THE L^2-BOUNDEDNESS OF PSEUDODIFFERENTIAL OPERATORS

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ABSTRACT. We give a new proof of the Calderon-Vaillancourt theorem. We also obtain the L^2-continuity of $a(x, D)$ if its symbol $a(x, \xi)$ satisfies some suitable conditions.

Introduction. We consider a pseudodifferential operator $a(x, D)$ on an open set $\Omega \subseteq R^n$. We want to find conditions on its symbol $a(x, \xi)$ that imply the L^2 continuity of $a(x, D)$. Several results for this problem can be found in the literature. We mention a few. The first is due to A. Calderon and R. Vaillancourt [1].

THEOREM. *Let $0 \leqslant \rho < 1$ and a: $R^n \times R^n \to \mathbf{C}$ be a continuous function whose derivatives $\partial_x^\alpha \partial_\xi^\beta a$ in the distribution sense satisfy the following condition*:

(∗)   *There is a constant $C > 0$ such that*

$$\left| \partial_x^\alpha \partial_\xi^\beta a(x, \xi) \right| \leqslant C \left(1 + |\xi| \right)^{\rho(|\alpha| - |\beta|)},$$

where $(x, \xi) \in R^n \times R^n$, $0 \leqslant |\beta| \leqslant 2[n/2] + n$ and $0 \leqslant |\alpha| \leqslant 2m$ with $m \in N$ and $m(1 - \rho) \geqslant \frac{5}{4} n$.

Then $a(x, D)$ is continuous from $L^2(R^n)$ to $L^2(R^n)$.

Coifman and Meyer [2] improved (∗) by assuming that it holds for $0 \leqslant |\alpha|$, $|\beta| \leqslant m$, $m \in N$ and $m \geqslant [n/2] + 1$. Moreover, they gave an example that shows $m \geqslant [n/2] + 1$ is the best possible. Kato [3], Kumano-go [4], R. Beals [5], Cordes [6] and others obtained similar results by assuming (∗) holds for different numbers of derivatives.

In this work we prove several new results about the L^2 continuity of $a(x, D)$ when $a(x, \xi)$ and some of its derivatives satisfy conditions similar to the above theorem.

§1 contains our results, and §2–§6 contain the proofs.

We use elementary tools such as integration by parts, the Fourier transform and Parseval's formula.

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1. Statement of results.

THEOREM 1. *Let* $\Omega = \prod_{j=1}^{n}(s_j, t_j)$ *be a bounded open box in* R^n *and* $a: \Omega \times R^n \to \mathbf{C}$ *a measurable function whose derivatives* $\partial_x^\alpha a$ *in the distribution sense satisfy the following condition*:

(1.1)   *There is a constant* $C > 0$ *such that* $\|\partial_x^\alpha a\|_{L^\infty(\Omega \times R^n)} \leqslant C$, *where* $\alpha = (\alpha_1, \dots, \alpha_n) \in N^n$ *with* $\alpha_j = 0$ *or* 1.

Then $a(x, D)$ *is continuous from* $L^2(R^n)$ *to* $L^2(\Omega)$ *with its norm bounded by* $C_n(1 + \max_{1 \leqslant j \leqslant n}(t_j - s_j))^{n/2}\|a\|$, *where* C_n *is a constant depending only on* n, *and* $\|a\|$ *is the smallest* C *such that* (1.1) *holds*.

An immediate consequence of Theorem 1 is the following well-known result.

COROLLARY 1.1. *Let* Ω *be a bounded open subset of* R^n *and* $a: \Omega \times R^n \to \mathbf{C}$ *a measurable function whose derivatives* $\partial_x^\alpha a$ *in the distribution sense satisfy the following condition*:

(1.2)   *There is a constant* $C > 0$ *such that* $\|\partial_x^\alpha a\|_{L^\infty(\Omega \times R^n)} \leqslant C$, *where* $\alpha = (\alpha_1, \dots, \alpha_n) \in N^n$ *with* $\alpha_j = 0$ *or* 1.

Then $a(x, D)$ *is continuous from* $L^2(R^n)$ *to* $L^2_{\mathrm{loc}}(\Omega)$.

THEOREM 2. *Let* $a: R^n \times R^n \to \mathbf{C}$ *be a continuous function whose derivatives* $\partial_x^\alpha \partial_\xi^\beta a$ *in the distribution sense satisfy the following condition*:

(1.3)   *There is a constant* $C > 0$ *such that* $\|\partial_x^\alpha \partial_\xi^\beta a\|_{L^\infty(R^n \times R^n)} \leqslant C$, *where* $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n) \in N^n$ *with* $\alpha_j = 0$ *or* 1, $\beta_j = 0$ *or* 1.

Then $a(x, D)$ *is continuous from* $L^2(R^n)$ *to* $L^2(R^n)$ *with its norm bounded by* $C_n\|a\|$, *where* C_n *is a constant depending only on* n *and* $\|a\|$ *is the smallest* C *such that* (1.3) *holds*.

REMARK. We mention that Theorem 2 is also obtained by Coifman and Meyer [2] and by Cordes [6]. We include it because we need it for the proofs of the next results, and because we believe our proof is simpler.

COROLLARY 2.1. *Let* $a: R^n \times R^n \to \mathbf{C}$ *be a measurable function whose derivatives* $\partial_\xi^\beta a$ *in the distribution sense satisfy the following condition*:

(1.4)   *There is a constant* $C > 0$ *such that*

$$\sup_{x \in R^n} \left(\int_{R^n} \left| \partial_\xi^\beta a(x, \xi) \right|^2 d\xi \right)^{1/2} \leqslant C,$$

where $\beta = (\beta_1, \dots, \beta_n) \in N^n$ *with* $\beta_j = 0$ *or* 1.

Then $a(x, D)$ *is continuous from* $L^2(R^n)$ *to* $L^2(R^n)$ *with its norm bounded by* $C_n\|a\|$, *where* C_n *is a constant depending only on* n, *and* $\|a\|$ *is the smallest* C *such that* (1.4) *holds*.

REMARK. A different sufficient condition for Corollary 2.1 can be found in Muramoto and Nagase [7].

COROLLARY 2.2. *Let* $a\colon R^n \times R^n \to \mathbf{C}$ *be a measurable function whose derivatives* $\partial_x^\alpha a$ *in the distribution sense satisfy the following condition:*

(1.5)   *There is a constant* $C > 0$ *such that*

$$\sup_{\xi \in R^n} \left(\int_{R^n} \left| \partial_x^\alpha a(x, \xi) \right|^2 dx \right)^{1/2} \leqslant C,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in N^n$ *with* $\alpha_j = 0$ *or* 1.

Then $a(x, D)$ *is continuous from* $L^2(R^n)$ *to* $L^2(R^n)$ *with its norm bounded by* $C_n |||a|||$, *where* C_n *is a constant depending only on* n, *and* $|||a|||$ *is the smallest* C *such that* (1.5) *holds.*

REMARK. A different sufficient condition for Corollary 2.2 can be found in Hörmander [8].

THEOREM 3. *Let* $\rho \in R$, $\rho < 1$, *and* $a\colon R^n \times R^n \to \mathbf{C}$ *be a continuous function whose derivatives* $\partial_x^\alpha \partial_\xi^\beta a$ *in the distribution sense satisfy the following condition:*

(1.6)   *There is a constant* $C > 0$ *such that*

$$\left| \partial_x^\alpha \partial_\xi^\beta a(x, \xi) \right| \leqslant C\left(1 + |\xi| \right)^{\rho(|\alpha| - |\beta|)},$$

where $(x, \xi) \in R^n \times R^n$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ $\in N^n$ *with* $\alpha_j = 0, 1$ *or* 2, $\beta_j = 0$ *or* 1.

Then $a(x, D)$ *is continuous from* $L^2(R^n)$ *to* $L^2(R^n)$ *with its norm bounded by* $C_{\rho, n} |||a|||$, *where* $C_{\rho, n}$ *is a constant depending only on* ρ *and* n, *and* $|||a|||$ *is the smallest* C *such that* (1.6) *holds.*

REMARK. When $-\infty < \rho < 1$, a different sufficient condition for Theorem 3 can be found in R. Beals [5].

THEOREM 4. *Let* $g_t(\xi) = (\log(1 + |\xi|))^{-t}$, $t > 0$. *Suppose* $a\colon R^n \times R^n \to \mathbf{C}$ *is a continuous function whose derivatives* $\partial_x^\alpha \partial_\xi^\beta a$ *in the distribution sense satisfy the following condition:*

(1.7)   *There is a constant* $C > 0$ *such that*

$$\left| \partial_x^\alpha \partial_\xi^\beta a(x, \xi) \right| \leqslant C\left(g_t(\xi)(1 + |\xi|) \right)^{|\alpha| - |\beta|},$$

where $(x, \xi) \in R^n \times R^n$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ $\in N^n$ *with* $0 \leqslant \alpha_j \leqslant [(4 + t)/2t] + 2$ *and* $\beta_j = 0$ *or* 1.

Then $a(x, D)$ *is continuous from* $L^2(R^n)$ *to* $L^2(R^n)$ *with its norm bounded by* $C_{t, n} |||a|||$, *where* $C_{t, n}$ *is a constant depending only on* t *and* n, *and* $|||a|||$ *is the smallest* C *such that* (1.7) *holds.*

THEOREM 5. *Let* $\rho \in R$, $\rho < 1$. *Suppose* $a\colon R \times R \to \mathbf{C}$ *is a continuous function whose derivatives* $\partial_x^\alpha \partial_\xi^\beta a$ *in the distribution sense satisfy the following condition:*

(1.8)   *There is a constant* $C > 0$ *such that*

$$\left| \partial_x^\alpha \partial_\xi^\beta a(x, \xi) \right| \leqslant C\left(1 + |\xi| \right)^{\rho(\alpha - \beta)},$$

where $(x, \xi) \in R \times R$, $\alpha = 0$ *or* 1, $\beta = 0$ *or* 1.

Then $a(x, D)$ is continuous from $L^2(R)$ to $L^2(R)$ with its norm bounded by $C_\rho|||a|||$, where C_ρ is a constant depending only on ρ, and $|||a|||$ in the smallest C such that (1.8) holds.

COROLLARY 5.1. Let $g_t(\xi) = (\log(1 + |\xi|))^{-t}$, $\xi \in R$, $t > 0$. Suppose a: $R \times R \to$ $\mathbf{C}$ is a continuous function whose derivatives $\partial_x^\alpha \partial_\xi^\beta a$ in the distribution sense satisfy the following condition:

(1.9)   There is a constant $C > 0$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)\right| \leqslant C\left(g_t(\xi)(1 + |\xi|)\right)^{\alpha - \beta},$$

where $(x, \xi) \in R \times R$, α, $\beta \in N$ and $0 \leqslant \alpha \leqslant [(4 + t)/2t]$ $+ 1$, $\beta = 0$ or 1.

Then $a(x, D)$ is continuous from $L^2(R)$ to $L^2(R)$ with its norm bounded by $C_t|||a|||$, where C_t is a constant depending only on t, and $|||a|||$ in the smallest C such that (1.9) holds.

2. Proof of Theorem 1. To prove Theorem 1, we need the following lemmas.

LEMMA 2.1. If $f(\xi) = 1/(1 + i\xi)$, $\xi \in R$, then the Fourier transform of f is

$$\hat{f}(t) = \begin{cases} 0, & t > 0, \\ 2\pi e^{-|t|}, & t < 0. \end{cases}$$

PROOF. Making use of the residue theorem, the reader can check it without difficulty.

More generally, we have

LEMMA 2.2. If

$$f(\xi) = \prod_{j=1}^n \frac{1}{1 + i\xi_j}, \qquad \xi = (\xi_1, \dots, \xi_n) \in R^n,$$

then the Fourier transform of f is

$$\hat{f}(t) = \prod_{j=1}^n g_j(t_j),$$

where $t = (t_1, \dots, t_n) \in R^n$ and

$$g_j(t_j) = \begin{cases} 0, & t_j > 0, \\ 2\pi e^{-|t_j|}, & t_j < 0. \end{cases}$$

In the following proposition we shall prove Theorem 1 in the special case $n = 1$ and under stronger conditions.

PROPOSITION 2.1. Let $\Omega = (s, t)$, $-\infty < s < t < \infty$. Suppose a: $\Omega \times R \to \mathbf{C}$ is a continuous function whose derivatives $\partial_x^\alpha a$ satisfy the following conditions:

(2.1) $\partial_x^\alpha a \in C^0(\Omega \times R)$, $\alpha = 0, 1$.

(2.2) There is a constant $C > 0$ such that $\|\partial_x^\alpha a\|_{L^\infty(\Omega \times R)} \leqslant C$, $\alpha = 0, 1$.

Then $a(x, D)$ is continuous from $L^2(R)$ to $L^2(\Omega)$ with its norm bounded by $C_\Omega|||a|||$, where C_Ω is a constant depending only on Ω and $|||a|||$ is the smallest C such that (2.2) holds.

PROOF. Let $0 < \varepsilon < \frac{1}{4}(s - t)$ and $\Omega_\varepsilon = (s + \varepsilon, t - \varepsilon)$. Our goal is to find the existence of an absolute constant $C_0 > 0$ such that

$$\|a(x, D)u\|_{L^2(\Omega_\varepsilon)} \leqslant C_0(1 + t - s - 2\varepsilon)^{1/2}|||a|||\,\|u\|_{L^2(R)} \quad \forall u \in C_0^\infty(R).$$

Then the proof of Proposition 2.1 follows by letting $\varepsilon \to 0$.

For $u \in C_0^\infty(R)$ we have

$$a(x, D)u(x) = \frac{1}{2\pi} \int_R e^{ix\xi}a(x, \xi)\hat{u}(\xi)\, d\xi, \qquad x \in \Omega.$$

Therefore, we have

$$\int_{\Omega_\varepsilon} |a(x, D)u(x)|^2\, dx$$

$$= \left(\frac{1}{2\pi}\right)^2 \int_{\Omega_\varepsilon} \int_R \int_R e^{ix(\xi - \lambda)}a(x, \xi)\hat{u}(\xi)\bar{a}(x, \lambda)\bar{\hat{u}}(\lambda)\, d\xi\, d\lambda\, dx$$

$$= \left(\frac{1}{2\pi}\right)^2 \int_{\Omega_\varepsilon} \int_R \int_R e^{ix\xi}a(x, \xi + \lambda)\hat{u}(\xi + \lambda)\bar{a}(x, \lambda)\bar{\hat{u}}(\lambda)\, d\xi\, d\lambda\, dx.$$

Because the integrand is in $L^1(\Omega_\varepsilon \times R \times R)$, by Fubini's theorem we have

$$(2.3) \quad \int_{\Omega_\varepsilon} |a(x, D)u(x)|^2\, dx$$

$$= \left(\frac{1}{2\pi}\right)^2 \int_R \int_R \left(\int_{\Omega_\varepsilon} e^{ix\xi}a(x, \xi + \lambda)\bar{a}(x, \lambda)\, dx\right)\hat{u}(\xi + \lambda)\bar{\hat{u}}(\lambda)\, d\xi\, d\lambda.$$

Making use of the identity

$$\frac{1}{1 + i\xi}(1 + \partial_x)(e^{ix\xi}) = e^{ix\xi},$$

we obtain

(2.4)

$$\int_{\Omega_\varepsilon} e^{ix\xi}a(x, \xi + \lambda)\bar{a}(x, \lambda)\, dx = \frac{1}{1 + i\xi}\int_{\Omega_\varepsilon} e^{ix\xi}a(x, \xi + \lambda)\bar{a}(x, \lambda)\, dx$$

$$+ \frac{1}{1 + i\xi}e^{i(t - \varepsilon)\xi}a(t - \varepsilon, \xi + \lambda)\bar{a}(t - \varepsilon, \lambda)$$

$$- \frac{1}{1 + i\xi}e^{i(s + \varepsilon)\xi}a(s + \varepsilon, \xi + \lambda)\bar{a}(s + \varepsilon, \lambda)$$

$$- \frac{1}{1 + i\xi}\int_{\Omega_\varepsilon} e^{ix\xi}\partial_x a(x, \xi + \lambda)\bar{a}(x, \lambda)\, dx$$

$$- \frac{1}{1 + i\xi}\int_{\Omega_\varepsilon} e^{ix\xi}a(x, \xi + \lambda)\partial_x\bar{a}(x, \lambda)\, dx.$$

Substituting (2.4) into (2.3), we have

(2.5)

$$\int_{\Omega_\varepsilon} |a(x, D) u(x)|^2 \, dx$$

$$= \left(\frac{1}{2\pi}\right)^2 \int_{\Omega_\varepsilon} \int_R \int_R \frac{e^{ix\xi}}{1 + i\xi} a(x, \xi + \lambda) \hat{u}(\xi + \lambda) \bar{a}(x, \lambda) \bar{\hat{u}}(\lambda) \, d\lambda \, d\xi \, dx$$

$$+ \left(\frac{1}{2\pi}\right)^2 \int_R \int_R \frac{e^{i(t-\varepsilon)\xi}}{1 + i\xi} a(t - \varepsilon, \xi + \lambda) \hat{u}(\xi + \lambda) \bar{a}(t - \varepsilon, \lambda) \bar{\hat{u}}(\lambda) \, d\lambda \, d\xi$$

$$- \left(\frac{1}{2\pi}\right)^2 \int_R \int_R \frac{e^{i(s+\varepsilon)\xi}}{1 + i\xi} a(s + \varepsilon, \xi + \lambda) \hat{u}(\xi + \lambda) \bar{a}(s + \varepsilon, \lambda) \bar{\hat{u}}(\lambda) \, d\lambda \, d\xi$$

$$- \left(\frac{1}{2\pi}\right)^2 \int_{\Omega_\varepsilon} \int_R \int_R \frac{e^{ix\xi}}{1 + i\xi} \partial_x a(x, \xi + \lambda) \hat{u}(\xi + \lambda) \bar{a}(x, \lambda) \bar{\hat{u}}(\lambda) \, d\lambda \, d\xi \, dx$$

$$- \left(\frac{1}{2\pi}\right)^2 \int_{\Omega_\varepsilon} \int_R \int_R \frac{e^{ix\xi}}{1 + i\xi} a(x, \xi + \lambda) \hat{u}(\xi + \lambda) \partial_x \bar{a}(x, \lambda) \bar{\hat{u}}(\lambda) \, d\lambda \, d\xi \, dx$$

$$= I_1 + I_2 + I_3 + I_4 + I_5.$$

We first estimate I_1.

$$(2.6) \quad I_1 = \left(\frac{1}{2\pi}\right)^2 \int_{\Omega_\varepsilon} \int_R \int_R \frac{e^{ix\xi}}{1 + i\xi} a(x, \xi + \lambda) \hat{u}(\xi + \lambda) \bar{a}(x, \lambda) \bar{\hat{u}}(\lambda) \, d\lambda \, d\xi \, dx.$$

For x fixed we study the following integral:

$$J_1 = \int_R \int_R \frac{e^{ix\xi}}{1 + i\xi} a(x, \xi + \lambda) \hat{u}(\xi + \lambda) \bar{a}(x, \lambda) \bar{\hat{u}}(\lambda) \, d\lambda \, d\xi$$

$$= \int_R g(\xi) h(\xi) \, d\xi,$$

where $g(\xi) = e^{ix\xi}/(1 + i\xi)$ and

$$h(\xi) = \int_R a(x, \xi + \lambda) \hat{u}(\xi + \lambda) \bar{a}(x, \lambda) \bar{\hat{u}}(\lambda) \, d\lambda.$$

Looking at the expression of $h(\xi)$ and taking into account that $u \in C_0^\infty(R)$ and $\|a(x, \cdot)\|_{L^\infty(R)} \le \||a\||$, we see that h is a kind of convolution of two functions both belonging to $L^1(R) \cap L^2(R)$. Therefore we have

(2.7) $$h \text{ is in } L^2(R),$$

and

(2.8) There is an absolute constant $C_1 > 0$ such that

$$\|\hat{h}\|_{L^1(R)} \le C_1 \|a(x, \cdot)\|_{L^\infty(R)}^2 \|u\|_{L^2(R)}^2 \le C_1 \||a\||^2 \|u\|_{L^2(R)}^2.$$

Obviously, g is in $L^2(R)$. Making use of Parseval's formula, (2.7), (2.8) and Lemma 2.1, we have

$$|J_1| = \left| \int_R g(\xi)h(\xi)\,d\xi \right| = \frac{1}{2\pi}\left| \int_R \hat{g}(\alpha)\overline{\hat{h}}(\alpha)\,d\alpha \right|$$

$$\leqslant \frac{1}{2\pi}\|\hat{g}\|_{L^\infty(R)}\|\hat{\overline{h}}\|_{L^1(R)}$$

$$\leqslant C_2\||a\||^2\|u\|_{L^2(R)}^2 \quad \text{for some absolute constant } C_2.$$

Now, going back to (2.6), we obtain

$$(2.9) \qquad |I_1| \leqslant C_2(t - s - 2\varepsilon)\||a\||^2\|u\|_{L^2(R)}^2.$$

By similar arguments, we have the following estimates:

$$(2.10) \qquad |I_2| \leqslant C_2\||a\||^2\|u\|_{L^2(R)}^2,$$

$$|I_3| \leqslant C_2\||a\||^2\|u\|_{L^2(R)}^2,$$

$$|I_4| \leqslant C_2(t - s - 2\varepsilon)\||a\||^2\|u\|_{L^2(R)}^2$$

and

$$|I_5| \leqslant C_2(t - s - 2\varepsilon)\||a\||^2\|u\|_{L^2(R)}^2.$$

Thus, it follows from (2.5), (2.9) and (2.10) that

$$\|a(x, D)u\|_{L^2(\Omega_\varepsilon)} \leqslant C_3(1 + t - s - 2\varepsilon)^{1/2}\||a\||\,\|u\|_{L^2(R)} \quad \forall u \in C_0^\infty(R),$$

where C_3 is an absolute constant. $\square$

More generally, we have

PROPOSITION 2.2. *Let $\Omega = \prod_{j=1}^n (s_j, t_j)$ be a bounded open box in R^n. Suppose a: $\Omega \times R^n \to \mathbf{C}$ is a continuous function whose derivatives $\partial_x^\alpha a$ satisfy the following conditions*:

$$(2.11) \qquad \partial_x^\alpha a \in C^0(\mathring{\Omega} \times R^n), \qquad \alpha = (\alpha_1, \ldots, \alpha_n), \alpha_j = 0 \text{ or } 1.$$

(2.12) *There is a constant $C > 0$ such that*

$$\|\partial_x^\alpha a\|_{L^\infty(\Omega \times R^n)} \leqslant C, \qquad \alpha = (\alpha_1, \ldots, \alpha_n), \alpha_j = 0 \text{ or } 1.$$

Then $a(x, D)$ is continuous from $L^2(R^n)$ to $L^2(\Omega)$ with its norm bounded by $C_n(1 + \max_{1 \leqslant j \leqslant n}(t_j - s_j))^{n/2}\||a\||$, where C_n is a constant depending only on n, and $\||a\||$ is the smallest C such that (2.12) holds.

PROOF. The proposition follows from Lemma 2.2 and an argument similar to the proof of Proposition 2.1. $\square$

Now, we are ready to prove Theorem 1.

PROOF OF THEOREM 1. We shall approach the symbol a by smooth symbols. Then we shall apply Proposition 2.2.

Let φ be in $C_0^\infty(R)$ such that the following conditions hold:

$$\varphi \geqslant 0, \quad \text{supp } \varphi \subseteq \{-1 \leqslant y \leqslant 1\} \quad \text{and} \quad \|\varphi\|_{L^1(R)} = 1.$$

If $0 < \varepsilon < \frac{1}{8}\min_{1 \le j \le n}(t_j - s_j)$, then we define $a_\varepsilon = \lambda_\varepsilon * a$, where

$$(2.13) \qquad \lambda_\varepsilon(x, \xi) = \frac{1}{\varepsilon^{2n}} \prod_{j=1}^{n} \varphi\left(\frac{x_j}{\varepsilon}\right)\varphi\left(\frac{\xi_j}{\varepsilon}\right),$$

with $x = (x_1, \ldots, x_n)$, $\xi = (\xi_1, \ldots, \xi_n) \in R^n$. We see that $a_\varepsilon \in C^\infty(R^n \times R^n)$ and $\lim_{\varepsilon \to 0} a_\varepsilon(x, \xi) = a(x, \xi)$ a.e. in $\Omega \times R^n$. Moreover, hypothesis (1.1) gives

$$(2.14) \qquad \|\partial_x^\alpha a_\varepsilon\|_{L^\infty(\Omega_{\varepsilon_0} \times R^n)} \le \|\|a\|\|,$$

where $0 < \varepsilon \le \varepsilon_0$, $\Omega_{\varepsilon_0} = \prod_{j=1}^{n}(s_j + 2_{\varepsilon_0}, t_j - 2\varepsilon_0)$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in N^n$ with $\alpha_j = 0$ or 1. By applying Proposition 2.2 to $a_\varepsilon(x, D)u$ with $u \in C_0^\infty(R^n)$, we get for $0 < \varepsilon \le \varepsilon_0$

$$(2.15) \quad \|a_\varepsilon(x, D)u\|_{L^2(\Omega_{\varepsilon_0})} \le C_n\left(1 - 4_{\varepsilon_0} + \max_{1 \le j \le n}(t_j - s_j)\right)^{n/2}\|\|a\|\|\,\|u\|_{L^2(R^n)}.$$

We now write

$$\|a_\varepsilon(x, D)u\|_{L^2(\Omega_{\varepsilon_0})}^2$$
$$= \left(\frac{1}{2\pi}\right)^{2n} \int_{\Omega_{\varepsilon_0}} \int_{R^n} \int_{R^n} e^{ix(\xi - \lambda)} a_\varepsilon(x, \xi)\hat{u}(\xi)\bar{a}_\varepsilon(x, \lambda)\bar{\hat{u}}(\lambda)\, d\lambda\, d\xi\, dx.$$

Then by the facts $\lim_{\varepsilon \to 0} a_\varepsilon = a$ a.e. in $\Omega \times R^n$, (2.14), $\hat{u} \in S(R^n)$, $|\Omega_{\varepsilon_0}| < \infty$, and the Lebesgue dominated convergence theorem, we obtain

$$\lim_{\varepsilon \to 0} \|a_\varepsilon(x, D)u\|_{L^2(\Omega_{\varepsilon_0})}^2$$
$$= \left(\frac{1}{2\pi}\right)^{2n} \lim_{\varepsilon \to 0} \int_{\Omega_{\varepsilon_0}} \int_{R^n} \int_{R^n} e^{ix(\xi - \lambda)} a_\varepsilon(x, \xi)\hat{u}(\xi)\bar{a}_\varepsilon(x, \lambda)\bar{\hat{u}}(\lambda)\, d\lambda\, d\xi\, dx$$
$$= \left(\frac{1}{2\pi}\right)^{2n} \int_{\Omega_{\varepsilon_0}} \int_{R^n} \int_{R^n} e^{ix(\xi - \lambda)} a(x, \xi)\hat{u}(\xi)\bar{a}(x, \lambda)\bar{\hat{u}}(\lambda)\, d\lambda\, d\xi\, dx$$
$$= \|a(x, D)u\|_{L^2(\Omega_{\varepsilon_0})}^2.$$

This and (2.15) imply

$$\|a(x, D)u\|_{L^2(\Omega_{\varepsilon_0})} \le C_n\left(1 - 4\varepsilon_0 + \max_{1 \le j \le n}(t_j - s_j)\right)^{n/2}\|\|a\|\|\,\|u\|_{L^2(R^n)}.$$

Then Theorem 1 follows by letting $\varepsilon_0 \to 0$. $\square$

PROOF OF COROLLARY 1.1. Since any compact subset of Ω can be covered by a union of finite open boxes in Ω, Corollary 1.1 follows from Theorem 1. $\square$

3. Proof of Theorem 2.

To prove Theorem 2, we need the following crucial lemma which is related to Wigner functions.

LEMMA 3.1. *For $u, \varphi \in L^2(R^n)$ we define*

$$g(x, \xi) = \int_{R^n} e^{-iy\xi}\varphi(x - y)u(y)\, dy,$$

and

$$h(x, \xi) = \int_{R^n} e^{ixy}\varphi(\xi + y)u(y)\, dy, \qquad (x, \xi) \in R^n \times R^n.$$

Then we have

$$\|g\|_{L^2(R^n \times R^n)} = \|h\|_{L^2(R^n \times R^n)} = (2\pi)^{n/2}\|\varphi\|_{L^2(R^n)}\|u\|_{L^2(R^n)}.$$

PROOF. Without loss of generality, we can assume that u, $\varphi \in C_0^\infty(R^n)$. Making use of Parseval's formula, we have

$$\int_{R^n} \int_{R^n} |g(x, \xi)|^2\, dx\, d\xi = \left(\frac{1}{2\pi}\right)^n \int_{R^n} \int_{R^n} \left|\widehat{g(\cdot, \xi)}(\alpha)\right|^2\, d\alpha\, d\xi.$$

By using the identity $\widehat{g(\cdot, \xi)}(\alpha) = \hat{\varphi}(\alpha)\hat{u}(\alpha + \xi)$ and Fubini's theorem, we obtain

$$\|g\|_{L^2(R^n \times R^n)} = (2\pi)^{n/2}\|\varphi\|_{L^2(R^n)}\|u\|_{L^2(R^n)}.$$

The second equality follows from a similar argument as above and by integrating with respect to ξ first instead of x. □

PROOF OF THEOREM 2. Without loss of generality, we can assume that the symbol a is in $C_0^\infty(R^n \times R^n)$ and its derivatives $\partial_x^\alpha \partial_\xi^\beta a$ satisfy (1.3). Then for $u \in C_0^\infty(R^n)$ and $x \in R^n$, we have

$$(3.1) \qquad a(x, D)u(x) = \left(\frac{1}{2\pi}\right)^n \int_{R^n} \int_{R^n} e^{i(x-y)\xi}a(x, \xi)u(y)\, dy\, d\xi.$$

Making use of the identity

$$\left(\prod_{s=1}^n \frac{1}{1 + i(x_s - y_s)}\right)\left(\prod_{s=1}^n (1 + \partial_{\xi_s})\right)(e^{i(x-y)\xi}) = e^{i(x-y)\xi},$$

we obtain

$$(3.2) \quad \int_{R^n} e^{i(x-y)\xi}a(x, \xi)\, d\xi = \int_{R^n} e^{i(x-y)\xi}b(x, \xi)\left(\prod_{s=1}^n \frac{1}{1 + i(x_s - y_s)}\right)\, d\xi,$$

where $b(x, \xi) = (\prod_{s=1}^n (1 - \partial_{\xi_s}))(a(x, \xi))$. Substituting (3.2) into (3.1), we have the following crucial equality:

$$(3.3) \quad a(x, D)u(x)$$

$$= \left(\frac{1}{2\pi}\right)^n \int_{R^n} \int_{R^n} e^{i(x-y)\xi}b(x, \xi)\left(\prod_{s=1}^n \frac{1}{1 + i(x_s - y_s)}\right)u(y)\, dy\, d\xi.$$

Let $v \in C_0^\infty(R^n)$. By (3.3) and

$$v(x) = \left(\frac{1}{2\pi}\right)^n \int_{R^n} e^{ix\lambda}\hat{v}(\lambda)\, d\lambda,$$

we have

(3.4)

$$\int_{R^n} a(x, D) u(x) v(x) \, dx$$

$$= \left(\frac{1}{2\pi}\right)^{2n} \int_{R^n} \int_{R^n} \int_{R^n} \int_{R^n} e^{i(x-y)\xi} e^{ix\lambda} b(x, \xi)$$

$$\times \left(\prod_{s=1}^{n} \frac{1}{1 + i(x_s - y_s)}\right) u(y) \hat{v}(\lambda) \, dy \, d\xi \, d\lambda \, dx$$

$$= \left(\frac{1}{2\pi}\right)^{2n} \int_{R^n} \int_{R^n} \int_{R^n} \left(\int_{R^n} \times e^{ix(\xi+\lambda)} b(x, \xi)\right.$$

$$\left. \times \left(\prod_{s=1}^{n} \frac{1}{1 + i(x_s - y_s)}\right) dx\right) e^{-iy\xi} u(y) \hat{v}(\lambda) \, dy \, d\xi \, d\lambda.$$

Integrating with respect to x first and making use of the identity

$$\left(\prod_{s=1}^{n} \frac{1}{1 + i(\xi_s + \lambda_s)}\right)\left(\prod_{s=1}^{n} (1 + \partial_x)\right)\left(e^{ix(\xi+\lambda)}\right) = e^{ix(\xi+\lambda)},$$

we get

(3.5) $$\int_{R^n} e^{ix(\xi+\lambda)} b(x, \xi)\left(\prod_{s=1}^{n} \frac{1}{1 + i(x_s - y_s)}\right) dx$$

$$= \left(\prod_{s=1}^{n} \frac{1}{1 + i(\xi_s + \lambda_s)}\right)$$

$$\times \int_{R^n} e^{ix(\xi+\lambda)}\left(\prod_{s=1}^{n} (1 - \partial_{x_s})\right)\left(b(x, \xi)\left(\prod_{s=1}^{n} \frac{1}{1 + i(x_s - y_s)}\right)\right) dx.$$

Let $T = \{\alpha \in N^n \mid \alpha = (\alpha_1, \ldots, \alpha_n)$ and $\alpha_j = 0$ or $1\}$. Then we have

(3.6) $$\left(\prod_{s=1}^{n} (1 - \partial_{x_s})\right)\left(b(x, \xi)\left(\prod_{s=1}^{n} \frac{1}{1 + i(x_s - y_s)}\right)\right)$$

$$= \sum_{\alpha \in T} (-1)^{|\alpha|} \partial_x^\alpha\left(b(x, \xi)\left(\prod_{s=1}^{n} \frac{1}{1 + i(x_s - y_s)}\right)\right)$$

$$= \sum_{\alpha \in T} (-1)^{|\alpha|} \sum_{\beta \leq \alpha} \partial_x^{\alpha-\beta} b(x, \xi) \partial_x^\beta\left(\prod_{s=1}^{n} \frac{1}{1 + i(x_s - y_s)}\right).$$

Substituting (3.6) into (3.5), we see that

(3.7) $$\int_{R^n} e^{ix(\xi+\lambda)} b(x, \xi)\left(\prod_{s=1}^{n} \frac{1}{1 + i(x_s - y_s)}\right) dx$$

$$= \sum_{\alpha \in T} \sum_{\beta \leq \alpha} (-1)^{|\alpha|}\left(\prod_{s=1}^{n} \frac{1}{1 + i(\xi_s + \lambda_s)}\right)$$

$$\times \int_{R^n} e^{ix(\xi+\lambda)} \partial_x^{\alpha-\beta} b(x, \xi) \partial_x^\beta\left(\prod_{s=1}^{n} \frac{1}{1 + i(x_s - y_s)}\right) dx.$$

We derive from (3.4) and (3.7) that

$$(3.8) \quad \int_{R^n} a(x,D)u(x)v(x)\,dx$$

$$= \sum_{\alpha \in T} \sum_{\beta \leqslant \alpha} (-1)^{|\alpha|} \left(\frac{1}{2\pi}\right)^{2n} \int_{R^n} \int_{R^n} \partial_x^{\alpha-\beta} b(x,\xi) g_\beta(x,\xi) h(x,\xi)\,dx\,d\xi,$$

where

$$g_\beta(x,\xi) = \int_{R^n} e^{i(x-y)\xi} \partial_x^\beta \left(\prod_{s=1}^n \frac{1}{1 + i(x_s - y_s)}\right) u(y)\,dy,$$

and

$$h(x,\xi) = \int_{R^n} e^{ix\lambda} \left(\prod_{s=1}^n \frac{1}{1 + i(\xi_s + \lambda_s)}\right) \hat{v}(\lambda)\,d\lambda.$$

Applying Lemma (3.1) to g_β, we get

$$(3.9) \quad \|g_\beta\|_{L^2(R^n \times R^n)} = (2\pi)^{n/2} \left(\int_{R^n} \left|\partial_x^\beta \left(\prod_{s=1}^n \frac{1}{1 + ix_s}\right)\right|^2 dx\right)^{1/2} \|u\|_{L^2(R^n)}$$

$$\leqslant (2\pi)^{n/2} \pi^{n/2} \|u\|_{L^2(R^n)}.$$

Similarly, we have

$$(3.10) \quad \|h\|_{L^2(R^n \times R^n)} = (2\pi)^{n/2} \left(\int_{R^n} \prod_{s=1}^n \frac{1}{1 + \xi_s^2}\,d\xi\right)^{1/2} \|\hat{v}\|_{L^2(R^n)}$$

$$= (2\pi)^n \pi^{n/2} \|v\|_{L^2(R^n)}.$$

From (1.3), we see that

$$(3.11) \quad \|\partial_x^{\alpha-\beta} b\|_{L^\infty(R^n \times R^n)} \leqslant 2^n \|a\|.$$

Thus, we derive from (3.8), (3.9), (3.10) and (3.11) that

$$\left|\int_{R^n} a(x,D)u(x)v(x)\,dx\right| \leqslant C_n \|a\| \|u\|_{L^2(R^n)} \|v\|_{L^2(R^n)},$$

where $u, v \in C_0^\infty(R^n)$ and C_n is a constant depending only on n. $\square$

PROOF OF COROLLARY 2.1. Without loss of generality, we can assume that the symbol a has compact support in $R^n \times R^n$ and that its derivatives $\partial_\xi^\beta a$ satisfy (1.4). Then we approach the symbol a by the smooth symbols a_ε defined by $a_\varepsilon = \lambda_\varepsilon * a$, where $0 < \varepsilon < 1$ and λ_ε is as in (2.13). By the definition of a_ε and taking into account that $a \in L^2(R^n \times R^n)$, we have the following properties:

$$(3.12) \quad a_\varepsilon \in C_0^\infty(R^n \times R^n),$$

$$(3.13) \quad \lim_{\varepsilon \to 0} \|a_\varepsilon - a\|_{L^2(R^n \times R^n)} = 0$$

and

$$(3.14) \quad \sup_{x \in R^n} \left(\int \left| \partial_\xi^\beta a_\varepsilon(x, \xi) \right|^2 d\xi \right)^{1/2} \leqslant |||a|||, \qquad \beta = (\beta_1, \ldots, \beta_n), \ \beta_j = 0 \text{ or } 1.$$

Let $u \in C_0^\infty(R^n)$ and $x \in R^n$. From (3.3), we have

$$(3.15) \quad a_\varepsilon(x, D)u(x)$$

$$= \left(\frac{1}{2\pi} \right)^n \int_{R^n} \int_{R^n} e^{i(x-y)\xi} b_\varepsilon(x, \xi) \left(\prod_{s=1}^n \frac{1}{1 + i(x_s - y_s)} \right) u(y) \, dy \, d\xi,$$

where

$$b_\varepsilon(x, \xi) = \left(\prod_{s=1}^n \left(1 - \partial_{\xi_s} \right) \right) (a_\varepsilon(x, \xi)).$$

Let $v \in C_0^\infty(R^n)$. Making use of (3.15), we write

$$(3.16) \quad \int_{R^n} a_\varepsilon(x, D)u(x)v(x) \, dx$$

$$= \left(\frac{1}{2\pi} \right)^n \int_{R^n} \int_{R^n} b_\varepsilon(x, \xi)v(x)$$

$$\times \left(\int_{R^n} e^{i(x-y)\xi} \left(\prod_{s=1}^n \frac{1}{1 + i(x_s - y_s)} \right) u(y) \, dy \right) dx \, d\xi$$

$$= \left(\frac{1}{2\pi} \right)^n \int_{R^n} \int_{R^n} g(x, \xi)h(x, \xi) \, dx \, d\xi,$$

where $g(x, \xi) = b_\varepsilon(x, \xi)v(x)$, and

$$h(x, \xi) = \int_{R^n} e^{i(x-y)\xi} \left(\prod_{s=1}^n \frac{1}{1 + i(x_s - y_s)} \right) u(y) \, dy.$$

By (3.14) and

$$b_\varepsilon(x, \xi) = \left(\prod_{s=1}^n \left(1 - \partial_{\xi_s} \right) \right) (a_\varepsilon(x, \xi)),$$

we see that

$$(3.17) \quad \| g \|_{L^2(R^n \times R^n)} \leqslant 2^n |||a||| \, \| v \|_{L^2(R^n)}.$$

According to Lemma (3.1), we have

$$(3.18) \quad \| h \|_{L^2(R^n \times R^n)} = (2\pi)^{n/2} \left(\int_{R^n} \prod_{s=1}^n \frac{1}{1 + x_s^2} \, dx \right)^{1/2} \| u \|_{L^2(R^n)}$$

$$\leqslant (2\pi)^n \| u \|_{L^2(R^n)}.$$

Therefore, we derive from (3.16), (3.17) and (3.18) that

$$(3.19) \quad \left| \int_{R^n} a_\varepsilon(x, D)u(x)v(x) \, dx \right| \leqslant 2^n |||a||| \, \| u \|_{L^2(R^n)} \| v \|_{L^2(R^n)}$$

$$\forall u, v \in C_0^\infty(R^n).$$

Now, we are ready to prove the L^2-boundedness of $a(x, D)$. For $u, v \in C_0^\infty(R^n)$, we write

$$\int_{R^n} a_\varepsilon(x, D) u(x) v(x)\, dx = \left(\frac{1}{2\pi}\right)^n \int_{R^n} \int_{R^n} e^{ix\xi} a_\varepsilon(x, \xi) \hat{u}(\xi) v(x)\, dx\, d\xi.$$

Making use of (3.13), we see that

$$(3.20) \quad \lim_{\varepsilon \to 0} \int_{R^n} \int_{R^n} e^{ix\xi} a_\varepsilon(x, \xi) \hat{u}(\xi) v(x)\, dx\, d\xi$$

$$= \int_{R^n} \int_{R^n} e^{ix\xi} a(x, \xi) \hat{u}(\xi) v(x)\, dx\, d\xi, \qquad u, v \in C_0^\infty(R^n).$$

Therefore we derive from (3.19) and (3.20) that

$$\left| \int_{R^n} a(x, D) u(x) v(x)\, dx \right| \leq 2^n \|\|a\|\| \|u\|_{L^2(R^n)} \|v\|_{L^2(R^n)} \quad \forall u, v \in C_0^\infty(R^n).$$

Thus

$$\|a(x, D)u\|_{L^2(R^n)} \leq 2^n \|\|a\|\| \|u\|_{L^2(R^n)}, \qquad u \in C_0^\infty(R^n). \quad \square$$

PROOF OF COROLLARY 2.2. Without loss of generality, we can assume that the symbol $a \in C_0^\infty(R^n \times R^n)$ and its derivatives $\partial_x^\alpha a$ satisfy (1.5). If $u, v \in C_0^\infty(R^n)$, we have

$$(3.21) \quad \int_{R^n} a(x, D) u(x) v(x)\, dx = \left(\frac{1}{2\pi}\right)^n \int_{R^n} \left(\int_{R^n} e^{ix\xi} a(x, \xi) \hat{u}(\xi)\, d\xi \right) v(x)\, dx$$

$$= \left(\frac{1}{2\pi}\right)^n \int_{R^n} \left(\int_{R^n} e^{ix\xi} a(x, \xi) v(x)\, dx \right) \hat{u}(\xi)\, d\xi$$

$$= \int_{R^n} g(\xi) \hat{u}(\xi)\, d\xi,$$

where

$$g(\xi) = \left(\frac{1}{2\pi}\right)^n \int_{R^n} e^{ix\xi} a(x, \xi) v(x)\, dx.$$

It follows from Corollary (2.1) that

$$(3.22) \quad \|g\|_{L^2(R^n)} \leq (2\pi)^{-n} 2^n \|\|a\|\| \|\hat{v}\|_{L^2(R^n)} = (2\pi)^{-n/2} 2^n \|\|a\|\| \|v\|_{L^2(R^n)}.$$

From (3.21) and (3.22), we have

$$\left| \int_{R^n} a(x, D) u(x) v(x)\, dx \right| \leq \|g\|_{L^2(R^n)} \|\hat{u}\|_{L^2(R^n)} \leq 2^n \|\|a\|\| \|u\|_{L^2(R^n)} \|v\|_{L^2(R^n)}.$$

Thus

$$\|a(x, D)u\|_{L^2(R^n)} \leq 2^n \|\|a\|\| \|u\|_{L^2(R^n)}, \qquad u \in C_0^\infty(R^n). \quad \square$$

4. Proof of Theorem 3. The case $\rho = 0$ has been proved in Theorem 2. It remains to prove the theorem for the cases $-\infty < \rho < 1$, $\rho \neq 0$. Throughout this section, we denote a constant depending only on ρ and n by $C_{\rho, n}$, which may vary from time to time. We may assume that a is in $C_0^\infty(R^n \times R^n)$ and its derivatives $\partial_x^\alpha \partial_\xi^\beta a$ satisfy

(1.6) for some $-\infty < \rho < 1$, $\rho \neq 0$. Making use of the partition of unity in [2] i.e. there exist $\tilde{\varphi}_0$, $\varphi \in C_0^\infty(R^n)$ such that

$$\operatorname{supp} \varphi \subseteq \left\{ \frac{1}{3} \leqslant |\xi| \leqslant 1 \right\} \quad \text{and} \quad \tilde{\varphi}_0 + \sum_{j=0}^\infty \varphi_j \equiv 1 \quad \text{with } \varphi_j(\xi) = \varphi(2^{-j}\xi), \ \xi \in R^n,$$

we write

$$a(x, \xi) = \tilde{\varphi}_0(\xi) a(x, \xi) + \sum_{j=0}^\infty \varphi_j(\xi) a(x, \xi)$$

$$= b_0(x, \xi) + \sum_{j=0}^\infty a_j(x, \xi), \qquad (x, \xi) \in R^n \times R^n.$$

Then

$$a(x, D)u = b_0(x, D)u + \sum_{j=0}^\infty a_j(x, D)u, \qquad u \in C_0^\infty(R^n).$$

It follows from Corollary 2.1 that $b_0(x, D)$ and $a_0(x, D)$ are continuous from $L^2(R^n)$ to $L^2(R^n)$ with their norms bounded by $C_{\rho,n}|\!|\!|a|\!|\!|$. Therefore we shall concentrate on estimating $\sum_{j=0}^\infty a_j(x, D)u$. Since the symbol a is in $C_0^\infty(R^n \times R^n)$, we obtain

$$\left(\sum_{j=1}^\infty a_j(x, D)u, \ \sum_{j=1}^\infty a_j(x, D)u \right) = \sum_{j,k=1}^\infty \left(a_j(x, D)u, a_k(x, D)u \right),$$

$$u \in C_0^\infty(R^n).$$

We shall give several lemmas to study the behavior of $(a_j(x, D)u, a_k(x, D)u)$. Then we shall see that Theorem 3 follows from these lemmas.

LEMMA 4.1. *Let $j \in N - \{0\}$ and $E_j = \{\xi \in R^n \,|\, 2^j/3 \leqslant |\xi| \leqslant 2^j\}$. Then*

$$\left\| a_j(x, D)u \right\|_{L^2(R^n)} \leqslant C_{\rho,n}|\!|\!|a|\!|\!| \, \|\hat{u}\|_{L^2(E_j)}, \qquad u \in C_0^\infty(R^n).$$

PROOF. Since $\operatorname{supp} a(x, \cdot) \subseteq E_j$ and

$$a_j(x, D)u(x) = \left(\frac{1}{2\pi} \right)^n \int_{R^n} e^{ix\xi} a(x, \xi) \hat{u}(\xi) \, d\xi, \qquad x \in R^n, \ u \in C_0^\infty(R^n),$$

it suffices to show the following property:

(4.1) $\left\| a_j(x, D)u \right\|_{L^2(R^n)} \leqslant C_{\rho,n}|\!|\!|a|\!|\!| \, \|\hat{u}\|_{L^2(E_j)}, \qquad \hat{u} \in C_0^\infty(E_j).$

Let $v \in C_0^\infty(R^n)$. Then

$$\int_{R^n} a_j(x, D)u(x) v(x) \, dx = \left(\frac{1}{2\pi} \right)^n \int_{R^n} \int_{R^n} e^{ix\xi} a_j(x, \xi) \hat{u}(\xi) v(x) \, d\xi \, dx$$

$$= \left(\frac{1}{2\pi} \right)^n \int_{R^n} \int_{R^n} e^{ix\xi} a_j(2^{-j\rho}x, 2^{j\rho}\xi) \hat{u}(2^{j\rho}\xi) v(2^{-j\rho}x) \, d\xi \, dx,$$

where the last equality is obtained by the change of variables $x \to 2^{-j\rho}x$, $\xi \to 2^{j\rho}\xi$. By the hypothesis (1.6), we have

$$\left| \partial_x^\alpha \partial_\xi^\beta \left(a_j(2^{-j\rho}x, 2^{j\rho}\xi) \right) \right| \leqslant C_{\rho,n}|\!|\!|a|\!|\!|,$$

where $(x, \xi) \in R^n \times R^n$, $j \geqslant 1$, and $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n) \in N^n$ with

$\alpha_t = 0$ or 1 and $\beta_t = 0$ or 1. So by the proof of Theorem 2, we have

$$\left| \int_{R^n} a_j(x, D) u(x) v(x)\, dx \right| \leqslant C_{\rho, n} \|\|a\|\| \|\hat{u}\|_{L^2(E_j)} \|v\|_{L^2(R^n)},$$

which implies (4.1). $\square$

Lemma 4.1 implies the following lemma for which we omit the proof.

LEMMA 4.2. *If $s \in N - \{0\}$ and $j, k \geqslant 1$, then*

$$\sum_{|j-k| \leqslant s} \left| a_j(x, D) u, a_k(x, D) u \right| \leqslant s^2 C_{\rho, n} \|\|a\|\|^2 \|u\|_{L^2(R^n)}^2 \quad \forall u \in C_0^\infty(R^n).$$

The following lemmas describe the properties of $(a_j(x, D) u, a_k(x, D) u)$, when $|j - k|$ is large.

LEMMA 4.3. *If $0 < \rho < 1$, then we have*

$$\left| (a_j(x, D) u, a_k(x, D) v) \right| \leqslant C_{\rho, n} \|\|a\|\|^2 2^{-j\varepsilon_\rho} 2^{-k\varepsilon_\rho} \|\hat{u}\|_{L^2(E_j)} \|\hat{v}\|_{L^2(E_k)},$$

where $u, v \in C_0^\infty(R^n)$, $j, k \geqslant 1$; $|j - k| \geqslant 4$ and $\varepsilon_\rho = (1 - \rho)/4$.

PROOF. Since $\operatorname{supp} a_j(x, \cdot)$ and $\operatorname{supp} a_k(x, \cdot)$ are contained in E_j and E_k respectively, it suffices to show the following property:

$$(4.2) \quad \left| (a_j(x, D) u, a_k(x, D) v) \right| \leqslant C_{\rho, n} \|\|a\|\|^2 2^{-j\varepsilon_\rho} 2^{-k\varepsilon_\rho} \|\hat{u}\|_{L^2(E_j)} \|\hat{v}\|_{L^2(E_k)},$$

where $\hat{u} \in C_0^\infty(E_j)$, $\hat{v} \in C_0^\infty(E_k)$, $j, k \geqslant 1$, $|j - k| \geqslant 4$ and $\varepsilon_\rho = (1 - \rho)/4$. Without loss of generality, we can assume that $j \geqslant k + 4$. Then we write

$$(4.3) \quad (a_j(x, D) u, a_k(x, D) v)$$

$$= \left(\frac{1}{2\pi} \right)^{2n} \int_{R^n} \int_{R^n} \int_{R^n} e^{ix(\xi - \lambda)} a_j(x, \xi) \hat{u}(\xi) \bar{a}_k(x, \lambda) \bar{v}(\lambda)\, d\lambda\, d\xi\, dx$$

$$= \left(\frac{1}{2\pi} \right)^{2n} 2^{j\rho n} \int \int \int e^{ix(\xi - \lambda)} a_j(2^{-j\rho} x, 2^{j\rho} \xi) \hat{u}(2^{j\rho} \xi)$$

$$\times \bar{a}_k(2^{-j\rho} x, 2^{j\rho} \xi) \bar{v}(2^{j\rho} \lambda)\, d\lambda\, d\xi\, dx,$$

where the last equality comes from the change of variables $x \to 2^{-j\rho} x$, $\xi \to 2^{j\rho} \xi$ and $\lambda \to 2^{j\rho} \lambda$.

In the last integral, we integrate with respect to x first. By using the following identity:

$$\left(\prod_{s=1}^{n} \frac{1}{1 + i(\xi_s - \lambda_s)} \right) \left(\prod_{s=1}^{n} (1 + \partial_{x_s}) \right) (e^{ix(\xi - \lambda)}) = e^{ix(\xi - \lambda)},$$

we write

(4.4).

$$\int_{R^n} e^{ix(\xi - \lambda)} a_j(2^{-j\rho} x, 2^{j\rho} \xi) \bar{a}_k(2^{-j\rho} x, 2^{j\rho} \lambda)\, dx$$

$$= \left(\prod_{s=1}^{n} \frac{1}{1 + i(\xi_s - \lambda_s)} \right)$$

$$\times \sum_{\alpha \in T} \sum_{\beta \leqslant \alpha} (-1)^{|\alpha|} \int_{R^n} e^{ix(\xi - \lambda)} \partial_x^{\alpha - \beta} \left(a_j(2^{-j\rho} x, 2^{j\rho} \xi) \right) \partial_x^{\beta} \left(a_k(2^{-j\rho} x, 2^{j\rho} \lambda) \right)\, dx,$$

where $T = \{\alpha \in N^n \mid \alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_t = 0$ or $1\}$. Substituting (4.4) into (4.3), we obtain

$$(4.5) \qquad \big(a_j(x, D)u, a_k(x, D)v\big) = \left(\frac{1}{2\pi}\right)^{2n} 2^{j\rho n} \sum_{\alpha \in T} \sum_{\beta \leqslant \alpha} (-1)^{|\alpha|} I_{\alpha, \beta},$$

where

$$(4.6) \quad I_{\alpha, \beta} = \int_{R^n} \int_{R^n} \int_{R^n} e^{ix(\xi - \lambda)} \partial_x^{\alpha - \beta}\big(a_j(2^{-j\rho}x, 2^{j\rho}\xi)\big) \partial_x^{\beta}\big(\bar{a}_k(2^{-j\rho}x, 2^{j\rho}\lambda)\big)$$

$$\times \left(\prod_{s=1}^{n} \frac{1}{1 + i(\xi_s - \lambda_s)}\right) \hat{u}(2^{j\rho}\xi) \bar{\hat{v}}(2^{j\rho}\lambda) \, d\lambda \, d\xi \, dx.$$

Let $\hat{g}(\xi) = \hat{u}(2^{j\rho}\xi)$, $\xi \in R^n$. Then

$$(4.7) \qquad\qquad \hat{g}(\xi) = \int_{R^n} e^{-iy\xi} g(y) \, dy.$$

Substituting (4.7) into (4.6), integrating with respect to ξ first and making use of the following identity:

$$\left(\prod_{s=1}^{n} \frac{1}{1 + i(x_s - y_s)}\right)\left(\prod_{s=1}^{n} \left(1 + \partial_{\xi_s}\right)\right)\left(e^{i(x-y)\xi}\right) = e^{i(x-y)\xi},$$

We obtain

$$(4.8) \qquad\qquad I_{\alpha, \beta} = \sum_{\gamma \in T} \sum_{\delta < \gamma} (-1)^{|\gamma|} J_{\alpha, \beta, \gamma, \delta},$$

where

$$(4.9) \quad J_{\alpha, \beta, \gamma, \delta} = \int_{R^n} \int_{R^n} \int_{R^n} \int_{R^n} e^{i(x-y)\xi} \partial_{\xi}^{\gamma - \delta} \partial_x^{\alpha - \beta}\big(a_j(2^{-j\rho}x, 2^{j\rho}\xi)\big)$$

$$\times \partial_{\xi}^{\delta}\left(\prod_{s=1}^{n} \frac{1}{1 + i(\xi_s - \lambda_s)}\right)\left(\prod_{s=1}^{n} \frac{1}{1 + i(x_s - y_s)}\right) g(y) e^{-ix\lambda}$$

$$\times \partial_x^{\beta}\big(\bar{a}_k(2^{-j\rho}x, 2^{j\rho}\lambda)\big) \bar{\hat{v}}(2^{j\rho}\lambda) \, dy \, d\lambda \, d\xi \, dx$$

$$= \int_{R^n} \int_{R^n} \partial_{\xi}^{\gamma - \delta} \partial_x^{\alpha - \beta}\big(a_j(2^{-j\rho}x, 2^{j\rho}\xi)\big) G(x, \xi) H_{\beta, \delta}(x, \xi) \, dx \, d\xi,$$

with

$$(4.10) \qquad G(x, \xi) = \int_{R^n} e^{i(x-y)\xi} \left(\prod_{s=1}^{n} \frac{1}{1 + i(x_s - y_s)}\right) g(y) \, dy,$$

and

$$(4.11) \quad H_{\beta, \delta}(x, \xi) = \int_{R^n} e^{-ix\lambda} \partial_x^{\beta}\big(\bar{a}_k(2^{-j\rho}x, 2^{j\rho}\lambda)\big)$$

$$\times \partial_{\xi}^{\beta}\left(\prod_{s=1}^{n} \frac{1}{1 + i(\xi_s - \lambda_s)}\right) \bar{\hat{v}}(2^{j\rho}\lambda) \, d\lambda.$$

By (4.5) and (4.8) we have

$$\left(a_j(x,D)u, a_k(x,D)v\right) = \left(\frac{1}{2\pi}\right)^{2n} 2^{j\rho n} \sum_{\alpha \in T} \sum_{\gamma \in T} \sum_{\beta \leqslant \alpha} \sum_{\delta \leqslant \gamma} (-1)^{|\alpha|+|\gamma|} J_{\alpha,\beta,\gamma,\delta}.$$

Therefore, (4.2) follows by showing that

(4.12) $$\left|J_{\alpha,\beta,\gamma,\delta}\right| \leqslant 2^{-j\rho n} C_{\rho,n} \||a\||^2 2^{-j\varepsilon_\rho} 2^{-k\varepsilon_\rho} \|\hat{u}\|_{L^2(E_j)} \|\hat{v}\|_{L^2(E_k)},$$

where $\alpha, \gamma \in T$, $\beta \leqslant \alpha$, $\delta \leqslant \gamma$ and $\varepsilon_\rho = (1-\rho)/4$. By Lemma 3.1 and (4.7), we get

(4.13) $$\|G\|_{L^2(R^n \times R^n)} = \pi^{n/2} 2^{-j\rho n/2} \|\hat{u}\|_{L^2(E_j)}.$$

By the hypothesis (1.6), we have

(4.14) $$\left|\partial_\xi^{\gamma-\delta} \partial_x^{\alpha-\beta}\left(a_j(2^{-j\rho}x, 2^{j\rho}\xi)\right)\right| \leqslant C_{\rho,n} \||a\||$$

for $(x,\xi) \in R^n \times R^n$, $\alpha, \beta, \gamma, \delta \in T$ and $\beta \leqslant \alpha$, $\delta \leqslant \gamma$.

It remains to estimate $\|H_{\beta,\delta}\|_{L^2(R^n \times R^n)}$. (4.11) can be written as

$$H_{\beta,\delta}(x,\xi) = \int_{R^n} e^{-ix\lambda} 2^{-j\rho|\beta|} \left(\partial_x^\beta \bar{a}_k\right)(2^{-j\rho}x, 2^{j\rho}\lambda)$$

$$\times \partial_\xi^\delta \left(\prod_{s=1}^n \frac{1}{1+i(\xi_s - \lambda_s)}\right) \bar{v}(2^{j\rho}\lambda)\, d\lambda$$

$$= \int_{R^n} e^{-ix\lambda} 2^{-j\rho|\beta|} \left(\partial_x^\beta \bar{a}_k\right)(2^{-j\rho}x, 2^{j\rho}\lambda) h_\xi(\lambda)\, d\lambda,$$

where

$$h_\xi(\lambda) = \partial_\xi^\delta \left(\prod_{s=1}^n \frac{1}{1+i(\xi_s - \lambda_s)}\right) \bar{v}(2^{j\rho}\lambda).$$

We first estimate $\|H_{\beta,\delta}(\cdot, \xi)\|_{L^2(R^n)}$, so let $w \in C_0^\infty(R^n)$, then

$$\int_{R^n} H_{\beta,\delta}(x,\xi) w(x)\, dx$$

$$= \int_{R^n} \int_{R^n} e^{-ix\lambda} 2^{-j\rho|\beta|} \left(\partial_x^\beta \bar{a}_k\right)(2^{-j\rho}x, 2^{j\rho}\lambda) h_\xi(\lambda) \cdot w(x)\, d\lambda\, dx$$

$$= \int_{R^n} \int_{R^n} e^{-ix\lambda} 2^{-j\rho|\beta|} \left(\partial_x^\beta \bar{a}_k\right)(2^{-k\rho}x, 2^{k\rho}\lambda)$$

$$\times h_\xi(2^{(-j+k)\rho}\lambda) w(2^{(j-k)\rho}x)\, d\lambda\, dx.$$

By (1.6) and by the assumption that $j \geqslant k+4$, we have

$$2^{-j\rho|\beta|} \left|\partial_x^{\alpha'} \partial_\xi^{\beta'}\left(\left(\partial_x^\beta \bar{a}_k\right)(2^{-k\rho}x, 2^{k\rho}\lambda)\right)\right| \leqslant C_{\rho,n} \||a\||,$$

where $(x,\lambda) \in R^n \times R^n$ and $\alpha', \beta', \beta \in T$.

Thus, we derive from Theorem 2 that

$$\left|\int_{R^n} H_{\beta,\delta}(x,\xi) w(x)\, dx\right| \leqslant C_{\rho,n} \||a\|| \|h_\xi(\cdot)\|_{L^2(R^n)} \|w\|_{L^2(R^n)}, \qquad w \in C_0^\infty(R^n),$$

which implies

$$\int_{R^n} \left| H_{\beta,\delta}(x,\xi) \right|^2 dx \le C_{\rho,n} \||a\||^2 \int_{R^n} \left| \partial_\xi^\delta \left(\prod_{s=1}^n \frac{1}{1 + i(\xi_s - \lambda_s)} \right) \right|^2 \left| \hat{v}(2^{j\rho}\lambda) \right|^2 d\lambda.$$

Then we have

$$(4.15) \quad \int_{2^{-j\rho}E_j} \left(\int_{R^n} \left| H_{\beta,\delta}(x,\xi) \right|^2 dx \right) d\xi$$

$$\le C_{\rho,n} \||a\||^2 \int_{2^{-j\rho}E_j} \left(\int_{R^n} \left| \partial_\xi^\delta \left(\prod_{s=1}^n \frac{1}{1 + i(\xi_s - \lambda_s)} \right) \right|^2 \left| \hat{v}(2^{j\rho}\lambda) \right|^2 d\lambda \right) d\xi.$$

Since $\delta = (\delta_1, \ldots, \delta_n) \in T$, we see that

$$(4.16) \quad \left| \partial_\xi^\delta \left(\prod_{s=1}^n \frac{1}{1 + i(\xi_s - \lambda_s)} \right) \right|^2 \le \prod_{s=1}^n \frac{1}{1 + (\xi_s - \lambda_s)^2}, \qquad \lambda, \xi \in R^n.$$

Then an elementary calculation gives

$$(4.17) \quad \int_{2^{-j\rho}E_j} \prod_{s=1}^n \frac{1}{1 + (\xi_s - \lambda_s)^2} d\xi \le 2^8 n^2 \pi^n 2^{-j(1-\rho)},$$

$$\text{for } \lambda \in 2^{-j\rho}E_k \text{ and } j \ge k + 4.$$

Thus, it follows from (4.15), (4.16), (4.17) and Fubini's theorem that

$$(4.18) \quad \left(\int_{2^{-j\rho}E_j} \int_{R^n} \left| H_{\beta,\delta}(x,\xi) \right|^2 dx\, d\xi \right)^{1/2} \le C_{\rho,n} \||a\|| 2^{-j(1-\rho)/2} 2^{-j\rho n/2} \| \hat{v} \|_{L^2(E_k)}.$$

By (4.9), and the fact that $\operatorname{supp} a_j(x, \cdot) \subseteq E_j$, we obtain

$$(4.19) \quad \left| J_{\alpha,\beta,\gamma,\delta} \right| \le C_{\rho,n} \||a\|| \, \|G\|_{L^2(R^n \times 2^{-j\rho}E_j)} \|H\|_{L^2\left(R^n \times 2^{-j\rho}E_j\right)}.$$

Thus (4.12) follows from (4.13), (4.18) and (4.19). □

LEMMA 4.4. *If* $-\infty < \rho < 0$, *then*

$$\left| \left(a_j(x,D)u, a_k(x,D)v \right) \right| \le C_{\rho,n} \||a\||^2 2^{-j/4} 2^{-k/4} \| \hat{u} \|_{L^2(E_j)} \| \hat{v} \|_{L^2(E_k)},$$

where $u, v \in C_0^\infty(R^n)$, $j, k \ge 1$ *and* $|j - k| \ge 4$.

PROOF. Without loss of generality, we can assume that $\hat{u} \in C_0^\infty(E_j)$, $\hat{v} \in C_0^\infty(E_k)$ and $j \ge k + 4$. Then

$$\left(a_j(x,D)u, a_k(x,D)v \right)$$

$$= \left(\frac{1}{2\pi} \right)^{2n} \int_{R^n} \int_{R^n} \int_{R^4} e^{ix(\xi-\lambda)} a_j(x,\xi) \hat{u}(\xi) \bar{a}_k(x,\lambda) \bar{\hat{v}}(\lambda)\, d\lambda\, d\xi\, dx$$

$$= \left(\frac{1}{2\pi} \right)^{2n} 2^{k\rho n} \int_{R^n} \int_{R^n} \int_{R^n} e^{ix(\xi-\lambda)} a_j(2^{-k\rho}x, 2^{k\rho}\xi) \hat{u}(2^{k\rho}\xi)$$

$$\times \bar{a}_k(2^{-k\rho}x, 2^{k\rho}\lambda) \bar{\hat{v}}(2^{k\rho}\lambda)\, d\lambda\, d\xi\, dx,$$

where the last equality comes from change of variables $x \to 2^{-k\rho}x$, $\xi \to 2^{k\rho}\xi$, $\lambda \to 2^{k\rho}\lambda$. We will use a technique similar to the proof of Lemma 4.3, but we integrate with respect to λ instead of integrating with respect to ξ. Then the lemma follows by using the inequality

$$\int_{2^{-k\rho}E_k} \prod_{s=1}^{n} \frac{1}{1 + |\xi_s - \lambda_s|^2} \, d\lambda \leqslant 2^8 n^2 \pi^n \cdot 2^{-j+k\rho} \quad \text{for } \xi \in 2^{-k\rho}E_j. \quad \square$$

Thus, Theorem 3 follows from Lemmas 4.1, 4.2, 4.3, and 4.4.

5. Proof of Theorem 4. By using the same partition of unity as in §4, we have

$$a(x, D)u(x) = b_0(x, D)u(x) + \sum_{j=0}^{\infty} a_j(x, D)u(x), \quad \text{for } u \in C_0^{\infty}(R^n).$$

By Corollary 2.1, it suffices to estimate $\|\sum_{j=5}^{\infty} a_j(x, D)u\|_{L^2(R^n)}$ with $u \in C_0^{\infty}(R^n)$. We write

$$\left\| \sum_{j=5}^{\infty} a_j(x, D)u \right\|_{L^2(R^n)}^2 = \sum_{j,k=5}^{\infty} \left(a_j(x, D)u, a_k(x, D)u \right).$$

The only difficulty in estimating $|(a_j(x, D)u, a_k(x, D)u)|$ occurs when $|j - k|$ is large. We give a lemma to handle this case.

LEMMA 5.1. *If $j, k \geqslant 5$ and $|j - k| \geqslant 4$, then*

$$\left| \left(a_j(x, D)u, a_k(x, D)v \right) \right| \leqslant C_{t,n} \|a\|^2 j^{-1-\varepsilon_t} k^{-1-\varepsilon_t} \|\hat{u}\|_{L^2(E_j)} \|\hat{v}\|_{L^2(E_k)},$$

where $u, v \in C_0^{\infty}(R^n)$, $\varepsilon_t > 0$ a constant depending only on t and $C_{t,n}$ is a constant depending only on n and t.

PROOF. Without loss of generality, we can assume that $j \geqslant k + 4$, $\hat{u} \in C_0^{\infty}(E_j)$ and $\hat{v} \in C_0^{\infty}(E_k)$. Then

$$\left(a_j(x, D)u, a_k(x, D)v \right)$$

$$= \left(\frac{1}{2\pi} \right)^{2n} \int_{R^n} \int_{R^n} \int_{R^n} e^{ix(\xi-\lambda)} a_j(x, \xi) \hat{u}(\xi) \bar{a}_k(x, \lambda) \bar{\hat{v}}(\lambda) \, d\lambda \, d\xi \, dx$$

$$= \left(\frac{1}{2\pi} \right)^{2n} (j^{-t}2^j)^n \int_{R^n} \int_{R^n} \int_{R^n} e^{ix(\xi-\lambda)} a_j(j^{-t}2^{-j}x, j^{-t}2^j\xi) \hat{u}(j^{-t}2^j\xi)$$

$$\times \bar{a}_k(j^{-t}2^{-j}x, j^{-t}2^j\lambda) \bar{\hat{v}}(j^{-t}2^j\lambda) \, d\lambda \, d\xi \, dx,$$

where the last equality comes from the change of variables $x \to j^{-t}2^{-j}x$, $\xi \to j^{-t}2^j\xi$, $\lambda \to j^{-t}2^j\lambda$. Integrating with respect to x first and making use of the following identity:

$$\prod_{s=1}^{n} \left(\frac{1}{1 + i(\xi_s - \lambda_s)r} \right) \left(\prod_{s=1}^{n} \left(1 + (i)^{-r+1}\partial_{x_s}^r \right) \right) \left(e^{ix(\xi-\lambda)} \right) = e^{ix(\xi-\lambda)},$$

$$\text{with } r = \left[\frac{(4+t)}{2t} \right] + 1,$$

we have

$$\left(a_j(x, D)u, a_k(x, D)v \right)$$

$$= \left(\frac{1}{2\pi} \right)^{2n} \left(j^{-t}2^j \right)^n \sum_{\alpha \in T} (i)^{(-r+1)|\alpha|}(-1)^{r|\alpha|} \sum_{\beta \leqslant r\alpha} C(r\alpha, \beta) \Gamma_{\alpha,\beta}$$

where

$$T = \left\{ \alpha \in N^n \,\middle|\, \alpha = (\alpha_1, \dots, \alpha_n) \text{ and } \alpha_m = 0 \text{ or } 1 \right\},$$

and

$$I_{\alpha,\beta} = \int_{R^n} \int_{R^n} \int_{R^n} e^{ix(\xi-\lambda)} \partial_x^{r\alpha-\beta} \left(a_j \left(j^t 2^{-j}x, \, j^{-t}2^j\xi \right) \right) \left(\prod_{s=1}^n \frac{1}{1 + i(\xi_s - \lambda_s)^r} \right)$$

$$\times \partial_x^{\beta} \left(\bar{a}_k \left(j^t 2^{-j}x, \, j^{-t}2^j\lambda \right) \right) \hat{u} \left(j^{-t}2^j\xi \right) \bar{\hat{v}} \left(j^{-t}2^j\lambda \right) d\lambda \, d\xi \, dx.$$

Then the lemma follows by an argument similar to the proof of Lemma (4.3) and by the following fact:

If $j \geqslant k + 4$, $r = [(4 + t)/2t] + 1$, $\varepsilon = r - (4 + t)/2t$ and $\delta = (\delta_1, \dots, \delta_n) \in N^n$ with $\delta_j = 0$ or 1, then

$$\int_{j^t 2^{-j}E_j} \left| \partial_\xi^\delta \left(\prod_{s=1}^n \frac{1}{1 + i(\xi_s - \lambda_s)^r} \right) \right|^2 d\xi \leqslant C_n r^{2n} j^{-4-2\varepsilon t} \quad \text{for } \lambda \in j^t 2^{-j}E_k,$$

where C_n is a constant depending only on n. $\square$

REMARK. Let Φ, φ be two positive functions defined on $R^n \times R^n$. We say a is in the Beals-Fefferman class $S_{\Phi,\varphi}^{0,0}$, if $a \in C^\infty(R^n \times R^n)$ and there exists a constant $C_{\alpha,\beta} > 0$ such that

$$\left| \partial_x^\alpha \partial_\xi^\beta a(x, \xi) \right| \leqslant C_{\alpha,\beta} (\Phi(x, \xi))^{-|\beta|} (\varphi(x, \xi))^{-|\alpha|},$$

$$\text{for } (x, \xi) \in R^n \times R^n, \, \alpha, \beta \in N^n.$$

In Hörmander [8], it is proved that $a(x, D)$ is continuous from $L^2(R^n)$ to $L^2(R^n)$ if $a \in S_{\Phi,\varphi}^{0,0}$ and Φ, φ satisfy the following conditions:

(1) There exists $\varepsilon_0 > 0$, $C_0 > 0$ such that if $(x, \xi), (y, n) \in R^n \times R^n$ and

$$\frac{|x - y|}{\varphi(x, \xi)} + \frac{|\xi - n|}{\Phi(x, \xi)} < \varepsilon_0,$$

then

$$C_0^{-1} < \frac{\varphi(y, n)}{\varphi(x, \xi)} < C_0 \quad \text{and} \quad C_0^{-1} < \frac{\Phi(y, n)}{\Phi(x, \xi)} < C_0.$$

(2) $\Phi(x, \xi) \cdot \varphi(x, \xi) \geqslant 1$, for $(x, \xi) \in R^n \times R^n$.

(3) There exist $C > 0$, $t > 0$ such that

$$\frac{\varphi(y, n)}{\varphi(x, \xi)} + \frac{\Phi(y, n)}{\Phi(x, \xi)} \leqslant C \left(1 + \Phi(x, \xi)|x - y| + \varphi(x, \xi)|\xi - n| \right)^t,$$

$$\text{for } (x, \xi), (y, n) \in R^n \times R^n.$$

We now construct a symbol a_0 which is in $C^\infty(R \times R)$ and satisfies the following condition:

For any α, $\beta \in N$, there exists a constant $C_{\alpha,\beta} > 0$ such that

$$\left| \partial_x^\alpha \partial_\xi^\beta a(x, \xi) \right| \leqslant C_{\alpha,\beta} \left((1 + |\xi|)(\log(1 + |\xi|))^{-1} \right)^{|\alpha| - |\beta|},$$

$(x, \xi) \in R \times R$. Suppose such an a_0 exists. Then by Theorem 4, $a_0(x, D)$ is continuous from $L^2(R)$ to $L^2(R)$. Also, we can see that $a_0 \in S_{\Phi_0,\varphi_0}^{0,0}$ with $\Phi_0(x, \xi) = (1 + |\xi|)(\log(1 + |\xi|))^{-1}$ and $\varphi_0(x, \xi) = (1 + |\xi|)^{-1}\log(1 + |\xi|)$, which can not satisfy (3). Therefore (1), (2) and (3) are not the necessary conditions for the L^2 continuity.

Construction of a_0. Let $\varphi \in C_0^\infty(R)$ such that $\varphi \geqslant 0$, $\varphi(0) = 1$ and $\operatorname{supp} \Phi \subseteq \{|\xi| \leqslant 1\}$. We define

$$a_0(x, \xi) = \sum_{j=4}^\infty e^{ij^{-1}2^j x} \varphi\left(j2^{-j}(\xi - 2^j) \right), \qquad (x, \xi) \in R \times R.$$

Then a_0 satisfy the condition that we want.

6. Proof of Theorem 5. We denote a constant depending only on ρ by C_ρ, which may vary from time to time. At first, we prove the following lemma.

LEMMA 6.1. *If $0 < \rho < 1$, then*

$$\left| \left(a_j(x, D)u, a_k(x, D)v \right) \right| \leqslant C_\rho 2^{-j\varepsilon_\rho} 2^{-k\varepsilon_\rho} \|\hat{u}\|_{L^2(E_j)} \|\hat{v}\|_{L^2(E_k)},$$

where $u, v \in C_0^\infty(R)$, $j, k \geqslant 1$, $|j - k| \geqslant 4$ and $\varepsilon_\rho = (1 - \rho)/4$.

PROOF. By following the proof of Lemma 4.3, we have

$$\left(a_j(x, D)u, a_k(x, D)v \right) = \left(\frac{1}{2\pi} \right)^2 2^{j\rho} \sum_{\alpha \in T} \sum_{\beta \leqslant \alpha} (-1)^\alpha I_{\alpha,\beta},$$

where $T = \{0, 1\}$ and

$$I_{\alpha,\beta} = \int_R \int_R \int_R e^{ix(\xi - \lambda)} \partial_x^{\alpha - \beta} \left(a_j(2^{-j\rho}x, 2^{j\rho}\xi) \right)$$

$$\times \partial_x^\beta \left(\bar{a}_k(2^{-j\rho}x, 2^{j\rho}\lambda) \right) \left(\frac{1}{1 + i(\xi - \lambda)} \right) u(2^{j\rho}\xi) \bar{v}(2^{j\rho}\lambda) \, d\lambda \, d\xi \, dx.$$

By the proof of Lemma 4.3, we have

$$|I_{\alpha,0}| \leqslant C_\rho 2^{-j\varepsilon_\rho} 2^{-k\varepsilon_\rho} \|\hat{u}\|_{L^2(E_j)} \|\hat{v}\|_{L^2(E_k)} \quad \text{with } \alpha = 0 \text{ or } 1.$$

To estimate $I_{0,1}$, we use a technique similar to the proof of Lemma 4.3, but we integrate with respect to λ instead of ξ. $\square$

We state the following results and omit their proofs.

LEMMA 6.2. *If $-\infty < \rho < 0$, then*

$$\left| \left(a_j(x, D)u, a_k(x, D)v \right) \right| \leqslant C_\rho 2^{-j/4} 2^{-k/4} \|\hat{u}\|_{L^2(E_j)} \|\hat{v}\|_{L^2(E_k)},$$

where $u, v \in C_0^\infty(R)$, $j, k \geqslant 1$ and $|j - k| \geqslant 4$.

Theorem 5 follows from Lemmas 4.1, 4.2, 6.1, and 6.2.

By using arguments similar to the proofs of Theorem 4 and Theorem 5, we obtain Corollary 5.1.

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