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The Lagrange Equations for Systems with Mass Varying Explicitly with Position: Some Applications to Offshore Engineering

The usual Lagrange equations of motion cannot be directly applied to systems with mass varying explicitly with position. In this particular context, a naive application, without any special consideration on non-conservative generalized forces, leads to equations of motions which lack (or exceed) terms of the form $1/2(\partial m/\partial q)\dot{q}^2$, where q is a generalized coordinate. This paper intends to discuss the issue a little further, by treating some applications in offshore engineering under the analytic mechanics point of view.
Keywords: *Lagrange equation, variable mass with position, offshore engineering applications*

Introduction

Despite the well known fact that the usual Lagrange equations of motion can be directly applied to mechanical systems with mass varying explicitly with time, being invariant with respect to that sort of possibility, this is not true if the mass variation is an explicit function of position. This subtle distinction has been discussed in Pesce (2003), where the Lagrange equations of motion were obtained in an extended form[1]. Two perspectives were there followed: systems with a material type of source, attached to particles continuously gaining or loosing mass and systems for which the variation of mass is of a "nonlinear control volume type", mass trespassing a control surface. This would be the case if, for some theoretical or practical reason, partitions into sub-systems were considered. In Pesce (2003), some interesting areas of application have been cited, as those related to, tethered satellite systems and lifting-crane problems, all of them concerning the deploying or the retrieving of cables. The textile industry has also been mentioned as an important source of variable-mass systems problems in mechanics.

Two problems were there chosen to exemplify the application of the extended Lagrange equations. The first one: the deployment of a heavy cable from a reel. The second one: the impact problem of a rigid body against a liquid free surface. In this latter example, the hydrodynamic impact force may be written as a function of the added-mass of the body entering the liquid. The added mass, in this case, is an explicit function of position and the variation is related to the changing in the size (and form) of the wetted surface.

The present paper re-addresses both problems in the offshore engineering context, presenting some numerical simulations and assessing the discrepancies that might be produced if the system were not properly modeled. Additionally, another hydro-mechanical problem is treated: the dynamics of a water column inside a pipe as an approximate model for the moon-pool problem, particularly relevant for mono-column oil production platforms.

Nomenclature

A = *sectional area of a pipe or cable*
C_f = *friction coefficient*
D = *diameter of a pipe*
F = *force*
F_R = *Froude number*
$\mathbf{f}$ = *active force*
g = *acceleration of gravity*
H = *pipe draught*
$\mathbf{h}$ = *reactive force*
L = *total length of cable; also used for Lagrangean function*
M = *mass*
M_{zz} = *added mass in the vertical direction*
m = *mass*
$\dot{m}$ = *mass flow rate*
$\mathbf{p}$ = *momentum*
p = *pressure*
Q = *generalized force*
q_j = *generalized coordinate*
q = *momentum flux through the mouth of a pipe*
R = *sphere radius or reel radius*
S = *surface*
T = *kinetic energy*
t = *time*
t = *dimensioless time*
$\mathbf{v}$ = *velocity*
W = *vertical impact velocity or*
z = *vertical coordinate*

Greek Symbols

α = instantaneous angle of the jets with respect to the horizontal
β = specific mass, dimensionless
δ = variation
γ = specific weight per unit length
Φ = Metchersky force
ϕ = velocity potential
η = free surface elevation, dimensionless
μ = mass density per unit length
ρ = water mass density
θ = angular displacement
ζ = free surface elevation or penetration depth of a body entering the water
Ω = fluid domain
ω = natural frequency

Subscripts

B relative to buoyancy
D relative to dynamic

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[1]This extended form also comprises the (hypothetical) case of an explicit variation with respect to velocity.

F relative to free surface
f relative to viscous friction
H relative to hydrostatic
I relative to impact
i relative to particle i
J relative to jets
j relative to generalized coordinate j
n relative to normal
R relative to control surface or to reel
S relative to suspended
W relative to wall
z relative to vertical coordinate z

The Extended Lagrange Equations

For the sake of motivation, consider for the moment a very simple and hypothetical problem of a particle of mass $m(x)$, explicitly dependent on the position x, acted on by a force dependent on position, time and velocity, *mass being expelled at null velocity*. The equation of motion is, simply,

$$m'(x)\dot{x}^2 + m(x)\ddot{x} = F(x,\dot{x},t) \ .$$

However, if a somewhat naive application of the usual Lagrange equation, were made, in the form,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = F(x,\dot{x},t) \ ,$$

one would obtain,

$$m'(x)\dot{x}^2/2 + m(x)\ddot{x} = F(x,\dot{x},t) \ ,$$

in an obvious disagreement with respect to the first and correct equation of motion derived from Newton's Law. The reason for such a somewhat unexpected discrepancy could be easily guessed: the usual form of Lagrange Equation is not the most general form that could be conceived, concerning a system presenting variation of mass, explicitly dependent on position. In this simple example of a one-degree-of-freedom system, we could infer that the correct 'Lagrange' equation should be written

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = F(x,\dot{x},t) - \frac{1}{2}\frac{dm}{dx}\dot{x} \ .$$

The extended Lagrange equations of motion can in fact be derived in a general case of a system of particles, for which mass is explicitly dependent on position (as well as on velocity), $m_i = m_i(q_j;\dot{q}_j;t)$. Consider a system of N particles of mass m_i. Let P_i be the corresponding position in a given inertial frame of reference and $\mathbf{p}_i = m_i\mathbf{v}_i$ the momentum. By extending Levi-Civita's form of Newton's law to cases when mass is gained or lost with no null velocity, the principle of virtual work applied to D'Alembert's Principle can be written

$$\sum_i \left(\frac{d\mathbf{p}_i}{dt} - \mathbf{F}_i\right)\cdot \delta P_i = \mathbf{0} \qquad (1)$$

where $\mathbf{F}_i = \mathbf{f}_i + \mathbf{h}_i$, being $\mathbf{f}_i$ the sum of all active forces acting on P_i, and $\mathbf{h}_i = \dot{m}_i\mathbf{v}_{oi}$, the reactive force, which is proportional to the rate of variation of mass with respect to time and to the velocity $\mathbf{v}_{oi}$ of the expelled (or gained) mass. Note that the reactive force known as Metchersky's force, in the Russian technical literature, is usually written as function of relative velocities, in the form $\Phi_i = \dot{m}_i(\mathbf{v}_{oi} - \mathbf{v}_i) = \mathbf{h}_i - \dot{m}_i\mathbf{v}_i$.

Under this latter interpretation, Eq. (1), would be written, Cveticanin (1993)),

$$\sum_i \left(m_i\frac{d\mathbf{v}_i}{dt} - (\mathbf{f}_i + \Phi_i)\right)\cdot \delta P_i = \mathbf{0} \qquad (2)$$

The extended Lagrange equation may be derived as, Pesce (2003),

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = \hat{Q}_j \ ; \qquad j = 1,....,M$$
$$\hat{Q}_j = \sum_i (\mathbf{f}_i + \dot{m}_i\mathbf{v}_{oi})\cdot\frac{\partial P_i}{\partial q_j} + \sum_i\left\{\frac{1}{2}\frac{d}{dt}\left(\frac{\partial m_i}{\partial \dot{q}_j}(\mathbf{v}_i)^2\right) - \frac{1}{2}\frac{\partial m_i}{\partial q_j}(\mathbf{v}_i)^2\right\} \ . \qquad (3)$$

where

$$\mathbf{v}_i = \mathbf{v}_i(q_j;\dot{q}_j;t);\ j = 1,...,M$$

is the velocity of a particle, q_j denotes a generalized coordinate and $\hat{Q}_j$, the respective non-conservative generalized force. This generalized force includes all active forces $\mathbf{f}_i$ and reactive forces $\dot{m}_i\mathbf{v}_{oi}$, due to addition or expelling of mass, with 'absolute' velocity $\mathbf{v}_{oi}$. Equation (3) recovers the derivation provided in Cveticanin (1993), which is valid for the simpler case of mass only explicitly dependent on position, not on velocity.

Three cases in offshore engineering where the present analysis might be relevant are exemplified. The first case is an approximate dynamic model to *the moon-pool problem*. The second one is an important problem in hydromechanics, *the impact of a rigid body against the water free surface*. The third one is the *deployment of a submarine cable from a laying-reel barge*. The second and third problems were already treated in Pesce (2003) and are re-addressed in the offshore engineering context, presenting some simulations and additional discussion.

The Dynamics of the Water Column inside Moon-Pools and Free-Surface Piercing Pipes

Moon-pools are commonly found in many floating offshore structures as in pipe-laying and work barges. Figure 1 presents a mono-column oil production platform, with a cylindrical moon-pool. Pipes and umbilical cables are suspended through the moon-pool to the sea bottom. The main purpose is to provide safer operational conditions, regarding the action of waves. Nevertheless, the water column inside the moon-pool may resonate due to the wave action and to the motions of the floating platform. Resonance in this case should be avoided. Another interesting related problem is the dynamics of free surface piercing pipes used as elements of hydro-electrical power devices driven by the action of waves; see e.g., Tannuri and Pesce (1995). In this latter case, however, resonance tuning is the key to a good performance. Either case, the nonlinear dynamics of the water column must be modeled properly. For the purpose of the present paper, we shall consider the simplest case of a free-surface piercing pipes opened to the atmosphere. Only the unforced problem will be addressed. The forced problem, due to the action of ocean waves, might then be readily assessed.

Figure 1. A mono-column, floating oil production platform. The risers and umbilical cables that connect the production plant to the well heads are suspended from the platform through the moon-pool.

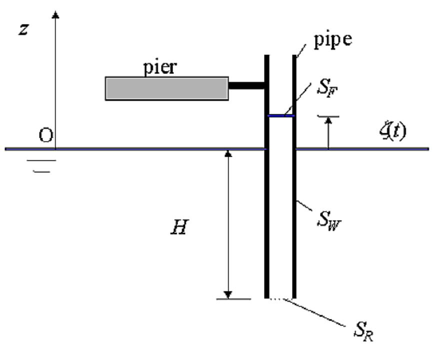

Figure 2. The free surface piercing, open pipe problem. Unit normal vectors are positive outwards the surfaces which enclose the mass of water inside the pipe.

Consider an open vertical circular pipe of internal radius R piercing a quiescent external free surface of an incompressible, inviscid liquid; Figure 2. Let H be the draft of the pipe. Let g be the acceleration of gravity. For simplicity, let $\zeta(t)$ describe the position of the free surface of the column of liquid in the interior of the pipe. Clearly, a simplified model with just one degree of freedom (one generalized coordinate) can be used, $\zeta(t)$. Other free surface vibration modes are not considered in this simplified model.

Before the Lagrangean approach is applied, the equation of motion is derived from the point of view of potential theory in hydrodynamics. This equation will serve as a basis of comparison.

The Classical Hydrodynamic Approach

Take the material sub-system as composed solely by the liquid inside the pipe. That is, the liquid that in a given instant fills the volume Ω bounded by $\partial\Omega = S = S_F \cup S_R \cup S_W$. S_F is the (material and non-permeable) free surface, $z = \zeta(t)$. S_W is the material, fixed and non-permeable surface, corresponding to the interior wetted surface of the pipe and S_R the non-material (permeable) fixed control surface at the lower end of the pipe, given by $z_R = -H$. An exchanging flux of mass clearly exists between the sub-system and the external fluid. Note that the vertical components of the outwardly positive normal unit vector are $n_z = 1$ on S_F and $n_z = -1$ on S_R. Let the flow be non-rotational and $\phi(z)$ the potential velocity function. The kinematic (Neuman) boundary condition on S_F is

$$\frac{\partial\phi}{\partial z} = \frac{\partial\zeta}{\partial t} = \dot{\zeta} \,.$$

The velocity potential, inside the pipe, can then be written

$$\phi(x, y, z, t) = z\dot{\zeta} \,.$$

Note that

$$\frac{\partial\phi}{\partial t} = z\ddot{\zeta} \,.$$

Let the fluid be unbounded in the far field. The dynamic pressure on S_R is given by

$$p_D(x, y)\big|_{S_R} = -\frac{1}{2}\rho\dot{\zeta}^2 \,.$$

Pressure on S_F is taken as null, as usual. Therefore, from momentum considerations, the dynamic equation for $\zeta(t)$ is readily derived. In fact, let Q_z be the linear momentum of the fluid inside the pipe. Then, from classical potential hydrodynamics, see, e.g. Newman (1978),

$$\frac{dQ_z}{dt} = \rho\frac{d}{dt}\int_S \phi n_z dS = F_H + F_D - \mathrm{q} \,, \tag{4}$$

where

$$\begin{aligned} F_H &= -\rho g A\zeta \\ F_D &= -\int_{S_R} p_D(x, y, z, t) n_z dS = -\rho A\frac{1}{2}\dot{\zeta}^2 \end{aligned} \tag{5}$$

are respectively the forces due to the differential hydrostatic pressure and to the hydrodynamic pressure applied to the water column, on S_R, and

$$\mathrm{q} = \rho\int_{S_R}\frac{\partial\phi}{\partial z}\left(\frac{\partial\phi}{\partial n} - U_n\right)dS = -\rho\int_{S_R}\left(\frac{\partial\phi}{\partial z}\right)^2 dS = -\rho A\dot{\zeta}^2 \tag{6}$$

is the flux of linear momentum across the fluid boundary, S_R. The mass of fluid inside the pipe at a given instant is an explicit function of position, $M = \rho A(\zeta + H)$. Therefore, the time rate of linear momentum inside the pipe can be directly calculated,

$$\frac{dQ_z}{dt} = \frac{d}{dt}\left(\rho A(\zeta + H)\dot{\zeta}\right) = \rho A(\zeta + H)\ddot{\zeta} + \rho A\dot{\zeta}^2 \,. \tag{7}$$

Note that this result could also be achieved by recalling that the derivative and integral signs are interchangeable on the fixed control surface S_R. Therefore,

$$\begin{aligned}\frac{dQ_z}{dt} &= \rho\frac{d}{dt}\int_S \phi n_z dS = \rho\frac{d}{dt}\int_{S_F}\phi n_z dS + \rho\frac{d}{dt}\int_{S_R}\phi n_z dS = \\ &= \rho\frac{d}{dt}\int_{S_F}\phi dS - \rho\frac{d}{dt}\int_{S_R}\phi dS = \rho\frac{d}{dt}\int_{S_F}\phi dS - \rho\int_{S_R}\frac{\partial\phi}{\partial t}dS = \\ &= \rho\frac{d}{dt}\int_{S_F}\zeta\dot{\zeta}dS + \rho\int_{S_R}H\ddot{\zeta}dS = \\ &= \rho A\left(\zeta\ddot{\zeta}+\dot{\zeta}^2+H\ddot{\zeta}\right) = \rho A(\zeta+H)\ddot{\zeta}+\rho A\dot{\zeta}^2\end{aligned} \quad (8).$$

Collecting terms from Eqs. (4) - (7), we obtain

$$\rho A(\zeta+H)\ddot{\zeta}+\rho A\dot{\zeta}^2 = -\rho A g\zeta - \frac{1}{2}\rho A\dot{\zeta}^2 + \rho A\dot{\zeta}^2 . \quad (9)$$

This reduces to the following nonlinear homogeneous equation

$$\ddot{\zeta} + \frac{1}{2}\frac{\dot{\zeta}^2}{(\zeta+H)} + g\frac{\zeta}{(\zeta+H)} = 0 . \quad (10)$$

Let $\eta(t) = \zeta(t)/H$ be the dimensionless free surface position. Defining the dimensionless time as, $\mathrm{t} = \omega t$, with $\omega = \sqrt{g/H}$, Eq. (10) may be written in dimensionless form as,

$$\ddot{\eta} + \frac{1}{2}\frac{\dot{\eta}^2}{(\eta+1)} + \frac{\eta}{(\eta+1)} = 0 . \quad (11)$$

The constant ω can be readily recognized as the dimensional natural frequency of the corresponding linear oscillator $\ddot{\eta}+\eta=0$, obtained from Eq. (11) in the case of small displacements and small velocities. Note also that the term that is quadratic in velocity is, in fact, conservative. This could be easily proved.

Equation (11) is valid for $-1<\eta$. A singular behavior, leading to infinity acceleration, arises when $\eta=-1$, i.e. $\zeta=-H$. Physically, this corresponds to the water-column surface reaching the bottom of the pipe, the mass of the system becoming zero. Beyond this point, a cavity would form, and a proper modeling should consider this other highly nonlinear phenomenon.

The Lagrange Equation Approach

From another point of view, the dynamics of the fluid inside the pipe may be modeled as a single degree of freedom (hydro-) mechanical system, such that

$$T = \frac{1}{2}\rho A(\zeta+H)\dot{\zeta}^2$$

is the kinetic energy. In this case, where out-fluxes of mass and kinetic energy do exist from the domain under analysis (the fluid inside the pipe), one must use the extended Lagrange equation; see Pesce (2003) and Casetta and Pesce (2006). One obtains

$$\begin{aligned}&\frac{d}{dt}\frac{\partial T}{\partial\dot{\zeta}} = \rho A(\zeta+H)\ddot{\zeta}+\rho A\dot{\zeta}^2 \\ &\frac{\partial T}{\partial\zeta} = \frac{1}{2}\rho A\dot{\zeta}^2\end{aligned} \quad . \quad (12)$$

Note that, if the system were defined starting from the kinetic energy, the mass dependence on position could not be promptly recognized. As can be clearly seen, the quantity

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{\zeta}} - \frac{\partial T}{\partial\zeta} = \rho A\left((\zeta+H)\ddot{\zeta}+\frac{1}{2}\rho A\dot{\zeta}^2\right), \quad (13)$$

that arises when the usual Euler-Lagrange equation is applied, is not the time rate of change of linear momentum inside the pipe, which is given by Eq. (13). To this quantity it should be added

$$\frac{1}{2}\rho A\dot{\zeta}^2 ,$$

that is exactly the quantity one would obtain from the additional term

$$\frac{1}{2}\sum_i\frac{\partial m_i}{\partial\zeta}\mathbf{v}_i^{\,2} ,$$

that appears on the right hand side of equation (3a). In fact,

$$\begin{aligned}\frac{1}{2}\sum_i\frac{\partial m_i}{\partial\zeta}\mathbf{v}_i^{\,2} &= \frac{1}{2}\sum_i\frac{\partial m_i}{\partial\zeta}\dot{\zeta}^2 = \frac{1}{2}\frac{\partial}{\partial\zeta}\left(\sum_i m_i\right)\dot{\zeta}^2 = \\ &= \frac{1}{2}\rho A\frac{\partial}{\partial\zeta}\left(\int_{-H}^{\zeta}dz\right)\dot{\zeta}^2 = \frac{1}{2}\rho A\frac{\partial}{\partial\zeta}(\zeta+H)\dot{\zeta}^2 = \\ &= \frac{1}{2}\rho A\dot{\zeta}^2\end{aligned} \quad (14)$$

To consistently apply the extended Lagrange Equation (3a), we must consider the equivalent non-conservative generalized force, according to Eq. (3b), that in this case reads,

$$\begin{aligned}\hat{F}_z &= f + \dot{m}v_o - \frac{1}{2}\sum_i\frac{\partial m_i}{\partial\zeta}\mathbf{v}_i^{\,2} = \\ &= (F_H+F_D) + \dot{m}v_o - \frac{1}{2}\frac{\partial m}{\partial\zeta}\dot{\zeta}^2 = \\ &= \left(-\rho A g\zeta - \frac{1}{2}\rho A\dot{\zeta}^2\right) + \left(\rho A\dot{\zeta}^2\right) - \left(\frac{1}{2}\rho A\dot{\zeta}^2\right) = \\ &= -\rho A g\zeta\end{aligned} \quad , \quad (15)$$

Note that, in this case, the term given by Eq. (14) is, quantitatively, half the momentum flux and exactly the same as that corresponding to the dynamic pressure. Note also that, curiously, only the (conservative) hydrostatic term is left.

Collecting results, from Eqs. (13) and (15), Eq. (3) *recovers the consistent dynamic equation*, given by Eqs. (10) or (11). Otherwise, *disregarding the term* given *by Eq. (14) would lead to the erroneous equation of motion*,

$$\ddot{\eta} + \frac{\dot{\eta}^2}{(\eta+1)} + \frac{\eta}{(\eta+1)} = 0 . \quad (16)$$

Apart the conceptual correctness, from the point of view of practical application, significant differences between Eq. (11) and Eq. (16) arise only if the motion is large enough.

Figure 3 presents a comparison between results obtained by using Eq. (11) and Eq. (16). The phase trajectories are closed curves, since no dissipation was considered. The quadratic terms in velocity are conservative, as already anticipated. For all initial displacements, the acceleration attains a maximum when the water column level reaches its minimum value (mass inside the pipe is minimum), as already mentioned. This is exactly what is observed in reality.

Despite the fact that the results could be consistently recovered, a rigorous generalization of Eq. (3) to continuum systems is not straightforward as it could appear through this simple example. A rigorous treatment of Hamilton Principles in Continuum Mechanics can be found in Seliger and Whitham (1968). However, to the present date, and to the author's knowledge, no theoretical extension has been made considering the case of continuum systems with variable mass as an explicit function of coordinates and velocities.

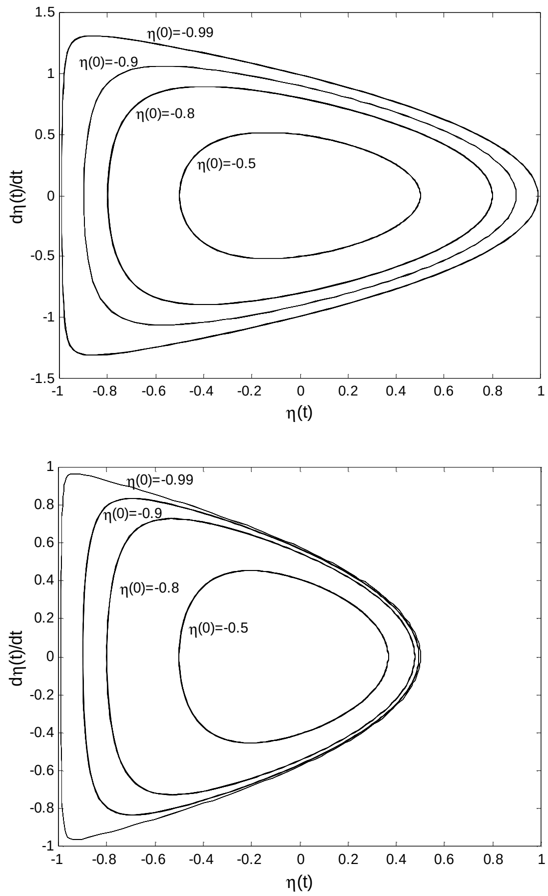

Figure 3. Phase portraits of the water column dynamics. Comparison between results from the consistent (left) and the erroneous equations (right). Initial conditions: $\eta(0)$ = -0,99;...;-0.5;$\dot\eta(0)$ = 0.

The Impact of a Rigid Body against the Water Surface

Consider a body impacting a quiescent free surface of a liquid. In the offshore engineering context, important examples that could be mentioned are the deployment of lifeboats from platforms, ship slamming and wave impacts against structures. Von Karman (1929) first addressed the simplest problem (of an impacting rigid body), in order to estimate the loading on seaplane floaters during “landing”.

The duration of the impact is so short that inertia forces dominate viscous ones. This makes consistent to treat the problem within potential flow theory. As well known, it is usual practice to treat potential hydrodynamic problems involving motion of solid bodies within the frame of system dynamics. This is done whenever a finite number of generalized coordinates can be used as a proper representation for the motion of the whole fluid. Terming this approach as 'hydro mechanical' the impact force acting upon the body, for a purely vertical impact, may be written - see, e.g., Faltinsen (1990), chapter 9 -,

$$F_z = -d(M_{zz}W)/dt\ ;$$

being W the (positively) downward vertical velocity and M_{zz} the corresponding added mass.

Note that in this case the added mass may be written as an explicit function of the position of the body and has to be determined at each instant of time, during the impact phenomenon. This is not an easy task, as the hydrodynamic problem is geometrically nonlinear due to the presence of the free surface and the due to the motion of the body. Usually, the added-mass *is defined only in the bulk of fluid*, excluding the jets. In this case, an out-flux of kinetic energy does exist from the domain under analysis (the bulk of fluid) to the jets. In other word, there is an ‘effective loss of added mass’ through the jets; see Casetta and Pesce (2006). In this case the extended Lagrange equation is the one that should be used. Otherwise, if the added mass is defined considering the whole liquid, including the bulk and the jets, such that the system under analysis turns to be conservative, i.e., there is no loss of kinetic energy, or equivalently, ‘no loss of added-mass’, the usual Lagrange equation must be used instead; see Casetta and Pesce (2006).

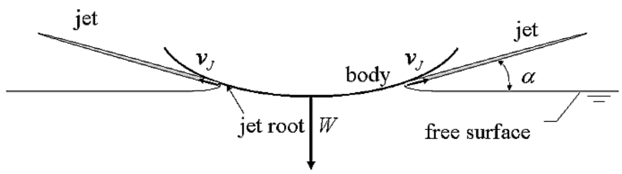

Figure 4. A rigid body impacting a quiescent free surface of a liquid.

The formulation of the impact problem under the Lagrangean formalism, should recall the explicit added mass dependence on the position of the body. However, *restraining the analysis to the bulk of the fluid*, an erroneous result would be obtained if the Lagrange equation were not properly applied, namely, the extended form given by Eq. (3); Pesce, (2003). Taking, for simplicity, the purely vertical impact case of an axi-symmetric rigid body against a free surface, let ζ be defined as the (positive downward) vertical displacement of the body into the water, measured from the quiescent free surface. Let $W(t)$ be the downward vertical velocity. The kinetic energy in the bulk of the liquid may be written as

$$\begin{aligned} T &= \frac{1}{2}M_{zz}W^2 \\ M_{zz} &= M_{zz}(\zeta)\,. \\ \zeta &= \int_{0^+}^{t} W dt \end{aligned} \tag{17}$$

The added mass, consistently defined *in the bulk of the liquid*, at each instant of time, takes into account the so-called wetted correction, due to the marching of the jet root. In this case, as already observed, the correct Lagrange equation approach is to use Eq. (3), such that the total vertical force applied by the body (and the jets) on the bulk of the fluid is given by

$$-F_z^{\ B} = -\frac{d}{dt}\left(\frac{\partial T}{\partial W}\right) + \frac{\partial T}{\partial \zeta} - \frac{1}{2}\frac{dM_{zz}}{d\zeta}W^2 - 2\dot{m}v_J \sin\alpha \tag{18}$$

The fourth term[2] corresponds to the reactive force applied by the jets, due to the momentum rate, where $\dot{m}$ is the effective flux of mass through the jets and v_J the absolute velocity of the fluid particles at the jet root; α is the instantaneous angle of the jets with respect to the horizontal. The force applied by the bulk of fluid on the body is then, simply,

$$F_z = -\frac{d}{dt}\left(\frac{\partial T}{\partial W}\right) + \frac{\partial T}{\partial \zeta} - \frac{1}{2}\frac{dM_{zz}}{d\zeta}W^2 . \quad (19)$$

Equation (19) transforms, as expected, into

$$\begin{aligned} F_z &= -\frac{d}{dt}\left(M_{zz}W\right) + \frac{1}{2}W^2\frac{dM_{zz}}{d\zeta} - \frac{1}{2}\frac{dM_{zz}}{d\zeta}W^2 = \\ &= -\frac{d}{dt}\left(M_{zz}W\right) \end{aligned} \quad (20).$$

The third term appearing on the right hand side of Eq. (20), if not considered, would lead to an *erroneous* assertive, according to which,

$$F_z = -\frac{1}{2}\frac{dM_{zz}}{dt}W - M_{zz}\frac{dW}{dt} . \quad (21)$$

As mentioned, Eq. (19) recovers the expected result. Note that in the present case the changing in the added mass is due to an actual changing of size and shape of the body in contact with the liquid. The computation of the function $M_{zz}(\zeta)$ is not an easy task, as the wetted surface of the body is not known a priori.

Equation (21) would be correct in form, however, if the analysis had considered the whole fluid domain, including not only the bulk but also the jets. In that case the added mass $M_{zz} = M_{zz}^{wfd}$ should be interpreted as a measure of kinetic energy of the whole fluid domain; see Casetta and Pesce (2006). Obviously, in that case, there would be no out-flux of kinetic energy – neither an 'out-flux of added mass'. In other words, there would be no loss of energy from the system and this is the key point. The extended Lagrange equation, for systems with mass explicitly dependent on position would be no longer applicable. One should then apply the usual form[3] of the Lagrange equation, as in Lamb (1932), art. 137. In this latter case the computation of the added mass corresponding to the whole fluid domain, $M_{zz}^{wfd}(t)$, would be even more difficult than that corresponding to the bulk of the fluid.

To finalize the present analysis, an analytical result will be shown, applying a still very useful approximate approach due to Wagner (1931). In this approach the added mass is defined in the bulk of the fluid only and the flux of kinetic energy to the jets must be properly considered. Under Wagner's approach the impact is modeled as a mathematical impulse idealization, enabling a time jump in velocity potential to occur. The impacting surface of the body is taken as the equivalent surface of a 'time-varying floating plate'. In other words, the interaction problem is treated as the 'continuous impact of a floating plate' whose area changes in time. The usual free-surface condition is replaced by an equipotential boundary condition, $\phi = 0$, that corresponds to the limit of infinity frequency in the sense of the wave radiation problem.

At the very start stage, the condition $\partial\phi/\partial t = 0$ is valid on an equipotential control surface that replaces the actual free surface, except at the surface-body intersection, where jets are formed. Actually, to impose such a condition at the body intersection is equivalent to disregard the flux of kinetic energy through the jets. A more detailed analysis is presented in Casetta and Pesce (2005) and in Pesce (2005).

Under Wagner's approximation, the equivalent floating plate of varying size has to be determined. For bodies of regular shape, as edges, cylinders and spheres, asymptotic techniques and singular perturbation methods can be applied successfully; see, e.g., Faltinsen and Zhao (1997) or Pesce et al (2003), for a brief review on this subject. For generic geometric forms however, numerical schemes have to be used to solve the nonlinear hydrodynamic problem.

As a simple example, we take the case of a sphere of radius R and mass m, reaching the free surface with initial velocity W_0. Let the dimensionless time be defined as

$$\mathrm{t} = W_0 t / R , \quad (22)$$

such that the dimensionless position, velocity and acceleration are given by

$$\eta = \zeta/R\ ;\ \dot{\eta} = \frac{d\eta}{d\mathrm{t}} = \frac{1}{W_0}\frac{d\zeta}{dt}\ ;\ \ddot{\eta} = \frac{d^2\eta}{d\mathrm{t}^2} = \frac{R}{W_0^{\,2}}\frac{d^2\zeta}{dt^2} . \quad (23)$$

Asymptotic techniques and similarity theory, applied to the impacting sphere problem to calculate the added mass function under Wagner's approach, together with the generally valid Eq. (19), leads to the following consistent dimensionless equation of motion, (Casetta (2004)),

$$\ddot{\eta} + \frac{\frac{9\sqrt{3}}{2\pi}\eta^{1/2}\dot{\eta}^2}{\beta + \frac{3\sqrt{3}}{\pi}\eta^{3/2}} = 0 \quad (24)$$

where

$$\beta = m/m_D$$

is the dimensionless mass ratio coefficient or specific mass, with $m_D = 4\rho\pi R^3/3$ the displaced mass of a totally immersed sphere. Note that this is the only parameter in Eq. (24).

However, if Eq. (21) were supposed to hold, the equation of motion would read,

$$\ddot{\eta} + \frac{1}{2}\frac{\frac{9\sqrt{3}}{2\pi}\eta^{1/2}\dot{\eta}^2}{\beta + \frac{3\sqrt{3}}{\pi}\eta^{3/2}} = 0 . \quad (25)$$

[2] This term is small. In fact, in the particular and important case of a circular cylinder of radius R, e.g., it can be proved, from the asymptotic analysis by Molin et al. (1996), that the vertical force, per unit length, applied by the jets on the bulk of fluid due to effective mass flux is of order $O(\varepsilon\pi\rho RW^2 \sin\alpha)$, where $\varepsilon = \sqrt{Wt/R}$ is a small parameter measuring a short scale of time. *Contrarily*, the energy flux is of order $G = O(\pi\rho RW^3)$ and $d(M_{zz}W)/dt = O(\varepsilon^{-2}\pi\rho RW^2)$.

[3] Recall that the usual form of Lagrange equation is invariant with respect to systems with mass varying as a function of time - as is the case if the whole fluid is taken as the domain; see, e.g., Pesce (2003), for a detailed discussion on this subject.

This would not be consistent with Wagner's approximation that considers the added mass defined in the bulk of the liquid and not in the whole fluid domain.

The dimensional impacting force is then given by

$$F_z(t) = \frac{mW_0^{\ 2}}{R}\frac{d^2\eta}{dt^2}. \tag{26}$$

Or else, if written in terms of the body weight, its is given by

$$F_z(t) = \left(F_R^{\ 2}\frac{d^2\eta}{dt^2}\right)mg\ , \tag{27}$$

where

$$F_R = W_0\big/\sqrt{gR} \tag{28}$$

is the 'impact Froude number'.

Note that Eq. (24) was asymptotically derived assuming small submergence, say $\eta < 0.2$. In this stage the impacting force reaches its maximum value. Moreover, the impacting force usually dominates the buoyancy force and that is the reason why buoyancy has not been considered.

As can be easily inspected from Eq. (24), the impacting force peak decreases with the mass ratio and increases with the square of the Froude number. In fact, from Eq. (22), the impacting force peak is of order

$$F_I = O(\beta^{-1}F_R^{\ 2}mg) = O(F_R^{\ 2}m_D g)\ .$$

On the other hand, the maximum buoyancy force (totally immersed sphere) is given by

$$F_B = m_D g = \beta^{-1}mg\ .$$

Therefore,

$$F_I\big/F_B \approx O(F_R^{\ 2}) >> 1\,,$$

for high-speed impacts. As a figure, if the sphere is dropped (in vacuum) to the free surface, from a height H, we obtain

$$F_R^{\ 2} = 2H/R\,.$$

Equation (24) is to be integrated under initial conditions $\eta(0) = 0$ and $\dot{\eta}(0) = 1$. Figure 5 exemplifies the large discrepancies, existing between the results obtained from both equations: the consistent equation, Eq. (24) and the erroneous one, Eq. (25). Note also that, for usual offshore and naval engineering applications, practical relevance exists for mass ratio values smaller than 1.

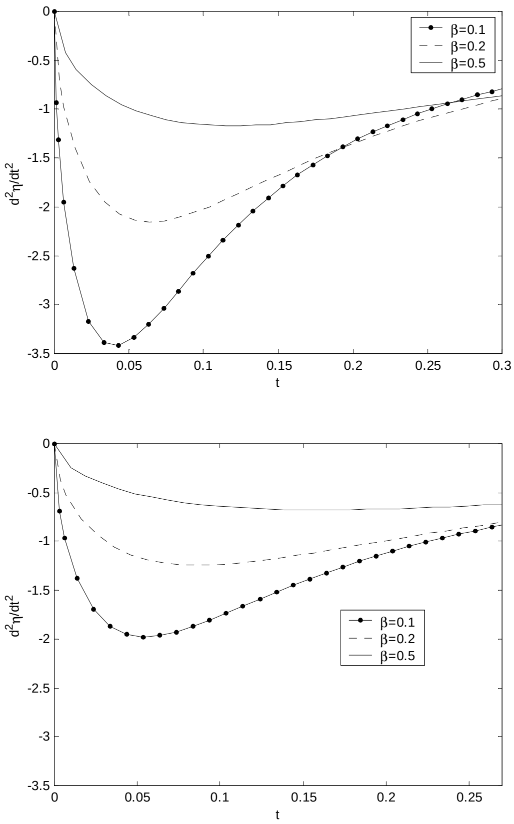

Figure 5. Dimensionless acceleration, of an impacting sphere of radius *R* vertically striking the water surface, as function of dimensionless time. Comparison between results obtained with the consistent (upper) and the erroneous equations (lower).

Legend: $\beta = m/m_D$; $t = W_0 t/R$.

The Deployment of a Submarine Cable from a Reel-Laying Barge

A common task in ocean and offshore engineering is the deployment of cables to the sea bottom. Power supply cables, umbilicals, telecommunication cables are just few examples of such systems to be mentioned. Usually, the cable is deployed from a reel, installed on the deck of a launching-vessel, sometimes through a moon-pool, as schematically illustrated in Figure 6.

This example will show how partition into sub-systems might lead to an erroneous use of the Lagrange equation. The cable is supposed to be acted on by the vessel, ocean current, sea waves, buoyancy and gravity. Initially, and for simplicity, consider only buoyancy and gravity actions, according to the scheme shown in Figure 7. Also for simplicity, the suspended part of the cable is considered fully immersed into the water.

The reel has radius R and moment of inertia I, around the axis of rotation. Let μ be the mass per unit of length of the cable, supposed non-extensible and infinitely flexible. Without loss of generality let θ be the generalized coordinate, measured from horizontal, such that at a given instant t the suspended length is $l(\theta) = R\theta$. Let also L be the total length of the cable such that $M = \mu L$ is the total cable mass. For simplicity we take the cable diameter very small compared to the radius of the reel such that the winding pitch is also small and that all turns can be accommodated into a single winding layer. Let also

$$m_S(\theta) = \mu l(\theta) = \mu R\theta$$

and

$$m_R(\theta) = m - m_S(\theta) = \mu(L - \theta R)$$

be, respectively, the suspended and the wound masses of the cable.

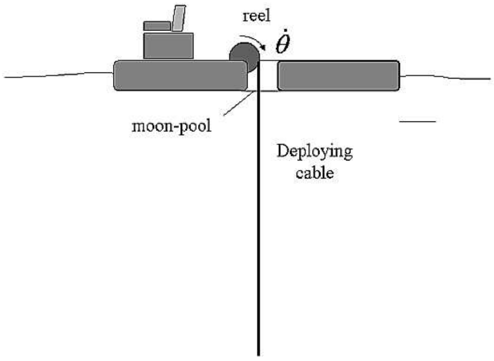

Figure 6. Cable being deployed from a barge, through a moon-pool.

Obviously, for this particular problem, the best and shortest way to directly apply the Lagrange equation would be to consider the whole (invariant mass) system. In this case, Kinetic Energy is simply

$$T = 1/2\left(I + mR^2\right)\dot{\theta}^2 .$$

Accordingly, potential energy is given by,

$$V = -1/2\left((m_s(\theta) - \rho AR\theta)gR\theta\right) = -1/2(1-\beta)\mu gR^2\theta^2 ,$$

where ρ is the density of water, A the area of the cross section of the cable and $\beta = \rho A/\mu$ is the mass density ratio. An extra non-conservative force has to be considered, to model the hydrodynamic friction force acting along the cable during the free-falling deployment. Otherwise, no limit speed would be achieved, and the rotation speed of the reel would increase indefinitely. This force may be written in the form

$$F_f(\theta,\dot{\theta}) = -1/2\, C_f \rho D(R\dot{\theta})^2 l(\theta) = -1/2\, C_f \rho DR^3\theta\dot{\theta}^2 ,$$

with the viscous friction force $C_f = O(10^{-3})$.

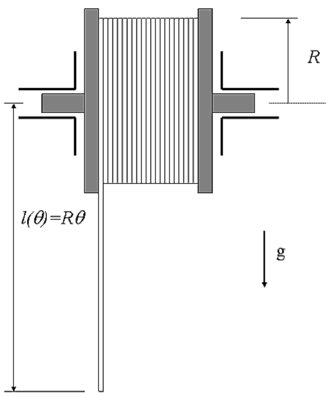

Figure 7. The simplest cable deployment problem.

The direct application of the usual Lagrange equation to this invariant mass system, in the form,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = F_f$$

leads to the consistent equation of motion

$$(I + mR^2)\ddot{\theta} - (1-\beta)\mu gR^2\theta = F_f(\theta,\dot{\theta}) . \qquad (29)$$

Suppose now that, for some practical reason, the analyst decides to take a sub-system composed by the reel and by the wound part of the cable, considering the suspended part of the cable as a second sub-system. Note that the suspended part of the cable can be considered as a material point gaining mass at rate

$$\dot{m}_S(\theta) = \mu R\dot{\theta} ,$$

with velocity

$$v = R\dot{\theta} .$$

The resultant of the active forces applied to the suspended part is

$$f(\theta) = (m_s(\theta) - \rho AR\theta)g - \tau(\theta) , \qquad (30)$$

being $\tau(\theta)$ the traction at the upper section. Applying the extended Levi-Civita form of Newton's law to the suspended part, we easily obtain

$$\frac{d}{dt}(m_S(\theta)R\dot{\theta}) = (1-\beta)\mu R\theta g - \tau(\theta) + \dot{m}_S(\theta)R\dot{\theta} + F_f(\theta,\dot{\theta}) . \; (31)$$

Hence, the traction applied by the wound part to the suspended part of the cable is simply

$$\tau(\theta) = \mu R\theta\left((1-\beta)g - R\ddot{\theta}\right) + F_f(\theta,\dot{\theta}) . \qquad (32)$$

Let, now,

$$J = I + m_R(\theta)R^2 = I + \mu R^2(L - R\theta)$$

be the moment of inertia of the first sub-system (reel + wound cable), such that the corresponding kinetic energy is given by

$$T_1 = 1/2\, J\dot{\theta}^2 .$$

Note that mass exits the wound part with velocity

$$v_o = R\dot{\theta}$$

at a rate

$$\dot{m}_S(\theta) = -\mu R\dot{\theta} .$$

If, *erroneously*, the usual Lagrange equation is applied to the first sub-system in the form,

$$\frac{d}{dt}\left(\frac{\partial T_1}{\partial \dot{\theta}}\right) - \frac{\partial T_1}{\partial \theta} = Q_\theta , \qquad (33)$$

with

$$Q_\theta = \left(\tau(\theta) + \dot{m}_R(\theta)R\dot{\theta}\right)R + F_f(\theta,\dot{\theta}) ,$$

the following and obviously *incorrect* equation of motion is obtained,

$$(I+mR^2)\ddot{\theta}+\frac{1}{2}\mu R^3\dot{\theta}^2-(1-\beta)\mu g R^2\theta=F_f(\theta,\dot{\theta})\ . \qquad (34)$$

Note the presence of an erroneous quadratic term in velocity, namely,

$$\frac{1}{2}\mu R^3\dot{\theta}^2\ .$$

This term is quadratic in the angular velocity of the reel. Therefore, apart the conceptual error, it could lead to significant discrepancies in the calculated traction, if the rotation speed is large enough.

However, if the correct form of the Lagrange equation, given by (3), is applied to this variable mass sub-system,

$$\frac{d}{dt}\left(\frac{\partial T_1}{\partial\dot{\theta}}\right)-\frac{\partial T_1}{\partial\theta}=\hat{Q}(\theta)\ , \qquad (35)$$

with

$$\hat{Q}_\theta=\left(\tau(\theta)+\dot{m}_R(\theta)R\dot{\theta}\right)R-\frac{1}{2}\frac{dm_R}{d\theta}R^2\dot{\theta}^2+F_f(\theta,\dot{\theta})\ ,$$

the *consistent* equation of motion, Eq. (29), previously derived when the whole system was considered, is readily recovered.

As an example, we take the case of a multi-functional electric cable being deployed vertically, in deep water. The cable has a diameter D=100mm and a weight per unit length, $\gamma=0.15\,\mathrm{kN/m}$. The reel has radius R=1.0m and inertia I=4t.m^2. The total length of the cable is L=3000m.

Figure 8 shows the simulation of an "immersed-free-fall" deployment. The depth is supposed to be 1500m and the simulation is carried out up to the instant the cable touches the soil. The solution, $\dot{\theta}(t)$ and $\tau(t)$, obtained from both equations, the consistent and the erroneous ones, are compared in Figure 8. Initial conditions were chosen as $\dot{\theta}(0)=0$ and $l(\theta(0))=10\,\mathrm{m}$ (the initial suspended length). As can be noticed, there is not a significant difference between both results, as the quadratic term in velocity is not dominant for this operation. Therefore, in this particular case, the importance of the present analysis is, in fact, much more theoretical than practical.

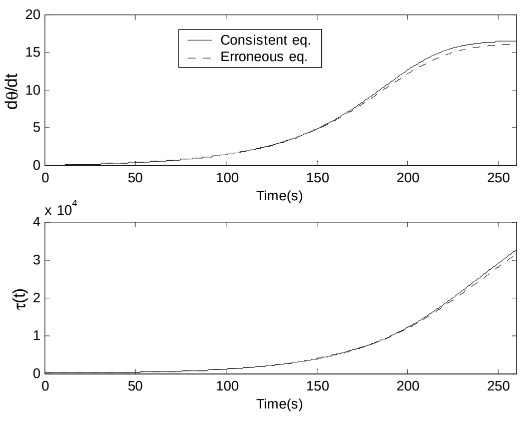

Figure 8. The "free-fall" deployment of a multi-functional electric cable from a reel barge, under no current.

Conclusions

Through simple modeling of typical problems in offshore engineering, this work exemplified how a non-proper use of the Lagrangean formalism may lead to important discrepancies in formulating the equations of motions. This would be always the case whenever one treats mechanical systems with mass explicitly dependent on position. Despite such a strong assertive, the corresponding extended form of the Lagrange equation is not well known, being absent in almost all textbooks in classical mechanics.

Acknowledgements

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References

Casetta, L. 2004, "Theoretical Investigation on the Hydrodynamic Impact Problem", (in Portuguese), Master Thesis, Escola Politécnica, University of São Paulo, 151p.

Caseta, L. And Pesce, C.P., 2006, "The Proper Definition of Added-Mass for the Water Entry Problem". 21st International Workshop on Water Waves and Floating Bodies 2nd–5th April 2006, Loughborough, England (http://www-staff.lboro.ac.uk/~mait/iwwwfb/index.htm).

Casetta, L. and Pesce, C.P., 2005, "A Noticeable Question of the Water Entry Problem: the Split of Kinetic Energy during the Initial Stage", COBEM2005, 18th International Congress of Mechanical Engineering, Brazilian Society of Engineering and Mechanical Sciences, Ouro Preto, Brazil.

Cveticanin, L., 1993, "Conservation Laws in Systems with Variable Mass", *Journal of Applied Mechanics*, Vo.60, pp. 954-958.

Faltinsen, O.M., 1990, *Sea Loads on Ships and Offshore Structures*. Cambridge Ocean Technology Series, Cambridge University Press, 328 p.

Faltinsen, O.M. and Zhao, R., 1997, "Water Entry of Ship Sections and Axi-symmetric Bodies". Agard Ukraine Inst on Hydromechanics. Workshop on High Speed Body Motion in Water.

Korobkin, A.A. and Pukhnachov, V.V., 1988, "Initial Stage of Water Impact". Ann Review Fluid Mech, Vol.20, pp.159-185.

Lamb, H., "Hydrodynamics", 1932, Dover Publications, N.Y., 6th Ed., 738 pp.

Molin, B., Cointe, R. and Fontaine, E., 1996, "On Energy Arguments Applied to the Slamming Force". 11th Int Workshop on Water Waves and Floating Bodies, Hamburg.

Newman, J.N., 1978, "Marine Hydrodynamics", The MIT Press, 402 pp.

Pesce, C.P., 2005, "A Note on The Classical Free Surface Hydrodynamic Impact Problem". *Advances in Engineering Mechanics*, Reflections and Outlooks; in honor of Theodore Y-T Wu; Ed. A. T. Chwang, M. H. Teng, D. T. Valentine, World Scientific Publishing Co., Singapore, pp. 390-407.

Pesce, C.P., 2003, "The Application of Lagrange Equations to Mechanical Systems with Mass Explicitly Dependent on Position", *Journal of Applied Mechanics*, Vol. 70, pp. 751-756.

Pesce, C.P., Casetta, L. and Crivelari, E.E., 2003, "The Free Surface Hydrodynamic Impact Problem: a Brief Review on Asymptotic Solutions and Experiments with a Hemisphere", Proceedings of the 17th International Congress of Mechanical Engineering, COBEM'2003, São Paulo, Brazil.

Seliger, R.L. and Whitham. G.B., 1968, "Variational Principles in Continuum Mechanics", Proc Roy Soc, A 305, pp. 1-25.

Tannuri, E.A. and Pesce, C.P., 1995, "Dinâmica da coluna d'água no interior de um tubo que penetra uma interface", In: Simpósio de Iniciação Científica da Universidade de São Paulo, 3. São Carlos, 1995. Resumos, São Paulo: Usp, v.2 p. 312.

Von Kármán, T., 1929, "The Impact of Seaplane Floats During Land". National Advisory Committee for Aeronautics (NACA). Technical Note no. 321.

Wagner, H., 1931, "Landing of Seaplanes". National Advisory Committee for Aeronautics (NACA). Technical Memorandum no. 622.