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THE LATTICE OF EQUATIONAL THEORIES PART IV: EQUATIONAL THEORIES OF FINITE ALGEBRAS

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0. INTRODUCTION

This paper is a continuation of [1], [2] and [3].

The lattice $\mathscr{L}_\Delta$ of equational theories of type Δ is antiisomorphic to the lattice of varieties of Δ-algebras. The variety, corresponding to an equational theory T, is denoted by $\mathrm{Mod}(T)$; its elements are called models of T. If K is any class of Δ-algebras, then $\mathrm{Eq}(K)$ denotes the equational theory corresponding to the variety $\mathrm{HSP}(K)$ (the variety generated by K). For any algebra A put $\mathrm{Eq}(A) = \mathrm{Eq}(\{A\})$; this equational theory is called the equational theory of A; it is just the set of equations satisfied in the algebra A.

In this paper we shall be interested in the equational theories of finite algebras. Our aim is to prove that for any type Δ, the set of the equational theories of finite Δ-algebras is definable in the lattice $\mathscr{L}_\Delta$ and that in the case of a finite type Δ, the equational theory of any finite Δ-algebra is definable up to automorphisms in $\mathscr{L}_\Delta$. This will answer a problem formulated by George McNulty.

For this purpose, we shall have to find a suitable encoding of finite algebras in $\mathscr{L}_\Delta$. The formulas ψ_{30} and ψ_{45}, the two most important formulas discovered in [3], enable us to carry most of the work over from $\mathscr{L}_\Delta$ to the lattice $\mathscr{F}_\Delta$ of full sets of Δ-terms. And so instead of in $\mathscr{L}_\Delta$ we shall encode the algebras in $\mathscr{F}_\Delta$. We shall not confine ourselves to finite algebras: in the case of a strictly large type Δ all algebras of cardinality $\leqq \mathrm{Max}(\aleph_0, \mathrm{Card}(\Delta))$ will be encoded, while in the case of a large but not strictly large type the same will be done for the algebras of cardinality $\leqq \mathrm{Max}(\aleph_0, \mathrm{Card}(\Delta \setminus \Delta_0))$ only.

For the terminology and notation see [1], [2] and [3].

Algebras are often identified with their underlying sets. If A is a Δ-algebra and $F \in \Delta$ is a symbol of an arity n, then the corresponding n-ary operation in A will be denoted by F_A.

Most of the lemmas are without proof; they are either evident or follow easily from the preceding ones.

I would like to correct one wrong place in Section 5 of [2]: the definition of the formula φ_{37} should be replaced by

$$\varphi_{37}(X_1, X_2, Y, A, B \equiv \varphi_{33}(X_1, X_2, Y) \mathbin{\&} (\exists Z(\varphi_{33}(X_1, X_2, Z) \mathbin{\&} \\ \mathbin{\&} Y \neq Z \mathbin{\&} \varphi_{36}(X_1, X_2, Z, A, B)) \text{ VEL } \exists U, A_0, B_0(\alpha_0(U) \mathbin{\&} \\ \mathbin{\&} U \leqslant A_0 \mathbin{\&} U \leqslant B_0 \mathbin{\&} \varphi_8(A_0, A) \mathbin{\&} \varphi_8(B_0, B) \mathbin{\&} A_0 \leqslant B_0)) .$$

1. STRICTLY LARGE TYPES

Throughout this section let Δ be a strictly large type.

Let $(F, i) \in \Delta^{(2)}$. The notion of an (F, i)-codelement is defined as follows:

(1) if Δ is finite, then (F, i)-codelements are the elements of $\mathscr{F}_\Delta$ of the form $(K_x(t))^*$ where $x \in V$ and $t \in x\begin{bmatrix} k \\ F, i \end{bmatrix}\begin{bmatrix} 1 \\ F, j \end{bmatrix}$ for some $k \geqq 2$ and some $j \in \{1, \ldots, n_F\} \setminus \{i\}$;

(2) if Δ is infinite and contains at least one nullary symbol, then (F, i)-codelements are elements of $\mathscr{F}_\Delta$ of the form $(G(C_1, \ldots, C_{n_G}))^*$ where $G \in \Delta \setminus \Delta_0$ and $C_1, \ldots, C_{n_G} \in \Delta_0$;

(3) if Δ is infinite and contains no nullary symbols, then (F, i)-codelements are elements of $\mathscr{F}_\Delta$ of the form $(G(x, x, \ldots, x))^*$ where $G \in \Delta$ and $x \in V$.

The set of (F, i)-codelements is denoted by $\mathrm{CEL}_{F,i}$.

1.1. Lemma. *Let $(F, i) \in \Delta^{(2)}$. Then $\mathrm{CEL}_{F,i}$ is a set of pairwise uncomparable elements of $\mathscr{F}_\Delta$; we have* $\mathrm{Card}(\mathrm{CEL}_{F,i}) = \mathrm{Max}(\aleph_0, \mathrm{Card}(\Delta))$.

Let $(F, i) \in \Delta^{(2)}$; let $G \in \Delta$ and let $A_1, \ldots, A_{n_G}, A$ be (F, i)-codelements. For every variable x there exists a unique pair a, b of terms such that $\mathrm{var}(a) \cup \mathrm{var}(b) \subseteq \{x\}$, $b^* = A$ and $a = G(a_1, \ldots, a_{n_G})$ where $a_1^* = A_1, \ldots, a_{n_G}^* = A_{n_G}$. The element $H_{F,i}(a, b)$ of $\mathscr{F}_\Delta$ (which does not depend on the choice of x) will be denoted by $[G, A_1, \ldots, A_{n_G}, A]_{F,i}$. The elements of $\mathscr{F}_\Delta$ of this form will be called (F, i)-definators.

1.2. Lemma. *Let $(F, i) \in \Delta^{(2)}$. If $[G, A_1, \ldots, A_{n_G}, A]_{F,i}$ and $[H, B_1, \ldots, B_{n_H}, B]_{F,i}$ are two (F, i)-definators and $[G, A_1, \ldots, A_{n_G}, A]_{F,i} \leqq [H, B_1, \ldots, B_{n_H}, B]_{F,i}$ then $G = H$, $A_1 = B_1, \ldots, A_{n_G} = B_{n_H}$ and $A = B$.*

Proof. As in the definition of codelements, it is necessary to distinguish three cases. However, each of them is easy.

For every $U \in \mathscr{F}_\Delta$ put $I^*(U) = \{t^*; t \in I(U)\}$.

By an (F, i)-codset we mean an element S of $\mathscr{F}_\Delta$ such that every element of $I^*(S)$ is an (F, i)-codelement. Elements of $I^*(S)$ are called (F, i)-codelements of S. There is a natural one-to-one correspondence between (F, i)-codsets and subsets of $\mathrm{CEL}_{F,i}$. The union of the sets in $\mathrm{CEL}_{F,i}$ is the largest (F, i)-codest, while the empty set is the least (F, i)-codset.

By an (F, i)-codalgebra we mean a pair S, R of elements of $\mathscr{F}_\Delta$ satisfying the following three conditions:

(1) S is a non-empty (F, i)-codset;

(2) every element of $I^*(R)$ is an (F, i)-definator of the form $[G, A_1, \ldots, A_{n_G}, A]_{F,i}$ where $G \in \Delta$ and $A_1, \ldots, A_{n_G}, A \in I^*(S)$;

(3) for every $G \in \Delta$ and every $A_1, \ldots, A_{n_G} \in I^*(S)$ there exists exactly one (F, i)-codelement A such that $[G, A_1, \ldots, A_{n_G}, A]_{F,i} \in I^*(R)$.

Given an (F, i)-codalgebra S, R, we can define an algebra Q of type Δ with the underlying set $I^*(S)$ as follows: if $G \in \Delta$ and $A_1, \ldots, A_{n_G} \in I^*(S)$ then $G_Q(A_1, \ldots$ $\ldots, A_{n_G}) = A$ where A is the only (F, i)-codelement with $[G, A_1, \ldots, A_{n_G}, A]_{F,i} \in$ $\in I^*(R)$. This algebra Q is said to be the Δ-algebra corresponding to the (F, i)-codalgebra S, R.

1.3. Lemma. *Let $(F, i) \in \Delta^{(2)}$. Every Δ-algebra whose underlying set is a subset of $\mathrm{CEL}_{F,i}$ corresponds to exactly one (F, i)-codalgebra. Consequently, a Δ-algebra Q is isomorphic to a Δ-algebra corresponding to an (F, i)-codalgebra, iff* $\mathrm{Card}(Q) \leqq$ $\leqq \mathrm{Max}(\aleph_0, \mathrm{Card}(\Delta))$.

Proof. Lemma follows from 1.2 and the definitions.

Definition. (i) $\chi_1(X, Y, Z, U) \equiv \varphi_{53}(X, U) \;\&\; Y \preccurlyeq U \;\&\; Z \preccurlyeq U \;\&\; \neg\omega_1(Y) \;\&$ $\& \neg\omega_1(Z) \;\&\; \exists A, B, C(\varphi_{56}(X, A, Y) \;\&\; \varphi_{56}(X, B, Z) \;\&\; \varphi_{56}(X, C, U) \;\&$ $\& \varphi_{59}(X, A, C) \;\&\; \varphi_{61}(X, C, B) \;\&\; \forall Z_1, U_1, Z_2, U_2((\varphi_{60}(X, A, Z_1, U_1) \;\&$ $\& \varphi_{60}(X, B, Z_2, U_2)) \rightarrow U_1 \neq U_2))$.

(ii) $\chi_2(X, Y, Z, U) \equiv \varphi_{53}(X, Y) \;\&\; Y \preccurlyeq U \;\&\; Z \preccurlyeq U \;\&\; (\omega_1(Y) \rightarrow U = Z) \;\&$ $\& (\omega_1(Z) \rightarrow U = Y) \;\&\; ((\neg\omega_1(Y) \;\&\; \neg\omega_1(Z)) \rightarrow (\chi_1(X, Y, Z, U) \;\&$ $\& \forall U_1(\chi_1(X, Y, Z, U_1) \rightarrow U \preccurlyeq U_1)))$.

(iii) $\chi_3(X, Y, A, B, Z) \equiv \exists U_1, U_2, U, C, D(\varphi_{60}(X, Y, A, U_1) \;\&\; \varphi_{60}(X, Y, B, U_2) \;\&$ $\& \chi_2(X, A, C, B) \;\&\; C \prec D \;\&\; \varphi_{59}(X, U, Y) \;\&\; \varphi_{56}(X, U, B) \;\&\; \varphi_{61}(X, U, Z) \;\&$ $\& \varphi_{56}(X, Z, D))$.

(iv) $\chi_4(X, Y, A, B, Z) \equiv \exists U(\chi_3(X, Y, A, B, U) \;\&\; \varphi_{69}(X, U, Z))$.

(v) $\chi_5(X, Y, A, B) \equiv \exists Z(\chi_4(X, Y, A, B, Z) \;\&\; \varphi_{72}(X, Z))$.

(vi) $\chi_6(X, Y, Z) \equiv \exists A, B, C, U_1, U_2, U_3, U_4, U(\varphi_{69}(X, A, Y) \;\&\; \varphi_4(Z) \;\&$ $\& \varphi_3(B, X) \;\&\; \varphi_3(B, C) \;\&\; X \neq C \;\&\; \varphi_{64}(X, X, U_1) \;\&\; \varphi_{64}(X, C, U_2) \;\&$ $\& \varphi_{65}(X, U_1, C, U_3) \;\&\; \varphi_{65}(X, U_3, Z, U_4) \;\&\; \varphi_{68}(X, Y, U_2, U_4, U))$.

(vii) $\chi_7(X, Y) \equiv \exists U_1, U_2(\varphi_{56}(X, Y, U_1) \;\&\; U_1 \prec U_2 \;\&$ $\& \forall Z, P, Q, R((\varphi_{56}(X, Z, U_2) \;\&\; \varphi_{59}(X, Y, Z) \;\&\; \chi_4(X, Z, U_1, U_2, P) \;\&$ $\& \varphi_{69}(X, P, Q) \;\&\; \chi_6(X, Q, R)) \rightarrow \exists U_3(U_3 \preccurlyeq U_1 \;\&\; \chi_5(X, Z, U_3, U_2))))$.

(viii) $\chi_8(X, Y) \equiv \chi_7(X, Y) \;\&\; \forall Z(\varphi_{59}(X, Z, Y) \rightarrow \chi_7(X, Z))$.

(ix) $\chi_9(X, Y, A, B, C) \equiv \exists Z(\chi_4(X, Y, A, B, Z) \;\&\; \chi_6(X, Z, C))$.

(x) $\chi_{10}(X, Y_1, Y_2) \equiv \exists Z, U_1, U_2(\varphi_{56}(X, Y_1, Z) \;\&\; \varphi_{56}(X, Y_2, Z) \;\&$ $\& \varphi_{60}(X, Y_1, Z, U_1) \;\&\; \varphi_{60}(X, Y_2, Z, U_2) \;\&\; (\alpha_0(U_1) \rightarrow U_1 = U_2) \;\&$ $\& \forall A, C(\chi_9(X, Y_1, A, Z, C) \rightarrow \chi_9(X, Y_2, A, Z, C)))$.

(xi) $\chi_{11} \equiv \exists A(\tau(A) \;\&\; \forall Z(\alpha(Z) \rightarrow Z \preccurlyeq A))$.

(xii) $\chi_{12}(X, Y) \equiv (\chi_{11} \rightarrow \exists Z, U, X_1, A(\varphi_{53}(X, Z) \;\&\; X \preccurlyeq Z \;\&\; \varphi_{29}(X_1, Z, U) \;\&$

$\& X \neq X_1 \& A \prec X \& A \prec X_1 \& \varphi_9(U, Y))) \& ((\neg\chi_{11} \& \exists A\alpha_0(A)) \rightarrow$
$\rightarrow (\exists Z(\bar{\alpha}_1(Z) \& \varphi_8(Y, Z)) \& \forall U(\varphi_{31}(Y, U) \rightarrow Y = U))) \& ((\neg\chi_{11} \& \neg\exists A\, \alpha_0(A)) \rightarrow$
$\rightarrow \exists Z(\alpha(Z) \& \varphi_9(Z, Y)))$.

(xiii) $\chi_{13}(X, Y, A, B) \equiv \exists U, U_0, C_1, C_2(\varphi_{56}(X, U, C_2) \& X \ll C_1 \& C_1 \prec C_2 \&$
$\& \chi_8(X, U) \& \varphi_{60}(X, U, C_2, A) \& \varphi_{60}(X, U, C_1, B) \& \chi_3(X, U, X, C_2, U_0) \&$
$\& \chi_7(X, U_0) \& \neg\omega_1(A) \& \forall P, Q((\varphi_{60}(X, U_0, P, Q) \& P \neq C_1) \rightarrow \chi_{12}(X, Q)) \&$
$\& \chi_4(X, U, C_1, C_2, Y))$.

(xiv) $\chi_{14}(X, Y) \equiv \exists U\varphi_{53}(X, U) \& \forall Z(\varphi_1(Z, Y) \rightarrow \chi_{12}(X, Z))$.

(xv) $\chi_{15}(X, Y, Z) \equiv \chi_{14}(X, Y) \& \exists Y_1, B\chi_{13}(X, Y_1, Z, B) \& \forall U(\varphi_{32}(X, U, Z) \rightarrow$
$\rightarrow \varphi_1(U, Y))$.

(xvi) $\chi_{16}(X, S, R) \equiv \chi_{14}(X, S) \& \neg\omega_0(S) \& \forall Z(\varphi_1(Z, R) \rightarrow$
$\rightarrow \exists A, B(\chi_{13}(X, Z, A, B) \& \chi_{15}(X, S, A) \& \varphi_1(B, S))) \& \forall A(\chi_{15}(X, S, A) \rightarrow$
$\rightarrow \exists!!\, B\, \exists Z(\chi_{13}(X, Z, A, B) \& \varphi_1(Z, R)))$.

(xvii) $\chi_{17}(X, S, R, Y, Z) \equiv \chi_{16}(X, S, R) \& \chi_8(X, Y) \& \exists P(\varphi_{56}(X, Y, P) \&$
$\& \varphi_{56}(X, Z, P)) \& \forall P, Q(\varphi_{60}(X, Z, P, Q) \rightarrow \varphi_1(Q, S)) \& ((\chi_{11} \text{ VEL } \neg\exists U\, \alpha_0(U)) \rightarrow$
$\rightarrow \forall P_1, P_2, C(\chi_4(X, Z, P_1, P_2, C) \rightarrow \neg\varphi_{62}(X, C))) \& \forall P_1, P_2(\chi_5(X, Y, P_1, P_2) \rightarrow$
$\rightarrow \exists Q(\varphi_{60}(X, Z, P_1, Q) \& \varphi_{60}(X, Z, P_2, Q))) \& \forall P, Q((\varphi_{60}(X, Y, P, Q) \&$
$\& \neg\omega_1(Q)) \rightarrow \exists Y_1, Z_1, P_1, D, A, B, Z_2(\varphi_{59}(X, Y_1, Y) \& \varphi_{56}(X, Y_1, P) \&$
$\& \varphi_{59}(X, Z_1, Z) \& \varphi_{56}(X, Z_1, P_1) \& P_1 \prec P \& \varphi_1(D, R) \& \chi_{13}(X, D, A, B) \&$
$\& \varphi_{56}(X, Z_2, P) \& \varphi_{59}(X, Z_1, Z_2) \& \varphi_{60}(X, Z_2, P, A) \& \chi_{10}(X, Y_1, Z_2) \&$
$\& \varphi_{60}(X, Z, P, B)))$.

(xviii) $\chi_{18}(X, S, R, U) \equiv \chi_{16}(X, S, R) \& \exists U_1\varphi_{69}(X, U_1, U) \&$
$\& \forall Y, Z((\chi_{17}(X, S, R, Y, Z) \& \varphi_{61}(X, Y, U)) \rightarrow \exists P, P_1, Q(\varphi_{56}(X, Y, P) \& P_1 \prec$
$\prec P \& \varphi_{60}(X, Z, P_1, Q) \& \varphi_{60}(X, Z, P, Q)))$.

1.4. Lemma. *Let Δ be a strictly large type. Then:*

(i) $\chi_1(X, Y, Z, U)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$, $x \in V$, *integers* $k, m, n \geqq 1$ *and terms* $a \in x\begin{bmatrix} k \\ F, i \end{bmatrix}$, $b \in x\begin{bmatrix} m \\ F, i \end{bmatrix}$, $c \in x\begin{bmatrix} n \\ F, i \end{bmatrix}$ *such that* $X = (F, i)^*$, $Y = a^*$, $Z = b^*$, $U = c^*$ *and* $n \geqq k + m$.

(ii) $\chi_2(X, Y, Z, U)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$, $x \in V$, *integers* $k, m \geqq 0$ *and terms* $a \in x\begin{bmatrix} k \\ F, i \end{bmatrix}$, $b \in x\begin{bmatrix} m \\ F, i \end{bmatrix}$, $c \in x\begin{bmatrix} k+m \\ F, i \end{bmatrix}$ *such that* $X = (F, i)^*$, $Y = a^*$, $Z = b^*$ *and* $U = c^*$.

(iii) $\chi_3(X, Y, A, B, Z)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$, $x \in V$, *a finite sequence* $a_1, \ldots, a_n$ *of terms, two integers* k, m $(1 \leqq k \leqq m \leqq n)$ *and terms* $a \in x\begin{bmatrix} k \\ F, i \end{bmatrix}$, $b \in x\begin{bmatrix} m \\ F, i \end{bmatrix}$ *such that* $X = (F, i)^*$, $Y = H_{F,i}(a_1, \ldots, a_n)$, $A = a^*$, $B = b^*$ *and* $Z = H_{F,i}(a_k, \ldots, a_m)$.

(iv) $\chi_4(X, Y, A, B, Z)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$, $x \in V$, *a finite sequence*

$a_1, \ldots, a_n$ *of terms, two integers* k, m $(1 \leqq k < m \leqq n)$ *and terms* $a \in x\begin{bmatrix} k \\ F, i \end{bmatrix}$, $b \in x\begin{bmatrix} m \\ F, i \end{bmatrix}$ *such that* $X = (F, i)^*$, $Y = H_{F,i}(a_1, \ldots, a_n)$, $A = a^*$, $B = b^*$ *and* $Z = H_{F,i}(a_m, a_k)$.

(v) $\chi_5(X, Y, A, B)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2}$, $x \in V$, *a finite sequence* $a_1, \ldots, a_n$ *of terms, two integers* k, m $(1 \leqq k < m \leqq n)$ *and terms* $a \in x\begin{bmatrix} k \\ F, i \end{bmatrix}$, $b \in x\begin{bmatrix} m \\ F, i \end{bmatrix}$ *such that* $X = (F, i)^*$, $Y = H_{F,i}(a_1, \ldots, a_n)$, $A = a^*$, $B = b^*$ *and* $a_k = a_m$.

(vi) $\chi_6(X, Y, Z)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$, $(G, j) \in \Delta^{(1)}$ *and two terms* a, b *such that* $X = (F, i)^*$, $Z = (G, j)^*$, $Y = H_{F,i}(a, b)$ *and* $a = G(b_1, \ldots, b_{n_G})$ *for some terms* $b_1, \ldots, b_{n_G}$ *with* $b_j = b$.

(vii) $\chi_7(X, Y)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$ *and a finite sequence* $a_1, \ldots, a_n$ *of terms such that* $X = (F, i)^*$, $Y = H_{F,i}(a_1, \ldots, a_n)$ *and the following is true: if* $a_n = G(b_1, \ldots, b_{n_G})$ *then* $b_1, \ldots, b_{n_G} \in \{a_1, \ldots, a_{n-1}\}$.

(viii) $\chi_8(X, Y)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$ *and a finite sequence* $a_1, \ldots, a_n$ *of terms such that* $X = (F, i)^*$, $Y = H_{F,i}(a_1, \ldots, a_n)$ *and the following is true: whenever* $a_j = G(b_1, \ldots, b_{n_G})$ *then* $b_1, \ldots, b_{n_G} \in \{a_1, \ldots, a_{j-1}\}$.

(ix) $\chi_9(X, Y, A, B, C)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$, $(G, j) \in \Delta^{(1)}$, $x \in V$, *a finite sequence* $a_1, \ldots, a_n$ *of terms, two integers* k, m $(1 \leqq k < m \leqq n)$ *and terms* $a \in x\begin{bmatrix} k \\ F, i \end{bmatrix}$, $b \in x\begin{bmatrix} m \\ F, i \end{bmatrix}$ *such that* $X = (F, i)^*$, $Y = H_{F,i}(a_1, \ldots, a_n)$, $A = a^*$, $B = b^*$, $C = (G, j)^*$ *and* $a_m = G(b_1, \ldots, b_{n_G})$ *for some terms* $b_1, \ldots, b_{n_G}$ *with* $b_j = a_k$.

(x) $\chi_{10}(X, Y_1, Y_2)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$ *and two finite sequence* $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ *of terms such that* $X = (F, i)^*$, $Y_1 = H_{F,i}(a_1, \ldots, a_n)$, $Y_2 = H_{F,i}(b_1, \ldots, b_n)$ *and the following is true: if* $a_n = G(a_{i_1}, \ldots, a_{i_k})$ *where* $k = n_G$ *and* $i_1, \ldots, i_k \in \{1, \ldots, n-1\}$ *then* $b_n = G(b_{i_1}, \ldots, b_{i_k})$.

(xi) χ_{11} *in* $\mathscr{F}_\Delta$ *iff* Δ *is finite.*

(xii) $\chi_{12}(X, Y)$ *in* $\mathscr{F}_\Delta$ *iff there is an* $(F, i) \in \Delta^{(2)}$ *such that* $X = (F, i)^*$ *and* Y *is an* (F, i)*-codelement.*

(xiii) $\chi_{13}(X, Y, A, B)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$ *and terms* a, b *such that* $X = (F, i)^*$, $A = a^*$, $B = b^*$ *and* $Y = H_{F,i}(a, b)$ *is an* (F, i)*-definator.*

(xiv) $\chi_{14}(X, Y)$ *in* $\mathscr{F}_\Delta$ *iff* $X = (F, i)^*$ *for some* $(F, i) \in \Delta^{(2)}$ *and* Y *is an* (F, i)*-codset.*

(xv) $\chi_{15}(X, Y, Z)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$ *and an* (F, i)*-definator* $H_{F,i}(a, b) = [G, A_1, \ldots, A_{n_G}, A]_{F,i}$ *such that* $X = (F, i)^*$, Y *is an* (F, i)*-codset,* $Z = a^*$ *and* $A_1, \ldots, A_{n_G} \in I^*(Y)$.

(xvi) $\chi_{16}(X, S, R)$ *in* $\mathscr{F}_\Delta$ *iff* $X = (F, i)^*$ *for some* $(F, i) \in \Delta^{(2)}$ *and* S, R *is an* (F, i)*-codalgebra.*

(xvii) $\chi_{17}(X, S, R, Y, Z)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$ *and two finite sequences* $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ *of terms such that* $X = (F, i)^*$, S, R *is an* (F, i)*-codalgebra,* $Y = H_{F,i}(a_1, \ldots, a_n)$, $Z = H_{F,i}(b_1, \ldots, b_n)$ *and the following are true: whenever*

$a_j = G(d_1, \ldots, d_{n_G})$ *then* $d_1, \ldots, d_{n_G} \in \{a_1, \ldots, a_{j-1}\}$; $\mathrm{Card}(\mathrm{var}(b_1) \cup \ldots \cup \mathrm{var}(b_n)) \leqq \leqq 1$; *there exists a homomorphism* h *of the* Δ*-algebra* W_Δ *into the* Δ*-algebra corresponding to* S, R *such that* $h(a_1) = b_1^*, \ldots, h(a_n) = b_n^*$.

(xviii) $\chi_{18}(X, S, R, U)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$ *and an equation* (a, b) *such that* $X = (F, i)^*$, S, R *is an* (F, i)*-codalgebra,* $U = H_{F,i}(a, b)$ *and* (a, b) *is satisfied in the* Δ*-algebra corresponding to* S, R.

Definition. (i) $\chi_{19}(X, S, R, T) \equiv \chi_{16}^\varepsilon(X, S, R) \;\&\; \forall A, B(\psi_{30}(X, A, B) \rightarrow (B \leqq T \leftrightarrow \leftrightarrow \chi_{18}^\varepsilon(X, S, R, A)))$.

(ii) $\chi_{20}(X, S, R, T) \equiv \exists U(\chi_{19}(X, S, R, U) \;\&\; T \leqq U)$.

(iii) $\chi_{21}(T) \equiv \exists X, S, R, A(\chi_{20}(X, S, R, T) \;\&\; \tau^\varepsilon(A) \;\&\; \forall U(\varphi_1^\varepsilon(U, S) \rightarrow A \leqq U))$.

1.5. Lemma. *Let* Δ *be a strictly large type. Then*:

(i) $\chi_{19}(X, S, R, T)$ *in* $\mathscr{L}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$ *and an* (F, i)*-codalgebra* S_0, R_0 *such that* $X = Z((F, i)^*)$, $S = Z(S_0)$, $R = Z(R_0)$ *and* T *is the equational theory of the* Δ*-algebra corresponding to* S_0, R_0.

(ii) $\chi_{20}(X, S, R, T)$ *in* $\mathscr{L}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$ *and an* (F, i)*-codalgebra* S_0, R_0 *such that* $X = Z((F, i)^*)$, $S = Z(S_0)$, $R = Z(R_0)$ *and the* Δ*-algebra corresponding to* S_0, R_0 *is a model of the equational theory* T.

(iii) $\chi_{21}(T)$ *in* $\mathscr{L}_\Delta$ *iff* T *is the equational theory of a finite* Δ*-algebra.*

Now let Δ be a finite, strictly large type. For every finite Δ-algebra A we shall construct a formula $f_A(T)$ with one free variable T in the following way: Denote by n the cardinality of A, by m the cardinality of Δ and put $A = \{a_1, \ldots, a_n\}$ and $\Delta = \{F_1, \ldots, F_m\}$. Denote by M the set of finite sequences $s = (F_i, a_{i_1}, \ldots, a_{i_{k+1}})$ such that $i \in \{1, \ldots, m\}$, k is the arity of F_i, $i_1, \ldots, i_{k+1} \in \{1, \ldots, n\}$ and $F_i(a_{i_1}, \ldots, a_{i_k}) = a_{i_{k+1}}$ holds in the algebra A. For every $s = (F_i, a_{i_1}, \ldots, a_{i_{k+1}}) \in M$ such that $k \geqq 1$ put

$$g_s \equiv \exists D, U(\varphi_1^\varepsilon(D, R) \;\&\; \chi_{13}^\varepsilon(X, D, U, X_{i_{k+1}}) \;\& \\ \&\; \varphi_{32}^\varepsilon(Y_{i,1}, X_{i_1}, U) \;\&\; \varphi_{32}^\varepsilon(Y_{i,k}, X_{i_k}, U)) .$$

For every $s = (F_i, a_j) \in M$ such that F_i is nullary put

$$g_s \equiv \exists D(\varphi_1^\varepsilon(D, R) \;\&\; \chi_{13}^\varepsilon(X, D, Y_i, X_j)) .$$

Denote by g the conjunction of the formulas g_s $(s \in M)$. For every $i \in \{1, \ldots, m\}$ such that F_i is of an arity $k \geqq 1$ put

$$h_i \equiv \varphi_3^\varepsilon(Y_i, Y_{i,1}) \;\&\; \ldots \;\&\; \varphi_3^\varepsilon(Y_i, Y_{i,k}) .$$

For every $i \in \{1, \ldots, m\}$ such that F_i is nullary put

$$h_i \equiv \alpha_0^\varepsilon(Y_i) .$$

Finally, put

$$f_A(T) \equiv \exists X, S, R\; \exists(X_1, \ldots, X_n)^{\neq} \\ \exists(Y_1, \ldots, Y_m, Y_{1,1}, \ldots, Y_{1,n_{F_1}}, \ldots, Y_{m,1}, \ldots, Y_{m,n_{F_m}})^{\neq} \\ (\chi_{19}(X, S, R, T) \;\&\; \forall U(\varphi_1^\varepsilon(U, S) \leftrightarrow \\ \leftrightarrow (U = X_1 \text{ VEL} \ldots \text{VEL } U = X_n)) \;\&\; h_1 \;\&\; \ldots \;\&\; h_m \;\&\; g) .$$

1.6. Lemma. *Let* Δ *be a finite, strictly large type; let* A *be a finite* Δ*-algebra; let* $T \in \mathscr{L}_\Delta$. *Then* $f_A(T)$ *in* $\mathscr{L}_\Delta$ *iff* $T = h(\mathrm{Eq}(A))$ *for some automorphism* h *of* $\mathscr{L}_\Delta$.

Throughout this section let Δ be a type such that $\Delta = \Delta_0 \cup \Delta_1$ and $\mathrm{Card}(\Delta_1) \geqq 2$.

By a codelement we mean an element of $\mathscr{F}_\Delta$ of the form $(FG^nFx)^*$ where $x \in V$, $n \geqq 2$ and $F, G \in \Delta_1$ are two different symbols. The set of (F, i)-codelements is denoted by CEL.

2.1. Lemma. CEL *is a set of pairwise uncomparable elements of* $\mathscr{F}_\Delta$; *we have* $\mathrm{Card}(\mathrm{CEL}) = \mathrm{Max}(\aleph_0, \mathrm{Card}(\Delta_1))$.

Let $H \in \Delta_1$ and let A, B be two codelements. For every variable x there exists a unique pair s_1, s_2 of elements of $\Delta^{(-)}$ such that $A = (s_1x)^*$ and $B = (s_2x)^*$. The element $(s_2Hs_1Hs_2x)^*$ of $\mathscr{F}_\Delta$ will be denoted by $[H, A, B]$. The elements of $\mathscr{F}_\Delta$ of this form will be called definators of the first kind.

Let $C \in \Delta_0$ and let A be a codelement. For every variable x there exists a unique element s of $\Delta^{(-)}$ such that $A = (sx)^*$. The element $(sC)^*$ of $\mathscr{F}_\Delta$ will be denoted by $[C, A]$. The elements of $\mathscr{F}_\Delta$ of this form will be called definators of the second kind.

Definators are elements of $\mathscr{F}_\Delta$ that are definators of either the first or the second kind.

2.2. Lemma. *If* $[H_1, A_1, B_1] \leqq [H_2, A_2, B_2]$ *then* $H_1 = H_2$, $A_1 = A_2$ *and* $B_1 = B_2$. *If* $[C_1, A_1] \leqq [C_2, A_2]$ *then* $C_1 = C_2$ *and* $A_1 = A_2$. *No definator of the first kind can be comparable with a definator of the second kind.*

By a codset we mean an element S of $\mathscr{F}_\Delta$ such that every element of $I^*(S) = \{t^*;\ t \in I(U)\}$ is a codelement. Elements of $I^*(S)$ are called codelements of S. There is a natural one-to-one correspondence between codsets and subsets of CEL. The union of the sets in CEL is the largest codset, while the empty set is the least codset.

By a codalgebra we mean a pair S, R of elements of $\mathscr{F}_\Delta$ satisfying the following three conditions:

(1) S is a nonempty codset;

(2) every element of $I^*(R)$ is a definator; if $[H, A, B] \in I^*(R)$ then $A, B \in I^*(S)$; if $[C, A] \in I^*(R)$ then $A \in I^*(S)$;

(3) for every $H \in \Delta_1$ and $A \in I^*(S)$ there exists exactly one $B \in I^*(S)$ with $[H, A, B] \in I^*(R)$; for every $C \in \Delta_0$ there exists exactly one $A \in I^*(S)$ with $[C, A] \in I^*(R)$.

Given a codalgebra S, R, we can define an algebra Q of type Δ with the underlying set $I^*(S)$ as follows: $H_Q(A) = B$ iff $[H, A, B] \in I^*(R)$; $C_Q = A$ iff $[C, A] \in I^*(R)$. This algebra Q is said to be the Δ-algebra corresponding to the codalgebra S, R.

2.3. Lemma. *Every* Δ*-algebra whose underlying set is a subset of* CEL *corresponds to exactly one codalgebra. A* Δ*-algebra* Q *is isomorphic to a* Δ*-algebra corresponding to a codalgebra, iff* $\mathrm{Card}(Q) \leqq \mathrm{Max}(\aleph_0, \mathrm{Car}(\Delta_1))$.

Definition. (i) $\chi_{22}(A, B, C) \equiv \exists X_1, X_2, Y, D(\varphi_{47}(X_1, X_2, Y, A, B, D)$ & & $\varphi_{47}(X_1, X_2, Y, D, A, C))$.

(ii) $\chi_{23}(Z) \equiv \exists A, B, X(\alpha_1(A)$ & $\varphi_{13}(X, B)$ & $X \neq A$ & $X \neq B$ & $\chi_{22}(A, B, Z))$.

(iii) $\chi_{24}(X, A, B, Y) \equiv \alpha_1(X)$ & $\chi_{23}(A)$ & $\chi_{23}(B)$ & $\exists C(\chi_{22}(X, A, U)$ & & $\chi_{22}(B, U, Y))$.

(iv) $\chi_{25}(X, A, Y) \equiv \alpha_0(X)$ & $\chi_{23}(A)$ & $X \leqslant Y$ & $\varphi_8(Y, A)$.

(v) $\chi_{26}(Y) \equiv \exists X, A, B\chi_{24}(X, A, B, Y)$ VEL $\exists X, A\chi_{25}(X, A, Y)$.

(vi) $\chi_{27}(Y) \equiv \forall A(\varphi_1(A, Y) \to \chi_{23}(A))$.

(vii) $\chi_{28}(S, R) \equiv \chi_{27}(S)$ & $\neg\omega_0(S)$ & $\forall Z(\varphi_1(Z, R) \to (\exists X, A, B(\chi_{24}(X, A, B, Z)$ & & $\varphi_1(A, S)$ & $\varphi_1(B, S))$ VEL $\exists X, A(\chi_{25}(X, A, Z)$ & $\varphi_1(A, S))))$ & $\forall X, A((\alpha_1(X)$ & & $\varphi_1(A, S)) \to \exists!!\, B\, \exists Z(\chi_{24}(X, A, B, Z)$ & $\varphi_1(Z, R)))$ & $\forall X(\alpha_0(X) \to$ $\to \exists!!\, A\, \exists Z(\chi_{25}(X, A, Z)$ & $\varphi_1(Z, R)))$.

(viii) $\chi_{29}(X_1, X_2, Y, S, R, A, B, D) \equiv \chi_{28}(S, R)$ & $\tau(A)$ & $\varphi_{41}(X_1, X_2, Y, B, D)$ & & $\exists D_0(D_0 \prec D$ & $\varphi_{45}(A, D_0))$ & $\forall Z, U, C(\varphi_{40}(X_1, X_2, Y, B, Z, U, C) \to$ $\to \varphi_1(C, S))$ & $\forall P, Q, H, Z_1, U_1, C_1, Z_2, U_2, C_2((\varphi_{46}(X_1, X_2, Y, P, A)$ & & $\varphi_{46}(X_1, X_2, Y, Q, A)$ & $\varphi_{38}(X_1, X_2, Y, H, P, Q)$ & $\varphi_{40}(X_1, X_2, Y, B, Z_1, U_1, C_1)$& & $\varphi_{40}(X_1, X_2, Y, B, Z_2, U_2, C_2)$ & $\varphi_{45}(Q, Z_1)$ & $Z_1 \prec Z_2) \to \exists X(\varphi_1(X, R)$ & & $\chi_{24}(H, C_1, C_2, X)))$ & $\forall C((\alpha_0(C)$ &$C \leqslant A) \to$ $\to \exists U, X, Z(\varphi_{40}(X_1, X_2, Y, B, X_1, U, X)$ & $\chi_{25}(C, X, Z)$ & $\varphi_1(Z, R)))$.

(ix) $\chi_{30}(X_1, X_2, Y, A, U_1, B, U_2, S, R) \equiv \varphi_{33}(X_1, X_2, Y)$ & $\varphi_{43}(X_1, A, U_1)$ & & $\varphi_{43}(X_1, B, U_2)$ & $\chi_{28}(S, R)$ & $\forall B_1, D_1, B_2, D_2, P_1, P_2, P_3, P_4, Q_1, Q_2, Q_3, Q_4$ $((\chi_{29}(X_1, X_2, Y, S, R, A, B_1, D_1)$ & $\chi_{29}(X_1, X_2, Y, S, R, B_2, D_2)$ & & $\varphi_{40}(X_1, X_2, Y, B_1, D_1, P_1, Q_1)$ & $\varphi_{40}(X_1, X_2, Y, B_2, D_2, P_2, Q_2)$ & & $\varphi_{40}(X_1, X_2, Y, B_1, X_1, P_3, Q_3)$ & $\varphi_{40}(X_1, X_2, Y, B_2, X_1, P_4, Q_4)$ & $Q_1 \neq Q_2) \to$ $\to (\neg\alpha_0(U_1)$ & $U_1 = U_2$ & $Q_3 \neq Q_4))$.

2.4. Lemma. *Let Δ be a large but not strictly large type. Then:*

i) *$\chi_{22}(A, B, C)$ in $\mathscr{F}_\Delta$ iff there are two sequences $s_1, s_2 \in \Delta^{(-)}$ and a variable x such that $A = (s_1 x)^*$, $B = (s_2 x)^*$, $C = (s_1 s_2 s_1 x)^*$.*

(ii) *$\chi_{23}(Z)$ in $\mathscr{F}_\Delta$ iff Z is a codelement.*

(iii) *$\chi_{24}(X, A, B, Y)$ in $\mathscr{F}_\Delta$ iff $X = F^*$ for some $F \in \Delta_1$, A, B are two codelements and $Y = [X, A, B]$.*

(iv) *$\chi_{25}(X, A, Y)$ in $\mathscr{F}_\Delta$ iff $X = C^*$ for some $C \in \Delta_0$, A is a codelement and $Y = [X, A]$.*

(v) *$\chi_{26}(Y)$ in $\mathscr{F}_\Delta$ iff Y is a definator.*

(vi) *$\chi_{27}(Y)$ in $\mathscr{F}_\Delta$ iff Y is a codset.*

(vii) *$\chi_{28}(S, R)$ in $\mathscr{F}_\Delta$ iff S, R is a codalgebra.*

(viii) *Let $F, G \in \Delta_1$, $F \neq G$, $x \in V$, $X_1 = F^*$, $X_2 = G^*$, $Y = (GFx)^*$. Then $\chi_{29}(X_1, X_2, Y, S, R, A, B, D)$ in $\mathscr{F}_\Delta$ iff S, R is a codalgebra, $A = (H_n \ldots H_1 y)^*$ for some $y \in V \cup \Delta_0$ and $H_1, \ldots, H_n \in \Delta_1$ $(n \geqq 0)$, and (B, D) is an (F, G, GF, x)-code of the sequence $h(y), h(H_1 y), \ldots, h(H_n \ldots H_1 y)$ for some homomorphism h of the algebra W_Δ into the Δ-algebra corresponding to the codalgebra S, R.*

(ix) *Let* $F, G \in \Delta_1$, $F \neq G$, $x \in V$, $X_1 = F^*$, $X_2 = G^*$, $Y = (GFx)^*$. *Then* $\chi_{30}(X_1, X_2, Y, A, U_1, B, U_2, S, R)$ *in* $\mathscr{F}_\Delta$ *iff* S, R *is a codalgebra*, (A, U_1) *is the fine F-code of a term* a, (B, U_2) *is the fine F-code of a term* b *and the equation* (a, b) *is satisfied in the* Δ*-algebra corresponding to* S, R.

Definition. (i) $\chi_{31}(X, A, U_1, B, U_2, S, R) \equiv \exists X_2, Y(\psi_{35}(X, X_2, Y) \; \& \; \& \; \chi^{\varepsilon}_{30}(X, X_2, Y, A, U_1, B, U_2, S, R))$.

(ii) $\chi_{32}(S, R, T) \equiv \chi^{\varepsilon}_{28}(S, R) \; \& \; \forall X, A, U_1, B, U_2, Y(\psi_{45}(X, A, U_1, B, U_2, Y) \to \to (\chi_{31}(X, A, U_1, B, U_2, S, R) \leftrightarrow Y \leqq T))$.

(iii) $\chi_{33}(T) \equiv \exists S, R, X_1, X_2, Y, A, D(\chi_{32}(S, R, T) \; \& \; \varphi^{\varepsilon}_{41}(X_1, X_2, Y, A, D) \; \& \; \& \; \forall U(\varphi^{\varepsilon}_1(U, S) \to \exists Z, B\varphi^{\varepsilon}_{40}(X_1, X_2, Y, A, Z, B, U)))$.

2.5. Lemma. *Let* Δ *be a large but not strictly large type. Then*:

(i) $\chi_{31}(X, A, U_1, B, U_2, S, R)$ *in* $\mathscr{L}_\Delta$ *iff there are* $F \in \Delta_1$, *terms* a, b *and a codalgebra* S_0, R_0 *such that* $X = Z(F^*)$, (A, U_1) *is the fine F-code of* a *in* $\mathscr{L}_\Delta$, (B, U_2) *is the fine F-code of* b *in* $\mathscr{L}_\Delta$, $S = Z(S_0)$, $R = Z(R_0)$ *and the equation* (a, b) *is satisfied in the* Δ*-algebra corresponding to* S_0, R_0.

(ii) $\chi_{32}(S, R, T)$ *in* $\mathscr{L}_\Delta$ *iff there is a codalgebra* S_0, R_0 *such that* $S = Z(S_0)$, $R = Z(R_0)$ *and* T *is the equational theory of the* Δ*-algebra corresponding to* S_0, R_0.

(iii) $\chi_{33}(T)$ *in* $\mathscr{L}_\Delta$ *iff* T *is the equational theory of a finite algebra.*

Now let Δ be a finite, large but not strictly large type. For every finite Δ-algebra A we shall construct a formula $f_A(T)$ with one free variable T in the following way. Denote by n the cardinality of A, by m_0 the cardinality of Δ_0, by m_1 the cardinality of Δ_1 and put $A = \{a_1, \ldots, a_n\}$, $\Delta_0 = \{C_1, \ldots, C_{m_0}\}$ and $\Delta_1 = \{F_1, \ldots, F_{m_1}\}$. Denote by M_1 the set of the triples $s = (F_i, a_j, a_k)$ such that $i \in \{1, \ldots, m_1\}$, $j, k \in \{1, \ldots, n\}$ and $F_i(a_j) = a_k$ holds in the algebra A; denote by M_0 the set of the pairs $s = (C_i, a_j)$ such that $i \in \{1, \ldots, m_0\}$, $j \in \{1, \ldots, n\}$ and $C_i = a_j$ holds in A. For every $s = (F_i, a_j, a_k) \in M_1$ put

$$g_s \equiv \exists D(\varphi^{\varepsilon}_1(D, R) \; \& \; \chi^{\varepsilon}_{24}(Y_i, X_j, X_k, D)) .$$

For every $s = (C_i, a_j) \in M_0$ put

$$g_s \equiv \exists D(\varphi^{\varepsilon}_1(D, R) \; \& \; \chi^{\varepsilon}_{25}(Z_i, X_j, D)) .$$

Denote by g the conjunction of the formulas g_s ($s \in M_1 \cup M_0$). Finally, put

$$f_A(T) \equiv \exists S, R \; \exists(X_1, \ldots, X_n)^{\neq} \; \exists(Y_1, \ldots, Y_{m_1})^{\neq} \; \exists(Z_1, \ldots, Z_{m_0})^{\neq}$$
$$(\chi_{32}(S, R, T) \; \& \; \forall U(\varphi^{\varepsilon}_1(U, S) \leftrightarrow (U = X_1 \text{ VEL} \ldots \text{VEL } U = X_n)) \; \&$$
$$\& \; \alpha^{\varepsilon}_1(Y_1) \; \& \ldots \& \; \alpha^{\varepsilon}_1(Y_{m_1}) \; \& \; \alpha^{\varepsilon}_0(Z_1) \; \& \ldots \& \; \alpha^{\varepsilon}_0(Z_{m_0}) \; \& \; g) .$$

2.6. Lemma. *Let* Δ *be a finite, large but not strictly large type; let* A *be a finite* Δ*-algebra; let* $T \in \mathscr{L}_\Delta$. *Then* $f_A(T)$ *in* $\mathscr{L}_\Delta$ *iff* $T = h(\mathrm{Eq}(A))$ *for some automorphism* h *of* $\mathscr{L}_\Delta$.

3.1. Lemma. *Let $\Delta = \Delta_0 \cup \{F\}$ for some unary symbol F and let $T \in \mathscr{L}_\Delta$. Then T is the equational theory of a finite algebra iff the following two conditions are satisfied:*

(1) *there are non-negative integers n, m such that $n < m$ and $(F^n x, F^m x) \in T$ (where $x \in V$);*

(2) *there exists a finite subset H of Δ_0 such that for every $F \in \Delta_0$ there is a $G \in H$ with $(F, G) \in T$.*

Proof. The direct implication is clear. Conversely, let (1) and (2) be satisfied. It is easy to see that the free algebra of rank 2 in the variety corresponding to T is finite; this algebra generates the variety, since Δ contains only nullary and unary symbols.

Definition. (i) $\chi_{34}(X) \equiv \exists A, B, C, P, Q(\psi_{59}(A, B, C) \;\&\; C \leqq X \;\&\; \psi_{63}(P) \;\&$
$\&\; \psi_{62}(P, Q) \;\&\; \forall U\, \exists Z, T((\alpha_0^\varepsilon(U) \;\&\; \neg\varphi_1^\varepsilon(U, Q)) \to (\varphi_1^\varepsilon(Z, Q) \;\&\; \psi_{34}(U, Z, T) \;\&$
$T \leqq X)))$.

(ii) $\chi_{35}(X) \equiv (\exists A, B(\alpha_0(A) \;\&\; \alpha_0(B) \;\&\; A \neq B) \;\&\; \chi_{34}(X))$ VEL $(\exists!!\, A\, \alpha_0(A) \;\&$
$\&\; \neg\omega_0(X) \;\&\; \neg\exists A, B\psi_{58}(A, B, X))$ VEL $(\neg\exists A\alpha_0(A) \;\&\; \neg\omega_0(X))$.

3.2. Lemma. (i) *Let $\Delta = \Delta_0 \cup \{F\}$ where $F \in \Delta_1$ and* $\mathrm{Card}(\Delta_0) \geqq 2$. *Then $\chi_{34}(X)$ in $\mathscr{L}_\Delta$ iff X is the equational theory of a finite algebra.*

(ii) *Let Δ be a small type containing a unary symbol. Then $\chi_{35}(X)$ in $\mathscr{L}_\Delta$ iff X is the equational theory of a finite algebra.*

3.3. Lemma. *Let $\Delta = \Delta_0$ and let $T \in \mathscr{L}_\Delta$. Then T is the equational theory of a finite algebra iff there exists a finite subset H of Δ_0 such that for every $F \in \Delta_0$ there is a $G \in H$ with $(F, G) \in T$.*

Definition. $\chi_{36}(X) \equiv \omega_1(X)$ VEL $\exists A, B(\psi_2(A) \;\&\; \psi_{53}(B) \;\&\; A = B \vee X)$.

3.4. Lemma. *Let $\Delta = \Delta_0$. Then $\chi_{36}(X)$ in $\mathscr{L}_\Delta$ iff X is the equational theory of a finite algebra.*

4. THE MAIN RESULTS

Definition. $\chi(X) \equiv (\chi_{21}(X) \;\&\; \psi_5 \;\&\; \exists A\, \bar{\alpha}_2^\varepsilon(A))$ VEL $(\chi_{33}(X) \;\&\; \psi_5$
$\&\; \neg\exists A\, \bar{\alpha}_2^\varepsilon(A))$ VEL $(\psi_4 \;\&\; \chi_{36}(X))$ VEL $(\chi_{35}(X) \;\&\; \neg\psi_4 \;\&\; \psi_5)$.

4.1. Theorem. *Let Δ be any type. Then $\chi(X)$ in $\mathscr{L}_\Delta$ iff X is the equational theory of a finite algebra. Consequently, the set of the equational theories of finite Δ-algebras is definable in $\mathscr{L}_\Delta$.*

Proof. Theorem follows from 1.5(iii), 2.5(iii), 3.2(ii) and 3.4.

4.2. Theorem. *Let Δ be a finite type and A a finite Δ-algebra. Then the equational theory* Eq(A) *is definable up to automorphisms in $\mathscr{L}_\Delta$.*

Proof. For large types the appropriate formula is constructed in Lemmas 1.6 and 2.6. If Δ is a finite small type, then every equational theory of type Δ is finitely based (see [4]) and so by Theorem 13.4 of [3] every element of $\mathscr{L}_\Delta$ is definable up to automorphisms.

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