

THE LANGLANDS CONJECTURE FOR GL_2 OF A LOCAL FIELD

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Let F be a p -field and let $W(F)$ be the absolute Weil group of G . Let $A_n(F)$ be the set of (equivalence classes of) continuous semisimple n -dimensional complex representations of $W(F)$ and let $A(GL_n(F))$ be the set of (equivalence classes of) irreducible admissible representations of $GL_n(F)$. By local classfield theory there is a natural bijection between the sets $A_1(F)$ and $A(GL_1(F))$, this latter set being just the set of quasi-characters of the multiplicative group F^\times of F ; we observe the convention of using this bijection to identify one-dimensional representations of $W(F)$ with quasi-characters of F^\times .

It is a conjecture of Langlands [JL] that there should exist a bijection $\sigma \rightarrow \pi(\sigma)$ between $A_2(F)$ and the subset of nonspecial representations in $A(GL_2(F))$, this bijection having the following properties.

1. For χ in $A_1(F)$, $\pi(\sigma \otimes \chi) = \pi(\sigma) \otimes \chi \circ \det$.
2. The one-dimensional representation $\det \sigma$ should (under our convention) be the central character of $\pi(\sigma)$.
3. $L(\sigma) = L(\pi(\sigma))$; $\epsilon(\sigma) = \epsilon(\pi(\sigma))$ where L, ϵ are the *local factors* associated to σ and $\pi(\sigma)$ [JL] with respect to some fixed character of F^+ .

In case the representation σ in $A_2(F)$ is reducible or imprimitive (i.e., induced from a proper subgroup of $W(F)$) the existence of $\pi(\sigma)$ is demonstrated in [JL]; in particular, this verifies the conjecture in case $p \neq 2$.

In case $p = 2$, Yoshida [Y] and Ree [R] have shown the existence of $\pi(\sigma)$ for certain primitive representations σ and Tunnell [T] has shown that the map $\sigma \rightarrow \pi(\sigma)$ is a bijection given that the existence of $\pi(\sigma)$ has already been established for all σ in $A_2(F)$, thus establishing the validity of the conjecture for $F = \mathbf{Q}_2$ as well as for fields F of residual characteristic two which contain the cube roots of unity.

We have recently verified the existence of $\pi(\sigma)$ for any primitive representation σ of $A_2(F)$ and we have shown that the map $\sigma \rightarrow \pi(\sigma)$ is indeed a bijection with the properties described above. We give here a sketch of our methods; a more detailed description of our results will appear elsewhere.

1. As above, let F be a p -field, $p = 2$ and let σ be a primitive two-dimensional representation of $W(F)$. Then [W] there exists a unique extension $K = K(\sigma)$

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of F , galois with galois group $\Gamma_{K/F}$ either A_3 or S_3 , such that the restriction, σ_K of σ to $\Gamma_{K/F}$ is imprimitive. We call $K(\sigma)$ the splitting field of σ . Given such a galois extension K/F we denote by $A_2(K/F)$ the set of representations in $A_2(F)$ with splitting field K .

Fix a character τ of F^+ and for any extension L/F , set $\tau_{L/F} = \tau \circ \text{Tr}_{L/F}$.

LEMMA 1.1. *If σ is in $A_2(K/F)$ then if $\Gamma_{K/F} \cong A_3$, $\epsilon(\sigma_K, \tau_{K/F}) = (\epsilon(\sigma, \tau))^3$ while if $\Gamma_{K/F} \cong S_3$ and H is any intermediate field to K/F with $[H: F] = 3$ then $\epsilon(\sigma_H, \tau_{H/F}) = \zeta(\epsilon(\sigma \otimes \omega, \tau))^3$ where ζ is a cube root of unity and ω is the nontrivial character of F^\times corresponding to the quadratic unramified extension F_0/F .*

LEMMA 1.2. *Let \mathfrak{A} be the field obtained by adjoining all n th roots of unity, $3 \nmid n$, to \mathbb{Q} . Let σ be a primitive two-dimensional representation of $W(F)$. Then*

1. *There exists a one-dimensional representation χ of $W(F)$ such that*
 - (i) $f(\sigma \otimes \eta) \geq f(\sigma \otimes \chi)$ for all η in $A_1(F)$ where $f(\sigma)$ is the exponent of the Artin conductor of σ [S];
 - (ii) $\det(\sigma \otimes \chi)$ has values in \mathfrak{A} .
2. *If χ is chosen with the above properties, then $\epsilon(\sigma \otimes \chi, \tau)$ lies in \mathfrak{A} .*

2. Let $G_F = \text{Gl}_2(F)$ and let π be an irreducible admissible supercuspidal [JL] representation of G_F . Then π will be called *unramified* if it may be induced from the subgroup $Z \cdot \text{Gl}_2(\mathcal{O}_F)$ where Z is the center of G_F and *ramified* otherwise. (It should be noted that this terminology is nonstandard; π is generally called unramified if its conductor is \mathcal{O}_F .)

LEMMA 2.1. *An irreducible supercuspidal representation π of G_F is unramified if and only if $\pi = \pi(\sigma)$ for some two-dimensional representation σ of $W(F)$ which is induced from $W(F_0)$.*

Now let π be a ramified irreducible supercuspidal representation of G_F . Then [K] π may be induced from a one-dimensional representation of a subgroup of G_F and as such is determined by a ramified quadratic extension E/F , a quasi-character ρ of E^\times , a quasi-character χ of F^\times and a character η of E^+ [GK]. Let K/F be tamely ramified. Then by lifting ρ to $E K^\times$ and χ to K^\times through the norm and lifting η to $E K^+$ through the trace, we obtain a representation π_K of G_K which we call a *tame lift* of π to G_K .

LEMMA 2.2. *If $[K: F] = 3$ then*

$$\epsilon(\pi_K, \tau_{K/F}) = [\epsilon(\pi, \tau)]^3.$$

LEMMA 2.3. *Let K/F be galois with prime cyclic galois group $\Gamma_{K/F}$. Then a representation π of G_K is a tame lift if and only if π is fixed under $\Gamma_{K/F}$.*

LEMMA 2.4. *Let π be an exceptional representation of G_F ; i.e., a supercuspidal irreducible representation not of the form $\pi(\sigma)$ for an imprimitive representation σ of Γ_F . Then there exists a unique extension K/F , galois with $\Gamma_{K/F}$ either A_3 or S_3 , such that π_K is not exceptional.*

With π, K as above we call K the *splitting field* for π and let $A(G_K, G_F)$ be the subset of $A(G_F)$ consisting of representations whose splitting field is K .

LEMMA 2.5. *Let π be an irreducible supercuspidal representation of G_F . Then there exists a quasi-character χ of F^\times such that $\pi \otimes \chi \circ \det$ has minimal conductor and the central character of $\pi \otimes \chi \circ \det$ takes values in \mathfrak{U} . With χ as above, $\epsilon(\pi \otimes \chi \circ \det, \tau)$ lies in \mathfrak{U} .*

LEMMA 2.6. *Let σ be an imprimitive two-dimensional representation of $W(F)$, let F_0/F be quadratic unramified and suppose that σ_{F_0} is irreducible. Then $[\pi(\sigma)]_{F_0} = \pi(\sigma_{F_0}) \otimes \omega \circ \det$ where ω is an unramified character of F_0^\times and $\omega^2 = 1$.*

3. THEOREM. *Let F be a 2-field and let K/F be galois with galois group either A_3 or S_3 . Then the map $\sigma \rightarrow \pi(\sigma)$ is defined on $A_2(K/F)$ and puts $A_2(K/F)$ into bijection with $A(G_K, G_F)$.*

PROOF (SKETCH). First let $\Gamma_{K/F} \cong A_3$. Pick σ in $A_2(K/F)$ such that $f(\sigma) \leq f(\sigma \otimes \chi)$ for quasi-characters χ of F^\times and such that $\det \sigma$ takes values in \mathfrak{U} . Then $\pi(\sigma_K)$ exists and is fixed by $\Gamma_{K/F}$. By Lemma 2.3, $\pi(\sigma_K) = \pi_K$ for some representation π in $A(G_K, G_F)$ and one may pick π such that its central character takes values in \mathfrak{U} . With this choice of π , $\pi = \pi(\sigma)$. In fact, it follows immediately that $\det \sigma$ is the central character of π . Also, one has $\epsilon(\pi \otimes \chi \circ \det, \tau) = \epsilon(\sigma \otimes \chi, \tau)$ for quasi-characters χ whose values lie in \mathfrak{U} by Lemmas 1.1 and 2.2 and for tamely ramified χ by a direct computation. One then deduces that $\epsilon(\pi \otimes \chi \circ \det, \tau) = \epsilon(\sigma \otimes \chi, \tau)$ for arbitrary χ whence $\pi = \pi(\sigma)$. By Lemma 1.2 one may construct $\pi(\sigma)$ for any σ in $A_2(K/F)$.

In the same manner, using Lemmas 2.3, 2.4, 2.5, one may construct an inverse map $\pi \rightarrow \sigma(\pi)$ from $A(G_K, G_F)$ to $A_2(K/F)$ thus demonstrating the theorem if $\Gamma_{K/F} \cong A_3$.

Now suppose that $\Gamma_{K/F} \cong S_3$, let H and F_0 be as in Lemma 1.1 and pick σ in $A_2(K/F)$ as above. Then σ_H is imprimitive and by Lemma 2.6, $[\pi(\sigma_H)]_K$ is fixed by $\Gamma_{K/F}$. It follows (with some work) that there exists a representation π in $A(G_K, G_F)$ such that $\pi_H = \pi(\sigma_H)$. Since $\det \sigma$ is the unique extension of $\det \sigma_H$ to $W(F)$ and since the cube roots of unity do not lie in F , it follows from Lemmas 1.1 and 2.2 that $\pi = \pi(\sigma)$. Just as above, one may construct an inverse to the map $\sigma \rightarrow \pi(\sigma)$ thus verifying the theorem in case $\Gamma_{K/F} \cong S_3$.

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