# THE LATTICE OF CONGRUENCES ON A BAND OF GROUPS 

by C. SPITZNAGEL

(Received 24 April, 1972)
It is implicit in a result of Kapp and Schneider [3] that, if $S$ is a completely simple semigroup, then the lattice $\Lambda(S)$ of congruences on $S$ can be embedded in the product of certain sublattices. In this paper we consider the problem of embedding $\Lambda(S)$ in a product of sublattices, when $S$ is an arbitrary band of groups. The principal tool is the $\theta$-relation of Reilly and Scheiblich [7]. The class of $\theta$-modular bands of groups is defined by means of a type of modularity condition on $\Lambda(S)$. It is shown that the $\theta$-modular bands of groups are precisely those for which a certain function is an embedding of $\Lambda(S)$ into a product of sublattices. The problem of embedding the inverse semigroup congruences into a certain product lattice is also considered.

1. Terminology and preliminary results. A semigroup that is a union of groups is called a band of groups, provided that Green's $\mathscr{H}$-relation is a congruence. It is rather well known [1, Theorem 4.6] that on any band of groups $S$ (and in fact on any union of groups), the $\mathscr{D}$-relation is the minimum semilattice congruence, and the $\mathscr{D}$-classes of $S$ are completely simple semigroups. The " fine structure" of such semigroups has recently been studied by Leech [5].

If $S$ is any regular semigroup, then the $\theta$-relation on $\Lambda(S)$, first studied by Reilly and Scheiblich in [7], is defined by $(\rho, \tau) \in \theta$ if and only if $\rho \cap\left(E_{S} \times E_{S}\right)=\tau \cap\left(E_{S} \times E_{S}\right)$. In [7] it is proved that, if $S$ is an inverse semigroup, then $\theta$ is a complete lattice congruence on $\Lambda(S)$. Scheiblich, in [8], later extended this result to regular semigroups.

The notation in this paper will be that of Clifford and Preston [1], with the exception of the following list of symbols.
$x^{-1}$ : the inverse of $x$ in $H_{x}$, in a band of groups.
$B(S)$ : the lattice of band congruences on $S$.
$M(S)$ : the lattice of idempotent-separating congruences on $S$.
$D(S)$ : the lattice of congruences on $S$ that are contained in $\mathscr{D}$.
$I(S)$ : the lattice of inverse semigroup congruences on $S$.
$Y(S)$ : the lattice of semilattice congruences on $S$.
$\Delta(S)$ : the $\theta$-class of $\mathscr{D}$.
$1_{s}$ : the universal congruence $S \times S$.
$0_{s}$ : the diagonal congruence $\Delta S^{2}=\{(x, x) \mid x \in S\}$.
$\beta$ : the minimum band congruence on $S$.
$\mu$ : the maximum idempotent-separating congruence on $S$.
$\sigma$ : the minimum group congruence on $S$.
$\delta$ : the minimum inverse semigroup congruence on $S$.
$\eta$ : the minimum semilattice congruence on $S$.

The congruences $\beta, \eta$, and $\sigma$ are discussed in [2], as well as in other places. The congruence $\mu$ is discussed in [6]. In [7] it is pointed out that, on any regular semigroup, the idempotentseparating congruences are precisely those that are contained in $\mathscr{H}$. Combining this with the result of Munn [6] that the congruences contained in $\mathscr{H}$ form a sublattice of $\Lambda(S)$ with a greatest and a least element, yields the result that $\mu$ exists on any regular semigroup, and that $\mu \subseteq \mathscr{H}$.

We have the following characterization of bands of groups, in terms of $\mu$ and $\beta$.
Lemma 1.1. Let $S$ be any regular semigroup. Then the following statements are equivalent.
(i) $S$ is a band of groups.
(ii) $\mu=\mathscr{H}=\beta$.
(iii) $\mu=\beta$.

Proof. In [2] it is shown that $\mathscr{H} \subseteq \beta$. Thus $\mu \subseteq \mathscr{H} \subseteq \beta$, in any regular semigroup. Now, if $S$ is a band of groups, $\mathscr{H}$ is a band congruence; so we must have $\mathscr{H}=\beta$. Also, each $\mathscr{H}$-class contains exactly one idempotent; so $\mathscr{H}$ is also idempotent-separating. Thus $\mathscr{H}=\mu$, and we see that (i) implies (ii). Since $\mu \subseteq \mathscr{H} \subseteq \beta$, it is clear that (ii) is equivalent to (iii). Now, if $\mu=\mathscr{H}=\beta$, then $\mathscr{H}$ is a band congruence. It then follows from [4, Lemma 2.2] that each $\mathscr{H}$-class contains an idempotent. So, by [1, Theorem 2.16], $S$ is a union of groups, and hence a band of groups. Thus (ii) implies (i).
2. The $\theta$-relation and $\Lambda(S)$. The following two lemmas are due to Scheiblich in [8].

Lemma 2.1. Let $S$ be a regular semigroup, and $\rho, \tau \in \Lambda(S)$, such that $\rho$ separates idempotents. Then $(\rho \vee \tau, \tau) \in \theta$.

Lemma 2.2. If $S$ is a regular semigroup, then $\theta$ is a complete lattice congruence on $\Lambda(S)$.
Now suppose that $S$ is a band of groups. It is then the case that $\mathscr{H}$ is idempotentseparating; so we have the following immediate corollary.

Corollary 2.3. Let $S$ be a band of groups. Then, for any $\rho \in \Lambda(S),(\rho \vee \mathscr{H}, \rho) \in \theta$.
We also note that a congruence $\tau$ on a regular semigroup is a band congruence if and only if $\tau$ contains $\beta$, the minimum band congruence. We therefore have

Proposition 2.4. Let $S$ be a regular semigroup. Then each $\theta$-class of $\Lambda(S)$ contains at most one band congruence. In addition, if $S$ is a band of groups, then each $\theta$-class contains exactly one band congruence.

Proof. Suppose that $\alpha$ and $\gamma$ are band congruences in the same $\theta$-class. Since $\beta \subseteq \alpha$ and $\beta \subseteq \gamma$, the $\alpha$ - and $\gamma$-classes are unions of $\beta$-classes. Also, by [4, Lemma 2.2], each $\beta$-class contains an idempotent. Now suppose that $x \alpha y$. Let $e$ and $f$ be idempotents such that $e \beta x, f \beta y$. Then $e \beta x \alpha y \beta f$, so that $e \alpha f$. Hence, since $(\alpha, \gamma) \in \theta$, we have $e \gamma f$. But then $x \beta e \gamma f \beta y$, so that $x \gamma y$. Thus $\alpha \subseteq \gamma$. Similarly, $\gamma \subseteq \alpha$, proving the first part. Now, if $S$ is a band of groups, we have $(\rho \vee \mathscr{H}, \rho) \in \theta$ for every $\rho \in \Lambda(S)$, by Corollary 2.3. Since $\beta=$ $\mathscr{H} \subseteq \rho \vee \mathscr{H}, \rho \vee \mathscr{H}$ is a band congruence in the $\theta$-class of $\rho$. This completes the proof.

The following proposition will prove to be useful.
Proposition 2.5. Let $S$ be a band of groups and let $\rho, \tau \in \Lambda(S)$. Then $(\rho, \tau) \in \theta$ if and only if $\rho \vee \mathscr{H}=\tau \vee \mathscr{H}$.

Proof. Suppose that $(\rho, \tau) \in \theta$. Combining this with $(\rho \vee \mathscr{H}, \rho) \in \theta$ and $(\tau \vee \mathscr{H}, \tau) \in \theta$, we obtain $(\rho \vee \mathscr{H}, \tau \vee \mathscr{H}) \in \theta$ by transitivity of $\theta$. Hence, since $\rho \vee \mathscr{H}$ and $\tau \vee \mathscr{H}$ are both band congruences, we have $\rho \vee \mathscr{H}=\tau \vee \mathscr{H}$, by Proposition 2.4. Conversely, if $\rho \vee \mathscr{H}=$ $\tau \vee \mathscr{H}$, then $(\rho \vee \mathscr{H}, \rho) \in \theta$ and $(\tau \vee \mathscr{H}, \tau) \in \theta$ imply that $(\rho, \tau) \in \theta$.

In [7] it is proved that the $\theta$-classes of a regular semigroup $S$ are very nice. We now record this for future reference.

Lemma 2.6. [7, Theorem 3.4(ii)] Let $S$ be a regular semigroup. Then each 0 -class is a complete modular sublattice of $\Lambda(S)$ (having a greatest and a least element).

The following proposition gives a necessary and sufficient condition for these greatest elements to be band congruences.

Proposition 2.7. Let $S$ be a regular semigroup. Then the greatest element of each $\theta$-class is a band congruence if and only if $S$ is a band of groups.

Proof. If $S$ is a band of groups, then $\mathscr{H}=\beta$. We have also seen that, if $\rho$ is any congruence, then $(\rho \vee \mathscr{H}, \rho) \in \theta$. So, if $\tau$ is the greatest element of the $\theta$-class of $\rho$, then $\mathscr{H} \subseteq \rho \vee \mathscr{H}$ $\subseteq \tau$, which implies that $\tau$ is a band congruence. Conversely, if the greatest element of each $\theta$-class is a band congruence, then in particular $\mu$, which is the greatest element of the $\theta$-class of $0_{S}$, is a band congruence. But $\mu \subseteq \mathscr{H} \subseteq \beta$; so we obtain $\mu=\mathscr{H}=\beta$, whence $S$ is a band of groups.

The $\theta$-relation is a useful means of viewing $\Lambda(S)$, particularly in the case that $S$ is a band of groups. For example, if $S$ is a band of groups, the $\theta$-class of $0_{S}$ consists of those congruences that partition the idempotents of $S$ in the same manner as $0_{S}$; that is, the $\theta$-class of $0_{S}$ is the set of idempotent-separating congruences on $S$. Its greatest element is $\mu=\mathscr{H}=\beta$. Similarly, the $\theta$-class of $1_{s}$ consists of all congruences that identify all idempotents of $S$; that is, it is the lattice of group congruences on $S$. The greatest element in this $\theta$-class is, of course, $1_{s}$, and the least element is $\sigma$, the minimum group congruence.

The $\theta$-relation, being a congruence, partitions $\Lambda(S)$, and, in view of Propositions 2.7 and $2.4, B(S)$ cross-sections the $\theta$-classes. This naturally leads to the problem of describing $\Lambda(S)$ in terms of $B(S)$ and some other sublattice; for $B(S)$ is isomorphic to $\Lambda(S / \beta)=\Lambda(S / \mathscr{H})$, and hence is more accessible than $\Lambda(S)$ itself. This problem is considered in the following section.
3. Embedding $\Lambda(S)$ in a product lattice. In this section it is shown that the lattice $D(S)$ on a band of groups $S$ can be embedded in the product lattice $B(S) \times M(S)$, but that the embedding does not always extend to an embedding of $\Lambda(S)$. A necessary and sufficient condition on $\Lambda(S)$ is then found, under which the natural extension of this map is an embedding of $\Lambda(S)$.

We begin with the following easy lemma, whose proof is omitted.
Lemma 3.1. Let $S$ be a band of groups and let $\rho \in \Lambda(S)$. Then $\rho \wedge \mathscr{H}$ is an idempotentseparating congruence; that is, $(\rho \wedge \mathscr{H}, \mathscr{H}) \in \theta$.

Lemma 3.2. Let $S$ be a band of groups, let $\rho \in \Lambda(S)$ and suppose that $(x, y) \in \rho$. Let $e \in E_{S} \cap H_{x}, f \in E_{S} \cap H_{y}$. Then $(e, f) \in \rho$.

Proof. We have $(e, f) \in \mathscr{H} \circ \rho \circ \mathscr{H} \subseteq \rho \vee \mathscr{H}$. Then, since $(\rho \vee \mathscr{H}, \rho) \in \theta$, by Corollary 2.3, we have $(e, f) \in \rho$.

Proposition 3.3. Let $S$ be a band of groups and let $\psi: \Lambda(S) \rightarrow B(S) \times M(S)$ be defined by $\psi(\rho)=(\rho \vee \mathscr{H}, \rho \wedge \mathscr{H})$. Then $\psi$ is one-to-one.

Proof. Suppose that $\rho, \tau \in \Lambda(S)$ are such that $(\rho \vee \mathscr{H}, \rho \wedge \mathscr{H})=(\tau \vee \mathscr{H}, \tau \wedge \mathscr{H})$. Then, from Proposition 2.5, we have $(\rho, \tau) \in \theta$, and also $\rho \wedge \mathscr{H}=\tau \wedge \mathscr{H}$. Suppose that $(x, y) \in \rho$, and let $e \in E_{S} \cap H_{x}, f \in E_{S} \cap H_{y}$. Then, by Lemma 3.2, we have $(e, f) \in \rho$; and, since $(\rho, \tau) \in \theta$, this implies that $(e, f) \in \tau$. Hence $x=x e \tau x f$, and $y=f y \tau e y$. But ey $\rho f y=$ $y \rho x=x e \rho x f$, so that ey $\rho x f$. Also, ey $\mathscr{H} x f$, since $\mathscr{H}$ is a congruence, and thus ey $(\rho \wedge \mathscr{H}) x f$. Since $\rho \wedge \mathscr{H}=\tau \wedge \mathscr{H}$, we then have ey( $\tau \wedge \mathscr{H}) x f$. Thus $x \tau x f(\tau \wedge \mathscr{H}) e y \tau y$, so that $(x, y) \in \tau$. Thus $\rho \subseteq \tau$. Likewise $\tau \subseteq \rho$, and the result follows.

We remark that, by this proposition, every congruence $\rho$ on a band of groups can be "factored" into a band congruence (namely $\rho \vee \mathscr{H}$ ), and an idempotent-separating congruence (namely $\rho \wedge \mathscr{H}$ ). The next proposition shows, to some extent, how the congruence $\rho$ can be recovered from this factorization.

Proposition 3.4. Let $S$ be a band of groups and let $\rho \in \Lambda(S)$. Then $\rho=\bar{\rho} \vee(\rho \wedge \mathscr{H})$, where $\bar{\rho}$ is the smallest element of the $\theta$-class of $\rho$.

Proof. It will suffice to show that $\psi(\rho)=\psi(\bar{\rho} \vee(\rho \wedge \mathscr{H}))$, where $\psi$ is as in Proposition 3.3. By Lemma 3.1, Corollary 2.3, and Lemma 2.2, we have $[\bar{\rho} \vee(\rho \wedge \mathscr{H})] \theta[\bar{\rho} \vee \mathscr{H}] \theta \bar{\rho} \theta \rho$, and hence, by Proposition 2.5, $\rho \vee \mathscr{H}=[\bar{\rho} \vee(\rho \wedge \mathscr{H})] \vee \mathscr{H}$. Thus it remains to show that $\rho \wedge \mathscr{H}=[\bar{\rho} \vee(\rho \wedge \mathscr{H})] \wedge \mathscr{H}$. But $\bar{\rho}, \rho \wedge \mathscr{H} \subseteq \rho$; so we have $\bar{\rho} \vee(\rho \wedge \mathscr{H}) \subseteq \rho$. Thus $[\bar{\rho} \vee(\rho \wedge \mathscr{H})] \wedge \mathscr{H} \subseteq \rho \wedge \mathscr{H}$. Also, $\rho \wedge \mathscr{H} \subseteq \bar{\rho} \vee(\rho \wedge \mathscr{H})$, and $\rho \wedge \mathscr{H} \subseteq \mathscr{H}$. So we have $\rho \wedge \mathscr{H} \subseteq[\bar{\rho} \vee(\rho \wedge \mathscr{H})] \wedge \mathscr{H}$. Thus $[\bar{\rho} \vee(\rho \wedge \mathscr{H})] \wedge \mathscr{H}=\rho \wedge \mathscr{H}$, and the result follows.

A more interesting question concerns the problem of when the function $\psi$ of Proposition 3.3 is an embedding. Needless to say, $\psi$ is not always an embedding. It is always $\wedge$-preserving, however, as the next proposition shows.

Proposition 3.5. Let $S$ be a band of groups and let $\psi: \Lambda(S) \rightarrow B(S) \times M(S)$ be as in Proposition 3.3. Then $\psi$ is $\wedge$-preserving; that is, $((\rho \wedge \tau) \vee \mathscr{H},(\rho \wedge \tau) \wedge \mathscr{H})=((\rho \vee \mathscr{H}) \wedge$ $(\tau \vee \mathscr{H}),(\rho \wedge \mathscr{H}) \wedge(\tau \wedge \mathscr{H}))$, for each $\rho, \tau \in \Lambda(S)$.

Proof. It is obvious that $(\rho \wedge \tau) \wedge \mathscr{H}=(\rho \wedge \mathscr{H}) \wedge(\tau \wedge \mathscr{H})$. For the other equality, since both $(\rho \wedge \tau) \vee \mathscr{H}$ and $(\rho \vee \mathscr{H}) \wedge(\tau \vee \mathscr{H})$ are band congruences, it suffices, by Proposition 2.4, to show that these congruences are $\theta$-related. But $[(\rho \wedge \tau) \vee \mathscr{H}] \theta(\rho \wedge \tau),(\rho \vee \mathscr{H})$ $\theta \rho$, and $(\tau \vee \mathscr{H}) \theta \tau$. And, since $\theta$ is a congruence, the last two relations imply that $[(\rho \vee \mathscr{H}) \wedge$ ( $\tau \vee \mathscr{H})] \theta(\rho \wedge \tau)$. The result then follows by the transitivity of $\theta$.

Corollary 3.6. Let $S$ be a band of groups. Then $B(S)$ is lattice-isomorphic with $\Lambda(S) / \theta$.
Proof. By Proposition 2.4, the map $\phi: \Lambda(S) / \theta \rightarrow B(S)$ defined by $\phi\left(\theta^{\mathrm{h}}(\rho)\right)=\rho \vee \mathscr{H}$ is a bijection. (It is well-defined by Proposition 2.5.) Since $\theta$ is a lattice congruence, we have $\phi\left(\theta^{\natural}(\rho) \vee \theta^{\natural}(\tau)\right)=\phi\left(\theta^{\natural}(\rho \vee \tau)\right)=(\rho \vee \tau) \vee \mathscr{H}=(\rho \vee \mathscr{H}) \vee(\tau \vee \mathscr{H})=\phi\left(\theta^{\natural}(\rho)\right) \vee \phi\left(\theta^{\natural}(\tau)\right) ;$ and $\phi\left(\theta^{\mathfrak{\natural}}(\rho) \wedge \theta^{\mathfrak{h}}(\tau)\right)=\phi\left(\theta^{\mathfrak{\natural}}(\rho \wedge \tau)\right)=(\rho \wedge \tau) \vee \mathscr{H}=(\rho \vee \mathscr{H}) \wedge(\tau \vee \mathscr{H})=\phi\left(\theta^{\mathfrak{h}}(\rho)\right) \wedge \phi\left(\theta^{\mathfrak{\natural}}(\tau)\right)$, by Proposition 3.5.

We now give a simple example to show that the function $\psi$ of Proposition 3.3 need not be $\vee$-preserving.

Example 3.7. Let $S=\{e, a, f, b\}$ be the semigroup given by the following table:

| - | $e$ | $a$ | $f$ | $b$ |
| :--- | :--- | :--- | :--- | :--- |
| $e$ | $e$ | $a$ | $f$ | $b$ |
| $a$ | $a$ | $e$ | $b$ | $f$ |
| $f$ | $f$ | $b$ | $f$ | $b$ |
| $b$ | $b$ | $f$ | $b$ | $f$. |

$S$ is then in fact a semilattice of the groups $\{e, a\}$ and $\{f, b\}$. It is not hard to show that $S$ has exactly five congruences; the classes of the three non-trivial congruences are listed below:

$$
\begin{aligned}
\sigma & :\{e, f\},\{a, b\} ; \\
\mathscr{H} & :\{e, a\},\{f, b\} ; \\
\alpha & :\{e\},\{a\},\{f, b\} .
\end{aligned}
$$

The congruence $\sigma$ is the minimum group congruence and $\alpha$ is the Rees congruence associated with the ideal $\{f, b\}$. We note that $\psi(\sigma)=(\sigma \vee \mathscr{H}, \sigma \wedge \mathscr{H})=\left(1_{s}, 0_{s}\right)$, and $\psi(\alpha)=(\alpha \vee \mathscr{H}$, $\alpha \wedge \mathscr{H})=(\mathscr{H}, \alpha)$, so that $\psi(\sigma) \vee \psi(\alpha)=\left(1_{s}, \alpha\right)$. But $\psi(\sigma \vee \alpha)=\psi\left(1_{S}\right)=\left(1_{s}, \mathscr{H}\right)$; so we see that $\psi$ is not $v$-preserving.

We now turn our attention to a portion of $\Lambda(S)$ on which $\psi$ is $v$-preserving. Let $S$ be a band of groups, and consider the function $\tilde{\psi}: D(S) \rightarrow B(S) \times M(S)$ defined by $\tilde{\psi}(\rho)=(\rho \vee \mathscr{H}$, $\rho \wedge \mathscr{H})$. That is, $\tilde{\psi}$ is the restriction to $D(S)$ of the function $\psi$ of Proposition 3.3. It follows immediately from Propositions 3.3 and 3.5 that $\psi$ is one-to-one and $\wedge$-preserving. The restriction $\tilde{\psi}$ behaves better than $\psi$, however, in the following sense.

Proposition 3.8. Let $S$ be a band of groups, and define $\tilde{\psi}: D(S) \rightarrow B(S) \times M(S)$ by $\tilde{\psi}(\rho)=(\rho \vee \mathscr{H}, \rho \wedge \mathscr{H})$. Then $\tilde{\psi}$ is $\vee$-preserving; that is, $((\rho \vee \tau) \vee \mathscr{H},(\rho \vee \tau) \wedge \mathscr{H})=$ $((\rho \vee \mathscr{H}) \vee(\tau \vee \mathscr{H}),(\rho \wedge \mathscr{H}) \vee(\tau \wedge \mathscr{H}))$ for each $\rho, \tau \in D(S)$.

Proof. It is clear that $(\rho \vee \tau) \vee \mathscr{H}=(\rho \vee \mathscr{H}) \vee(\tau \vee \mathscr{H})$. For the other equality, we note that $(\rho \wedge \mathscr{H}) \vee(\tau \wedge \mathscr{H})$ is the smallest congruence containing $\rho \wedge \mathscr{H}$ and $\tau \wedge \mathscr{H}$.

But $(\rho \vee \tau) \wedge \mathscr{H}$ is certainly such a congruence. Hence we have $(\rho \wedge \mathscr{H}) \vee(\tau \wedge \mathscr{H}) \subseteq$ $(\rho \vee \tau) \wedge \mathscr{H}$. On the other hand, suppose that $(x, y) \in(\rho \vee \tau) \wedge \mathscr{H}$. Then $x \mathscr{H} y$, and $(x, y) \in \rho \vee \tau=\bigcup_{n=1}^{\infty}(\rho \circ \tau)^{n}$. Thus there exist a positive integer $n$ and elements $x_{i}, x_{i}^{\prime}(i=$ $1, \ldots, n$ ) of $S$ such that

$$
x \rho x_{1} \tau x_{1}^{\prime} \rho x_{2} \tau x_{2}^{\prime} \rho \ldots \rho x_{n} \tau x_{n}^{\prime}=y .
$$

Furthermore, since $\rho, \tau \subseteq \mathscr{D}$, all of the $x_{i}$ and $x_{i}^{\prime}$ are in $D_{x}=D_{y}$. Now let $e$ be the idempotent in $H_{x}=H_{y}$. Then

$$
x=\operatorname{exe} \rho e x_{1} \text { e } \tau e x_{1}^{\prime} e \rho \ldots \rho e x_{n} e \tau e x_{n}^{\prime} e=e y e=y
$$

But $D_{x}=D_{y}$ is a completely simple semigroup, and so, for each $i$, $e x_{i} e, e x_{i}^{\prime} e \in e D_{x} e=H_{x}$. Thus we in fact have $(x, y) \in \bigcup_{n=1}^{\infty}[(\rho \wedge \mathscr{H}) \circ(\tau \wedge \mathscr{H})]^{n}=(\rho \wedge \mathscr{H}) \vee(\tau \wedge \mathscr{H})$, completing the proof.

As a corollary, we now have
Theorem 3.9. Let $S$ be a band of groups. Then $D(S)$ is lattice-isomorphic with a sublattice of the product lattice $B(S) \times M(S)$; specifically, $\psi: D(S) \rightarrow B(S) \times M(S)$ is an embedding.

Since a completely simple semigroup has the property that $\mathscr{D}=1_{S}$, and thus $D(S)=\Lambda(S)$, the following corollary is obvious.

Corollary 3.10. Let $S$ be a completely simple semigroup. Then $\Lambda(S)$ is lattice-isomorphic with a sublattice of the product lattice $B(S) \times M(S)$; specifically, $\psi: \Lambda(S) \rightarrow B(S) \times M(S)$ is an embedding.

We shall now find a necessary and sufficient condition on $\Lambda(S)$, where $S$ is an arbitrary band of groups, under which $\psi$ is actually an embedding.

Recall that an arbitrary lattice $L$ is called modular if, whenever $a, b, c \in L$ with $a \geqq b$, then $a \wedge(c \vee b)=(a \wedge c) \vee b$. It is well known that a lattice $L$ is modular if and only if the conditions $a \geqq b, a \wedge c=b \wedge c$, and $a \vee c=b \vee c$, for elements $a, b, c \in L$, imply that $a=b$. This motivates the following definition.

Definition 3.11. Let $L$ be a lattice, and $\zeta$ a lattice congruence on $L$. We say that $L$ is $\zeta$-modular if the conditions $a \geqq b,(a, b) \in \zeta, a \wedge c=b \wedge c$, and $a \vee c=b \vee c$, for elements $a, b, c \in L$, imply that $a=b$.

For convenience, if $S$ is a semigroup, and $\zeta$ is a lattice congruence on $\Lambda(S)$, we agree to call $S \zeta$-modular, provided that $\Lambda(S)$ is $\zeta$-modular. Since $\theta$ is a lattice congruence on $\Lambda(S)$, we may speak of $\theta$-modularity of $S$. It is in this specialization of the above definition that we are interested.

As examples, we note that all bands are $\theta$-modular; for all their congruences are band congruences, and so the $\theta$-classes are trivial. All groups are $\theta$-modular, for the lattice of congruences on a group consists of a single $\theta$-class, which is in fact modular, by Lemma 2.6. Of course, not all bands of groups are $\theta$-modular, as Example 3.7 readily shows. We shall see shortly that the class of $\theta$-modular bands of groups is particularly interesting.

We begin with a technical lemma.
Lemma 3.12. Let $S$ be a $\theta$-modular band of groups. Then, for any $\rho, \tau \in \Lambda(S), \rho \vee[(\rho \wedge \mathscr{H})$ $\vee(\tau \wedge \mathscr{H})]=\rho \vee[(\rho \vee \tau) \wedge \mathscr{H}]$.

Proof. We first note that $(\rho \wedge \mathscr{H}) \vee(\tau \wedge \mathscr{H}) \subseteq(\rho \vee \tau) \wedge \mathscr{H}$, since $(\rho \wedge \mathscr{H}) \vee(\tau \wedge \mathscr{H})$ is the smallest congruence containing $\rho \wedge \mathscr{H}$ and $\tau \wedge \mathscr{H}$, and since $(\rho \vee \tau) \wedge \mathscr{H}$ is such a congruence. Thus $\rho \vee[(\rho \wedge \mathscr{H}) \vee(\tau \wedge \mathscr{H})] \subseteq \rho \vee[(\rho \vee \tau) \wedge \mathscr{H}]$. Now note that $\rho \vee$ $[(\rho \wedge \mathscr{H}) \vee(\tau \wedge \mathscr{H})]=[\rho \vee(\rho \wedge \mathscr{H})] \vee(\tau \wedge \mathscr{H})=\rho \vee(\tau \wedge \mathscr{H})$. Now $\rho \vee(\tau \wedge \mathscr{H})$ and $\rho \vee[(\rho \vee \tau) \wedge \mathscr{H}]$ are $\theta$-related, for, by Lemma 3.1, $(\tau \wedge \mathscr{H}) \theta(\rho \vee \tau) \wedge \mathscr{H}$, and then, by Lemma 2.2, $[\rho \vee(\tau \wedge \mathscr{H})] \theta \rho \vee[(\rho \vee \tau) \wedge \mathscr{H}]$. Thus, by $\theta$-modularity, it will suffice to show that $\tau \vee[\rho \vee(\tau \wedge \mathscr{H})]=\tau \vee[\rho \vee[(\rho \vee \tau) \wedge \mathscr{H}]]$ and $\tau \wedge[\rho \vee(\tau \wedge \mathscr{H})]=\tau \wedge[\rho \vee$ $[(\rho \vee \tau) \wedge \mathscr{H}]]$. Now we have already seen that $\rho \vee(\tau \wedge \mathscr{H}) \subseteq \rho \vee[(\rho \vee \tau) \wedge \mathscr{H}]$. Thus $\tau \vee \rho \subseteq \tau \vee[\rho \vee(\tau \wedge \mathscr{H})] \subseteq \tau \vee[\rho \vee[(\rho \vee \tau) \wedge \mathscr{H}]] \subseteq \tau \vee[\rho \vee(\rho \vee \tau)]=\tau \vee(\rho \vee \tau)=$ $\tau \vee \rho$, implying the first equality above. For the other equality, we have $\tau \wedge[\rho \vee(\tau \wedge \mathscr{H})] \subseteq$ $\tau \wedge[\rho \vee[(\rho \vee \tau) \wedge \mathscr{H}]] \subseteq \tau \wedge[\rho \vee \mathscr{H}]$. Hence it suffices to show that $\tau \wedge[\rho \vee \mathscr{H}] \subseteq$ $\tau \wedge[\rho \vee(\tau \wedge \mathscr{H})]$. For this, it is sufficient to show that $\tau \wedge(\rho \vee \mathscr{H}) \subseteq \rho \vee(\tau \wedge \mathscr{H})$; for then $\tau \wedge(\rho \vee \mathscr{H})=\tau \wedge[\tau \wedge(\rho \vee \mathscr{H})] \subseteq \tau \wedge[\rho \vee(\tau \wedge \mathscr{H})]$. So suppose that $(x, y) \in$ $\tau \wedge(\rho \vee \mathscr{H})$. Let $e \in E_{S} \cap H_{x}, f \in E_{S} \cap H_{y}$ and $g \in E_{S} \cap H_{x y}$. Since $(x, y) \in \rho \vee \mathscr{H}$, we have $(e, f) \in \rho \vee \mathscr{H}$, by Lemma 3.2. Hence, since $(\rho, \rho \vee \mathscr{H}) \in \theta$, we have $(e, f) \in \rho$. Thus $e=$ ee $\rho$ ef $\mathscr{H}$ xy $\mathscr{H} g$, so that $(e, g) \in \rho \vee \mathscr{H}$. Again, since $(\rho, \rho \vee \mathscr{H}) \in \theta$, we have $(e, g) \in \rho$, and hence also ( $f, g) \in \rho$, by the transitivity of $\rho$. Moreover, using the fact that the $\mathscr{D}$-class $D_{g}$ is completely simple, we have $\operatorname{gxg} \mathscr{H}$ gyg. Thus $x=\operatorname{exe} \rho g x g(\tau \wedge \mathscr{H}) \operatorname{gyg} \rho f y f=y$, so that $(x, y) \in \rho \vee(\tau \wedge \mathscr{H})$. This completes the proof.

Proposition 3.13. Let $S$ be a $\theta$-modular band of groups. Then the function $\psi: \Lambda(S) \rightarrow$ $B(S) \times M(S)$ defined by $\psi(\rho)=(\rho \vee \mathscr{H}, \rho \wedge \mathscr{H})$ is $\vee$-preserving; that is,

$$
((\rho \vee \tau) \vee \mathscr{H},(\rho \vee \tau) \wedge \mathscr{H})=((\rho \vee \mathscr{H}) \vee(\tau \vee \mathscr{H}),(\rho \wedge \mathscr{H}) \vee(\tau \wedge \mathscr{H}))
$$

for each $\rho, \tau \in \Lambda(S)$.
Proof. It is obvious that $(\rho \vee \tau) \vee \mathscr{H}=(\rho \vee \mathscr{H}) \vee(\tau \vee \mathscr{H})$. For the other equality, we have already noted in the proof of Lemma 3.12 that $(\rho \wedge \mathscr{H}) \vee(\tau \wedge \mathscr{H}) \subseteq(\rho \vee \tau) \wedge \mathscr{H}$. Also, both $(\rho \wedge \mathscr{H}) \vee(\tau \wedge \mathscr{H})$ and $(\rho \vee \tau) \wedge \mathscr{H}$ are contained in $\mathscr{H}$, and are therefore $\theta$ related. Thus, by $\theta$-modularity, it will suffice to show that $\rho \vee[(\rho \wedge \mathscr{H}) \vee(\tau \wedge \mathscr{H})]=\rho \vee$ $[(\rho \vee \tau) \wedge \mathscr{H}]$, and $\rho \wedge[(\rho \wedge \mathscr{H}) \vee(\tau \wedge \mathscr{H})]=\rho \wedge[(\rho \vee \tau) \wedge \mathscr{H}]$. The first of these equalities is the content of Lemma 3.12. Also, since $\rho \wedge \mathscr{H} \subseteq(\rho \wedge \mathscr{H}) \vee(\tau \wedge \mathscr{H})$, we have $\rho \wedge \mathscr{H}=\rho \wedge(\rho \wedge \mathscr{H}) \subseteq \rho \wedge[(\rho \wedge \mathscr{H}) \vee(\tau \wedge \mathscr{H})] \subseteq \rho \wedge[(\rho \vee \tau) \wedge \mathscr{H}]=[\rho \wedge(\rho \vee \tau)] \wedge \mathscr{H}$ $=\rho \wedge \mathscr{H}$, from which the second equality follows.

Combining Propositions 3.3,3.5, and 3.13, we obtain
Theorem 3.14. Let $S$ be a $\theta$-modular band of groups. Then the function $\psi: \Lambda(S) \rightarrow$ $B(S) \times M(S)$ defined by $\psi(\rho)=(\rho \vee \mathscr{H}, \rho \wedge \mathscr{H})$ is an embedding.

The converse of this theorem is also true.
Theorem 3.15. Let $S$ be a band of groups, and suppose that $\psi: \Lambda(S) \rightarrow B(S) \times M(S)$ as defined above is an embedding. Then $S$ is $\theta$-modular.

Proof. Let $\rho \subseteq \tau$ be $\theta$-related congruences, and suppose that $\alpha$ is a congruence such that $\rho \vee \alpha=\tau \vee \alpha$ and $\rho \wedge \alpha=\tau \wedge \alpha$. Clearly $\rho \wedge \mathscr{H} \subseteq \tau \wedge \mathscr{H}$; and, since $\psi$ is an embedding, we have $(\rho \wedge \mathscr{H}) \vee(\alpha \wedge \mathscr{H})=(\rho \vee \alpha) \wedge \mathscr{H}=(\tau \vee \alpha) \wedge \mathscr{H}=(\tau \wedge \mathscr{H}) \vee(\alpha \wedge \mathscr{H})$. Also, $(\rho \wedge \mathscr{H}) \wedge(\alpha \wedge \mathscr{H})=(\rho \wedge \alpha) \wedge \mathscr{H}=(\tau \wedge \alpha) \wedge \mathscr{H}=(\tau \wedge \mathscr{H}) \wedge(\alpha \wedge \mathscr{H})$. Hence, since, by Lemma 2.6, the $\theta$-class of $\mathscr{H}$ is a modular sublattice of $\Lambda(S)$, we conclude that $\rho \wedge \mathscr{H}=$ $\tau \wedge \mathscr{H}$. Also, since $\rho \theta \tau$, we have $\rho \vee \mathscr{H}=\tau \vee \mathscr{H}$, by Proposition 2.5. Since $\psi$ is one-to-one, we conclude that $\rho=\tau$. Thus $S$ is $\theta$-modular.

The above two theorems characterize $\theta$-modular bands of groups as being those whose lattice of congruences can be naturally embedded in a certain product lattice. The class of $\theta$-modular bands of groups is studied further in [9].
4. The inverse semigroup congruences. In this final section, we study the connection between the $\theta$-relation and the sublattice $I(S)$ of inverse semigroup congruences on a band of groups $S$.

Proposition 4.1. Let $S$ be a band of groups, and let $\tau \in Y(S)$. Let $\rho$ be a congruence $\theta$-related to $\tau$. Then $\rho \in I(S)$.

Proof. It will suffice to show that $S / \rho$ is a semilattice of groups. (See Exercise 2 on page 129 in [1].) Write $S=\bigcup_{\alpha \in S / \tau} S_{\alpha}$, where the $S_{\alpha}$ are the $\tau$-classes of $S$. Since $\mathscr{D}=\eta \subseteq \tau$, each $S_{\alpha}$ is a union of $\mathscr{D}$-classes, and is hence a regular subsemigroup of $S$. Since $\tau$ is a semilattice congruence (and thus a fortiori a band congruence), it follows from Propositions 2.7 and 2.4 that $\tau$ is the greatest element of its $\theta$-class. In particular then, $\rho \subseteq \tau$. Since $\rho \subseteq \tau$, it follows that the sets $\rho^{\natural}\left[S_{\alpha}\right]$ are disjoint subsemigroups of $S / \rho$. Now, since $S$ is a semilattice of the $S_{a}$ and $\rho^{\natural}$ is a homomorphism, it follows that $S / \rho$ is a semilattice of the $\rho^{\natural}\left[S_{\alpha}\right]$. Moreover, since $(\rho, \tau) \in \theta, \rho$ identifies all the idempotents in the $\tau$-class $S_{\alpha}$. Hence $\rho^{\natural}\left[S_{\alpha}\right]$ is a group, and it follows that $S / \rho$ is an inverse semigroup.

The converse of this proposition is also true.
Proposition 4.2. Let $S$ be a band of groups, and let $\rho \in I(S)$. Then there is some congruence $\tau \in Y(S)$ such that $(\rho, \tau) \in \theta$.

Proof. $S / \rho$ is an inverse semigroup which is a union of groups; that is, $S / \rho$ is a semilattice of groups. Let $Y=(S / \rho) / \mathscr{D}_{S / \rho}$ be the structure semilattice of $S / \rho$, and let $\phi$ denote $\mathscr{D}_{S / \rho}^{\natural}: S / \rho \rightarrow Y$. Let $\tau$ be the congruence on $S$ determined by $\phi \circ \rho^{\natural}$. Then clearly $\tau$ is a semilattice congruence. Moreover, we have $(\tau, \rho) \in \theta$. For, if $e, f \in E_{S}$, then $e \rho f$ clearly implies $e \tau f$. And conversely, if $e \tau f$, then $\phi \circ \rho^{\natural}(e)=\phi \circ \rho^{\natural}(f)$; but, since the $\mathscr{D}$-classes of $S / \rho$ are groups, $\phi$ is an idempotent-separating homomorphism, and so we must have $\rho^{\natural}(e)=\rho^{\natural}(f)$; that is, $e \rho f$.

As an immediate corollary, we now deduce
Theorem 4.3. Let $S$ be a band of groups. Then the $\theta$-saturation of $Y(S)$ is $I(S)$; that is, the inverse semigroup congruences on $S$ are precisely those that are $\theta$-related to some semilattice congruence.

We now give an alternative characterization of the inverse semigroup congruences on a band of groups.

Proposition 4.4. Let $S$ be a band of groups. Then a congruence $\rho$ is an inverse semigroup congruence if and only if $\rho \vee \mathscr{D}=\rho \vee \mathscr{H}$.

Proof. Suppose that $\rho \vee \mathscr{D}=\rho \vee \mathscr{H}$. Since $(\rho, \rho \vee \mathscr{H}) \in \theta$, we have $(\rho, \rho \vee \mathscr{D}) \in \theta$. But $\eta=\mathscr{D} \subseteq \rho \vee \mathscr{D}$, so that $\rho \vee \mathscr{D}$ is a semilattice congruence. It then follows from Proposition 4.1 that $\rho$ is an inverse semigroup congruence. To prove the converse, we first note that $\rho \circ \mathscr{D}=\rho \circ \mathscr{H}$. For certainly $\rho \circ \mathscr{H} \subseteq \rho \circ \mathscr{D}$. On the other hand, if $(x, y) \in \rho \circ \mathscr{D}$, say $x \rho z \mathscr{D} y$, let $y^{\prime}$ be the inverse of $z^{-1}$ in $H_{y}$. Since $S / \rho$ is an inverse semigroup, we have uniqueness of inverses in $S / \rho$, and thus $\rho^{\mathfrak{k}}\left(y^{\prime}\right)=\rho^{\natural}(z)$; that is, $z \rho y^{\prime}$. Thus $x \rho y^{\prime} \mathscr{H} y$, so that $(x, y) \in \rho \circ \mathscr{H}$. We thus have $\rho \circ \mathscr{H}=\rho \circ \mathscr{D}$. Hence $\rho \vee \mathscr{D}=\bigcup_{n=1}^{\infty}(\rho \circ \mathscr{D})^{n}=\bigcup_{n=1}^{\infty}(\rho \circ \mathscr{H})^{n}=\rho \vee \mathscr{H}$, completing the proof.

We now have the following corollary.
Corollary 4.5. Let $S$ be a band of groups. Then $\delta$, the minimum inverse semigroup congruence on $S$, is the least element of the $\theta$-class of $\mathscr{D}$.

Proof. Since $\mathscr{D}=\eta$ is an inverse semigroup congruence, we must have $\delta \subseteq \mathscr{D}$. Hence, by Proposition 4.4, $\delta \vee \mathscr{H}=\delta \vee \mathscr{D}=\mathscr{D}$. But $(\delta, \delta \vee \mathscr{H}) \in \theta$; so $(\delta, \mathscr{D}) \in \theta$. But, by Proposition 4.1, every congruence in the $\theta$-class of $\mathscr{D}$ is an inverse semigroup congruence. Hence $\delta$ must be the least element of this $\theta$-class, since it is to be contained in all inverse semigroup congruences.

A natural question to ask at this point is whether one obtains an embedding theorem for $I(S)$ similar to Theorem 3.9. The answer is that one does not, as is illustrated by the semigroup of Example 3.7. We shall show that $\theta$-modularity of the semigroup $S / \delta$ is a necessary and sufficient condition for such a result.

Now let $S$ be an arbitrary semigroup. If $\rho, \gamma \in \Lambda(S)$ and $\gamma \subseteq \rho$, then the relation $\rho / \gamma$ on $S / \gamma$ defined by $\rho / \gamma=\left\{\left(\gamma^{\natural}(x), \gamma^{\natural}(y)\right) \mid(x, y) \in \rho\right\}$ is a congruence. Moreover, the lattice $\gamma \vee \Lambda(S)$ is isomorphic with $\Lambda(S / \gamma)$ under the map $\gamma \vee \tau \rightarrow(\gamma \vee \tau) / \gamma$. In particular, if $\gamma \subseteq \rho, \tau$, then $(\rho \wedge \tau) / \gamma=(\rho / \gamma) \wedge(\tau / \gamma)$ and $(\rho \vee \tau) / \gamma=(\rho / \gamma) \vee(\tau / \gamma)$. These facts are readily verified, as is pointed out in [7].

We now have
Lemma 4.6. Let $S$ be a band of groups. Then $\mathscr{H}_{S / \delta}=\mathscr{D} / \delta$.
Proof. Suppose that $\delta^{\natural}(x) \mathscr{D} / \delta \delta^{\natural}(y)$. Then $x \mathscr{D} y$, so that $\delta^{\natural}(x) \mathscr{D}_{S / \delta} \delta^{\natural}(y)$. But $S / \delta$ is an inverse semigroup; that is, $S / \delta$ is a semilattice of groups. Hence $\mathscr{D}_{S / \delta}=\mathscr{H}_{S / \delta}$, and we thus have $\delta^{\natural}(x) \mathscr{H}_{s / \delta} \delta^{\natural}(y)$. Conversely, suppose that $\delta^{\natural}(x) \mathscr{H}_{s / \delta} \delta^{\natural}(y)$. Then $\mathscr{D}^{\natural}(x)=(\mathscr{D} / \delta)^{\natural}$ $\left(\delta^{\mathfrak{h}}(x)\right) \mathscr{H}_{S / \mathscr{G}}(\mathscr{D} / \delta)^{\mathfrak{h}}\left(\delta^{\natural}(y)\right)=\mathscr{D}^{\mathrm{h}}(y)$. But $S / \mathscr{D}$ is a semilattice; so its $\mathscr{H}$-relation is trivial. Hence we get $\mathscr{D}^{\natural}(x)=\mathscr{D}^{\natural}(y)$; that is, $x \mathscr{D} y$. Thus $\delta^{\natural}(x) \mathscr{D} / \delta \delta^{\natural}(y)$, and the result follows.

Proposition 4.7. Let $S$ be a band of groups such that $S / \delta$ is $\theta$-modular. Then the function $\hat{\psi}: I(S) \rightarrow Y(S) \times \Delta(S)$ defined by $\hat{\psi}(\rho)=(\rho \vee \mathscr{D}, \rho \wedge \mathscr{D})$ is an embedding.

Proof. We first note that the function $\hat{\psi}$ is indeed well-defined; for $\rho \vee \mathscr{D}$ contains the minimum semilattice congruence $\mathscr{D}$, and is thus itself a semilattice congruence. And, by Corollary 4.5, $\rho \wedge \mathscr{D} \theta \rho \wedge \delta=\delta \theta \mathscr{D}$ since $\rho$ is an inverse semigroup congruence. Since $S / \delta$ is $\theta$-modular, the function $\psi: \Lambda(S / \delta) \rightarrow B(S / \delta) \times M(S / \delta)$ defined by $\psi(\rho / \delta)=\left(\rho / \delta \vee \mathscr{H}_{S / \delta}\right.$, $\left.\rho / \delta \wedge \mathscr{H}_{S / \delta}\right)$ is an embedding. Now, by Lemma 4.6, we have $\rho / \delta \vee \mathscr{H}_{S / \delta}=\rho / \delta \vee \mathscr{D} / \delta=$ $(\rho \vee \mathscr{D}) / \delta$, and likewise $\rho / \delta \wedge \mathscr{H}_{S / \delta}=(\rho \wedge \mathscr{D}) / \delta$. But $I(S)=\delta \vee \Lambda(S)$ is isomorphic to $\Lambda(S / \delta)$, under the isomorphism $\rho \rightarrow \rho / \delta$. Thus the composition $\rho \rightarrow \rho / \delta \xrightarrow{\psi}((\rho \vee \mathscr{D}) / \delta$, $(\rho \wedge \mathscr{D}) / \delta) \rightarrow(\rho \vee \mathscr{D}, \rho \wedge \mathscr{D})$ is an embedding. This completes the proof.

Before proving the converse of this proposition, we need the following lemma.
Lemma 4.8. Let $S$ be any regular semigroup, and $\rho, \tau, \alpha \in \Lambda(S)$ such that $\alpha \subseteq \rho, \tau$. Then $\rho \theta \tau$ if and only if $\rho / \alpha \theta \tau / \alpha$.

Proof. We note first that, by [4, Lemma 2.2], $E_{S / \alpha}=\left\{\alpha^{\natural}(e) \mid e \in E_{S}\right\}$. Hence, if $\rho \theta \tau$, we have $\alpha^{\natural}(e) \rho / \alpha \alpha^{\natural}(f) \Leftrightarrow e \rho f \Leftrightarrow e \tau f \Leftrightarrow \alpha^{\natural}(e) \tau / \alpha \alpha^{\natural}(f)$, so that $\rho / \alpha \theta \tau / \alpha$. Conversely, if $\rho / \alpha \theta \tau / \alpha$, then $e \rho f \Leftrightarrow \alpha^{\natural}(e) \rho / \alpha \alpha^{\natural}(f) \Leftrightarrow \alpha^{\natural}(e) \tau / \alpha \alpha^{\natural}(f) \Leftrightarrow e \tau f$, and so $\rho \theta \tau$.

Proposition 4.9. Let $S$ be a band of groups, and suppose that the function $\hat{\psi}: I(S) \rightarrow$ $Y(S) \times \Delta(S)$ defined by $\hat{\psi}(\rho)=(\rho \vee \mathscr{D}, \rho \wedge \mathscr{D})$ is an embedding. Then $S / \delta$ is $\theta$-modular.

Proof. Suppose that $\rho / \delta \subseteq \tau / \delta, \rho / \delta \theta \tau / \delta$, and that, for some $\alpha / \delta \in \Lambda(S / \delta), \rho / \delta \vee \alpha / \delta=$ $\tau / \delta \vee \alpha / \delta$ and $\rho / \delta \wedge \alpha / \delta=\tau / \delta \wedge \alpha / \delta$. We then have $(\rho \vee \alpha) / \delta=(\tau \vee \alpha) / \delta$, so that $\rho \vee \alpha=$ $\tau \vee \alpha$; and likewise, $\rho \wedge \alpha=\tau \wedge \alpha$. Then, since $\hat{\psi}$ is $\vee$-preserving, we have $(\rho \wedge \mathscr{D}) \vee(\alpha \wedge \mathscr{D})=(\rho \vee \alpha) \wedge \mathscr{D}=(\tau \vee \alpha) \wedge \mathscr{D}=(\tau \wedge \mathscr{D}) \vee(\alpha \wedge \mathscr{D})$. Moreover, $(\rho \wedge \mathscr{D})$ $\wedge(\alpha \wedge \mathscr{D})=(\rho \wedge \alpha) \wedge \mathscr{D}=(\tau \wedge \alpha) \wedge \mathscr{D}=(\tau \wedge \mathscr{D}) \wedge(\alpha \wedge \mathscr{D})$. Also, $\rho / \delta \subseteq \tau / \delta$ implies $\rho \subseteq \tau$, so that $\rho \wedge \mathscr{D} \subseteq \tau \wedge \mathscr{D}$. Now $\rho \wedge \mathscr{D}, \tau \wedge \mathscr{D}$, and $\alpha \wedge \mathscr{D}$ are inverse semigroup congruences contained in $\mathscr{D}$, and are hence in $\Delta(S)$. But $\Delta(S)$ is modular by Lemma 2.6 ; so we have $\rho \wedge \mathscr{D}=\tau \wedge \mathscr{D}$. Also, since $\rho / \delta \theta \tau / \delta$, we have $\rho \theta \tau$, by Lemma 4.8, and hence, by Propositions 2.5 and 4.4, $\rho \vee \mathscr{D}=\rho \vee \mathscr{H}=\tau \vee \mathscr{H}=\tau \vee \mathscr{D}$. Since $\hat{\psi}$ is one-to-one, we conclude that $\rho=\tau$, and hence $\rho / \delta=\tau / \delta$, completing the proof.

Combining Propositions 4.7 and 4.9 , we immediately deduce
Theorem 4.10. Let $S$ be a band of groups. Then $\hat{\psi}: I(S) \rightarrow Y(S) \times \Delta(S)$ defined by $\hat{\psi}(\rho)=(\rho \vee \mathscr{D}, \rho \wedge \mathscr{D})$ is an embedding if and only if $S / \delta$ is $\theta$-modular.

This paper is a portion of the author's doctoral dissertation, written at the University of Kentucky. I would like to express to my adviser, Dr Carl Eberhart, my appreciation of his many helpful suggestions and comments.

## REFERENCES

1. A. H. Clifford and G. B. Preston, The algebraic theory of semigroups, Vol. I, Amer. Math. Soc. Mathematical Surveys No. 7 (Providence, R.I., 1961).
2. J. M. Howie and G. Lallement, Certain fundamental congruences on a regular semigroup, Proc. Glasgow Math. Assoc. 7 (1966), 145-159.
3. K. M. Kapp and H. Schneider, Completely O-simple semigroups (New York, 1969).
4. G. Lallement, Congruence et équivalences de Green sur un demi-group régulier, C. R. Acad. Sci. Paris, Série A, 262 (1966), 613-616.
5. J. Leech, The structure of bands of groups; to appear.
6. W. D. Munn, A certain sublattice of the lattice of congruences on a regular semigroup, Proc. Cambridge Philos. Soc. 60 (1964), 385-391.
7. N. R. Reilly and H. E. Scheiblich, Congruences on regular semigroups, Pacific J. Math. 23 (1967), 349-360.
8. H. E. Scheiblich, Certain congruence and quotient lattices related to completely 0 -simple and primitive regular semigroups, Glasgow Math. J. 10 (1969), 21-24.
9. C. Spitznagel, $\theta$-modular bands of groups, Trans. Amer. Math. Soc.; to appear.

## University of Kentucky

Lexington, Kentucky 40506

## AND

John Carroll University
Cleveland, Ohio 44118

