamma$, there are exactly $\gamma+1$ different $C_{\alpha\beta}$'s. We now specify the relation between them. First we inspect the simple case $\alpha+\beta=\gamma=\mu+\nu=1$. From (3.9) we conclude that

$$c_{10}(x)=\frac{1}{\sqrt{2}}\mu_x^{-\frac{1}{2}}b_{10}(x)+\frac{i}{\sqrt{2}}\mu_x^{\frac{1}{2}}b_{01}(x) \tag{4.64}$$

and

$$c_{01}(x)=\frac{1}{\sqrt{2}}\mu_x^{-\frac{1}{2}}b_{10}(x)-\frac{i}{\sqrt{2}}\mu_x^{\frac{1}{2}}b_{01}(x). \tag{4.65}$$

We also have the inverse relation

$$b_{10}(x)=\frac{1}{\sqrt{2}}\mu_x^{\frac{1}{2}}c_{10}(x)+\frac{1}{\sqrt{2}}\mu_x^{\frac{1}{2}}c_{01}(x) \tag{4.66}$$

and

$$b_{01}(x)=\frac{-i}{\sqrt{2}}\mu_x^{-\frac{1}{2}}c_{10}(x)+\frac{i}{\sqrt{2}}\mu_x^{-\frac{1}{2}}c_{01}(x). \tag{4.67}$$

The transformations (4.64)–(4.67) by themselves are valid for a wide variety of $b_{01}(x)$, $b_{10}(x)$; they are defined, for instance on tempered distributions. These relations do not require that the $b_{01}(x)$ and $b_{10}(x)$ be local. We now make the locality explicit by means of a C_0^∞ function $\zeta(x)$ which has its support in X_0 and which equals one on a neighborhood of $\mathcal{B}$. We rewrite (4.66) as

$$b_{10}(x)=\frac{1}{\sqrt{2}}\zeta(x)\mu_x^{\frac{1}{2}}c_{10}(x)+\frac{1}{\sqrt{2}}\zeta(x)\mu_x^{\frac{1}{2}}c_{01}(x), \tag{4.68}$$

and similarly for $b_{01}(x)$. We note that the operators μ_x^τ, which were introduced in Section 3.1 have integral kernels $k_\tau(x)$ satisfying (3.1.5)–(3.1.6).

$$(\mu_x^\tau f)(x)=\int k_\tau(x-z)\,f(z)\,dz.$$

By substituting (4.68) into (4.64) – (4.65), we obtain the equations

$$c_{10}(x)=(K_+c_{10})(x)+(K_-c_{01})(x) \tag{4.69}$$

and

$$c_{01}(x)=(K_-c_{10})(x)+(K_+c_{01})(x), \tag{4.70}$$

where we define

$$K_\pm=\tfrac{1}{2}\mu_x^{-\frac{1}{2}}\zeta\mu_x^{\frac{1}{2}}\pm\tfrac{1}{2}\mu_x^{\frac{1}{2}}\zeta\mu_x^{-\frac{1}{2}}. \tag{4.71}$$

Let $k_\pm(x,z)$ be the (tempered distribution) kernel of $K_\pm$. For $x\notin X_0=[0,1)$, the property (3.1.6) of k_τ implies that $k_\pm(x,z)$ is a C^∞ function of x and z and that for a constant M_{14},

$$|k_{\pm}(x,z)| \leqslant M_{14} \exp(-m_0|x| - m_0|z|). \tag{4.72}$$

We note that the locality requirement

$$\operatorname{supp} b_{10}(x) \subset \mathcal{B}, \qquad \operatorname{supp} b_{01}(x) \subset \mathcal{B},$$

imposes implicit restrictions on—and relations between—$c_{01}(x)$ and $c_{10}(x)$; for instance, if either one is nonzero, they must both be nonzero. Thus we cannot expect, and we do not find, that the identity transformation (4.69)–(4.70) yields the identity on all functions as in (4.64)–(4.67). In fact for the kernels of (4.71), we have

$$k_{+}(x,z) \neq \delta(x-z), \quad \text{and} \quad k_{-}(x,z) \neq 0.$$

We now define localized kernels and estimate their norms. Let

$$k^i_{\pm}(x,z) = \begin{cases} k_{\pm}(x,z), & x \in X_i \\ 0 \quad , & \text{otherwise,} \end{cases} \tag{4.73}$$

and let $K^i_{\pm}$ be the operator with kernel $k^i_{\pm}$.

LEMMA. 4.9. *For some constant M_{15}, the operator norm of K^i_{+} is bounded by*

$$\|K^i_{+}\| \leqslant M_{15} e^{-m_0|i|}, \tag{4.74}$$

and the Hilbert–Schmidt norm of k^i_{-} is bounded by

$$\|k^i_{-}\|_2 \leqslant M_{15} e^{-m_0|i|}. \tag{4.75}$$

Proof. If $i \neq 0$, the bound (4.72) immediately yields

$$\|K^i_{\pm}\| \leqslant \|k^i_{\pm}\|_2 \leqslant M_{15} \exp(-m_0|i|),$$

so we need only consider the case $i=0$. Since

$$\mu_x^{-\frac{1}{2}} \zeta \mu_x^{\frac{1}{2}} = \zeta + \mu_x^{-\frac{1}{2}} [\zeta, \mu_x^{\frac{1}{2}}],$$

the lemma follows from (4.71) and an estimate on the Hilbert–Schmidt norm of the operator

$$A = \mu_x^{-\frac{1}{2}} [\zeta, \mu_x^{\frac{1}{2}}].$$

Let $a(x,z)$ be the kernel of A. The Fourier transform of $a(x,z)$ is

$$\tilde{a}(p,q) = \mu(p)^{-\frac{1}{2}} \tilde{\zeta}(p-q) \{\mu(q)^{\frac{1}{2}} - \mu(p)^{\frac{1}{2}}\},$$

where $\tilde{\zeta}$ is the Fourier transform of ζ. Since

$$|\mu(p)^{\frac{1}{2}} - \mu(q)^{\frac{1}{2}}| = \frac{|\mu(p) - \mu(q)|}{\mu(p)^{\frac{1}{2}} + \mu(q)^{\frac{1}{2}}} \leqslant \frac{\mu(p-q)}{\mu(p)^{\frac{1}{2}} + \mu(q)^{\frac{1}{2}}}$$

we have
$$|\tilde{a}(p,q)| \leqslant \frac{\mu(p-q)\,|\tilde{\zeta}(p-q)|}{\mu(p)^{\frac{1}{2}}\{\mu(p)^{\frac{1}{2}} + \mu(q)^{\frac{1}{2}}\}} \in L_2.$$

Thus
$$\|a\|_2 = \|\tilde{a}\|_2 < \infty,$$

and the proof is complete.

Proof of Lemma 4.3. Let $L = \{L_1\}$ be a localization index in the case $\gamma = 1$, which localizes the single particle in X_{L_1}. Then

$$c_{10}^L(x) = (K_+^{L_1} c_{10})(x) + (K_-^{L_1} c_{01})(x) \tag{4.76}$$

and
$$c_{01}^L(x) = (K_-^{L_1} c_{10})(x) + (K_+^{L_1} c_{01})(x). \tag{4.77}$$

The bound of Lemma 4.3 for c_{10}^L follows from

$$\|c_{10}^L\| \leqslant \|K_+^{L_1} c_{10}\| + \|K_-^{L_1} c_{01}\| \leqslant \|K_+^{L_1}\|\,\|c_{10}\| + \|K_-^{L_1} c_{01}\| \tag{4.78}$$

$$\leqslant \|K_+^{L_1}\|\,\|c_{10}\| + \|k_-^{L_1}\|_2\,\|c_{01}\|$$

by Lemma 4.8

$$\leqslant M_{15}(\|c_{10}\| + \|c_{01}\|)\exp(-m_0|L_1|)$$

by Lemma 4.9

$$\leqslant 2M_{15}\exp(-m_0|L_1|)\,\|C\| = 2M_{15}\,e^{-m_0 D(L)}\|C\|$$

by Lemma 4.2. A similar result holds for c_{01}^L.

We now generalize this method to deal with any $c_{\alpha\beta}^L$, and we need an expansion similar to (4.76) – (4.77). We first obtain such an expansion for operators C_γ which have the form

$$C_\gamma = \sum_{\alpha+\beta=\gamma} C_{\alpha\beta} = :C_1^\gamma:, \tag{4.79}$$

where C_1 is a degree one expression

$$C_1 = A^*(c_{10}) + A(c_{01}) = \phi(b_{10}) + \pi(b_{01}), \tag{4.80}$$

and supp $b_{10} \subset \mathcal{B}$, supp $b_{01} \subset \mathcal{B}$. For our special case

$$C_\gamma = \sum_{\alpha+\beta=\gamma} \int A^*(x_1) \dots A^*(x_\alpha) A(x_{\alpha+1}) \dots A(x_{\alpha+\beta}) \binom{\alpha+\beta}{\alpha}$$
$$\times\, c_{10}(x_1) \dots c_{10}(x_\alpha)\, c_{01}(x_{\alpha+1}) \dots c_{01}(x_{\alpha+\beta})\, dx \tag{4.81}$$

or

$$C_\gamma = \sum_{\mu+\nu=\gamma} \int :\phi(y_1)\ldots\phi(y_\mu)\,\pi(y_{\mu+1})\ldots\pi(y_{\mu+\nu}): \binom{\mu+\nu}{\nu} \times b_{10}(y_1)\ldots b_{10}(y_\mu)\, b_{01}(y_{\mu+1})\ldots b_{01}(y_{\mu+\nu})\,dy. \tag{4.82}$$

We now apply (4.69) to the creation variables, expanding $c_{10}(x_1), \ldots, c_{10}(x_\alpha)$. We apply (4.70) to the annihilation variables, expanding $c_{01}(x_{\alpha+1}) \ldots c_{01}(x_{\alpha+\beta})$. The kernels $c_{\alpha\beta}(x)$ for (4.79)

$$c_{\alpha\beta}(x) = \binom{\alpha+\beta}{\alpha} \prod_{j=1}^{\alpha} c_{10}(x_j) \prod_{j=\alpha+1}^{\beta} c_{01}(x_j) \tag{4.83}$$

can be written

$$c_{\alpha\beta}(x) = \binom{\alpha+\beta}{\alpha} \prod_{j=1}^{\alpha} \{(K_+ c_{10})(x_j) + (K_- c_{01})(x_j)\} \prod_{j=\alpha+1}^{\alpha+\beta} \{(K_+ c_{01})(x_j) + (K_- c_{10})(x_j)\}. \tag{4.84}$$

In a more compact form

$$c_{\alpha\beta}(x) = \sum_{\delta+\varepsilon=\alpha+\beta} \int l_{\alpha\beta,\,\delta\varepsilon}(x,z)\, c_{\delta\varepsilon}(z)\,dz, \tag{4.85}$$

where the kernel $l_{\alpha\beta,\,\delta\varepsilon}(x,z)$ is a sum of tensor products of the kernels $k_\pm(x,z)$ times some numerical factors. We can write

$$l_{\alpha\beta,\,\delta\varepsilon}(x,z) = \text{Sym.} \sum_{\mu=0}^{\delta} \binom{\delta}{\mu} \binom{\varepsilon}{\alpha-\mu} \prod_{j=1}^{\mu} k_+(x_j, z_j) \times \prod_{j=\mu+1}^{\alpha} k_-(x_j, z_{j+\delta-\mu}) \prod_{j=\alpha+1}^{\delta+\alpha-\mu} k_-(x_j, z_{j+\mu-\alpha}) \prod_{j=\delta+\alpha-\mu+1}^{\delta+\varepsilon} k_+(x_j, z_j) \tag{4.86}$$

where the symmetrization operation symmetrizes the kernel in the α creation variables $x_1, \ldots, x_\alpha$ and in the β annihilation variables $x_{\alpha+1}, \ldots, x_{\alpha+\beta}$.

We note some properties of the monomials contributing to the sum (4.86). In each kernel $k_+(x_j, z_j)$ that occurs, either z_j is a creation variable for $c_{\delta\varepsilon}$ and x_j is a creation variable for $c_{\alpha\beta}$, or else z_j is an annihilation variable for $c_{\delta\varepsilon}$ and x_j is an annihilation variable for $c_{\alpha\beta}$. In other words, the K_+ operators in (4.86) do not connect creation variables to annihilation variables. On the other hand, the kernels $k_-(x_j, z_{j'})$ connect an annihilation (creation) variable in $c_{\delta\varepsilon}$ to a creation (annihilation) variable in $c_{\alpha\beta}$. In majorizing (4.86) we will later use the operator norm to dominate k_+ kernels, and Lemma 4.8 to deal with the k_- kernels.

The next step in our argument is to prove that the transformation (4.85)–(4.86) which we derived for a class of $c_{\alpha\beta}(x)$ defined in (4.83), is correct for any $c_{\alpha\beta}(x)$ that might occur in the expansion of an operator $C \in \mathfrak{A}_0(\mathcal{B})$. We note that by a generalized polarization identity, (4.85)–(4.86) is valid for kernels arising from operators

$$C_\gamma = :C_1^{(1)} C_1^{(2)} \dots C_1^{(\gamma)}:, \tag{4.87}$$

where
$$C_1^{(j)} = A^*(c_{10}^{(j)}) + A(c_{01}^{(j)}) = \phi(b_{10}^{(j)}) + \pi(b_{01}^{(j)}), \quad j = 1, 2, \dots, \gamma, \tag{4.88}$$

and supp $b_{10}^{(j)} \subset \mathcal{B}$, supp $b_{01}^{(j)} \subset \mathcal{B}$.

Here we use the identity (for real $a_1, a_2 \dots, a_n$),

$$2^{n-1} n! \, a_1 a_2 \dots a_n = \sum_{\varepsilon_j = \pm 1} \varepsilon_2 \dots \varepsilon_n (a_1 + \varepsilon_2 a_2 + \dots + \varepsilon_n a_n)^n. \tag{4.89}$$

Kernels that are finite sums of kernels of the operators (4.87) are the only kernels $c_{\alpha\beta}(x)$ that occur in the expansion of operators $C \in \mathfrak{A}_0(\mathcal{B})$. Hence the identity transformation (4.85)–(4.86) holds for all our kernels.

We now return to the localized kernels $c_{\alpha\beta}(x)$ and the proof of Lemma 4.3. Using (4.85) we can write

$$c_{\alpha\beta}^L(x) = \sum_{\delta+\varepsilon=\alpha+\beta} \int l_{\alpha\beta,\delta\varepsilon}^L(x,z) \, c_{\delta\varepsilon}(z) \, dz, \tag{4.90}$$

where the localized kernel $l_{\alpha\beta,\delta\varepsilon}^L(x,z)$ is defined by

$$l_{\alpha\beta,\delta\varepsilon}^L(x,z) = \operatorname{Sym} \sum_{\mu=0}^{\delta} \binom{\delta}{\mu} \binom{\varepsilon}{\alpha-\mu} \prod_{j=1}^{\mu} k_+^{L_j}(x_j, z_j) \prod_{j=\mu+1}^{\delta+\alpha-\mu} k_-^{L_j}(x_j, z_{j'}) \prod_{j=\delta+\alpha-\mu+1}^{\delta+\varepsilon} k_+^{L_j}(x_j, z_j) \tag{4.91}$$

and $L = \{L_1, \dots, L_{\alpha+\beta}\}$ is the localization multi-index. Each kernel (4.91), before symmetrization, is the sum of $\delta+1$ terms with a numerical coefficient dominated by $2^{\delta+\varepsilon} = 2^{\alpha+\beta}$. The kernels from one term, obtained by fixing μ, are a tensor product of k_+ kernels and of k_- kernels. The k_- factor

$$\prod_{j=\mu+1}^{\delta+\alpha-\mu} k_-^{L_j}(x_j, z_{j'}) = \prod_{j=\mu+1}^{\alpha} k_-^{L_j}(x_j, z_{j+\delta-\mu}) \prod_{j=\alpha+1}^{\delta+\alpha-\mu} k_-^{L_j}(x_j, z_{j+\mu-\alpha})$$

is the tensor product of Hilbert–Schmidt kernels and has the form of a tensor product of kernels $k_C k_A$ of Lemma 4.8. The first product in (4.91) of k_+ kernels is a bounded operator O_α on $\mathcal{J}_\alpha$ and the second product in (4.91) of k_+ kernels is a bounded operator O_β on $\mathcal{J}_\beta$. Thus applying Lemma 4.8 to the $k_C k_A$ kernels and using the operator norm on the k_+ kernels, we find that each term $l_{\alpha\beta,\delta\varepsilon}^L(x,z)_\mu$ in (4.91) with fixed μ is a kernel such that

$$\int l_{\alpha\beta,\delta\varepsilon}^L(x,z)_\mu \, c_{\delta\varepsilon}(z) \, dz$$

is the kernel of an operator from $L_2(R^\beta)$ to $L_2(R^\alpha)$ with a norm dominated by

$$2^{\alpha+\beta} \|O_\alpha\| \, \|k_C\|_2 \, \|k_A\|_2 \, \|O_\beta\| \, \|c_{\delta\varepsilon}\| = 2^{\alpha+\beta} \prod_{j=1}^{\alpha_1} \|K_+^{L_j}\| \prod_{j=\alpha_1+1}^{\delta+\alpha+\alpha_1} \|k_-^{L_j}\|_2 \prod_{j=\delta-\alpha-\alpha_1+1}^{\delta+\varepsilon} \|K_+^{L_j}\| \, \|c_{\delta\varepsilon}\|.$$

By Lemma 4.9, this is less than

$$2^{\alpha+\beta} M_{15}^{\alpha+\beta} \prod_{j=1}^{\alpha+\beta} e^{-m_0|L_j|} \|c_{\delta\varepsilon}\| = (2M_{15})^{\alpha+\beta} e^{-m_0 D(L)} \|c_{\delta\varepsilon}\|,$$

and by Lemma 4.2,
$$\leqslant (4M_{15})^{\alpha+\beta} e^{-m_0 D(L)} \|C\|.$$

Summing over μ in (4.91) and over $\delta+\varepsilon=\alpha+\beta$ in (4.90) yields fewer than

$$(\delta+1)(\alpha+\beta+1) \leqslant (\alpha+\beta+1)^2 \leqslant e^{\alpha+\beta}$$

such terms. Since symmetrization does not increase our bound,

$$\|c_{\alpha\beta}^L\| \leqslant (4eM_{15})^{\alpha+\beta} e^{-m_0 D(L)} \|C\|,$$

which completes the proof of Lemma 4.3.

5. The vacuum self energy per unit volume is finite

The lower bound E_g on the operator $\hat{H}(g) = H_0 + H_{I,g}$ is called the vacuum energy, because

$$\hat{H}(g)\,\Omega_g = E_g \Omega_g, \tag{5.1}$$

and
$$E_g = (\Omega_g, \hat{H}(g)\,\Omega_g)$$

is the value of the energy operator $\hat{H}(g)$ in its vacuum state. This actually represents the *shift* in vacuum energy between the problem with and without interaction,

$$E_g = E_g - E_0,$$

where $E_0 = 0$ is the lower bound of H_0. The scale of $\hat{H}(g)$ is arbitrarily specified so that

$$(\Omega_0, \hat{H}(g)\,\Omega_0) = 0. \tag{5.2}$$

It is customary and convenient to add a constant to the Hamiltonian $\hat{H}(g)$ so that its vacuum energy is zero

$$H(g) = \hat{H}(g) - E_g, \tag{5.3}$$

and
$$H(g)\,\Omega_g = 0. \tag{5.4}$$

This shift is one of the standard renormalizations of quantum field theory, and we shall call $H(g)$ the renormalized Hamiltonian. We omit the renormalization to the mass m_0 and to the coupling constant λ since perturbation theory indicates that they are finite for our

$(\varphi^4)_2$ model, even in the limit $g(x)\to 1$. We note that it is the renormalized Hamiltonians $H(g)$, and not the unrenormalized Hamiltonians $\hat{H}(g)$ that have a limit as $g(x)\to 1$. In Section 2 we proved that as $g(x)\to 1$,

$$e^{-iH(g)t}\to e^{-iHt}$$

in a certain weak sense, and $0\leqslant H$.

In this section, we bound the rate at which E_g can diverge. According to perturbation theory, E_g is proportional to the volume of space in which the particles interact. However, the perturbation expansion for E_g can be shown by methods such as [18] not to converge and at best to be asymptotic. Thus we cannot get a rigorous bound on E_g from perturbation theory. Instead, we use a modification of Nelson's method [22]. It is this method which produced the estimates (cf. II, (2.1.16–18)) stating that the vacuum self energy for a fixed volume cutoff is finite [22, 8, 4].

THEOREM 5.1. *Let* $0\leqslant g(x)\leqslant 1$ *and for some constant* M_0,

$$\left|\frac{dg(x)}{dx}\right|\leqslant M_0.$$

Then

$$-M\leqslant E_g\leqslant 0, \tag{5.5}$$

where M *is a positive constant proportional to the volume of the set* $(\operatorname{supp} g)_1$, *that is the set of points within distance one of* $(\operatorname{supp} g)$.

Remarks 1. This theorem states that the average vacuum energy density, namely the vacuum self energy per unit volume, is finite for the theory with $g(x)=1$. Thus it shows that perturbation theory, though inapplicable, actually predicts the correct answer.

2. Furthermore, the theorem assures us that the expected number of (bare) particles per unit volume and the expected free energy per unit volume are both finite in the vacuum Ω. We define these expectation values as limits as $g(x)\to 1$ of the expectation value in Ω_g. Since

$$(\Omega_g, N\Omega_g)\leqslant\frac{1}{m_0}(\Omega_g, H_0\Omega_g), \tag{5.6}$$

it is sufficient to bound the free energy per unit volume. Since $H_0=2\hat{H}(g)-\hat{H}(2g)$,

$$(\Omega_g, H_0\Omega_g)\leqslant(\Omega_g, (2\hat{H}(g)-E_{2g})\,\Omega_g)=2E_g-E_{2g}. \tag{5.7}$$

In particular, if we choose g to be $g_n(x)$ defined in (2.3), then (5.5) says that for M independent of n,

$$-nM\leqslant E_{2g_n}. \tag{5.8}$$

Therefore, by (5.6–7)
$$(\Omega_{g_n}, N\Omega_{g_n}) \leqslant \text{const. } n, \tag{5.9}$$

$$(\Omega_{g_n}, H_0\Omega_{g_n}) \leqslant \text{const. } n, \tag{5.10}$$

$$\omega_n(N) \leqslant \text{const. } n, \tag{5.11}$$

and
$$\omega_n(H_0) \leqslant \text{const. } n. \tag{5.12}$$

3. A similar result holds for the Hamiltonians H_V defined in II, (2.1.7). Namely, there is a positive constant M independent of V, such that as $V \to \infty$

$$-MV \leqslant H_V. \tag{5.13}$$

Our proof yields (5.13) as well as (5.5), although a somewhat easier proof may be based on a modification of Federbush's proof for V fixed [4, 19].

4. The theorem also holds for any interaction Hamiltonian in two dimensions of the form

$$H_{I,g} = \int :P(\varphi(x)):g(x)\,dx,$$

where P is a polynomial of even degree whose leading coefficient is positive.

5. The same proof also shows that for $g(x)$ satisfying the hypotheses of the theorem, and for $\varepsilon > 0$, there exists a constant M proportional to $(\operatorname{supp} g)_1$ such that

$$0 \leqslant \varepsilon N + H_{I,g} + M.$$

Proof of Theorem 5.1. We use a partition of unity $\zeta_i(x)$ constructed from a positive C^∞ function $\zeta(x)$ with support in the interval $|x| < 1$, such that

$$\zeta_i(x) = \zeta(x - i), \tag{5.14}$$

and
$$\sum_i \zeta_i(x) = 1. \tag{5.15}$$

Thus we decompose g and $H_{I,g}$ into a sum of local parts

$$g_i = \zeta_i g \tag{5.16}$$

$$H_{I,g} = \sum_i H_{I,g_i} \tag{5.17}$$

Since we use the Feynman Kac formula to bound $\hat{H}(g)$, we wish to approximate $\hat{H}(g)$ by a Hamiltonian with a finite number of modes. Thus we study the Hamiltonian

$$H(g)_{K,V} = H_{0,K,V} + H_{I,g,K,V} \tag{5.18}$$

$$H_{0,K,V} = \int_{-K}^{K} a^*(k)\, a(k)\, \mu(k_V)\, dk \tag{5.19}$$

$$H_{I,g,K,V} = \lambda \int :\varphi_{K,V}(x)^4:\, g(x)\, dx \tag{5.20}$$

and
$$\varphi_{K,V}(x) = \left(\frac{1}{2V}\right)^{\frac{1}{2}} \sum_{k \in \Gamma_V} e^{-ikx} (a_V^*(k) + a_V(-k))\, \mu(k)^{-\frac{1}{2}} \xi(k/K). \tag{5.21}$$

See II, section 2.1 for the notation used above. Here $\xi(x) \geqslant 0$ is a fixed C_0^∞ function, equal to one on a neighborhood of zero, and vanishing for $|x| > 1$. As in II, section 2.2, we find that

$$\hat{H}(g)_{K,V} \,|\, \mathfrak{F}_{K,V}^{\perp} \geqslant E_{K,V}, \tag{5.22}$$

where $E_{K,V}$ is the lower bound of $\hat{H}(g)_{K,V}$ on the full Fock space $\mathfrak{F}$. Therefore, we need only find the lower bound of $\hat{H}(g)_{K,V} \,|\, \mathfrak{F}_{K,V}$, which we now study. For simplicity of notation, we sometimes suppress the V cutoffs and define

$$H_I(K, j) = H_{I,g_j,K,V}. \tag{5.23}$$

Thus it is sufficient to obtain a lower bound for

$$\{H_{0,K,V} + \sum_j H_I(K, j)\} \,|\, \mathfrak{F}_{K,V} \tag{5.24}$$

which is independent of K and V, and which is at most proportional to the number of nonzero terms $H_I(K, j)$. On $\mathfrak{F}_{K,V}$

$$H_{0,K,V} = \sum_{\substack{k \in \Gamma_V \\ |k| \leqslant K}} a_V^*(k)\, a_V(k)\, \mu(k) \tag{5.25}$$

We note that $H_I(K, j)$ is the sum of five monomials in creation and annihilation operators, each of which has the kernel proportional to

$$b_{K,j}(k_1, \ldots, k_4) = \lambda (2V)^{-2} (g\zeta_j)^{\sim} \left(\sum_{l=1}^{4} k_l\right) \prod_{l=1}^{4} [\mu(k_l)^{-\frac{1}{2}} \xi(k_l/K)], \tag{5.26}$$

corresponding to an expansion in creation and annihilation operators for volume V,

$$H_I(K, j) = \sum_{k_1, \ldots, k_4 \in \Gamma_V} \sum_{\alpha=0}^{4} \binom{4}{\alpha} b_{K,j}(k_1, \ldots, k_4)\, a_V^*(k_1) \ldots a_V^*(k_\alpha)\, a_V(-k_{\alpha+1}) \ldots a_V(-k_4). \tag{5.27}$$

We decompose the single particle Fock space $\mathfrak{F}_{V,1}$ of $\mathfrak{F}_V$ into an orthogonal direct sum

$$\mathfrak{F}_{V,1} = \bigoplus_i \mathfrak{F}_{V,1,i}, \tag{5.28}$$

where $\mathfrak{F}_{V,1,i}$ consists of the Fourier series of functions vanishing off the interval

$$[i-\tfrac{1}{2}, i+\tfrac{1}{2}] \cap [-V/2, V/2]. \tag{5.29}$$

Furthermore, we have the Parseval equality

$$\int_{-V/2}^{V/2} f(x)\, h(x)\, dx = \sum_{k \in \Gamma_V} \tilde{f}(k)\, \tilde{h}(-k) = \sum_{k,\, k' \in \Gamma_V} \tilde{f}(k)\, \tilde{h}(k')\, \delta(k; -k'),$$

where

$$\delta(k;k') = \begin{cases} 1, & k = k' \\ 0, & k \neq k'. \end{cases}$$

From this, we see that

$$\sum_{k \in \Gamma_V} \tilde{f}(k)\, \tilde{h}(-k) = \sum_{i,j} \int_{-V/2}^{V/2} f_i(x)\, h_j(x)\, dx = \sum_i \int_{-V/2}^{V/2} f_i(x)\, h_i(x)\, dx$$
$$= \sum_i \sum_{k \in \Gamma_V} (f_i)^\sim (k)\, (h_i)^\sim (-k). \tag{5.30}$$

Thus (5.30) provides a decomposition into localized parts,

$$f_i \in \mathfrak{F}_{V,1,i}.$$

Using this decomposition, we can decompose any r-fold tensor product

$$\mathfrak{F}_{V,1} \otimes_s \mathfrak{F}_{V,1} \otimes_s \dots \otimes_s \mathfrak{F}_{V,1},$$

into an orthogonal direct sum, and corresponding to (5.30) we have

$$\sum_{k_j \in \Gamma_V} \tilde{f}(k_1, \dots, k_r)\, \tilde{h}(-k_1, \dots, -k_r) = \sum_i \sum_{k_j \in \Gamma_V} (f_i)^\sim (k_1, \dots, k_r)\, (h_i)^\sim (-k_1, \dots, -k_r), \tag{5.31}$$

where $i = \{i_1, \dots, i_r\}$ is a localization index and

$$f_i \in \mathfrak{F}_{V,1,i_1} \otimes_s \dots \otimes_s \mathfrak{F}_{V,1,i_r}.$$

We therefore can write

$$b_{K,j}(k_1, \dots, k_4) = \sum_{i_1, \dots, i_4} b_{K,j,i_1,\dots,i_4}(k_1, \dots, k_4) \tag{5.32}$$

corresponding to the direct sum (5.28), with

$$b_{\mathrm{K},j,i_1,\dots,i_4} \in \mathfrak{F}_{V,1,i_1} \otimes_s \cdots \otimes_s \mathfrak{F}_{V,1,i_4}.$$

Let H_I (K, j, i_1, ..., i_4) be the part of $H_I(\mathrm{K}, j)$ with the kernels (5.32). We notice that the kernel (5.32) has five localization indices. If the indices are incompatible, that is, if any of the differences $|j-i_l|$ is large, then the kernel must be small. We need the same result for a somewhat more general kernel. For $k_l \in \Gamma_V$, let

$$b_{\mathrm{K},j,s}(k_1, \dots, k_4) = b_{\mathrm{K},j}(k_1, \dots, k_4) \exp\{-\sum_{l=1}^{4} s_l \mu(k_l)\}, \tag{5.33}$$

where $s = \{s_1, \dots, s_4\}$.

LEMMA 5.2. *For each integer r, each $\varepsilon > 0$, and each positive number T, there is a constant $c = c(r, T)$ such that for $0 \leq s_l \leq T$, and $1 \leq \Lambda \leq \mathrm{K}$,*

$$\|b_{\Lambda,j,s,i} - b_{\mathrm{K},j,s,i}\|_{2,V} \leq c\Lambda^{-\frac{1}{2}+\varepsilon} \prod_{l=1}^{4} (1 + |j - i_l|)^{-r}. \tag{5.34}$$

Note $i = \{i_1, ..., i_4\}$. Note that $b_{\Lambda,j,s,i}$ is the ith component of $b_{\Lambda,j,s}$ defined in (5.33). Here the norm $\|\cdot\|_{2,V}$ refers to the Fourier series L_2 norm, which equals $(V/2\pi)^2$ times the usual L_2 norm of (5.26) with each k_l replaced by k_{lV}.

Before proving this lemma, we make some general remarks. Let $f(x)$ be a function with a smooth Fourier transform. That is, assume that for all α,

$$\|D^\alpha \hat{f}\|_2 < \infty. \tag{5.35}$$

This corresponds to a rapidly decreasing $f(x)$. Now let

$$f_V(x) \in L_2([-V/2, V/2]) \tag{5.36}$$

be a periodic function with Fourier coefficients

$$V^{-\frac{1}{2}} \hat{f}(k), \quad k \in \Gamma_V.$$

Then

$$f_V(x) = \sum_{n=-\infty}^{\infty} f(x + nV), \tag{5.37}$$

as can be checked by computing the Fourier coefficient of the right side of (5.37). In fact (5.37) is valid for $f \in L_1$. Using the rapid decrease of $f(x)$, we have for any integer r, a bound on the norms,

$$\|(1 + |x|)^r f_V\|_{2,V} \tag{5.38}$$

in $L_2([V/2, V/2])$, which is independent of V, as long as V is bounded away from zero. Furthermore, the norms (5.38) can be estimated in terms of the norms (5.35).

Proof of Lemma 5.2. We have $b_{K,j,s}$ given by (5.33). Multiplication of the $b_{K,j,s}$ by $\exp(ij\sum_{l=1}^{4} k_l)$ maps the $(i_1, \dots, i_4)$ component of $b_{K,j,s}$ into the $(i_1-j, \dots, i_4-j)$ component of

$$\lambda(2V)^{-2}(g(\cdot+j)\zeta)^{\sim}(\sum_{l=1}^{4} k_l)\prod_{l=1}^{4}\mu(k_l)^{-\frac{1}{2}}\xi(k_l/K)\exp(-\sum_{l=1}^{4} s_l\mu(k_l)).$$

These functions are of the form f_V, and we now see that the corresponding functions f satisfy estimates of the form (5.35), which are independent of K and j. Let $P = k_1 + \dots + k_4$, and

$$F(K, \Lambda, s) = (1+P^2)^{-\frac{1}{2}}\prod_{l=1}^{4}[\mu(k_l)^{-\frac{1}{2}}e^{-s_l\mu(k_l)}]\{\prod_{j=1}^{4}\xi(k_j/K) - \prod_{j=1}^{4}\xi(k_j/\Lambda)\}.$$

Then $D^r F(K, \Lambda, s) \in L_2$ and

$$\|D^r\{(b_{K,j,s} - b_{\Lambda,j,s})\exp(ij\sum_{l=1}^{4} k_l)\}\|_2$$
$$\leqslant \sum_{\alpha}\binom{r}{\alpha}\|D^\alpha[(1+P^2)^{\frac{1}{2}}(g(\cdot+j)\zeta)^{\sim}(P)]\|_\infty \times \|D^{r-\alpha}F(K, \Lambda, s)\|_2. \quad (5.39)$$

We observe that $$\xi(k/K) = 1 = \xi(k/\Lambda)$$
unless $$|k| > \text{const. } \Lambda.$$

Thus $D^{p-\alpha}F(K, \Lambda, s) = 0$ unless at least one of the k's has magnitude greater than const. Λ. By checking the order of convergence of the integrals, we see that for each r, the L_2 norm of $D^{r-\alpha}F(K, \Lambda, s)$ is $O(\Lambda^{-\frac{1}{2}+\varepsilon})$, independent of K and s as long as they lie in their allowed ranges. (See [26].) (1) In order to see that the L_∞ norms in (5.39) are bounded independently of j, we note that

$$|D^\beta(1+P^2)^{\frac{1}{2}}| \leqslant \text{const. }(1+|P|)$$

for a constant independent of β if $|\beta| \leqslant |r|$, and $|r|$ is fixed. Thus for $|\alpha| \leqslant r$.

$$\|D^\alpha\{(1+P^2)^{\frac{1}{2}}(g(\cdot+j)\zeta)(P)\}\|_\infty \leqslant \sup_{\beta\leqslant|r|}\text{const. }\|(1+|P|)D^\beta(g(\cdot+j)\zeta)^{\sim}(P)\|_\infty$$
$$\leqslant \sup_{\beta\leqslant|r|}\text{const. }(\|x^\beta g(\cdot+j)\zeta\|_1 + \|D(x^\beta g(\cdot+j)\zeta)\|_1)$$
$$\leqslant \text{const. }\sup_{\beta\leqslant r}((1+M_0)\|x^\beta\zeta\|_1 + \|D(x^\beta\zeta)\|_1) \quad (5.40)$$

by the assumed bounds on $g(x)$ and $|Dg(x)|$. Since $\zeta \in C^\infty$ and has compact support, (5.40) is dominated by $c = c(|r|, M_0, T)$. Thus (5.39) satisfies a bound

$$\|D^r\{(b_{K,j,s} - b_{\Lambda,j,s})\exp(i\sum_{l=1}^{4} k_l j)\}\|_2 \leqslant c(r, T)\Lambda^{-\frac{1}{2}+\varepsilon},$$

(1) A direct calculation is also possible.

which is independent of j and K. This is an estimate of the form (5.35), so the corresponding norms (5.38) satisfy similar bounds. The rapid decrease of these functions could be written as the estimate (5.34), which completes the proof.

We now bound $|E_g|$ through the formula

$$|E_g| = t^{-1} \log \|e^{-tH(g)}\| = t^{-1} \log \{ \sup_{\|\theta\|=1=\|\chi\|} |(\theta, e^{-tH(g)}\chi)|\}. \tag{5.41}$$

Since

$$\lim_{K, V \to \infty} E_{g, K, V} = E_g,$$

it is sufficient to bound $|E_{g,K,V}|$ uniformly in K and V. We use the Feynman Kac formula to represent the operator $\exp(-tH(g)_{K,V})$ on $\mathfrak{F}_{K,V}$. For $\theta, \chi \in \mathfrak{F}_{K,V}$

$$(\theta, e^{-tH(g)_{K,V}}\chi) = \int_{C_{K,V}} \exp\left[-\int_0^t H_{I,g,K,V}(q(s))\, ds\right] \bar{\theta}(q(0))\, \chi(q(t))\, dq(\cdot). \tag{5.42}$$

Here $C_{K,V}$ is the path space corresponding to $\mathfrak{F}_{K,V}$. It consists of continuous paths $q(\cdot)$ taking values $q(s)$ in a Euclidean space of high dimension—one dimension for each mode permitted by the cutoffs K and V. The integral is a Wiener type integral on $C_{K,V}$ coming from $H_{0,K,V}$. The transition probabilities of the Wiener process are the kernel of $\exp(-tH_{0,K,V})$. We bound the inner product (5.42) with Hölder's inequality, as in [22, 8]. For some sufficiently large numbers p and t chosen independently of g, K and V, we have

$$|E_{g,K,V}| \leq (pt)^{-1} \log \left\{ \int_{C_{K,V}} \exp - \left[p \int_0^t H_{I,g,K,V}(q(s))\, ds \right] dq(\cdot) \right\}. \tag{5.43}$$

Thus we must show that

$$\int_{C_{K,V}} \exp\left[-p \int_0^t H_{I,g,K,V}(q(s))\, ds \right] dq(\cdot) \tag{5.44}$$

grows at most exponentially in the number of j's, with the exponential constant independent of K and V. We have

$$:\varphi_{\Lambda,V}(x)^4: = (\varphi_{\Lambda,V}^2(x) - 3\,C_0)^2 - 6\,C_0^2,$$

where C_0 is the no particle expectation value

$$C_0 = (\Omega_0, \varphi_{\Lambda,V}(x)^2 \Omega_0) = \frac{1}{2V} \sum_{k \in \Gamma_V} \frac{1}{\mu(k)}\, \xi^2(k/\Lambda) \leq C(\log \Lambda)$$

for $\Lambda \geq 2$. Thus

$$-6\,C^2 \lambda \|\zeta\|_1 (\log \Lambda)^2 \leq H_I(\Lambda, j),$$

and for a new constant c_1 depending on t

$$1 - c_1(\log \Lambda)^2 \leqslant \int_0^t H_I(\Lambda, j)\,(q(s))\,ds. \tag{5.45}$$

Let
$$\delta H(\Lambda, j) = H_I(\mathrm{K}, j) - H_I(\Lambda, j) \tag{5.46}$$

and
$$I(\Lambda, j) = \int_0^t \delta H(\Lambda, j)\,(q(s))\,ds. \tag{5.47}$$

Also let Pr denote the measure defined by the integral

$$\int_{C_{\mathrm{K},V}} dq(\cdot).$$

We define $\mathfrak{J}$ as the set of integers j for which $g\zeta_j \neq 0$. Let $\mathfrak{J}_0$ denote any subset of $\mathfrak{J}$. We consider a finite sequence $\{\Lambda_j : j \in \mathfrak{J}_0\}$ such that

$$2 \leqslant \Lambda_j \leqslant \mathrm{K}.$$

We define for such a sequence $\{\Lambda_j\}$

$$P(\{\Lambda_j\}) = Pr\{|I(\Lambda_j, j)| \geqslant 1, \text{ for all } j \in \mathfrak{J}_0\}. \tag{5.48}$$

The main step remaining is to prove

LEMMA 5.3. *For constants c, d independent of* K *and* V,

$$P(\{\Lambda_j\}) \leqslant \prod_{j \in \mathfrak{J}_0} [c \exp(-d\Lambda_j^{\frac{1}{2}-\varepsilon})]. \tag{5.49}$$

Let us for the moment assume Lemma 5.3 and derive the bound on (5.37). We represent $C_{\mathrm{K},V}$ as a disjoint union of measurable subsets X_ν, and we estimate the integral by the maximum of its integrand on each X_ν.

$$\left| \int_{C_{\mathrm{K},V}} F(q(\cdot))\,dq(\cdot) \right| \leqslant \sum_\nu Pr\{X_\nu\} \sup_{q(\cdot) \in X_\nu} |F(q(\cdot))|. \tag{5.50}$$

Let us assign to each path $q(\cdot) \in C_{\mathrm{K},V}$ a sequence of integers ν_j with $j \in \mathfrak{J}$. We define

$$\nu_j(q(\cdot)) = \nu_j \tag{5.51}$$

as the smallest integer in the interval $[d_1, \mathrm{K}]$ such that

$$|I(\nu_j, j)(q(\cdot))| \leqslant 1. \tag{5.52}$$

Here $d_1 > 2$ is a constant which we choose depending only on the c and d in Lemma 5.3.

We take d_1 and K to be integers. Since $I(\mathrm{K}, j)=0$, there is a smallest integer ν_j with the required property.

For each sequence of integers $\{\nu_j\}$, namely

$$\nu_j \in [d_1, \mathrm{K}], \quad j \in \mathfrak{J},$$

let $X(\{\nu_j\})$ be the set of all paths in $C_{\mathrm{K},V}$ which, by (5.51), correspond to the sequence $\{\nu_j\}$. Then $C_{\mathrm{K},V}$ is a disjoint union of measurable sets

$$C_{\mathrm{K},V} = \bigcup_{\{\nu_j\}} X(\{\nu_j\}), \tag{5.53}$$

where the various ν_j range over $[d_1, \mathrm{K}]$.

Furthermore, we note that

$$\int_0^t H_{I,g,\mathrm{K},V}(q(s))\,ds = \sum_{j\in\mathfrak{J}} \left\{ \int_0^t H_I(\nu_j, j)\,(q(s))\,ds + I(\nu_j, j) \right\}, \tag{5.54}$$

and by (5.45)

$$1 - c_1(\log \nu_j)^2 \leqslant \int_0^t H_I(\nu_j, j)\,(q(s))\,ds.$$

Therefore for $q(\cdot) \in X(\{\nu_j\})$, (5.54) yields

$$-c_1 \sum_{j\in\mathfrak{J}} (\log \nu_j)^2 \leqslant \int_0^t H_{I,g,\mathrm{K},V}(q(s))\,ds. \tag{5.55}$$

To each path $q(\cdot) \in C_{\mathrm{K},V}$ we now assign a second sequence of integers $\{\Lambda_j\}$, where j ranges over a subset $\mathfrak{J}_0$ of $\mathfrak{J}$. Let $\mathfrak{J}_0$ consist of those k for which

$$\nu_k > d_1 + 1, \tag{5.56}$$

and for $k \in \mathfrak{J}_0$ define

$$\Lambda_k = \nu_k - 1.$$

To each subset $X(\{\nu_j\})$ of $C_{\mathrm{K},V}$, there corresponds one sequence $\{\Lambda_k\}$. For $k \in \mathfrak{J}_0 \subset \mathfrak{J}$,

$$|I(\Lambda_k, k)\,(q(\cdot))| \geqslant 1,$$

and so

$$Pr\{X(\{\nu_j\})\} \leqslant P(\{\Lambda_k\}). \tag{5.57}$$

Thus using (5.50), (5.55), (5.57) and Lemma 5.3, the desired integral (5.44) is bounded by

$$\sum_{\{\nu_j\}} Pr(X\{\nu_j\}) \exp\,[c_1 p \sum_{j\in\mathfrak{J}} \log^2 \nu_j] \leqslant \sum_{\{\nu_j\}} \prod_{j\in\mathfrak{J}_0} [c \exp\,(-d\Lambda_j^{\frac{1}{2}-\varepsilon})] \prod_{j\in\mathfrak{J}} [\exp\,(c_1 p \log^2 \nu_j)]. \tag{5.58}$$

Each ν_j in the above sum ranges over $d_1 \leqslant \nu_j \leqslant \mathrm{K}$, and we now treat the various cases in which some of the ν_j equal d_1. Let M be the number of elements of $\mathfrak{J}$ and M_0 be the number of elements of $\mathfrak{J}_0$, leaving $(M-M_0)$ of the ν_j equal to d_1. Since there are 2^M subsets of $\mathfrak{J}$, $\mathfrak{J}_0$ can be chosen in 2^M different ways. Restricting the sum (5.58) to $\{\nu_j\}$ which give one fixed subset $\mathfrak{J}_0$, this restricted sum is bounded by

$$\exp\,[(M-M_0)\,c_1 p \log^2 d_1] \sum_{\{\Lambda_j\}} \prod_{j\in\mathfrak{J}_0} [\exp\,(\log c + c_1 p \log^2 \nu_j - d\Lambda_j^{\frac{1}{4}-\varepsilon})]$$
$$\leqslant \exp\,[(M-M_0)\,c_1 p \log^2 d_1]\exp\,[M_0 d_2]. \tag{5.59}$$

Here we have chosen the arbitrary constant d_1 sufficiently large so that for $\nu > d_1$,

$$c_1 p \log^2 \nu + \log c < \tfrac{1}{2}\, d(\nu-1)^{\frac{1}{4}-\varepsilon} = \tfrac{1}{2}\, d\Lambda^{\frac{1}{4}-\varepsilon},$$

in which case

$$\sum_{\Lambda=d_1}^{K-1} \exp\,(-\tfrac{1}{2}\, d\Lambda^{\frac{1}{4}-\varepsilon}) \leqslant \sum_{\Lambda=0}^{\infty} \exp\,(-\tfrac{1}{2}\, d\Lambda^{\frac{1}{4}-\varepsilon}) = \exp\,[d_2].$$

We are considering the case of M_0 different Λ_j's so the bound (5.59) follows. Thus if

$$a = d_2 + c_1 p \log^2 d_1,$$

(5.59) is dominated by $\exp\,[aM]$ and (5.44) is less than

$$2^M \exp\,[aM],$$

which proves the theorem.

Proof of Lemma 5.3. For any set of positive even integers l_j,

$$P(\{\Lambda_j\}) \leqslant \int_{C_{K,V}} \prod_{j\in\mathfrak{J}_0} I(\Lambda_j, j)^{l_j} dq(\cdot). \tag{5.60}$$

We approximate the Riemann integral defining $I(\Lambda_j, j)$ by Riemann sums. In order to bound (5.60) it is sufficient to do so with $I(\Lambda_j, j)$ replaced by a Riemann sum. We replace all $L=\sum_j l_j$ factors by Riemann sums and then expand. The result will be a sum of M^L terms, if each Riemann sum has M terms. Each term is a product of L factors, and each factor occurs at a sharp time. To write down a typical term, we choose mesh times $s_1, \ldots, s_L$, and for the αth factor we choose an index $j(\alpha)$. The index j is chosen exactly l_j times. Then (5.60) is replaced by

$$P(\{\Lambda_j\}) \leqslant t^L \sup \int_{C_{K,V}} \delta H(\Lambda_{j(1)}, j(1))\,(q(s_1)) \ldots \delta H(\Lambda_{j(L)}, j(L))\,(q(s_L))\, dq(\cdot), \tag{5.61}$$

where the supremum is taken over all choices of mesh times, mesh lengths and all choices of $j(\alpha)$ consistent with the l_j. We can rearrange the factors in the integral, so that is no loss of generality to assume that

$$s_1 < s_2 < \ldots < s_L.$$

By the definition of the Wiener integral, (5.61) equals the no particle expectation value

$$t^L \sup (\Omega_0, \exp(-s_1 H_{0,K,V})\, \delta H(\Lambda_{j(1)}, j(1)) \exp(-(s_2 - s_1) H_{0,K,V}) \\ \ldots \exp(-(s_L - s_{L-1}) H_{0,K,V})\, \delta H(\Lambda_{j(L)}, j(L))\, \Omega_0). \tag{5.62}$$

The next step is to evaluate (5.62) as a sum of elementary terms. Each elementary term will itself be a sum of explicit functions. This reduction of (5.62) is accomplished by repeated application of the commutation relation

$$[a_V(-k), a_V^*(k')] = \delta(-k; k') = \begin{cases} 1 & \text{if } -k = k' \\ 0 & \text{otherwise} \end{cases}$$

and

$$[a_V(-k), a_V(k')] = 0,$$

for $k, k' \in \Gamma_V$. We use this relation to bring all the annihilation operators $a_V(k)$ to the right and the creation operators $a_V(k)$ to the left. Furthermore, these commutation relations yield the relations for $k \in \Gamma_{K,V}$,

$$a_V(k) \exp(-sH_{0,K,V}) = \exp(-sH_{0,K,V})\, a_V(k) \exp[-s\mu(k)],$$

which are used to bring the annihilation operators to the right past the factors $\exp(-sH_{0,K,V})$. Since

$$a_V(k)\, \Omega_0 = 0,$$

moving the annihilation operators to the right does reduce (5.62) to the desired sum of elementary terms. Furthermore, each variable k must occur in a delta function $\delta(-k; k')$ in any non zero term.

In order to keep track of the distinct terms and the commutation rules used, we introduce a graph G for each term. The value of the term will be a number $N(G)$ assigned to G. Thus we shall write

$$(\Omega_0, \prod_{\alpha=1}^{L} \{\exp[-(s_\alpha - s_{\alpha-1}) H_{0,K,V}]\, \delta H(\Lambda_{j(\alpha)}, j(\alpha))\}\, \Omega_0) = \sum_G N(G), \tag{5.63}$$

where $s_0 = 0$, and where the product of noncommuting operators is defined by

$$\prod_{\alpha=1}^{L} \{A_\alpha\} = A_1 A_2 \ldots A_L.$$

We construct G as follows. We start with L vertices, labeled by the index α, $1 \leqslant \alpha \leqslant L$. From each vertex we draw four lines (called legs), labeled by (α, ϱ), $1 \leqslant \varrho \leqslant 4$. Each factor

$\delta H(\Lambda_{j(\alpha)}, j(\alpha))$ in (5.63) is the difference between two interaction Hamiltonians, and as such is the sum of five monomials as in (5.27). For each factor we choose one such monomial with ν_α creators and $4-\nu_\alpha$ annihilators. This monomial will correspond to a vertex in G, and we draw ν_α legs from the αth vertex to the left, and $4-\nu_\alpha$ legs pointing to the right. Each leg (α, ϱ) corresponds to a momentum variable $k_{\alpha\varrho}$ in the kernel $B(\Lambda_{j(\alpha)}, j(\alpha))$ at vertex α. Since our factors are noncommuting, we order the vertices linearly from left to right, as in (5.63), with vertex 1 on the left and vertex L on the right. Each time that the use of the commutation relations introduces a delta function, we say that the variables $-k_{\alpha\varrho}$ and $k_{\alpha'\varrho'}$ occurring in it are *contracted*. Graphically we join together the annihilation leg (α, ϱ) and the creation leg (α', ϱ') which correspond to these variables. We have already noticed that for a nonzero term, each variable occurs in exactly one delta function, and so for the corresponding graph each creating leg (pointing left) is connected to a unique annihilating leg (pointing right). Thus only terms with $\nu_1=0$, $\nu_L=4$ will contribute. Since annihilating legs all moved to the right, the annihilating legs at vertex α are always contracted with creating legs on their right.

The possible graphs are all graphs which can be constructed in this manner. Each graph consists of L vertices and $2L$ lines (each line being two joined legs). Each vertex is the endpoint of four lines and each line joins two distinct vertices. This last fact results from Wick ordering the interaction. Evidently there are fewer than

$$(4L)!! = (4L-1)(4L-3)\dots 3\cdot 1$$

possible graphs.

We let $N(G)$ denote the contribution to (5.63) from a given graph G and this is obtained as follows: To the vertex α of G we assign the kernel

$$B(\Lambda_{j(\alpha)}, j(\alpha); k_{\alpha 1}, \dots, k_{\alpha 4}) = b_{K, j(\alpha)}(k_{\alpha 1}, \dots, k_{\alpha 4}) - b_{\Lambda_{j(\alpha)}, j(\alpha)}(k_{\alpha 1}, \dots, k_{\alpha 4}), \tag{5.64}$$

where $b_{K,j}$ is defined in (5.26). We write down the product of the L kernels (5.64) for G and multiply this by a product of $2L$ delta functions

$$\delta(-k_{\alpha\varrho}; k_{\alpha'\varrho'}),$$

one delta function for each line, obtained by pairing the annihilator (α, ϱ) with the creator (α', ϱ'), $\alpha' > \alpha$. We then multiply by a product of $2L$ energy factors

$$\exp\left[-(s_{\alpha'} - s_\alpha)\mu(k_{\alpha\varrho})\right],$$

one for each line, which arise from the commutations past $\exp(-sH_{0,K,V})$ factors. These factors could actually be included in the kernels $B(\Lambda_{j(\alpha)}, j(\alpha))$ by defining for each graph G

$$\hat{B}(\Lambda_{j(\alpha)}, j(\alpha)) = B(\Lambda_{j(\alpha)}, j(\alpha)) \exp\left[-\sum_{\varrho=1}^{4} (s_{\alpha'} - s_\alpha)\,\mu(k_{\alpha\varrho})/2\right]. \tag{5.65}$$

The numerical value of G is the sum of this expression over the $4L$ variables $k_{\alpha\varrho} \in \Gamma_{K,V}$ times some numerical factors.

$$N(G) = \sum_{\substack{k_{\beta\sigma}\in\Gamma_{K,V}\\ 1\leqslant\beta\leqslant L\\ 1\leqslant\sigma\leqslant 4}} \left[\sum_{\alpha=1}^{L}\binom{4}{\nu_\alpha}\hat{B}(\Lambda_{j(\alpha)}, j(\alpha); k_{\alpha\varrho})\right]\left[\prod_{\substack{1\leqslant\alpha\leqslant L\\ 1\leqslant\varrho\leqslant 4}} \delta(-k_{\alpha\varrho}; k_{\alpha'\varrho'})\right]. \tag{5.66}$$

Let us now motivate the type of bound we expect (in the simple case $s_1 = \ldots = s_L = 0$). We expect that different regions of localization will be independent, by the localization property of free fields (cluster property). Since each localization index j must be chosen exactly l_j times, exact localization would mean for $s_1 = \ldots = s_L = 0$ that (5.63) equals

$$\prod_{j\in\mathfrak{J}_0} (\Omega_0, \delta H(\Lambda_j, j)^{l_j}\Omega_0), \tag{5.67}$$

which can be shown to be dominated by

$$\prod_{j\in\mathfrak{J}_0} [c^{l_j}(2l_j)!\,\Lambda_j^{-l_j/2}], \tag{5.68}$$

for some constant c independent of K, V, $\mathfrak{J}_0$, Λ_j and l_j. In fact, we will show that although the localization is not exact, it is approximate in the sense that (5.68) actually bounds (5.63), and this holds even in the case that the s_j are nonzero. In other words, the action of $\exp(-sH_0)$ does not destroy, in any essential way, the localization of the particles.

LEMMA 5.4. *The expectation value* (5.63) *is dominated by*

$$\prod_{j\in\mathfrak{J}_0} [c^{l_j}(2l_j)!\,\Lambda_j^{-l_j(1-\varepsilon)/2}],$$

where the constant c depends only on T providing the bounds

$$0 \leqslant g(x) \leqslant 1,$$

$$|Dg(x)| \leqslant M_0,$$

and

$$0 \leqslant s_\alpha \leqslant T$$

are satisfied.

Let us assume this lemma and complete the proof of Lemma 5.3. We apply Lemma 5.4 to (5.63), and get a bound on (5.61) independent of the variables in the supremum. Thus

$$Pr(\{\Lambda_j\}) \leq (ct)^L \prod_{j\in\mathfrak{J}_0} (2l_j)!\Lambda_j^{-l_j(1-\varepsilon)/2}.$$

We now choose l_j to be the positive even integer nearest to $(4ct/\Lambda_j^{\frac{1}{2}-\varepsilon})^{-\frac{1}{2}} = O(\Lambda_j^{(1-\varepsilon)/4})$. Then using Sirling's formula to estimate the factorials, with $d \leq (4ct)^{-\frac{1}{2}}$, with a new constant c_1, and with a new ε,

$$Pr(\{\Lambda_j\}) \leq \prod_{j\in\mathfrak{J}_0} [c_1 \exp(-d\Lambda_j^{\frac{1}{4}-\varepsilon})].$$

This completes the proof.

Proof of Lemma 5.4. By repeated application of the Schwarz inequality in the variables $k_{\alpha\varrho}$, $N(G)$ in (5.66) can be dominated by

$$|N(G)| \leq 2^{4L} \prod_{\alpha=1}^{L} \|\hat{B}(\Lambda_{j(\alpha)}, j(\alpha))\|_2 = 2^{4L} \prod_{j\in\mathfrak{J}_0} \|\hat{B}(\Lambda_j, j)\|_2^{l_j} \leq \prod_{j\in\mathfrak{J}_0} [c^{l_j}\Lambda_j^{-l_j(1-\varepsilon)/2}],$$

by Lemma 5.2. However, we need a sharper bound (5.82) which reflects the localization of particles. Thus we classify the graphs in a way which reflects this property.

For a particular graph G, each vertex α, $1 \leq \alpha \leq L$, has an associated localization number $j(\alpha) \in \mathfrak{J}_0$. For $\varrho = 1, 2, 3, 4$, let us define $j(\alpha, \varrho)$ by

$$j(\alpha, \varrho) = j(G; \alpha, \varrho) = j(\alpha'), \tag{5.69}$$

where the ϱth leg at vertex α is connected in G to the ϱ'th leg at vertex α'. We can also define

$$\delta(\alpha, \varrho) = j(\alpha) - j(\alpha, \varrho), \tag{5.70}$$

which measures the difference in configuration space localization between the two vertices connected by the pair of legs (α, ϱ), (α', ϱ') corresponding to a line in G.

The functions (5.69) uniquely specify the functions (5.70) and vice versa, for any graph G contributing to (5.63). However, (5.69) does not uniquely specify the graph. We now show that there are at most

$$\prod_{j\in\mathfrak{J}_0} [4^{l_j}(2l_j)!] \tag{5.71}$$

out of the (fewer than $(4L)!!$) different graphs G with the same function $j(G, \alpha, \varrho)$.

Let us suppose that in G there are

$$l_{ij} = l_{ij}(G)$$

lines connecting a j-localized vertex to an i-localized vertex, and to simplify the following formulae, we count each line with $i = j$ twice. Thus

$$\sum_{j\in\mathfrak{J}_0} l_{ij} = 4l_i, \tag{5.72}$$

and
$$\sum_{i,\, j \in \mathcal{J}_0} l_{ij} = 4L. \tag{5.73}$$

Note that the G's which contribute to (5.65) are exactly those for which the off diagonal l_{ij}'s vanish, and $l_{ij} = 4l_i \delta_{ij}$.

With the functions $j(\alpha, \varrho)$ given and with i and j held fixed, there are two sets of l_{ij} legs specified, and for the graphs we are considering each leg in the first set must be contracted to a leg in the second set. For a vertex α of a leg in the first set, $j(\alpha) = j$ and for a vertex α' of a leg in the second set, $j(\alpha') = i$. If $i \neq j$ then these two sets of legs are disjoint and the contractions can be made in at most $(l_{ij})!$ ways. If $i = j$ then the two sets of legs coincide and the contractions can be made in at most $(l_{ii})!!$ ways. Once the contractions have been made for all $i \leqslant j$, the graph G is uniquely determined, and so there are at most

$$\prod_{\substack{i<j \\ i,\, j \in \mathcal{J}_0}} [l_{ij}\,!] \prod_{i \in \mathcal{J}_0} [l_{ii}\,!!] \tag{5.74}$$

graphs with given functions $j(\alpha, \varrho)$.

If l is even
$$l\,! \leqslant 2^l [(\tfrac{1}{2}l)\,!]^2$$
$$l\,!! \leqslant 2^{l/2} (\tfrac{1}{2}l)\,!$$

and there are similar estimates for l_{ij} odd.

Using (5.72) – (5.73), we thus have that (5.74) is dominated by

$$\prod_{i,\, j \in \mathcal{J}_0} [2^{l_{ij}/2} (\tfrac{1}{2} l_{ij})\,!] = 4^L \prod_{j \in \mathcal{J}_0} \{ \prod_{i \in \mathcal{J}_0} [(\tfrac{1}{2} l_{ij})\,!] \} \leqslant 4^L \prod_{j \in \mathcal{J}_0} [(2l_j)\,!],$$

which is the desired bound (5.71).

Thus
$$\Big|\Big(\Omega_g, \prod_{\alpha=1}^{L} \delta H(\Lambda_{j(\alpha)}, j(\alpha))\, \Omega_0\Big)\Big| \leqslant \{ \prod_{j \in \mathcal{J}_0} [(4)^{l_j} (2l_j)\,!] \} \sum_{\{\delta(\alpha, \varrho)\}} \sup |N(G)|. \tag{5.75}$$

Here the supremum is over all graphs G with fixed $j(G, \alpha, \varrho) = j(\alpha, \varrho)$, or fixed $\delta(\alpha, \varrho)$, and the sum is over all possible integers $\delta(\alpha, \varrho)$ of graphs contributing to (5.63). We now bound $N(G)$ in (5.66).

Each of the $2L$ independent sums over momentum variables $k_{\alpha\varrho} \in \Gamma_V$ can be regarded as an inner product in the one particle space $\mathfrak{F}_{V,1}$, or the equivalent inner product in the space of Fourier series on $[-V/2, V/2]$. We now make an expansion by writing the inner product on $\mathfrak{F}_{V,1}$ as a sum of inner products on the $\mathfrak{F}_{V,1,i}$, using (5.28). Because of the support properties, many terms vanish giving as in (5.31)

$$N(G) = \sum_{\substack{i_{\beta\sigma}, k_{\beta\sigma} \\ 1 \leqslant \beta \leqslant L \\ 1 \leqslant \sigma \leqslant 4}} \prod_{\alpha=1}^{L} \hat{B}(\Lambda_{j(\alpha)}, j(\alpha), i_{\alpha\varrho}; k_{\alpha\varrho}) \binom{4}{\nu_\alpha} \prod_{\substack{1 \leqslant \alpha \leqslant L \\ 1 \leqslant \varrho \leqslant 4}} \delta(-k_{\alpha\varrho}; k_{\alpha'\varrho'})\, \delta(i_{\alpha\varrho}; i_{\alpha'\varrho'}). \tag{5.76}$$

The sum runs over all possible choices of the localization indices $i_{\beta\sigma}$ and the momenta $k_{\beta\sigma}$. We now apply the Schwarz inequality repeatedly in the $2L$ independent $k_{\alpha\varrho}$ variables. This gives the bound

$$|N(G)| \leqslant \sum_{\substack{i_{\beta\sigma} \\ 1\leqslant\beta\leqslant L \\ 1\leqslant\sigma\leqslant 4}} \prod_{\alpha=1}^{L} 16 \|\hat{B}(\Lambda_{j(\alpha)}, j(\alpha), i_{\alpha\varrho})\|_2 \prod_{\substack{1\leqslant\alpha\leqslant L \\ 1\leqslant\varrho\leqslant 4}} \delta(i_{\alpha\varrho}; i_{\alpha'\varrho'}). \tag{5.77}$$

We choose V sufficiently large to enclose the support of the spatial cutoff $g(x)$. Thus by Lemma 5.2 independently of V,

$$\|\hat{B}(\Lambda_{j(\alpha)}, j(\alpha), i_{\alpha 1}, \dots, i_{\alpha 4})\|_2 \leqslant c\Lambda_{j(\alpha)}^{-(1-\varepsilon)/2} \prod_{1\leqslant\varrho\leqslant 4} (1+|j(\alpha)-i_{\alpha\varrho}|)^{-r}, \tag{5.78}$$

where c depends on λ, M_0, r and T. Using the fact that each index $j=j(\alpha)$ occurs l_j times in G, we have by (5.77)–(5.78),

$$|N(G)| \leqslant \prod_{j\in\mathcal{J}_0} [16\, c^{l_j} \Lambda_j^{-l_j(1-\varepsilon)/2}] \sum_{\substack{i_{\beta\sigma} \\ 1\leqslant\beta\leqslant L \\ 1\leqslant\sigma\leqslant 4}} \prod_{\substack{1\leqslant\alpha\leqslant L \\ 1\leqslant\varrho\leqslant 4}} (1+|j(\alpha)-i_{\alpha\varrho}|)^{-r} \delta(i_{\alpha\varrho}; i_{\alpha'\varrho'}). \tag{5.79}$$

Since each leg (α, ϱ) is paired with another leg (α', ϱ'), the factors

$$(1+|j(\alpha)-i_{\alpha\varrho}|)^{-r}\delta(i_{\alpha\varrho}; i_{\alpha'\varrho'})$$

occur in pairs,

$$(1+|j(\alpha)-i_{\alpha\varrho}|)^{-r}(1+|j(\alpha')-i_{\alpha'\varrho'}|)^{-r}\delta(i_{\alpha\varrho}; i_{\alpha'\varrho'})^2$$
$$= (1+|j(\alpha)-i_{\alpha\varrho}|)^{-r}(1+|j(\alpha')-i_{\alpha\varrho}|)^{-r}\delta(i_{\alpha\varrho}; i_{\alpha'\varrho'}), \tag{5.80}$$

where
$$\delta(\alpha,\varrho) = j(\alpha) - j(\alpha') = j(\alpha) - j(\alpha,\varrho). \tag{5.81}$$

As
$$(1+|x|)^{-2}(1+|y|)^{-2} \leqslant (1+|x-y|)^{-1}(1+|x|)^{-1},$$

(5.80) is dominated by

$$(1+|\delta(\alpha,\varrho)|)^{-r/2}(1+|j(\alpha)-i_{\alpha\varrho}|)^{-r/2}.$$

Hence by (5.79)

$$|N(G)| \leqslant \prod_{j\in\mathcal{J}_0} (16\, cc_1)^{l_j} \Lambda_j^{-l_j(1-\varepsilon)/2} \prod_{\substack{1\leqslant\alpha\leqslant L \\ 1\leqslant\varrho\leqslant 4}} (1+|\delta(\alpha,\varrho)|)^{-r/2}, \tag{5.82}$$

where we have chosen $r > 2$ and used the fact that

$$\sum_{\substack{i_{\beta\sigma} \\ 1\leqslant\beta\leqslant L \\ 1\leqslant\sigma\leqslant 4}} \prod_{\substack{1\leqslant\alpha\leqslant L \\ 1\leqslant\varrho\leqslant 4}} (1+|j(\alpha)-i_{\alpha\varrho}|)^{-r/2} = c_1^L,$$

with
$$c_1 = [\sum_{n=-\infty}^{\infty} (1+|n|)^{-r/2}]^4.$$

The bound (5.82) gives the dependence of G on the localization of its vertices. Thus by (5.75) and (5.82),

$$\left|(\Omega_0, \prod_{\alpha=1}^{L} \delta H(\Lambda_{j(\alpha)}, j(\alpha))\, \Omega_0)\right| \leq \prod_{j\in \mathcal{J}_0} [(64\, cc_1^2)^{l_j}(2l_j)!\, \Lambda_j^{-l_j(1-\varepsilon)/2}],$$

which establishes the lemma.

References

[1]. ARAKI, H., The type of von Neumann algebra associated with the free field. *Progr. of Theor. Phys.*, 32 (1964), 956–965.

[2]. DIXMIER, J., *Les algèbres d'opérateurs dans l'espace Hilbertien (algèbres de von Neumann)*. Gauthiers-Villars, Paris, 1957.

[3]. EPSTEIN, H., GLASER, V. & JAFFE, A., Nonpositivity of the energy density in quantized field theories. *Nuovo Cimento*, 36 (1965), 1016–1022.

[4]. FEDERBUSH, P., A partially alternate derivation of a result of Nelson. *J. Math. Phys.*, 10 (1969), 50–52.

[5]. FELL, J., The dual spaces of C^*-algebras. *Trans. Amer. Math. Soc.*, 94 (1960), 365–403.

[6]. GALINDO, A., On a class of perturbations in quantum field theory. *Proc. Nat. Acad. Sci. (U.S.A.)*, 48 (1962), 1128–1134.

[7]. GLIMM, J., A Stone–Weierstrass theorem for C^*-algebras. *Ann. of Math.*, 72 (1960), 216–244.

[8]. —— Boson fields with nonlinear selfinteraction in two dimensions. *Commun. Math. Phys.*, 8 (1968), 12–25.

[9]. —— *Varenna summer school-lectures, 1968.* Academic Press.

[10]. GLIMM, J. & JAFFE, A., A $\lambda(\phi^4)_2$ quantum field theory without cutoffs I. *Phys. Rev.*, 176 (1968), 1945–1951.

[11]. —— Singular perturbations of self adjoint operators. *Comm. Pure Appl. Math.*, 22 (1969), 401–414.

[12]. —— The $\lambda(\phi^4)_2$ quantum field theory without cutoffs II. The field operators and the approximate vacuum. *Ann. of Math.*, 91 (1970). 362–401.

[13]. —— An infinite renormalization of the Hamiltonian is necessary. *J. Math. Phys.*, 10 (1969), 2213–2214.

[14]. GRIFFIN, E., Some contributions to the theory of rings of operators II. *Trans. Amer. Math. Soc.*, 79 (1955), 389–400.

[15]. HAAG, R. & KASTLER, D., An algebraic approach to quantum field theory. *J. Math. Phys.*, 5 (1964), 848–861.

[16]. HALL, D. & WIGHTMAN, A., A theorem on invariant functions with applications to relativistic quantum field theory. *Mat.-Fys. Medd. Kongl. Danske Vidensk. Selskab*, 31 (1957) No. 5.

[17]. HEPP, K., Théorie de la renormalisation, Springer Verlag, New York, 1969,

[18]. JAFFE, A., Divergence of perturbation theory for bosons. *Commun. Math. Phys.*, 1 (1965), 127–149.

[19]. —— *Varenna summer school lectures, 1968.* Academic Press.

[20]. KRISTENSEN, P., MEJLBO, L. & POULSEN, E., Tempered distributions in infinitely many dimensions I. *Commun. Math. Phys.*, 1 (1965), 175–214.

[21]. NAIMARK, M., *Normed rings.* Noordhoff, Groningen, 1959.

[22]. NELSON, E., A quartic interaction in two dimensions. In *Mathematical theory of elementary particles*, edited by R. Goodman and I. Segal, M.I.T. Press 1966.

[23]. SEGAL, I., Nonlinear functions of weak processes I. *J. Functional Anal.*, 4 (1969), 404–456.

[24]. STREIT, L., A generalization of Haag's theorem. *Nuovo Cimento*, to appear.
[25]. WATSON, G., *A treatise on the theory of Bessel functions*. Cambridge 1944.
[26]. WEINBERG, S., High energy behavior in quantum field theory. *Phys. Rev.*, 118 (1960), 838–849.
[27]. CANNON, J. & JAFFE, A., Lorentz covariance of the $\lambda\phi_2^4$ quantum field theory. *Commun. Math. Phys.*, 17 (1970), 261–321.
[28]. ROSEN, L., A $\lambda\phi^{2n}$ field theory without cutoffs. *Commun. Math. Phys.*, 16 (1970), 157–183.

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