THE LAMPERTI TRANSFORMS OF SELF-SIMILAR GAUSSIAN PROCESSES AND THEIR EXPONENTIALS

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Abstract. We present results on the second order behavior and the expected maximal increments of Lamperti transforms of self-similar Gaussian processes and their exponentials. The Ornstein Uhlenbeck processes driven by fractional Brownian motion (fBM) and its exponentials have been recently studied in [20] and [21], where we essentially make use of some particular properties, e.g., stationary increments of fBM. Here the treated processes are fBM, bi-fBM and sub-fBM; the latter two are not of stationary increments. We utilize decompositions of self-similar Gaussian processes and effectively evaluate the maxima and correlations of each decomposed process. We also present discussion on the usage of the exponential stationary processes for stochastic modelling.

1. Introduction

In this paper we consider stationary processes constructed from self-similar Gaussian processes, among which we especially focus on the fractional Brownian motion and its variants, namely the bi- and the sub- versions . For $H \in (0,1)$, a fractional Brownian motion (fBM) $B^H := \{B^H(t)\}_{t\in\mathbb{R}}$ is a centered Gaussian process with $B^H(0) = 0$ and

$$\mathrm{Cov}\left(B^H(s), B^H(t)\right) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right),\ (t,s) \in \mathbb{R}^2.$$

It is well known that fBM has both stationary increments and self-similarity with index H, i.e., for any $c > 0$ $\{B^H(ct)\}_{t\in\mathbb{R}} \stackrel{d}{=} \{c^H B^H(t)\}_{t\in\mathbb{R}}$, where $\stackrel{d}{=}$ denotes equality in all finite dimensional distributions. The fBM has the only stationary increments among self similar Gaussian processes, of which a considerable number of theoretical studies have been conducted (see e.g. [11, 22]). These studies include p-variation of its paths with $p < H$, and the long memory property of the increments for $H \in (\frac{1}{2}, 1)$, as often observed in real-life data.

For Brownian motion ($H = 1/2$), we may generate the stationary processes in two ways: one is by the stochastic integration of exponential function with respect to Brownian motion which yields the famous Ornstein-Uhlenbeck process, and the other is the Lamperti transform which is introduced in a seminal paper [13]. These two transforms are well-known to be law equivalent; that is, both processes have the same finite dimensional distributions deduced from the corresponding strictly stationary Gaussian process. However, when we replace BM by fBM in the construction, in [10] the authors proved that these two transforms produce *different* stationary Gaussian processes.

In general, these two transforms yield different stationary processes, which reflect the different focus of constructions; the stationarity by Ornstein-Uhlenbeck (OU) transform is based on the stationary increments property, while the stationarity by Lamperti transform is based on the self-similarity. The OU processes driven by fBM have recently been studied in [20], and the research

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is continued in [21], where the continuous time autoregressive moving average processes driven by fBM are intensively studied.

The purpose of this article is to study the Lamperti transform of fBM. As we have mentioned, for the Lamperti transform to be stationary it is sufficient that the underlying process be self-similar; thus we may study carefully the Lamperti transform for some more general self-similar Gaussian processes. We remark that a reason for the focus on stationary processes is that the such property is inevitable for statistical applications; for example, the statistical treatment of non-stationary processes requires non-standard asymptotic theory which requires considerable amount of technical complexity in practice.

We pay attention to the following two processes, which are variants of fBM.

Definition 1.1. *Let $H \in (0,1) \cap K \in (0,2)$ such that $HK \in (0,1)$. A bifractional Brownian motion ($bfBM$) is a centered Gaussian process $B^{H,K} := \{B^{H,K}(t)\}_{t\in\mathbb{R}_+}$ with $B^{H,K}(0) = 0$ and*

$$\mathrm{Cov}\left(B^{H,K}(s), B^{H,K}(t)\right) = \frac{1}{2^K}\left\{\left(t^{2H} + s^{2H}\right)^K - |t-s|^{2HK}\right\},\ (t,s) \in \mathbb{R}_+^2.$$

A sub-fractional Brownian motion ($sfBM$) is a centered Gaussian process $S^H := \{S^H(t)\}_{t\in\mathbb{R}_+}$ with $S^H(0) = 0$ and

$$\mathrm{Cov}(S^H(s), S^H(t)) = \frac{1}{2 - 2^{2H-1}}\left(s^{2H} + t^{2H} - \frac{1}{2}\{(s+t)^{2H} + |s-t|^{2H}\}\right),\ (t,s) \in \mathbb{R}_+^2.$$

Note that we multiply $1/\sqrt{2 - 2^{2H-1}}$ to the original process so that we equalize all variances $\mathrm{Var}(S^{HK}(s)) = s^{2HK} = \mathrm{Var}(B^{HK}(s)) = \mathrm{Var}(B^{H,K}(s))$.

The process $B^{H,K}$ was introduced in [12], aiming to broaden the modelling related to fBM; it namely discards the whole stationarity of increments, which as the authors remarked that the fBM is inadequate for large increments in modelling turbulence. The process is known to be HK self-similar, Hölder continuous of order δ for any $\delta < HK$ and $B^{H,1}$ corresponds to fBM. Notably, $B^{H,K}$ for $K \neq 1$ does not have stationary increments; as fBM, $B^{H,K}$ is not a semimartingale except for $B^{1/2,1}$. Other interesting properties have been investigated, e.g., the variational property [28], the path properties [31] and relation to the solution of some stochastic partial differential equations [17], to name just a few. The process S^H is derived from certain particle systems by [6], which contributes as an intermediate process between the standard BM and fBM in the sense of the correlation decay of increments. The process S^H is H self-similar, Hölder continuous and could have long memory increments such that $S^{1/2}$ corresponds to BM. For $H \neq 1/2$ it is not a semimartingale, nor of stationary increments. Similarly, as $B^{H,K}$, some generic properties have been intensively studied, e.g., in [32]. The reason for our study of these two processes is that, besides the interesting properties stated above, their structures which one can see in covariance functions are simple and useful for applications; meanwhile, they retain several important properties of fBM, e.g., both processes are quasi-helix in the sense of J.P. Kahane [15, 16]. We also mention that our methodology in this article may work for other extensions of fBM.

Each self-similar process has the correspondence with a stationary process by the following well-known transform (see [13]): for $H > 0$ a stochastic process $\{X(t)\}_{t\ge 0}$ is H-self-similar if and only if for all $\lambda > 0$, the process

$$\widehat{X}(t) = e^{-\lambda t} X(e^{\frac{\lambda}{H} t}) \tag{1.1}$$

is stationary. See book [11] devoted to the self-similar processes, in which the significance of Lamperti transforms is well illustrated.

In this article, we discuss the Lamperti transform of self-similar Gaussian processes; we denote the Lamperti transform of B^H, $B^{H,K}$ and of S^H by

$$\widehat{B}^H(t) := e^{-\lambda t} B^H(e^{\frac{\lambda}{H} t}),$$

$$\widehat{B}^{H,K}(t) := e^{-\lambda t} B^{H,K}(e^{\frac{\lambda}{HK}t}),$$
$$\widehat{S}^{H}(t) := e^{-\lambda t} S^{H}(e^{\frac{\lambda}{H}t}),$$

respectively. We remark that the Ornstein-Uhlenbeck transform is another prominent way to produce a stationary process from fBM ; it is by an exponential integration with respect to fBM:

$$Y^H(t) := \int_{-\infty}^{t} e^{-\lambda(t-u)} dB^H(u), \tag{1.2}$$

and is known as a fractional Ornstein-Uhlenbeck process. Although for $H = \frac{1}{2}$ (Brownian motion case) $\widehat{B}^H \stackrel{d}{=} Y^H$ holds, for $H \neq \frac{1}{2}$ the finite dimensional distributions of $\widehat{B}^H$ and Y^H are different [10]. In this article, we will study the correlation decay and the expected maximal increments of $\widehat{B}^H$ and the two related processes $\widehat{B}^{H,K}$ and $\widehat{S}^H$; which exhibit quite different behavior from those of Y^H.

This article is organized as follows. In Section 2, we list some preliminaries, including important decompositions for the bfBM and the sfBM. We present our main results in Section 3. A discussion on the role of the exponential stationary processes is given in Section 4. All the proofs are given in the final Section 5.

2. Some preliminaries

In this section we present some tools for our purpose. Firstly we describe known results about the decompositions for $B^{H,K}$ and S^H, which are the key to analyze the expected maximal increments of the processes. For the decompositions we introduce another centered Gaussian process X^K (defined by [17]),

$$X^K(t) = \int_0^{\infty} (1 - e^{-ut}) u^{-\frac{1+K}{2}} dB(u), \quad K \in (0,1) \cup (1,2), \tag{2.1}$$

such that the covariance function satisfies

$$\operatorname{Cov}(X^K(t), X^K(s)) = \begin{cases} \frac{\Gamma(1-K)}{K} \left[t^K + s^K - (t+s)^K\right] & \text{if } K \in (0,1), \\ \\ \frac{\Gamma(2-K)}{K(K-1)} \left[(t+s)^K - t^K - s^K\right] & \text{if } K \in (1,2). \end{cases}$$

By definition the process X^K is self-similar with index $K/2$, and its paths are shown to be absolutely-continuous on $[0, \infty)$ and infinitely-differentiable on $(0, \infty)$ in [17] for $K \in (0,1)$, which are also extended to the case $K \in (1,2)$ by [3] and [27]. We prepare some normalizing constants c_i, $i = 1, 2, \ldots, 5$ as

$$c_1 = \sqrt{\frac{2^{-K}K}{\Gamma(1-K)}},\ c_2 = 2^{\frac{1-K}{2}},\ c_3 = \sqrt{\frac{K(K-1)}{2^K\Gamma(2-K)}},\ c_4 = \sqrt{\frac{H}{\Gamma(1-2H)}},\ c_5 = \sqrt{\frac{H(2H-1)}{\Gamma(2-2H)}}.$$

Now the decompositions are as follows.

$\Diamond$ Decompositions of bfBM $B^{H,K}$ by X^K and fBM B^H:

(**B1**, by [17]) For $H \in (0,1) \cap K \in (0,1)$ it follows that

$$\{c_1 X^K(t^{2H}) + B^{H,K}(t)\} \stackrel{d}{=} \{c_2 B^{HK}(t)\}, \tag{2.2}$$

where $B^{H,K}$ and B of integrator in the definition of X^K are independent.

(**B2**, by [4]) For $H \in (0,1)$ and $K \in (1,2)$ with $HK \in (0,1)$, bfBM $B^{H,K}$ has the decomposition,

$$\{B^{H,K}(t)\} \stackrel{d}{=} \{c_2 B^{HK}(t) + c_3 X^K(t^{2H})\}, \tag{2.3}$$

where B^{HK} and B of integrator in the definition of X^K are independent.

$\Diamond$ Decomposition of sfBM S^H by X^H and fBM B^H by [27] (cf. [3]):

(**S1**) For $H \in (0, \frac{1}{2})$, S^H has a decomposition,

$$\{d_H S^H(t)\} \stackrel{d}{=} \{c_4 X^{2H}(t) + B^H(t)\} \tag{2.4}$$

where $d_H = \sqrt{2 - 2^{2H-1}}$ and B^H and B of the integrator in X^{2H} are independent.

(**S2**) For $H \in (\frac{1}{2}, 1)$, it follows that

$$\{c_5 X^{2H}(t) + d_H S^H(t)\} \stackrel{d}{=} \{B^H(t)\} \tag{2.5}$$

where S^H and B of the integrator in X^{2H} are independent.

Now we characterize the sizes of covariance functions for B^H, $B^{H,K}$ and S^H and bound the probability for the maximum of the processes $B^{H,K}$ and S^H. As a basis process for comparisons, we consider a Gaussian Markov process $\overline{B}^H := \{\overline{B}^H(t)\}_{t\in[0,1]}$, which is a centered Gaussian process with $\overline{B}^H(0) = 0$ and

$$\mathrm{Cov}(\overline{B}^H(s), \overline{B}^H(t)) = s^{2H}, \qquad 0 < s < t \le 1,$$

such that its increments are independent. This process is found in, e.g., [29, Lemma 5.7] or [24, Theorem 3.1] where they intensively use the process to investigate properties of fBM. Moreover, we impose two additional self-similar Gaussian processes to our analysis. The first one $\overline{S}^H$ is the original (non-normalized) sfBM, i.e., $\overline{S}^H := d_H S^H$, which is a centered Gaussian with covariance

$$\mathrm{Cov}(\overline{S}^H(s), \overline{S}^H(t)) = s^{2H} + t^{2H} - \frac{1}{2}\{(s+t)^{2H} + |s-t|^{2H}\}.$$

The other one is the process $\overline{X}^H$, $H \in (0, \frac{1}{2}) \cup H \in (\frac{1}{2}, 1)$ defined by

$$\overline{X}^H := \begin{cases} \sqrt{\frac{2H}{\Gamma(1-2H)} \frac{1}{2-2^{2H}}}\, X^{2H} & \text{if } H \in (0, \frac{1}{2}), \\ \sqrt{\frac{2H(2H-1)}{\Gamma(2-2H)} \frac{1}{2^{2H}-2}}\, X^{2H} & \text{if } H \in (\frac{1}{2}, 1), \end{cases}$$

such that its covariance function is

$$\mathrm{Cov}(\overline{X}^H(s), \overline{X}^H(t)) = \frac{1}{|2 - 2^{2H}|}\, \big| t^{2H} + s^{2H} - (t+s)^{2H} \big|.$$

The process $\overline{X}^H$ is the standardized version of X^{2H}, namely $\mathrm{Var}(\overline{X}^H(s)) = s^{2H}$.

Firstly, we present the following relations of sizes of covariance with $\overline{B}^H$ as our standard, and from which we derive the bounds for the probabilities of maxima for self-similar Gaussian processes. All covariance functions are easily shown to be positive on $s, t \in [0,1]$ and all variances are equal except that of $\overline{S}^H$.

Lemma 2.1. *Let $H \in (0,1)$, $K \in (0,1) \cup (1,2)$ and $HK \in (0,1)$, and we write*

$$\begin{aligned} \mathrm{Cov}(B^H(s), B^H(t)) &=: \Upsilon_H(s,t), \\ \mathrm{Cov}(B^{H,K}(s), B^{H,K}(t)) &=: \Upsilon_{H,K}(s,t), \\ \mathrm{Cov}(\overline{B}^H(s), \overline{B}^H(t)) &=: \Upsilon_{\overline{H}}(s,t), \\ \mathrm{Cov}(\overline{S}^H(s), \overline{S}^H(t)) &=: \mathcal{S}_{\overline{H}}(s,t), \\ \mathrm{Cov}(S^H(s), S^H(t)) &=: \mathcal{S}_H(s,t), \\ \mathrm{Cov}(\overline{X}^H(s), \overline{X}^H(t)) &=: \chi_{\overline{H}}(s,t). \end{aligned}$$

Then we have the following relations. (1) *bfBM case : for $s, t \in [0,1]$,*

$HK \setminus K$	$K \in (0,1)$	$K \in (1,2)$
$HK \in (0, \frac{1}{2})$	$\Upsilon_{H,K}(s,t) \le \Upsilon_{\overline{HK}}(s,t)$	(i)
$HK \in (\frac{1}{2}, 1)$	$\Upsilon_{H,K}(s,t) \ge \Upsilon_{\overline{HK}}(s,t)$	

In the above table for the range (i), *if* $H \in (\frac{1}{2(2K-1)}, \frac{1}{2})$ *then* $\Upsilon_{H,K}(s,t) \ge \Upsilon_{\overline{HK}}(s,t)$. *If* $HK \in (\frac{1}{2}, 1)$ *or* $H \in (\frac{1}{2(2K-1)}, \frac{1}{2}) \cap K \in (1,2)$

$$P\left(\max_{0\le t\le 1} B^{H,K}(t) \ge a\right) \le 2P\left(\overline{B}^{HK}(1) \ge a\right),\ a \ge 0.$$

(2) *sfBM case : for* $s,t \in [0,1]$,

$$\Upsilon_H(s,t) \gtreqqless \mathcal{S}_H(s,t) \gtreqqless \Upsilon_{\overline{H}}(s,t) \gtreqqless \mathcal{S}_{\overline{H}}(s,t) \qquad \text{if } H \gtreqqless \frac{1}{2}$$

which yields for $H \in (\frac{1}{2}, 1)$ *and for* $a > 0$,

$$P\left(\max_{0\le t\le 1} S^H(t) \ge a\right) \le 2P\left(\overline{B}^H(1) \ge a\right),\ a \ge 0.$$

(3) X^H *or* $\overline{X}^H$ *case : for* $s,t \in [0,1]$,

$$\chi_{\overline{H}}(s,t) \ge \Upsilon_{\overline{H}}(s,t) \ge \Upsilon_H(s,t) \qquad \text{if } H \in (0, \frac{1}{2}),$$
$$\chi_{\overline{H}}(s,t) \ge \Upsilon_H(s,t) \ge \Upsilon_{\overline{H}}(s,t) \qquad \text{if } H \in (\frac{1}{2}, 1),$$

which yields

$$P\left(\max_{0\le t\le 1} \overline{X}^H(t) \ge a\right) \le 2P\left(\overline{B}^H(1) \ge a\right),\ a \ge 0.$$

We remark that at $a = 0$ probability inequalities are trivially satisfied. However, since our goal is the expected maxima of processes, the bounds for tail probabilities (for large a) are significant. Since $\overline{B}^H(1)$ follows the standard normal distribution, we can explicitly calculate the upper bound as

$$P\left(\overline{B}^H(1) \ge a\right) = \frac{1}{\sqrt{2\pi}} \int_a^\infty e^{-x^2/2} dx.$$

Note that if we extend the processes on the unit interval $s,t \in [0,1]$ to the whole real line, their covariance functions may be negative, e.g., $\Upsilon_H(-s,s) \le 0$ for $H \in (\frac{1}{2}, 1)$ and $s > 0$. Therefore, the study of covariance relations for these extended ones would be a future topic. Moreover, since results of Lemma 2.1 do not cover all comparisons of the covariances, the complete characterizations of their sizes would be in itself interesting. It would presumably depend on values of both H and K.

Next we consider the maximal increments of X^K; we recall Lemma 2.2 in [20],

Lemma 2.2. *Let* $H \in [\frac{1}{2}, 1), a \ge 0, r \ge 0$.

$$P\left(\max_{0\le t\le 1} B_t^H \ge a\right) \le \sqrt{\frac{2}{\pi}} \int_a^\infty e^{-x^2/2} dx.$$
$$P\left(\max_{0\le t\le 1} \left|B_t^H\right| \ge a\right) \le \frac{2\sqrt{2}}{\sqrt{\pi}} \int_a^\infty e^{-x^2/2} dx.$$

$$E\left[\left(\max_{0\le t\le r}|B_t^H|\right)^m\right]\le\begin{cases} r^{Hm}\,\frac{2\sqrt{2}}{\sqrt{\pi}}\,(m-1)!! & \text{if } m \text{ is odd}\\ r^{Hm}\,2\,(m-1)!! & \text{if } m \text{ is even.}\end{cases}$$

Finally we give the maximal inequalities for the self-similar Gaussian process X^K which is useful for the analysis of expected maximal increments. Note that, since X^K is not of stationary increments, we cannot employ the tools used in the previous works.

Lemma 2.3. *Let $m=1,2,\dots$ and $(s,t)\in\mathbb{R}_+^2$. Let X^K be a Gaussian process defined by (2.1). Then m-th power of maximal increments satisfy*

$$E\Big[\max_{s\le t\le s+r}|X^K(t)-X^K(s)|^m\Big]\le r^m\,s^{\frac{m}{2}(K-1)-1}\,C_K^m\,(m-1)!! \qquad \text{if } K\in(0,1), \tag{2.6}$$

$$E\Big[\max_{s\le t\le s+r}|X^K(t)-X^K(s)|^m\Big]\le r^m\,s^{\frac{m}{2}(K-2)}\,C_K^m\,(m-1)!! \qquad \text{if } K\in(1,2), \tag{2.7}$$

where C_K is a positive constant depending on K.

It is interesting to observe that, for X^K, the maximum increments on the interval $[s,s+r]$ for $r>0$ is a decreasing function in s, which is similar to that of expected squared increments $E[(X^K(s+r)-X^K(s))^2]$.

3. Main results

3.1. **Correlation decay.** In this section, we rigorously investigate the autocovariane functions of our target processes. We begin with the following correlation decay of $\widehat{B}^H$, which is cited from [10].

Proposition 3.1. *Let $H\in(0,1]$ and $t,s\in\mathbb{R}$.*

$$\begin{aligned}\mathrm{Cov}\left(\widehat{B}^H(t),\widehat{B}^H(t+s)\right)&=\frac{1}{2}e^{\lambda|s|}\left\{1+e^{-2\lambda|s|}-\left(1-e^{-\frac{\lambda}{H}|s|}\right)^{2H}\right\}\\ &=\frac{1}{2}\left\{e^{-\lambda|s|}+\sum_{n=1}^{\infty}(-1)^{n-1}\binom{2H}{n}e^{-\lambda(\frac{n}{H}-1)|s|}\right\}.\end{aligned}$$

Thus the leading term of the correlating decay of $\widehat{B}^H$ is, for $|s|\to\infty$,

$$\mathrm{Cov}(\widehat{B}^H(t),\widehat{B}^H(t+s))=\begin{cases}\frac{1}{2}e^{-\lambda|s|}+O\left(e^{-\lambda(\frac{1}{H}-1)|s|}\right) & \text{if } H\in(0,\frac{1}{2})\\[2ex] He^{-\lambda(\frac{1}{H}-1)|s|}+O\left(e^{-\lambda|s|}\right) & \text{if } H\in[\frac{1}{2},1).\end{cases}$$

In the following, we denote the Lamperti transform of fBM by LfBM, and that for bfBM and sfBM are denoted by LbfBM and LsfBM respectively. From the lemma below, we see that the correlation decay of LbfBM $\widehat{B}^{H,K}$ and LsfBM $\widehat{S}^H$ are both different from that of LfBM $\widehat{B}^H$.

Lemma 3.2. *Let $H\in(0,1)$ and $K\in(0,2)$ such that $HK\in(0,1)$ and $t,s\ge 0$.*
(1) *The correlation of $\widehat{B}^{H,K}$ has an expansion*

$$\begin{aligned}\mathrm{Cov}(\widehat{B}^{H,K}(t),\widehat{B}^{H,K}(t+s))&=\frac{1}{2^K}e^{\lambda s}\left\{\left(1+e^{-\frac{2\lambda}{K}s}\right)^K-\left(1-e^{-\frac{\lambda}{HK}s}\right)^{2HK}\right\}\\ &=\frac{1}{2^K}e^{\lambda s}\left\{\sum_{n=1}^{\infty}\binom{K}{n}e^{-\frac{2\lambda}{K}ns}-\sum_{n=1}^{\infty}(-1)^n\binom{2HK}{n}e^{-\frac{\lambda}{HK}ns}\right\}.\end{aligned}$$

Hence as $s\to\infty$, the asymptotic behavior is

$$\mathrm{Cov}(\widehat{B}^{H,K}(t),\widehat{B}^{H,K}(t+s))=\begin{cases}\frac{K}{2^K}e^{\lambda s(1-\frac{2}{K})}+O\left(e^{\lambda s(1-\frac{4}{K})}\vee e^{\lambda s(1-\frac{1}{HK})}\right) & \text{if } H\in(0,\frac{1}{2})\\[2ex] \frac{2HK}{2^K}e^{\lambda s(1-\frac{1}{HK})}+O\left(e^{\lambda s(1-\frac{2}{K})}\right) & \text{if } H\in(\frac{1}{2},1).\end{cases}$$

(2) *The correlation of* $\widehat{S}^H$ *has an expansion*

$$\begin{aligned}\mathrm{Cov}(\widehat{S}^H(t), \widehat{S}^H(t+s)) &= \frac{e^{\lambda s}}{2-2^{2H-1}}\Big(e^{-2\lambda s}+1-\frac{1}{2}\Big\{\big(1+e^{-\frac{\lambda}{H}s}\big)^{2H}+\big(1-e^{-\frac{\lambda}{H}s}\big)^{2H}\Big\}\Big)\\ &= \frac{e^{-\lambda s}}{2-2^{2H-1}}\Big(1-\sum_{n=1}^{\infty}\binom{2H}{2n}e^{2\lambda s(1-\frac{n}{H})}\Big)\\ &= \frac{e^{-\lambda s}}{2-2^{2H-1}}+O(e^{\lambda s(1-\frac{2}{H})}),\ \text{ as } s\to\infty.\end{aligned}$$

Note that for $\widehat{B}^{H,K}$, $K=1$, the result reduces to that of Proposition 3.1 for $\widehat{B}^H$. Similarly for $\widehat{S}^H$ with $H=1/2$, the result reduces to that by BM. However for $K\neq 1$ and $H\neq 1/2$, this is not the case as one sees in Remark 3.3. In [31, Proposition 2.1], they also analyze the correlation decay for the Lamberti transform of $B^{H,K}$ with different parameterizations; our result is consistent with theirs, if we set $\lambda = HK$.

Remark 3.3. *In view of Proposition 3.1 and Lemma 3.2 , although all of processes exhibit the short memory property, autocorrelations decay in different ways. For* $\lambda>0$*, the autocorrelation of s-distant points for* $\widehat{B}^H$ *decreases faster than or the same as* $e^{-\lambda s}$ *as* $s\to\infty$*, whereas* $\widehat{B}^{H,K}$ *has the more flexible asymptotic, i.e., it decreases as* $e^{-\lambda c_{H,K}s}$ *where* $c_{H,K}$ *is any positive number adjusted by* H *and* K*. Moreover, the autocorrelation of s-distant points for* $\widehat{S}^H$ *decreases as* $e^{-\lambda s}$ *only.*

We mention that, for a stationary Gaussian process Y, the correlation decay of the power process $Y^m, m=1,2,\dots$ and of the exponential process $Z:=e^Y$ obey the following relation, which are straightforward generalizations of Proposition 2.2 in [20] (for (a) in Lemma 3.4) and Lemma 2.2 in [21] (for (b) in Lemma 3.4). In fact Proposition 2.2 in [20] is derived for fractional Ornstein-Uhlenbeck processes, but the result holds for stationary Gaussian processes in exactly the same way.

Lemma 3.4. *Let* $m=1,2,\dots$ *and let* $\{Y(t)\}_{t\in\mathbb{R}}$ *be a stationary Gaussian with variance* $\sigma^2 := \mathrm{Var}(Y(0))$.
(a) *Assume that* $\mathrm{Cov}(Y(0),Y(s))\to 0$ *as* $s\to\infty$*, then for* $s\to\infty$*,*

$$\begin{aligned}&\mathrm{Cov}\left((Y(t))^m,(Y(t+s))^m\right)\\ &= \begin{cases} m^2((m-2)!!)^2\sigma^{2(m-1)}\mathrm{Cov}(Y(0),Y(s))+O\left((\mathrm{Cov}(Y(0),Y(s)))^2\right) & \text{if } m \text{ is odd,}\\ \frac{1}{2}\left(\frac{m!(m-3)!!}{(m-2)!}\right)^2\sigma^{2(m-2)}(\mathrm{Cov}(Y(0),Y(s)))^2+O\left((\mathrm{Cov}(Y(0),Y(s)))^4\right) & \text{if } m \text{ is even.}\end{cases}\end{aligned}$$

(b) *Let* $Z:=e^Y$ *be the exponential stationary process determined by* $Y(t)$*. Then*

$$\mathrm{Cov}(Z(0),Z(s)) \gtreqless 0 \quad \textit{if and only if} \quad \mathrm{Cov}(Y(0),Y(s)) \gtreqless 0.$$

Moreover, assume that as $s\to\infty$, $\mathrm{Cov}(Y(0),Y(s))\to 0$*. Then it follows that*

$$\mathrm{Cov}(Z(0),Z(s)) = e^{\sigma^2}\left\{\mathrm{Cov}(Y(0),Y(s))+o(\mathrm{Cov}(Y(0),Y(s)))\right\}. \tag{3.1}$$

Since the correlations of the LfBM, LbfBM and LsfBM are all positive-correlated, we thus have

Proposition 3.5. *Let* $H\in(0,1]$*. We denote LfBM by* $\widehat{B}^H$ *and the associated exponential process by* $\widetilde{B}^H := e^{\widehat{B}^H}$*. Then for fixed* $t\in\mathbb{R}$*,* $m=1,3,\dots$ *and* $s\to\infty$*,*

$$\begin{aligned}&\mathrm{Cov}((\widehat{B}^H(t))^m,(\widehat{B}^H(t+s))^m)\\ &= m^2((m-2)!!)^2\begin{cases}\frac{1}{2}e^{-\lambda|s|}+O\left(e^{-\lambda(\frac{1}{H}-1)|s|}\right), & \text{if } H\in(0,\frac{1}{2}),\\ e^{-\lambda(\frac{1}{H}-1)|s|}+O(e^{-\lambda|s|}), & \text{if } H\in[\frac{1}{2},1).\end{cases}\end{aligned}$$

Then for fixed $t \in \mathbb{R}$, $m = 2, 4, \dots$ and $s \to \infty$,

$$\begin{aligned}&\mathrm{Cov}((\widehat{B}^H(t))^m, (\widehat{B}^H(t+s))^m)\\ &= \frac{1}{2}\left(\frac{m!(m-3)!!}{(m-2)!}\right)^2 \begin{cases} \frac{1}{2}e^{-\lambda|s|} + O\left(e^{-\lambda(\frac{1}{H}-1)|s|}\right) & \text{if } H \in (0, \frac{1}{2}),\\ e^{-\lambda(\frac{1}{H}-1)|s|} + O(e^{-\lambda|s|}) & \text{if } H \in [\frac{1}{2}, 1).\end{cases}\end{aligned}$$

Moreover, for fixed $t \in \mathbb{R}$ and $s \to \infty$,

$$\begin{aligned}&\mathrm{Cov}(\widetilde{B}^H(t),\, \widetilde{B}^H(t+s))\\ &= \begin{cases} \frac{1}{2}e^{1-\lambda|s|} + O\left(e^{-\lambda(\frac{1}{H}-1)|s|}\right) & \text{if } H \in (0, \frac{1}{2}),\\ He^{1-\lambda(\frac{1}{H}-1)|s|} + O\left(e^{-\lambda|s|}\right) & \text{if } H \in [\frac{1}{2}, 1).\end{cases}\end{aligned}$$

Proposition 3.6. *Let $H \in (0,1)$ and $K \in (0,2)$ such that $HK \in (0,1)$. Denote LbfBM by $\widehat{B}^{H,K}$ and its exponential by $\widetilde{B}^{H,K} := e^{\widehat{B}^{H,K}}$. Then for $m = 1, 3, 5, \dots$ and $t \in \mathbb{R}$, the covariance decay as $s \to \infty$ is given by*

$$\begin{aligned}&\mathrm{Cov}((\widehat{B}^{H,K}(t))^m, (\widehat{B}^{H,K}(t+s))^m),\\ &= m^2((m-2)!!)^2 \begin{cases} \frac{K}{2^K}\, e^{\lambda s(1-\frac{2}{K})} + O\left(e^{2\lambda s(1-\frac{2}{K})} \vee e^{\lambda s(1-\frac{1}{HK})}\right) & \text{if } H \in (0, \frac{1}{2}),\\ \frac{2HK}{2^K}\, e^{\lambda s(1-\frac{1}{HK})} + O\left(e^{\lambda s(1-\frac{2}{K})} \vee e^{2\lambda s(1-\frac{1}{HK})}\right) & \text{if } H \in (\frac{1}{2}, 1),\end{cases}\end{aligned}$$

and that for $m = 2, 4, 6, \dots$ is

$$\begin{aligned}&\mathrm{Cov}((\widehat{B}^{H,K}(t))^m, (\widehat{B}^{H,K}(t+s))^m)\\ &= \frac{1}{2}\left(\frac{m!(m-3)!!}{(m-2)!}\right)^2 \begin{cases} \frac{K^2}{2^{2K}}\, e^{2\lambda s(1-\frac{2}{K})} + O\left(e^{3\lambda s(1-\frac{2}{K})} \vee e^{\lambda s(2-\frac{2}{K}-\frac{1}{HK})}\right) & \text{if } H \in (0, \frac{1}{2}),\\ \frac{(2HK)^2}{2^{2K}}\, e^{2\lambda s(1-\frac{1}{HK})} + O\left(e^{\lambda s(2-\frac{2}{K}-\frac{1}{HK})} \vee e^{3\lambda s(1-\frac{1}{HK})}\right) & \text{if } H \in (\frac{1}{2}, 1).\end{cases}\end{aligned}$$

Moreover, for fixed $t \in \mathbb{R}$ and $s \to \infty$,

$$\mathrm{Cov}(\widetilde{B}^{H,K}(t), \widetilde{B}^{H,K}(t+s)) = \begin{cases} \frac{Ke}{2^K} e^{\lambda s(1-\frac{2}{K})} + o\left(e^{\lambda s(1-\frac{2}{K})}\right) & \text{if } H \in (0, \frac{1}{2}),\\ \frac{2HKe}{2^K} e^{\lambda s(1-\frac{1}{HK})} + o\left(e^{\lambda s(1-\frac{1}{HK})}\right) & \text{if } H \in (\frac{1}{2}, 1).\end{cases}$$

Proposition 3.7. *Let $H \in (0,1)$ and denote LsfBM by $\widehat{S}^H$ its exponential by $\widetilde{S}^H := e^{\widehat{S}^H}$. Then for $t \in \mathbb{R}$ and $m = 1, 3, 5, \dots$, the correlation decay by $s \to \infty$ is*

$$\mathrm{Cov}((\widehat{S}^H(t))^m, (\widehat{S}^H(t+s))^m) = m^2\big((m-2)!!\big)^2 \frac{e^{-\lambda s}}{2 - 2^{2H-1}} + O(e^{\lambda s(1-\frac{2}{H})})$$

and for $m = 2, 4, 6, \dots$,

$$\mathrm{Cov}((\widehat{S}^H(t))^m, (\widehat{S}^H(t+s))^m) = \frac{1}{2}\Big(\frac{m!(m-3)!!}{(m-2)!}\Big)^2 \frac{e^{-2\lambda s}}{(2 - 2^{2H-1})^2} + O(e^{-\frac{2\lambda s}{H}} \vee e^{-4\lambda s}).$$

Moreover, for $t \in \mathbb{R}$ and $s \to \infty$

$$\mathrm{Cov}(\widetilde{S}^H(t), \widetilde{S}^H(t+s)) = \frac{e^{1-\lambda s}}{2 - 2^{2H-1}} + O(e^{\lambda s(1-\frac{2}{H})}).$$

3.2. **Expected maximal increments.** We present maximal inequalities for the Lamperti processes LfBM $\widehat{B}^H$, LbfBM $\widehat{B}^{H,K}$ and LsfBM $\widehat{S}^H$ treated in this article and their exponentials $\widetilde{B}^H$, $\widetilde{B}^{H,K}$ and $\widetilde{S}^H$. The idea is to make use of the stationarity of Lamperti processes and decompositions of self-similar processes. As far as exponentials of stationary processes generated by self-similar Gaussian processes are concerned, expect for that by BM, only a few results are known, e.g., that of fractional Ornstein-Uhlenbeck process or CARMA processes; see [20] or [21].

We start with LfBM $\widetilde{B}^H$ and its exponential.

Proposition 3.8. *Let $H \in [\frac{1}{2}, 1)$ and $m = 1, 2, \dots$ and denote LfBM by $\widehat{B}^H$ and its exponential by $\widetilde{B}^H := e^{\widehat{B}^H}$. Then for $s \in \mathbb{R}$ and $r \in (0, 1)$,*

$$\frac{E\left[\max_{s\le t\le s+r} |\widehat{B}^H(t) - \widehat{B}^H(s)|^m\right]}{m!} \le C^m \frac{r^{Hm}}{\sqrt{m!}}, \tag{3.2}$$

and

$$E\left[\max_{s\le t\le s+r} |\widetilde{B}^H(t) - \widetilde{B}^H(s)|\right] \le C' r^H, \tag{3.3}$$

where C and C' are positive constants which are taken uniformly in m.

Next we analyze LbfBM $\widehat{B}^{H,K}$ and its exponential $\widetilde{B}^{H,K}$, using of decompositions (2.2) and (2.3).

Proposition 3.9. *Let $H \in (0, 1)$, $K \in (0, 2)$ such that $HK \in (\frac{1}{2}, 1)$ and $m = 1, 2, \dots$ denote LbfBM by $\widehat{B}^{H,K}$ and its exponential by $\widetilde{B}^{H,K} := e^{\widehat{B}^{H,K}}$, then for $s \in \mathbb{R}$ and $r \in (0, 1)$,*

$$\frac{E\left[\max_{s\le t\le s+r} |\widehat{B}^{H,K}(t) - \widehat{B}^{H,K}(s)|^m\right]}{m!} \le C^m \frac{r^{HKm}}{\sqrt{m!}}, \tag{3.4}$$

and

$$E\left[\max_{s\le t\le s+r} |\widetilde{B}^{H,K}(t) - \widetilde{B}^{H,K}(s)|\right] \le C' r^{HK}, \tag{3.5}$$

where C and C' are positive constants and we take these constants uniformly in m.

Finally, we present results for LsfBM $\widehat{S}^H$ and its exponential $\widehat{S}^H$; similarly as before we utilize decompositions (2.5).

Proposition 3.10. *Let $H \in (\frac{1}{2}, 1)$ and $m = 1, 2, \dots$ denote LsfBM by $\widehat{S}^H$ and its exponential by $\widetilde{S}^H := e^{\widehat{S}^H}$, then for $s \in \mathbb{R}$ and $r \in (0, 1)$,*

$$\frac{E\left[\max_{s\le t\le s+r} |\widehat{S}^H(t) - \widehat{S}^H(s)|^m\right]}{m!} \le C^m \frac{r^{Hm}}{\sqrt{m!}}, \tag{3.6}$$

and

$$E\left[\max_{s\le t\le s+r} |\widetilde{S}^H(t) - \widetilde{S}^H(s)|\right] \le C' r^H, \tag{3.7}$$

where C and C' are positive constants and we take these constant uniformly in m.

Remark 3.11. *1. All Lamperti transforms and their exponentials have analogous bounds for their expected maxima of small increments. The results are naturally understandable, since they are derived from the self-similar Gaussian processes and each bound reflects the corresponding self-similar parameter of the underlying process.*
2. There is literature to discuss the maximum distribution and inequality of fBM; one can see such results in [23], in the monographs [11, 22]), and in a recent overview [26]. Most of them combine

a Gaussian-Markov process $\overline{B}^H$ with the Slepian's lemma or combine martingale inequalities with a Gaussian martingale process $M(t)$ such that its variance is ct^{2-2H} for some constant $c > 0$. All the derived bounds have relations with the self-similar parameter H. Our results on maximal increments for fBM are comparable with these existing literature, since fBM is of stationary increments with $B^H(0) = 0$. However, our results on maximal increments for other two processes, bfBM and sfBM, are different, since they are not stationary increments, and have not been studied so far. In view of Propositions 3.9, our results for maximal increments have similar relations with self-similar parameters as in that for fBM; in this sense the results are nearly optimal.
3. As for the distribution of maximum of stationary Gaussian processes, a large number of studies have been conducted: the tail probability of maximum, ([14], [19], and [7]), the inequality for distributions of maximum of two different Gaussian processes ([30]). Variations of these are found in the monograph for general Gaussian processes [1], and also in [2]. In [8] and [9] the authors evaluate the supremum distribution of Gaussian processes with the stationary increments based on extreme value theory and apply the result to queueing analysis. However, these are not applicable to our purpose. Note that our target processes (Lamperti transforms and their exponentials) are based on self-similar processes and every bound for maximal increments is as a whole controlled by the self-similar parameter.

4. Discussion

This section discusses the role of the exponential stationary processes in stochastic modelling. Consider the exponential processes $\widetilde{B}^H, \widetilde{B}^{H,K}, \widetilde{S}^H(t)$, and denote each of them by a common $\widetilde{Z}(t)$, then by the results presented in Section 3, $\widetilde{Z}(t)$ has features: (1) *strictly stationary in* t; (2) *positive valued*; (3) *positive correlated in any two time instants* s, t ; (4) *the correlation decay in the time lag* $[t, t+s]$ *is fast in* s*, indeed it is of exponential decay*; and (5) *for an arbitrarily fixed* s*, the expected maximum increments can form a summable sequence,*

$$E\left[\max_{s\le t\le s+b^{-k}} |\widetilde{Z}(t) - \widetilde{Z}(s)|\right] \le C' b^{-kH},\ k = 1, 2, \dots$$

where we may choose any suitable $b > 1$, uniformly over all s.

Therefore, the mean 1 process,

$$\frac{\widetilde{Z}(t)}{E[\widetilde{Z}(t)]},$$

can be used as a mother process to generate a certain multifractal stochastic infinite-product process, which is related to the burst phenomenon of Internet communications; see Section 3 of [20] (this paper studied the exponential OU transform of fBM), and an earlier paper [18] (this paper studied the general schemes to generate infinite-product processes).

Moreover, positive stationary processes are often required in applications since many real life data are non-negative. For instance, in the continuous time stochastic volatility models ([5]), the log-price of risky asset $P(t)$ is represented as

$$dP(t) = (\mu + \beta\sigma(t))dt + \sqrt{\sigma(t-)}dW(t),$$

where $\sigma(t)$ is a positive stationary process and W is BM. The simplest one for σ is the exponential of the ordinary OU process. More complex alternatives include OU by non-negative Lévy processes and their variations. The solutions of different SDEs involving W are also considered (e.g., Hull-White model or Vasicek model). In financial time-series, both stationarity and positivity (sometimes long memory or jumps) are essential for the volatility processes $\sigma(t)$. Then noticing that the OU process is the Lamperti transform of BM, the exponential of other transformed processes $\widetilde{Z}(t)$ could be good candidates. They are simply defined and model correlation decay more flexibly than that by OU; in addition we could theoretically characterize the sign of auto-correlation functions.

5. Proofs

Proof of Lemma 2.1. (1) Without loss of generality we let $t \geq s$. We observe a function

$$\begin{aligned} f_1(t;s) &:= \Upsilon_{H,K}(s,t) - \Upsilon_{\widetilde{HK}}(s,t) \\ &= \frac{1}{2^K}\{(t^{2H}+s^{2H})^K - (2s^{2H})^K - (t-s)^{2HK}\} \end{aligned}$$

and its partial derivative with t,

$$f_1'(t;s) = \frac{2HK}{2^K} t^{2HK-1}\{(1+(s/t)^{2H})^{K-1} - (1-s/t)^{2HK-1}\},$$

such that $f_1(t;s)$ is a function of t with parameter s. Then noticing the sign in the brace of $f_1'(t;s)$, for $K \in (1,2) \cap HK \in (\frac{1}{2},1)$ we have $f_1'(t;s) \geq 0$ which yields $f_1(t;s) \geq f_1(s;s) = 0$. On the contrary for $K \in (0,1) \cap HK \in (0,\frac{1}{2})$, it follows that $f_1'(t;s) \leq 0$, which concludes $f_1(t;s) \leq f_1(s;s) = 0$.

In order to obtain results for (i) $K \in (0,1) \cap HK \in (\frac{1}{2},1)$ and (ii) $K \in (1,2) \cap HK \in (0,\frac{1}{2})$, we further analyze the sign of $f_1'(t;s)$; namely we analyze

$$g(x) = (1+x^{2H})^{K-1} - (1-x)^{2HK-1}, \ x \in [0,1]$$

with $g(0) = 0$ and $g(1) = 2^{K-1}$. (i) Noticing $H \in (\frac{1}{2},1)$ and the derivative

$$g'(x) = 2H(K-1)(1+x^{2H})^{K-2}x^{2H-1} + (2HK-1)(1-x)^{2HK-2},$$

we have $g'(x) \geq 0$ for $2H(1-K) \geq 2HK-1$, which implies $f_1'(s,t) \geq 0$. Hence we obtain the result (i). (ii) Noticing $H \in (\frac{1}{2},1)$, we observe that

$$\begin{aligned} g''(x) &= 2H(K-1)(K-2)(1+x^{2H})^{K-3}2H(x^{2H-1})^2 \\ &\quad +2H(K-1)(2H-1)(1+x^{2H})^{K-2}x^{2H-2} \\ &\quad -(2HK-1)(2HK-2)(1-x)^{2HK-3} \leq 0. \end{aligned}$$

Now the concavity of $g(x)$ implies $g(x) \geq 0$.

Finally the last inequality follows from Slepian's lemma,

$$P\Big(\max_{0\leq t\leq 1} B^{H,K}(t) \geq a\Big) \leq P\Big(\max_{0\leq t\leq 1} \overline{B}^{HK}(t) \geq a\Big)$$

and the symmetric property and the reflection principle of a Gaussian Markov process as in the proof of Lemma 2.3 in [20]. Notice that $\overline{B}^{HK}$ is a deterministic time change of B.

(2) Without loss of generality we let $t \geq s$. The inequality of the right hand side is implied by

$$\mathcal{S}_{\widetilde{H}}(s,t) - \Upsilon_{\widetilde{H}}(s,t) = t^{2H} - \frac{1}{2}\{(t+s)^{2H} + (t-s)^{2H}\} \gtreqqless 0 \qquad \text{if } H \lesseqqgtr \frac{1}{2}.$$

Regarding the inequality of the center, we let

$$\begin{aligned} f_2(t;s) &:= \mathcal{S}_H(s,t) - \Upsilon_{\widetilde{H}}(s,t) \\ &= \frac{1}{2-2^{2H-1}}[t^{2H} - s^{2H} - \frac{1}{2}\{(s+t)^{2H} - (2s)^{2H} + (t-s)^{2H}\}], \end{aligned}$$

which we regard as a function of t given s. Since the differential with t yields

$$f_2'(t;s) = \frac{2H}{2-2^{2H-1}}\{t^{2H-1} - \frac{(s+t)^{2H-1}+(t-s)^{2H-1}}{2}\} \gtreqqless 0 \qquad \text{if } H \gtreqqless \frac{1}{2},$$

noticing $f_2(s;s) = 0$ we conclude that

$$\mathcal{S}_H(s,t) \gtreqqless \Upsilon_{\widetilde{H}}(s,t) \qquad \text{if } H \gtreqqless \frac{1}{2}.$$

In order to analyze

$$\begin{aligned}&\mathcal{S}_H(s,t)-\Upsilon_H(s,t)\\&=\frac{2^{2H-1}}{2-2^{2H-1}}\Big\{\frac{s^{2H}+t^{2H}}{2}-\Big(\frac{t+s}{2}\Big)^{2H}+\Big(\frac{t-s}{2}\Big)^{2H}-\frac{(t-s)^{2H}}{2}\Big\},\end{aligned}$$

we define a function of a with parameter b as

$$f_3(a;b)=\frac{a^{2H}+(a+b)^{2H}}{2}-\Big(\frac{2a+b}{2}\Big)^{2H},\ a\ge 0,\, b\ge 0,$$

such that its derivative satisfies

$$f_3'(a;b)=2H\Big(\frac{a^{2H-1}+(a+b)^{2H-1}}{2}-\Big(\frac{2a+b}{2}\Big)^{2H-1}\Big)\lesseqqgtr 0\quad\text{for } H\gtreqqless\frac12,$$

from which we know that f_3 is non-increasing (resp. non-decreasing) for $H\in(\frac12,1)$ (resp. $H\in(0,\frac12)$) as a function of a. Now putting $b=t-s$, we observe that

$$\mathcal{S}_H(s,t)-\Upsilon_H(s,t)=\frac{2^{2H-1}}{2-2^{2H-1}}\big(g(s)-g(0)\big)\lesseqqgtr 0\quad\text{for } H\gtreqqless\frac12.$$

The probability of maximal increments is bounded in the same manner as before.
(3) For $H\in(0,\frac12)$, the result is implied by

$$\chi_{\overline{H}}(s,t)-\Upsilon_{\overline{H}}(s,t)=\frac{1}{2-2^{2H}}\big[t^{2H}-s^{2H}-\{(t+s)^{2H}-(2s)^{2H}\}\big]\ge 0.$$

For $H\in(\frac12,1)$, it suffices to observe

$$\chi_{\overline{H}}(s,t)-\Upsilon_H(s,t):=\frac{2-2^{2H-1}}{2(2^{2H}-2)}(\Upsilon_H(s,t)-\mathcal{S}_H(s,t))\ge 0.$$

Hence the maxima of the process is bounded by that of $\overline{B}^H$ for $H\in(0,\frac12)\cup H\in(\frac12,1)$ similarly as in the proof for (1). $\square$

Proof of Lemma 2.3. In the proof, constants c_i^K, $i=1,2,\dots$ will denote positive constants depending on $K\in(0,2)$ for which the exact values are irrelevant and may vary from line to line.
(1) The law of the iterated logarithm for B at 0 and ∞ assures the existence of the pathwise integral and the integral by parts for X^K, $K\in(0,1)$, which yields

$$\begin{aligned}X^K(t)-X^K(s)&=\int_0^\infty(e^{-us}-e^{-ut})u^{-\frac{1+K}{2}}dB(u)\\&=\int_0^\infty(se^{-us}-te^{-ut})u^{-\frac{1+K}{2}}B(u)du\\&\quad+\frac{1+K}{2}\int_0^\infty(e^{-ut}-e^{-us})u^{-\frac{3+K}{2}}B(u)du,\quad t\ge s>0.\end{aligned}$$

By applying the inequality $1-e^{-x}\le x$, $x\ge 0$ and the triangle inequality several times we obtain

$$\begin{aligned}|X^K(t)-X^K(s)|&\le\int_0^\infty|se^{-us}-te^{-ut}|u^{-\frac{1+K}{2}}|B(u)|du\\&\quad+\frac{1+K}{2}\int_0^\infty|e^{-ut}-e^{-us}|u^{-\frac{3+K}{2}}|B(u)|du\\&\le(t-s)\int_0^\infty e^{-ut}u^{-\frac{1+K}{2}}|B(u)|du+s\int_0^\infty|e^{-u(t-s)}-1|e^{-us}u^{-\frac{1+K}{2}}|B(u)|du\\&\quad+\frac{1+K}{2}\int_0^\infty|e^{-u(t-s)}-1|e^{-us}u^{-\frac{3+K}{2}}|B(u)|du\end{aligned}$$

$$\leq (t-s)\int_0^\infty \Big(\frac{3+K}{2}+us\Big)u^{-\frac{1+K}{2}}e^{-us}|B(u)|du,$$

which yields via Hölder's inequality,

$$\begin{aligned}
&\max_{s\leq t\leq s+r}|X^K(t)-X^K(s)|^m\\
&\leq \max_{s\leq t\leq s+r}(t-s)^m\Big(\int_0^\infty \Big(\frac{3+K}{2}+us\Big)^2u^{-\frac{1+K}{2}}e^{-us}du\Big)^{\frac{m}{2}}\\
&\times\Big(\int_0^\infty e^{-us}u^{-\frac{1+K}{2}}|B(u)|^2du\Big)^{\frac{m}{2}}\\
&\leq r^m\big(c_1^K s^{\frac{K-1}{2}}\big)^{\frac{m}{2}}\int_0^\infty e^{-\frac{m}{2}su}u^{-\frac{m}{4}(1+K)}|B(u)|^m du,
\end{aligned}$$

where in the last step we directly bound the fisrt integral by

$$\begin{aligned}
2\int_0^\infty\left\{\left(\frac{3+K}{2}\right)^2+s^2u^2\right\}e^{-us}u^{-\frac{1+K}{2}}du &= 2\left(\frac{3+K}{2}\right)^2 s^{\frac{K-1}{2}}\Gamma\left(\frac{1-K}{2}\right)+2s^{\frac{K-1}{2}}\Gamma\left(\frac{5-K}{2}\right)\\
&=: c_1^K s^{\frac{K-1}{2}}.
\end{aligned}$$

In the middle term of inequalities, the first deterministic integral exists with $K\in(0,1)$. Noticing $E[|B(t)|^m]\leq t^{\frac{m}{2}}(m-1)!!$ (see p. 605 of [20]), we take expectation to obtain

$$\begin{aligned}
E\big[\max_{s\leq t\leq s+r}|X^K(t)-X^K(s)|^m\big] &\leq r^m\big(c_1^K\, s^{\frac{K-1}{2}}\big)^{\frac{m}{2}}\int_0^\infty e^{-\frac{m}{2}su}u^{-\frac{m}{4}(1+K)}E[|B(u)|^m]du\\
&\leq r^m\big(c_1^K\, s^{\frac{K-1}{2}}\big)^{\frac{m}{2}}(m-1)!!\int_0^\infty e^{-\frac{m}{2}su}u^{\frac{m}{4}(1-K)}du\\
&\leq r^m\big(c_1^K\, s^{\frac{K-1}{2}}\big)^{\frac{m}{2}}(m-1)!!\Big(\frac{s}{2}\Big)^{\frac{m}{4}(K-1)-1}\int_0^\infty e^{-v}v^{\frac{1-K}{4}}dv\\
&\leq r^m\big(c_2^K\, s^{\frac{(K-1)}{2}}\big)^m s^{-1}(m-1)!!,
\end{aligned}$$

where in the third step we use a change of variables formula and the fact $e^{-v}v^{\frac{1-K}{4}}\leq 1$ for $v>0$. Hence the result is concluded.

(2) In case $K\in(1,2)$, another representation for X^K (refer to [17, Theorem 2] or [27, Remark 3.1]),

$$X^K(t)=\int_0^t Y^K(u)du\qquad\text{where}\qquad Y^K(t)=\int_0^\infty u^{\frac{1-K}{2}}e^{-ut}dB(u)$$

yields

$$\begin{aligned}
|X^K(t)-X^K(s)| &\leq \Big(\int_s^t|Y^K(u)|^m du\Big)^{\frac{1}{m}}\Big(\int_s^t 1^{\frac{m}{m-1}}du\Big)^{\frac{m-1}{m}}\\
&\leq (t-s)^{\frac{m-1}{m}}\Big(\int_s^t|Y^K(u)|^m du\Big)^{\frac{1}{m}},
\end{aligned}$$

where we use the Hölder's inequality. Accordingly

$$\begin{aligned}
E[\max_{s\leq t\leq s+r}|X^K(t)-X^K(s)|^m] &\leq r^{m-1}\int_s^{s+r}E[|Y^K(u)|^m]du\\
&\leq (\Gamma(2-k)2^{K-2})^{\frac{m}{2}}(m-1)!!r^{m-1}\int_s^{s+r}u^{\frac{m}{2}(K-2)}du\\
&\leq r^m(s^{\frac{K-2}{2}}c_3^K)^m(m-1)!!,
\end{aligned}$$

where we use $E[|Y^K(u)|^m]=\big(\Gamma(2-K)2^{K-2}\big)^{\frac{m}{2}}(m-1)!!u^{\frac{m}{2}(K-2)}$. □

Proof of Lemma 3.2. Let X be a centered Gaussian process and then the covariance of Lamperti transform $\widehat{X}(u) = e^{-\lambda u} X(e^{\frac{\lambda}{H} u})$ for $s, t > 0$ has the expression

$$\mathrm{Cov}\left(\widehat{X}(t), \widehat{X}(t+s)\right) = e^{-\lambda(2t+s)} \mathrm{Cov}\left(X(e^{\frac{\lambda}{H} t}), X(e^{\frac{\lambda}{H}(t+s)})\right).$$

Substituting $B^{H,K}$ and S^H into X respectively and using expressions in Definition 1.1, we obtain the first expressions of covariances for $B^{H,K}$ and S^H, which do not depend on $t \geq 0$ by the stationarity. To obtain asymptotic behavior, it suffices to apply the binomial expansions before taking $s \to \infty$. □

Proof of Proposition 3.8. Throughout the proof, d_i, $i = 1, 2, \ldots$ will denote positive constants for which the exact values are irrelevant and may vary from line to line. We start to see the distribution of increments for $s \leq t \leq s + r$:

$$\begin{aligned}
\widehat{B}^H(t) - \widehat{B}^H(s) &\overset{d}{=} \widehat{B}^H(t-s) - \widehat{B}^H(0) \\
&= e^{-\lambda(t-s)} B^H(e^{\frac{\lambda}{H}(t-s)}) - B^H(1) \\
&= (e^{-\lambda(t-s)} - 1) B^H(e^{\frac{\lambda}{H}(t-s)}) + B^H(e^{\frac{\lambda}{H}(t-s)}) - B^H(1),
\end{aligned}$$

where we use the stationarity of $\widehat{B}^H$. Then we have

$$\begin{aligned}
&E\Big[\big(\max_{s \leq t \leq s+r} |\widehat{B}^H(t) - \widehat{B}^H(s)|\big)^m\Big] \\
&\leq E\Big[\big(\max_{s \leq t \leq s+r} \big|(e^{-\lambda(t-s)} - 1) B^H(e^{\frac{\lambda}{H}(t-s)})\big| + \max_{s \leq t \leq s+r} \big|B^H(e^{\frac{\lambda}{H}(t-s)}) - B^H(1)\big|\big)^m\Big] \\
&\leq \sum_{k=0}^m \binom{m}{k} E\Big[\big(\max_{s \leq t \leq s+r} (1 - e^{-\lambda(t-s)}) \big|B^H(e^{\frac{\lambda}{H}(t-s)})\big|\big)^k \\
&\qquad\qquad \times \big(\max_{s \leq t \leq s+r} \big|B^H(e^{\frac{\lambda}{H}(t-s)}) - B^H(1)\big|\big)^{m-k}\Big] \\
&\leq \sum_{k=0}^m \binom{m}{k} (1 - e^{-\lambda r})^k \Big(E\big[\big(\max_{s \leq t \leq s+r} |B^H(e^{\frac{\lambda}{H}(t-s)})|\big)^{2k}\big]\Big)^{1/2} \\
&\qquad\qquad \times \Big(E\big[\big(\max_{s \leq t \leq s+r} |B^H(e^{\frac{\lambda}{H}(t-s)} - 1)|\big)^{2(m-k)}\big]\Big)^{1/2}.
\end{aligned} \tag{5.1}$$

From Lemma 2.2, the calculation proceeds as

$$\begin{aligned}
E\Big[\big(\max_{s \leq t \leq s+r} |B^H(e^{\frac{\lambda}{H}(t-s)})|\big)^{2k}\Big] &= E\Big[\big(\max_{0 \leq t \leq r} |B^H(e^{\frac{\lambda}{H} t})|\big)^{2k}\Big] \\
&\leq \big(e^{\frac{\lambda}{H} r}\big)^{2kH} 2\,(2k-1)!! \\
&\leq \big(e^{\lambda r}\big)^{2k} 2^k\, k!
\end{aligned}$$

and

$$\begin{aligned}
E\Big[\big(\max_{s \leq t \leq s+r} |B^H(e^{\frac{\lambda}{H}(t-s)} - 1)|\big)^{2(m-k)}\Big] &= E\Big[\big(\max_{0 \leq t \leq r} |B^H(e^{\frac{\lambda}{H} t} - 1)|\big)^{2(m-k)}\Big] \\
&= E\Big[\big(\max_{0 \leq t \leq (e^{\frac{\lambda}{H} r} - 1)} |B^H(t)|\big)^{2(m-k)}\Big] \\
&\leq \big(e^{\frac{\lambda}{H} r} - 1\big)^{2(m-k)H} 2\,(2(m-k)-1)!! \\
&\leq \big(e^{\frac{\lambda}{H} r} - 1\big)^{2(m-k)H} 2^{m-k}\,(m-k)!,
\end{aligned}$$

where we use relations

$$(2n)!! = 2^n n! \quad \text{and} \quad (2n-1)!! = (2n)!/(2^n n!).$$

Put these two bounds in (5.1) to obtain

$$\begin{aligned}
&\frac{E\Big[\big(\max_{s\le t\le s+r}|\widehat{B}^H(t)-\widehat{B}^H(s)|\big)^m\Big]}{m!}\\
&\le \frac{1}{\sqrt{m!}}\sum_{k=0}^{m}\binom{m}{k}\sqrt{\frac{k!(m-k)!}{m!}}\big\{(1-e^{-\lambda r})\sqrt{2}e^{\lambda r}\big\}^k\big\{\sqrt{2}(e^{\frac{\lambda}{H}r}-1)^H\big\}^{m-k}\\
&\le \frac{1}{\sqrt{m!}}\big\{(1-e^{-\lambda r})\sqrt{2}e^{\lambda r}+\sqrt{2}(e^{\frac{\lambda}{H}r}-1)^H\big\}^m\\
&= \frac{(\sqrt{2}e^{\lambda r})^m}{\sqrt{m!}}\big\{(1-e^{-\lambda r})+(1-e^{-\frac{\lambda}{H}r})^H\big\}^m\\
&\le \frac{d_1^m}{\sqrt{m!}}(\lambda r+(\lambda/H\cdot r)^H)^m\\
&\le \frac{d_2^m}{\sqrt{m!}}r^{Hm},
\end{aligned}$$

where in the last step we use $r\in(0,1)$ and $H\in[\frac{1}{2},1)$. Now we obtain the first result (3.2).

Next we show the maxima for $\widetilde{B}^H$, i.e., (3.3). The stationarity of $\widetilde{B}^H$ gives

$$\begin{aligned}
\max_{s\le t\le s+r}|\widetilde{B}^H(t)-\widetilde{B}^H(s)| &= \max_{s\le t\le s+r}|e^{\widehat{B}^H(t)}-e^{\widehat{B}^H(s)}|\\
&\stackrel{d}{=} \max_{s\le t\le s+r}|e^{\widehat{B}^H(t-s)}-e^{\widehat{B}^H(0)}|\\
&= e^{\widehat{B}^H(0)}\max_{0\le t\le r}|e^{\widehat{B}^H(t)-\widehat{B}^H(0)}-1|.
\end{aligned}$$

By Schwartz inequality and stationarity

$$E\Big[\max_{s\le t\le s+r}|\widetilde{B}^H(t)-\widetilde{B}^H(s)|\Big]\le e^{\mathrm{Var}(\widehat{B}^H(0))}\Big(E\Big[\max_{0\le t\le r}|e^{\widehat{B}^H(t)-\widehat{B}^H(0)}-1|^2\Big]\Big)^{1/2}. \tag{5.2}$$

Since $\widehat{B}^H$ has a continuous version, it is bounded on $0\le s\le r$ and we can use the expansion $e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}$ on $\widehat{B}^H$. Then elementary calculations show that

$$\begin{aligned}
&\max_{0\le s\le r}\big|e^{\widehat{B}^H(s)-\widehat{B}^H(0)}+1\big|^2\\
&=\max_{0\le s\le r}\big|e^{2(\widehat{B}^H(s)-\widehat{B}^H(0))}-2e^{\widehat{B}^H(s)-\widehat{B}^H(0)}+1\big|\\
&=\max_{0\le s\le r}\Big|\sum_{m=1}^{\infty}\frac{\{2(\widehat{B}^H(s)-\widehat{B}^H(0)\}^m}{m!}-2\sum_{m=1}^{\infty}\frac{(\widehat{B}^H(s)-\widehat{B}^H(0))^m}{m!}\Big|\\
&=\max_{0\le s\le r}\Big|(\widehat{B}^H(s)-\widehat{B}^H(0))^2+\sum_{m=3}^{\infty}\frac{\{2(\widehat{B}^H(s)-\widehat{B}^H(0))\}^m}{m!}-2\sum_{m=3}^{\infty}\frac{(\widehat{B}^H(s)-\widehat{B}^H(0))^m}{m!}\Big|\\
&\le \max_{0\le s\le r}(\widehat{B}^H(s)-B^H(0))^2+\sum_{m=3}^{\infty}\max_{0\le s\le r}\frac{\{2|\widehat{B}^H(s)-\widehat{B}^H(0)|\}^m}{m!}\\
&+\sum_{m=3}^{\infty}\frac{\max_{0\le s\le r}|\widehat{B}^H(s)-B^H(0)|^m}{m!}.
\end{aligned}$$

Due to the first result (3.2), it follows that

$$E\Big[\max_{0\le s\le r}|e^{\widehat{B}^H(s)-\widehat{B}^H(0)}-1|^2\Big] \le E\Big[\max_{0\le s\le r}(\widehat{B}^H(s)-\widehat{B}^H(0))^2\Big]$$

$$
\begin{aligned}
&+\sum_{m=3}^{\infty}\frac{E[\max_{0\le s\le r}2^m|\widehat{B}^H(s)-\widehat{B}^H(0)|^m]}{m!}\\
&+\sum_{m=3}^{\infty}\frac{E[\max_{0\le s\le r}|\widehat{B}^H(s)-\widehat{B}^H(0)|^m]}{m!}\\
&=\sqrt{2}d_1^2r^{2H}+\sum_{m=3}^{\infty}\frac{d_1^m2^mr^{Hm}}{\sqrt{m!}}+\sum_{m=3}^{\infty}\frac{d_1^mr^{Hm}}{\sqrt{m!}}\\
&=\sqrt{2}d_1^2r^{2H}+r^{3H}\sum_{m=3}^{\infty}\frac{d_1^m2^mr^{H(m-3)}}{\sqrt{m!}}+r^{3H}\sum_{m=3}^{\infty}\frac{d_1^m}{\sqrt{m!}}r^{H(m-3)}\\
&=r^{2H}(d_2+d_3r^H)
\end{aligned}
$$

for all $r\in(0,1)$, where in the last step, we use Stirling's formula for the convergence of infinite sums. Substitute this into (5.2) and observe that

$$E\big[\max_{0\le s\le r}\big|\widehat{B}^H(s)-\widehat{B}^H(0)\big|\big]\le d_6e^{\mathrm{Var}(\widehat{B}^H(0))}\sqrt{d_4+d_5}\cdot r^H.$$

Now putting

$$C':=d_6e^{\mathrm{Var}(\widehat{B}^H(0))}\sqrt{d_4+d_5},$$

we obtain the result. □

Proof of Proposition 3.9. For the proof we rely on the following lemma.

Lemma 5.1. *Let $H\in(0,1)$, $K\in(0,2)$ and $HK\in(\frac{1}{2},1)$. Then for $r\in(0,1)$ we have*

$$E\Big[\big(\max_{1\le t\le 1+r}|B^{H,K}(t)|\big)^m\Big]\le(1+r)^{HKm}2(m-1)!! \tag{5.3}$$

and

$$E\Big[\big(\max_{1\le t\le 1+r}|B^{H,K}(t)-B^{H,K}(1)|\big)^m\Big]\le r^{HKm}D_1^m\sqrt{m!}, \tag{5.4}$$

where D_1 is a bounded positive constant which we can take uniformly in m.

Proof of Lemma 5.1. We observe from Lemma 2.1 that for $a\ge 0$,

$$
\begin{aligned}
P\Big(\max_{1\le t\le 1+r}|B^{H,K}(t)|\ge a\Big) &= P\Big(\max_{(1+r)^{-1}\le t\le 1}|B^{H,K}(t)|\ge a(1+r)^{-HK}\Big)\\
&\le P\Big(\max_{0\le t\le 1}|B^{H,K}(t)|\ge a(1+r)^{-HK}\Big)\\
&\le P\Big(\max_{0\le t\le 1}B^{H,K}(t)\ge a(1+r)^{-HK}\Big)\\
&\quad +P\Big(\min_{0\le t\le 1}B^{H,K}(t)\le -a(1+r)^{-HK}\Big)\\
&= 2P\Big(\max_{0\le t\le 1}B^{H,K}(t)\ge a(1+r)^{-HK}\Big)\\
&\le 4P\Big(\overline{B}^{HK}(1)\ge a(1+r)^{-HK}\Big),
\end{aligned}
$$

where in the third step we use the symmetry of a Gaussian process. This together with Lemma 2.2 yields

$$
\begin{aligned}
E\left[\Big(\max_{1\le t\le 1+r}|B^{H,K}(t)|\Big)^m\right] &\le \int_0^\infty my^{m-1}P\Big(\max_{1\le t\le 1+r}|B^{H,K}(t)|\ge y\Big)dy\\
&\le 4\int_0^\infty my^{m-1}P\Big(\overline{B}^{HK}(1)\ge(1+r)^{-HK}y\Big)dy
\end{aligned}
$$

$$
\begin{aligned}
&\le 2\sqrt{\frac{2}{\pi}}\int_0^\infty m y^{m-1}dy\int_{(1+r)^{-HK}y}^\infty e^{-x^2/2}dx\\
&= (1+r)^{HKm}2\sqrt{\frac{2}{\pi}}\int_0^\infty x^m e^{-x^2/2}dx\\
&\le (1+r)^{HKm}2(m-1)!!.
\end{aligned}
$$

Hence we prove (5.3).

Next we show (5.4) for $K\in(0,1)$ with the triangular inequality

$$
\begin{aligned}
|B^{H,K}(t)-B^{H,K}(s)| \le\ & c_1|X^K(t^{2H})-X^K(s^{2H})|\\
&+|B^{H,K}(t)-B^{H,K}(s)+c_1(X^K(t^{2H})-X^K(s^{2H}))|
\end{aligned}
\tag{5.5}
$$

and the relation

$$
\max_{1\le t\le 1+r}\{B^{H,K}(t)-B^{H,K}(1)+c_1(X^K(t^{2H})-X^K(1))\}\stackrel{d}{=}\max_{1\le t\le 1+r}\{c_2(B^{HK}(t)-B^{HK}(1))\},
$$

which follows from the decomposition (2.2) and the continuity of the processes $B^{H,K}$, B^{HK} and X^K. From these facts the expected maximal increments satisfy

$$
\begin{aligned}
&E\Big[\big(\max_{1\le t\le 1+r}|B^{H,K}(t)-B^{H,K}(1)|\big)^m\Big]\\
&\le\sum_{k=0}^m\binom{m}{k}\sqrt{E\Big[\big(\max_{1\le t\le 1+r}|B^{HK}(t)-B^{HK}(1)|\big)^{2k}\Big]}\\
&\qquad\times c_1^{m-k}\sqrt{E\Big[\big(\max_{1\le t\le 1+r}|X^K(t^{2H})-X^K(1)|\big)^{2(m-k)}\Big]}\\
&\le\sum_{k=0}^m\binom{m}{k}c_2^k\sqrt{E\Big[\big(\max_{0\le t\le r}|B^{HK}(t)|\big)^{2k}\Big]}c_1^{m-k}\sqrt{E\Big[\big(\max_{1\le t\le(1+r)^{2H}}|X^K(t)-X^K(1)|\big)^{2(m-k)}\Big]}\\
&\le\sum_{k=0}^m\binom{m}{k}c_2^kc_1^{m-k}\sqrt{r^{2HKk}2(2k-1)!!}\sqrt{\big\{C_K((1+r)^{2H}-1)\big\}^{2(m-k)}(2(m-k)-1)!!}\\
&=\sum_{k=0}^m\binom{m}{k}c_2^kc_1^{m-k}r^{HKk}\big\{C_K((1+r)^{2H}-1)\big\}^{m-k}\sqrt{2^kk!}\sqrt{2^{m-k}(m-k)!}\\
&\le\sqrt{2}^m\sum_{k=0}^m\binom{m}{k}(c_2r^{HK})^k\big\{c_1C_K((1+r)^{2H}-1)\big\}^{m-k}\sqrt{m!}\\
&=\sqrt{2}^m\big\{c_2r^{HK}+c_1C_K((1+r)^{2H}-1)\big\}^m\sqrt{m!},
\end{aligned}
$$

where in the third step we used inequalities in Lemma 2.2 for B^{HK} and Lemma 2.3 for X^K with $K\in(0,1)$. Note that in the first step we can not use the independence of $B^{H,K}$ and X^K since we take the power of (5.5). Now putting $\sqrt{2}c_2+\sqrt{2}c_1C_K((1+r)^{2H}-1)/r^{HK}=D_1$, we obtain the first result.

Finally we show (5.4) for $K\in(1,2)$ by using

$$
\max_{1\le t\le 1+r}|B^{H,K}(t)-B^{H,K}(1)|^m\stackrel{d}{=}\max_{1\le t\le 1+r}|c_2(B^{HK}(t)-B^{HK}(1))+c_3(X^K(t^{2H})-X^K(1))|^m,
$$

which follows from the decomposition (2.3). Noticing the independence of B^{HK} and X^K, we have

$$
E\Big[\max_{1\le t\le 1+r}|B^{H,K}(t)-B^{H,K}(1)|^m\Big]
$$

$$
\begin{aligned}
&\le \sum_{k=0}^{m}\binom{m}{k} c_2^k c_3^{m-k} E\Big[\max_{1\le t\le 1+r}|B^{HK}(t)-B^{HK}(1)|^k\Big] E\Big[\max_{1\le t\le 1+r}|X^K(t^{2H})-X^K(1)|^{m-k}\Big]\\
&= \sum_{k=0}^{m}\binom{m}{k} c_2^k c_3^{m-k} E\Big[\max_{0\le t\le r}|B^{HK}(t)|^k\Big] E\Big[\max_{1\le t\le (1+r)^{2H}}|X^K(t)-X^K(1)|^{m-k}\Big]\\
&\le \sum_{k=0}^{m}\binom{m}{k} c_2^k c_3^{m-k}\cdot r^{HKk} 2(k-1)!!\cdot((1+r)^{2H}-1)^{m-k} C_K^{m-k}(m-k-1)!!\\
&\le 2\sum_{k=0}^{m}\binom{m}{k}(c_2 r^{HK})^k\big\{c_3 C_K((1+r)^{2H}-1)\big\}^{m-k}(m-1)!!\\
&\le 2(m-1)!!\big\{c_2 r^{HK}+c_3 C_K((1+r)^{2H}-1)\big\}^m,
\end{aligned}
$$

where in the third step the inequality and constant C_K in Lemmas 2.2 and 2.3 are used. The result is obtained by setting $2^{\frac{1}{m}}\big\{c_2+c_3C_K((1+r)^{2H}-1)/r^{HK}\big\}=D_1$ and the fact $\{(m-1)!!\}^2\le m!$. $\square$

(*Proof of Proposition 3.9*). First we prove the former result (3.4) with an application of (5.3) and (5.4) in Lemma 5.1. Similarly as in the proof of Proposition 3.8, it follows that

$$
\begin{aligned}
&E\big[\max_{s\le t\le s+r}|\widehat{B}^{H,K}(t)-\widehat{B}^{H,K}(s)|^m\big]\\
&= E\Big[\max_{s\le t\le s+r}|(e^{-\lambda(t-s)}-1)B^{H,K}(e^{\frac{\lambda}{HK}(t-s)})+B^{H,K}(e^{\frac{\lambda}{HK}(t-s)})-B^{H,K}(1)|^m\Big]\\
&\le \sum_{k=0}^{m}\binom{m}{k}|1-e^{-\lambda r}|^k\sqrt{E\big[\max_{0\le t\le r}|B^{H,K}(e^{\frac{\lambda}{HK}t})|^{2k}\big]}\\
&\qquad\times\sqrt{E\big[\max_{0\le t\le r}|B^{H,K}(e^{\frac{\lambda}{HK}t})-B^{H,K}(1)|^{2(m-k)}\big]}\\
&\le \sum_{k=0}^{m}\binom{m}{k}|1-e^{-\lambda r}|^k\sqrt{(e^{\frac{\lambda}{HK}r})^{2HKk}2(2k-1)!!}\sqrt{(e^{\frac{\lambda}{HK}r}-1)^{2HK(m-k)}D_1^{2(m-k)}\sqrt{(2(m-k))!}}\\
&\le \sum_{k=0}^{m}\binom{m}{k}|1-e^{-\lambda r}|^k(e^{\lambda r})^k\sqrt{2^k k!}\{D_1(e^{\frac{\lambda}{HK}r}-1)^{HK}\}^{m-k}\sqrt{2^{m-k}(m-k)!}\\
&\le \sum_{k=0}^{m}\binom{m}{k}\big(\sqrt{2}(e^{\lambda r}-1)\big)^k\{\sqrt{2}D_1(e^{\frac{\lambda}{HK}r}-1)^{HK}\}^{m-k}\sqrt{k!}\sqrt{(m-k)!}\\
&\le \sqrt{m!}\sqrt{2}^m\{e^{\lambda r}-1+D_1(e^{\frac{\lambda}{HK}r}-1)^{HK}\}^m\\
&\le \sqrt{m!}\sqrt{2}^m r^{HKm}\{(e^{\lambda r}-1)/r^{HK}+D_1(e^{\frac{\lambda}{HK}r}-1)^{HK}/r^{HK}\}^m\\
&\le \sqrt{m!}\sqrt{2}^m r^{HKm}\{e^{\lambda r}\lambda r^{1-HK}+D_1e^{\lambda r}(\lambda/HK)^{HK}\}^m\\
&\le r^{HKm}\sqrt{m!}\,d^m,
\end{aligned}
$$

where d is some bounded positive constant. In the third step we use inequalities

$$2(2n-1)!!\le 2^n n! \quad\text{and}\quad (2n)!=2^n n!(2n-1)!!\le 2^{2n}(n!)^2.$$

Now dividing both sides with $m!$, we obtain the result.

Next we prove (3.5) in the same way as the proof of $\widetilde{B}^H$ in Proposition 3.8 and we replace $\widehat{B}^H$ with $\widehat{B}^{H,K}$ in the proof to obtain

$$E\Big[\max_{s\le t\le s+r}|\widetilde{B}^{H,K}(t)-\widetilde{B}^{H,K}(s)|\Big]\le e^{\mathrm{Var}(\widehat{B}^{H,K}(0))}\Big(E\Big[\max_{0\le t\le r}|e^{\widehat{B}^{H,K}(t)-\widehat{B}^{H,K}(0)}-1|^2\Big]\Big)^{1/2}. \tag{5.6}$$

Again we follow the proof of $\widetilde{B}^H$ in Proposition 3.8 and obtain

$$\begin{aligned} E\Big[\max_{0\le s\le r}\big(e^{\widehat{B}^{H,K}(s)-\widehat{B}^{H,K}(0)}-1\big)^2\Big] &\le E\Big[\max_{0\le s\le r}\big(\widehat{B}^{H,K}(s)-\widehat{B}^{H,K}(0)\big)^2\Big] \\ &\quad+\sum_{m=3}^{\infty}\frac{E\big[\max_{0\le s\le r}|2(\widehat{B}^{H,K}(s)-\widehat{B}^{H,K}(0))|^m\big]}{m!} \\ &\quad+\sum_{m=3}^{\infty}\frac{E\big[\max_{0\le s\le r}|\widehat{B}^{H,K}(s)-\widehat{B}^{H,K}(0)|^m\big]}{m!} \\ &= \sqrt{2}C^2r^{2HK}+r^{3HK}\sum_{m=3}^{\infty}\frac{C^m2^mr^{HK(m-3)}}{\sqrt{m!}} \\ &\quad+r^{3HK}\sum_{m=3}^{\infty}\frac{C^m}{\sqrt{m!}}r^{HK(m-3)} \\ &\le r^{2HK}(d_1+d_2r^{HK}) \end{aligned}$$

for all $r\in(0,1)$, where d_i, $i=1,2$ are positive constants and we use (3.4) in the second step. In the last step, we use the Stirling formula for the convergence of infinite sums. Substitute this into (5.6) and observe that

$$E\big[\max_{0\le s\le r}|\widehat{B}^{H,K}(s)-\widehat{B}^{H,K}(0)|\big]\le d_3e^{\mathrm{Var}(\widehat{B}^{H,K}(0))}\sqrt{d_1+d_2}\cdot r^{HK},$$

where $d_3>0$ is a constant. Then putting

$$C'=d_3e^{\mathrm{Var}(\widehat{B}^{H,K}(0))}\sqrt{d_1+d_2},$$

we obtain the result. □

Proof of Proposition 3.10. Again we start with the related lemma.

Lemma 5.2. *Let $H\in(\frac{1}{2},1)$ and $m=1,2,\dots,$ then for $r\in(0,1)$*

$$E\big[\max_{0\le t\le 1+r}|S^H(t)|^m\big]\le(1+r)^{Hm}2(m-1)!! \tag{5.7}$$

and

$$E\big[\max_{1\le t\le 1+r}|S^H(t)-S^H(1)|^m\big]\le r^{Hm}D_1^m\sqrt{m!}, \tag{5.8}$$

where $D_1>0$ is a constant which we can take uniformly in m.

Proof of Lemma 5.2. We omit the proof of (5.7) since it is the same as that of (5.3) in Lemma 5.1 if we replace $B^{H,K}$ by S^H. Next we prove (5.8). The triangular inequality yields

$$\begin{aligned} \max_{1\le t\le 1+r}|d_H(S^H(t)-S^H(1))| &\le \max_{1\le t\le 1+r}|c_5(X^{2H}(t)-X^{2H}(1))| \\ &\quad+\max_{1\le t\le 1+r}|c_5(X^{2H}(t)-X^{2H}(1))+d_H(S^H(t)-S^H(1))|. \end{aligned}$$

Due to the decomposition (2.5) with the continuity of concerning processes, we have

$$\begin{aligned} &E\big[\max_{1\le t\le 1+r}|d_H(S^H(t)-S^H(1))|^m\big] \\ &\le E\Big[\big(\max_{1\le t\le 1+r}|c_5(X^{2H}(t)-X^{2H}(1))| \\ &+\max_{1\le t\le 1+r}|c_5(X^{2H}(t)-X^{2H}(1))+d_H(S^H(t)-S^H(1))|\big)^m\Big] \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{k=0}^m \binom{m}{k} \sqrt{E\Big[(\max_{1\le t\le 1+r} |B^H(t)-B^H(1)|)^{2k}\Big]} \\
&\qquad \times c_5^{m-k} \sqrt{E\Big[(\max_{1\le t\le 1+r} |X^{2H}(t)-X^{2H}(1)|)^{2(m-k)}\Big]} \\
&\le \sum_{k=0}^m \binom{m}{k} c_5^{m-k} \sqrt{r^{2Hk} 2(2k-1)!!} \sqrt{r^{2(m-k)} C_K^{2(m-k)} (2(m-k)-1)!!} \\
&\le \sum_{k=0}^m \binom{m}{k} (r^H)^k \sqrt{2^k k!}\, (c_5 C_K r)^{m-k} \sqrt{2^{m-k}(m-k)!} \\
&\le \sqrt{2}^m \sqrt{m!} (r^H + c_5 C_K r)^m \\
&:= r^H \sqrt{2}^m \sqrt{m!} d^m,
\end{aligned}
$$

where d is a positive constant and in the third step we use Lemma 2.2 and (2.7) of Lemma 2.3 with stationary increments of B^H. Now dividing both sides with d_H^m, we obtain the result. □

(*Proof of Proposition 3.10*). In a similar manner as before, we proceed the proof by using the stationarity and the results of Lemma 5.2. It follows that

$$
\begin{aligned}
&E\Big[(\max_{s\le t\le s+r} |\widehat{S}^H(t)-\widehat{S}^H(s)|)^m\Big] \\
&\le E\Big[\big(\max_{s\le t\le s+r} |(e^{-\lambda(t-s)}-1) S^H(e^{\frac{\lambda}{H}(t-s)}) + S^H(e^{\frac{\lambda}{H}(t-s)}) - S^H(1)|\big)^m\Big] \\
&\le \sum_{k=0}^m \binom{m}{k} (1-e^{-\lambda r})^k \sqrt{E\Big[\max_{s\le t\le s+r} |S^H(e^{\frac{\lambda}{H}(t-s)})|^{2k}\Big]} \sqrt{E\Big[\max_{s\le t\le s+r} |S^H(e^{\frac{\lambda}{H}(t-s)}) - S^H(1)|^{2(m-k)}\Big]} \\
&\le \sum_{k=0}^m \binom{m}{k} (1-e^{-\lambda r})^k \sqrt{(1+e^{\frac{\lambda}{H}r})^{2Hk} 2(2k-1)!!} \sqrt{(e^{\frac{\lambda}{H}r}-1)^{2(m-k)H} D_1^{2(m-k)} \sqrt{2(m-k)!!}} \\
&\le \sum_{k=0}^m \binom{m}{k} \Big\{(1+e^{\frac{\lambda}{H}r})^H (1-e^{-\lambda r})\Big\}^k \sqrt{2^k k!} \Big\{(e^{\frac{\lambda}{H}r}-1)^H D_1\Big\}^{m-k} \sqrt{2^{m-k}(m-k)!} \\
&\le \sqrt{m!}\sqrt{2}^m \Big\{(1+e^{\frac{\lambda}{H}r})(1-e^{-\lambda r}) + D_1 (e^{\frac{\lambda}{H}r}-1)^H\Big\}^m \\
&\le r^{Hm}\sqrt{m!}\sqrt{2}^m \Big\{(1+e^{\frac{\lambda}{H}r})(1-e^{-\lambda r})/r^H + D_1 (e^{\frac{\lambda}{H}r}-1)^H/r^H\Big\}^m \\
&\le r^{Hm}\sqrt{m!}\sqrt{2}^m d_1^m,
\end{aligned}
$$

where $d_1 > 0$ is a constant and in the 4th step we apply (5.7) and (5.8) of Lemma 5.2. Hence we obtain the first result (3.6).

The proof of (3.7) is almost the same as that of $\widetilde{B}^H$ in Proposition 3.8 if we replace Lemma 5.1 with Lemma 5.2, and we omit it. □

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