

The Largest Degrees of Irreducible Characters of the Symmetric Group

By John McKay

Abstract. The largest irreducible degrees and the partitions associated with them are tabulated for the symmetric group Σ_n for n up to 75. Analytic upper and lower bounds are derived for the largest degree.

Introduction. A question has been raised by Bivins and others [2]—namely:

For what irreducible representations of the symmetric group Σ_n does the degree attain its maximal value and how does this maximum behave for large n ?

This was apparently motivated by the practical considerations of number overflow in the computer but the same question arises in connection with sorting [1].

Each irreducible representation is associated with a partition $a = (a_1, a_2, \dots, a_k)$, $a_1 \geq a_2 \geq \dots \geq a_k > 0$, of n . (We shall use $a \in n$ to mean that a is one of the p_n partitions of n .) Its degree is given by [6, p. 61]:

$$d_a = n! \frac{\prod_{i < j} (b_i - b_j)}{\prod_i (b_i!)}$$

where $b_i = a_i + k - i$. A combinatorial interpretation of d_a is that it is the number of ways the votes for k candidates can be counted one at a time such that the final total number of votes cast is n and at all stages in the counting $n_1 \geq n_2 \geq \dots \geq n_k$, where n_i is the current number of votes for candidate i ($i = 1, \dots, k$) with finally $n_i = a_i$ ($i = 1, \dots, k$).

Computation of the Maximal Degree. The calculations were made at Edinburgh University on a 4K 12-bit word length PDP8 computer using a multi-length routine for expansile integer multiplication. The strategy is straightforward. For increasing n , partitions of n are generated in natural order (n first and 1^n last) as described in [11]. If a partition, a , precedes or coincides with its conjugate then the degree d_a is computed as in the procedure *degree* of [9] but exponent arithmetic is used retaining integers throughout and avoiding unnecessary overflow. A description of exponent arithmetic appears in [10] but this description is slightly different from that used in this application, and the algorithm given there is a little garbled.

Three arrays are declared, *ex*, *hfac*, *lfac* [2: N], where N is the largest integer occurring as a natural factor (here N is at most 75); *ex*[n] contains the exponent of n in the result and for all $n \leq N$, *hfac*[n] contains the largest prime factor of n and *lfac*[n] contains the other factor. After initialization, the expression is evaluated by modifying the exponents in *ex*. For example, to divide by $k!$:

for $i := 2$ step 1 until k do $ex[i] := ex[i] - 1$;

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Maximal Irreducible Degrees of the Symmetric Group

m_n	n	m_n	n
1	1	247 48435 71200	26
1	2	1276 09121 64000	27
2	3	5742 41047 38000	28
3	4	29528 41929 42320	29
6	5	1 86513 49218 90240	30
16	6	9 24182 73851 90400	31
35	7	50 38573 19942 59200	32
90	8	268 40130 62455 29600	33
216	9	1579 81237 60723 20000	34
768	10	7821 85911 50700 00000	35
2310	11	40971 64298 37000 00000	36
7700	12	2 22250 51347 85087 15200	37
21450	13	15 92694 28320 99526 65600	38
69498	14	93 35226 29027 57090 91840	39
2 92864	15	589 65081 68506 18031 30880	40
11 53152	16	3660 86379 16673 31465 21600	41
48 73050	17	24558 61544 13590 64616 32000	42
163 36320	18	1 40647 43140 34029 84224 96480	43
646 64600	19	8 26287 24406 18222 00507 44960	44
2494 20600	20	50 02839 28761 42234 84343 20000	45
11189 39184	21	309 91868 81321 01700 50024 84000	46
54628 65408	22	2036 88735 12400 42742 34055 68000	47
2 85421 58568	23	13910 87091 49402 51649 95795 35360	48
11 74870 79424	24	1 00788 28728 27294 45059 89182 25920	49
54 75915 90000	25	7 21304 41781 17167 52220 04203 52000	50

	m_n								n			
	54	86245	62826	89907	32913	48475	90400		51			
	360	27173	44007	80906	66116	28632	57600		52			
	2416	32801	79788	35907	70622	12235	61800		53			
	16032	08919	82658	76501	24498	76481	40000		54			
	1	12332	94008	00148	07351	23185	00477	31500	55			
	7	80924	18237	44344	89607	49414	47168	50000	56			
	57	59492	68858	65309	68032	60594	83410	40000	57			
	392	04228	54325	17105	67342	81079	91024	00000	58			
	2843	60991	01639	97708	94957	04013	43897	60000	59			
	23219	99844	17184	55788	71179	66465	14524	16000	60			
	1	98964	36084	33813	49744	27586	95268	29039	61600	61		
	14	84932	70650	29909	32159	91941	84305	99280	64000	62		
	112	80848	15471	49092	37752	38783	18899	58910	11200	63		
	822	90818	64439	40221	23814	78702	63130	68681	13280	64		
	6474	45118	59060	42071	22906	42354	58681	10615	19360	65		
	49264	88872	06925	77842	72444	27860	67020	29690	57200	66		
	4	02557	12513	54748	85330	10840	14788	82368	98346	54000	67	
	30	47316	79121	25109	10697	47261	28840	64586	73715	20000	68	
	234	41791	16438	06987	94867	83935	00955	83550	21660	16000	69	
	1788	61125	56865	99443	44127	54238	97069	70842	13760	00000	70	
	14061	79814	66342	15100	92845	75298	46541	20312	24000	00000	71	
	1	30752	27432	79523	21538	98976	09524	06388	52853	50440	96000	72
	10	99941	83391	42975	66548	10097	63063	04543	75434	51852	80000	73
	93	93814	29772	20073	46466	22546	26656	28282	24403	04999	04000	74
	755	91730	44948	11890	68765	20714	81759	17862	44539	84930	00000	75

n	$\sqrt{n!}$	$= a \cdot 10^b$		$m_n/\sqrt{n!}$	partition
	a	b			
1	1.00	0		1.000	1
2	1.41	0		0.707	2
3	2.45	0		0.816	2,1
4	4.90	0		0.612	3,1
5	1.10	1		0.548	3,1 ²
6	2.68	1		0.596	3,2,1
7	7.10	1		0.493	4,2,1
8	2.01	2		0.448	4,2,1 ²
9	6.02	2		0.359	4,3,1 ²
10	1.90	3		0.403	4,3,2,1
11	6.32	3		0.366	5,3,2,1
12	2.19	4		0.352	5,3,2,1 ²
13	7.89	4		0.272	5,4,2,1 ²
14	2.95	5		0.235	6,4,2,1 ²
15	1.14	6		0.256	5,4,3,2,1
16	4.57	6		0.252	6,4,3,2,1
17	1.89	7		0.258	6,4,3,2,1 ²
18	8.00	7		0.204	7,4,3,2,1 ²
19	3.49	8		0.185	7,5,3,2,1 ²
20	1.56	9		0.160	7,5,3,2 ² ,1
21	7.15	9		0.157	7,5,3,2 ² ,1 ²
22	3.35	10		0.163	7,5,4,3,2,1
23	1.61	11		0.178	7,5,4,3,2,1 ²
24	7.88	11		0.149	8,5,4,3,2,1 ²
25	3.94	12		0.139	8,6,4,3,2,1 ²

n	$\sqrt{n!}$ a	$= a \cdot 10^b$ b	$m_n / \sqrt{n!}$	partition
26	2.01	13	0.123	8,6,4,3,2,1 ³
27	1.04	14	0.122	8,6,4,3,2 ² ,1 ²
28	5.52	14	0.104	8,6,5,3,2 ² ,1 ²
29	2.97	15	0.099	8,6,5,4,3,2,1
30	1.63	16	0.115	8,6,5,4,3,2,1 ²
31	9.07	16	0.102	9,6,5,4,3,2,1 ²
32	5.13	17	0.098	9,7,5,4,3,2,1 ²
33	2.95	18	0.091	9,7,5,4,3,2,1 ³
34	1.72	19	0.092	9,7,5,4,3,2 ² ,1 ²
35	1.02	20	0.077	9,7,6,4,3,2 ² ,1 ²
36	6.10	20	0.067	9,7,6,4,3 ² ,2,1 ²
37	3.71	21	0.060	10,8,6,4,3,2 ² ,1 ²
38	2.29	22	0.070	9,7,6,5,4,3,2,1 ²
39	1.43	23	0.065	10,7,6,5,4,3,2,1 ²
40	9.03	23	0.065	10,8,6,5,4,3,2,1 ²
41	5.78	24	0.063	10,8,6,5,4,3,2,1 ³
42	3.75	25	0.066	10,8,6,5,4,3,2 ² ,1 ²
43	2.46	26	0.057	11,8,6,5,4,3,2 ² ,1 ²
44	1.63	27	0.051	11,8,6,5,4,3,2 ² ,1 ³
45	1.09	28	0.046	11,9,7,5,4,3,2 ² ,1 ²
46	7.42	28	0.042	11,9,7,5,4,3,2 ² ,1 ³
47	5.09	29	0.040	10,8,7,6,5,4,3,2,1 ²
48	3.52	30	0.039	11,8,7,6,5,4,3,2,1 ²
49	2.47	31	0.041	11,9,7,6,5,4,3,2,1 ²
50	1.74	32	0.041	11,9,7,6,5,4,3,2,1 ³

n	$\sqrt{n!}$ a	= a · 10 ^b b	$m_n/\sqrt{n!}$	partition
51	1.25	33	0.044	11,9,7,6,5,4,3,2 ² ,1 ²
52	8.98	33	0.040	12,9,7,6,5,4,3,2 ² ,1 ²
53	6.54	34	0.037	12,9,7,6,5,4,3,2 ² ,1 ³
54	4.80	35	0.033	12,10,8,6,5,4,3,2 ² ,1 ²
55	3.56	36	0.032	12,10,8,6,5,4,3,2 ² ,1 ³
56	2.67	37	0.029	12,10,8,6,5,4,3 ² ,2,1 ³
57	2.01	38	0.029	12,10,8,6,5,4,3 ² ,2 ² ,1 ²
58	1.53	39	0.026	12,10,8,7,5,4,3 ² ,2 ² ,1 ²
59	1.18	40	0.024	12,10,8,7,6,5,4,3,2,1 ²
60	9.12	40	0.025	12,10,8,7,6,5,4,3,2,1 ³
61	7.12	41	0.028	12,10,8,7,6,5,4,3,2 ² ,1 ²
62	5.61	42	0.026	13,10,8,7,6,5,4,3,2 ² ,1 ²
63	4.45	43	0.025	13,10,8,7,6,5,4,3,2 ² ,1 ³
64	3.56	44	0.026	13,10,9,7,6,5,4,3,2 ² ,1 ³
65	2.87	45	0.023	13,11,9,7,6,5,4,3,2 ² ,1 ³
66	2.33	46	0.021	13,11,9,7,6,5,4,3 ² ,2,1 ³
67	1.91	47	0.021	13,11,9,7,6,5,4,3 ² ,2 ² ,1 ²
68	1.57	48	0.019	14,11,9,7,6,5,4,3 ² ,2 ² ,1 ²
69	1.31	49	0.018	14,11,9,7,6,5,4,3 ² ,2 ² ,1 ³
70	1.09	50	0.016	14,11,9,8,6,5,4,3 ² ,2 ² ,1 ³
71	9.22	50	0.015	14,11,9,8,6,5,4 ² ,3,2 ² ,1 ³
72	7.83	51	0.017	13,11,9,8,7,6,5,4,3,2 ² ,1 ²
73	6.69	52	0.016	14,11,9,8,7,6,5,4,3,2 ² ,1 ²
74	5.75	53	0.016	14,11,9,8,7,6,5,4,3,2 ² ,1 ³
75	4.98	54	0.015	14,11,10,8,7,6,5,4,3,2 ² ,1 ³

When complete, the result is reduced to a product of prime powers by decomposing the factors in decreasing order of magnitude into prime factors. The final numerical result may then be obtained. The computation of the cumulated product is speeded up by storing those prime powers that can be contained in a single-length word (viz. up to $2^{12} - 1$). An ALGOL algorithm for the reduction is given by:

comment *den* is not needed if the result is known to be an integer, otherwise the result is given by *num/den*;

num := *den* := 1;

for *k* := *N* **step** -1 **until** 2 **do**

begin **if** *ex* [*k*] ≠ 0 **then**

begin **if** *lfac* [*k*] > 1 **then**

begin

ex [*hfac* [*k*]] := *ex* [*hfac* [*k*]] + *ex* [*k*];

ex [*lfac* [*k*]] := *ex* [*lfac* [*k*]] + *ex* [*k*];

ex [*k*] := 0; **go to** *a*

end;

if *ex* [*k*] > 0 **then** *num* := *num* × *k* ↑ *ex* [*k*];

if *ex* [*k*] < 0 **then** *den* := *den* × *k* ↑ *ex* [*k*]

end else

a: **end**

The partial product *num* is stored as a multi-length integer and repeatedly multiplied by single-length integers to obtain the final value for the degree d_a . The machine used had no hardware multiplication and the worst case ($2^{12} - 1 \times 2^{12} - 1$) single-length × single-length multiplication took approximately ½ millisc = 500 instructions to complete.

The degrees were printed in decimal using Lunnon's [8] multi-length arithmetic package for Atlas.

The tables extend those of Comét [4] (up to $n = 30$) and those of Baer and Brock [1] (up to $n = 36$). They do not seem to reveal any simple recurrence between the partitions associated with the maximal degree for Σ_n and those for Σ_k ($k < n$). It is notable, however, that frequently a partition for the maximal degree for Σ_n differs from that for the maximal degree for Σ_{n-1} in a single part only.

Bounds for max d_a . Upper and lower bounds for $m_n = \max_{a \in n} d_a$ are easy to find using group-theoretic facts concerning the characters of Σ_n . For a lower bound we have [5, p. 23] that the mean value of d_a is given by s_n/p_n where s_n is the number of solutions to $x^2 = 1$, $x \in \Sigma_n$; viz.,

$$s_n = \sum_{k=0}^{\lfloor \frac{1}{2}n \rfloor} \frac{n!}{2^k k! (n-2k)!}.$$

The character column orthogonality relation on the degrees gives $n! = \sum_{a \in n} d_a^2$; hence, $s_n/p_n \leq m_n \leq (n!)^{1/2}$.

Now Chowla, Herstein, and Moore [3] give the asymptotic $s_n \sim 2^{-\frac{1}{2}}(n/e)^{\frac{1}{2}n} e^{n^{\frac{1}{2}} - \frac{1}{4}}$ and this, together with well-known approximations for p_n and $n!$, gives

$$l_n = \frac{4n\sqrt{3}}{e^{(\pi/3)\sqrt{6n}}} \cdot \frac{(n/e)^{\frac{1}{2}n} e^{n^{\frac{1}{2}}}}{2^{\frac{1}{2}} e^{\frac{1}{4}}} \leq m_n \leq (2\pi n)^{\frac{1}{4}} (n/e)^{\frac{1}{2}n} = u_n$$

from which

$$\frac{u_n}{l_n} = \frac{ke^{tn^{\frac{1}{2}}}}{n^{3/4}}, \quad k = (\pi e/288)^{\frac{1}{4}}, \quad t = \frac{\pi\sqrt{6}}{3} - 1.$$

Remark. The referee has brought to my attention the work of Logan and Shepp [7] who have solved a continuous analogue of this problem and find their result not inconsistent with the partition tabulated here for $n = 75$.

Added in Proof. It has been conjectured from the tables given here that $m_n \leq \sqrt{n!}/n$, but Eric Regener has computed that the smallest value of n for which the conjecture is false is $n = 81$ which has a maximal partition of 15, 12, 10, 9, 7, 6, 5, 4, 3², 2², 1³.

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