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THE LATTICE OF INTUITIONISTIC FUZZY CONGRUENCES

Kul Hur[1] **and Su Youn Jang**

Division of Mathematics and Informational Statistic
Wonkwang University, Iksan, Chonbuk, Korea 570-749
kulhur@wonkwang.ac.kr
suyoun123@yahoo.co.kr

Hee Won Kang

Dept. of Mathematics Education
Woosuk University
Hujong-Ri Samrae-Eup, Wanju-kun Chonbuk, Korea 565-701
khwon@woosuk.ac.kr

Abstract. First, we prove that the set of intuitionistic fuzzy congruences on a semigroup satisfying the particular condition is a modular lattice [Theorem 2.9]. Secondly, we prove that the set of all intuitionistic fuzzy congruences on a regular semigroup contained in $(\chi_{\mathcal{H}}, \chi_{\mathcal{H}^c})$ forms a modular lattice [Proposition 3.5]. And also we show that the set of all intuitionistic fuzzy idempotent separating congruences on a regular semigroup forms a modular lattice[Theorem 3.6]. Moreover, we prove that the lattice of intuitionistic fuzzy congruences on a regular semigroup is a disjoint union of some modular sublattices of the lattice[Corollary 3.15]. Finally, we show that the lattice of intuitionistic fuzzy congruences on a group and the lattice of intuitionistic fuzzy normal subgroups satisfying the particular condition are lattice isomorphic[Theorem 4.6].

[1]Corresponding author

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1. Introduction

In 1965, Zadeh [33] introduced the concept of fuzzy sets as the generalization of ordinary subsets. After that time,several researchers [1,23,24-27,29,31] have applied the notion of fuzzy sets to congruence. In particular, Das[10] and Yijia[32] investigated the set of all fuzzy congruences in the view of lattice theory.

In 1986, Atanassov[2] introduced the concept of intuitionistic fuzzy sets as the generalization of fuzzy sets. After that time, many researchers [3,5-8,11,12,14-16] applied the notion of intuitionistic fuzzy sets to relation, group theory and topology. Recently, Hur and his colleagues [17-21] studied intuitionistic fuzzy equivalence relations and various intuitionistic fuzzy congruences.

In this paper, first, we prove that the set of intuitionistic fuzzy congruences on a semigroup satisfying the particular condition is a modular lattice [Theorem 2.9]. Secondly, we prove that the set of all intuitionistic fuzzy congruences on a regular semigroup contained in $(\chi_{\mathcal{H}}, \chi_{\mathcal{H}^c})$ forms a modular lattice [Proposition 3.5]. And also we show that the set of all intuitionistic fuzzy idempotent separating congruences on a regular semigroup forms a modular lattice[Theorem 3.6]. Moreover, we prove that the lattice of intuitionistic fuzzy congruences on a regular semigroup is a disjoint union of some modular sublattices of the lattice[Corollary 3.15]. Finally, we show that the lattice of intuitionistic fuzzy congruences on a group and the lattice of intuitionistic fuzzy normal subgroups satisfying the particular condition are lattice isomorphic[Theorem 4.6].

2. Preliminaries

In this section, we list some basic concepts and well-known results which are needed in the later sections.

For sets X, Y and $Z, f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping* if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Throughout this paper, we will denote the unit interval $[0, 1]$ as I. And for a lattice, refer to [4,22]. For any ordinary relation R on a set X, we will denote the characteristic function of R as χ_R.

Definition 2.1[2,7]. *Let X be a nonempty set. A complex mapping $A = (\mu_A,\ \nu_A) : X \to I \times I$ is called an intuitionistic fuzzy set (in short, IFS) in X if $\mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$, where the mapping $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of each $x \in X$ to A, respectively. In particular, $0_\sim$ and $1_\sim$ denote the intuitionistic fuzzy empty set and the intuitionistic fuzzy whole set in X defined by $0_\sim(x) = (0, 1)$ and $1_\sim(x) = (1, 0)$ for each $x \in X$, respectively.*

We will denote the set of all IFSs in X as IFS(X).

Definitions 2.2[2]. *Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSs on X. Then*

(1) $A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.
(2) $A = B$ iff $A \subset B$ and $B \subset A$.
(3) $A^c = (\nu_A, \mu_A)$.
(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.
(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.
(6) $[\ ]A = (\mu_A, 1 - \mu_A)$, $< > A = (1 - \nu_A, \nu_A)$.

Definition 2.3[7]. *Let $\{A_i\}_{i \in J}$ be an arbitrary family of IFSs in X, where $A_i = (\mu_{A_i}, \nu_{A_i})$ for each $i \in J$. Then*

(1) $\bigcap A_i = (\bigwedge \mu_{A_i}, \bigvee \nu_{A_i})$.
(2) $\bigcup A_i = (\bigvee \mu_{A_i}, \bigwedge \nu_{A_i})$.

Definition 2.4[6]. *Let X be a set. Then a complex mapping $R = (\mu_R,\ \ \nu_R) : X \times X \to I \times I$ is called an intuitionistic fuzzy relation (in short, IFR) on X if $\mu_R(x, y) + \nu_R(x, y) \leq 1$ for each $(x, y) \in X \times X$, i.e., $R \in$ IFS $(X \times X)$.*

We will denote the set of all IFRs on a set X as IFR(X).

Definition 2.5[6]. *Let $R \in IFR(X)$. Then the inverse of R, R^{-1} is defined by $R^{-1}(x,y) = R(y,x)$ for any $x, y \in X$.*

Definition 2.6[6,11]*Let X be a set and let $R, Q \in IFR(X)$. Then the composition of R and Q, $Q \circ R$, is defined as follows : for any $x, y \in X$,*

$$\mu_{Q\circ R}(x,y) = \bigvee_{z\in X} [\mu_R(x,z) \wedge \mu_Q(z,y)]$$

and

$$\nu_{Q\circ R}(x,y) = \bigwedge_{z\in X} [\nu_R(x,z) \vee \nu_Q(z,y)].$$

Definition 2.7[6].*An Intutionistic fuzzy Relation R on a set X is called an intutionsitic fuzzy equivalence relation (in short, IFER) on X if it satisfies the following conditions :*

(i) it is intutionsitic fuzzy reflexive, i.e.,$R(x,y) = (1,0)$ for any $x, y \in X$.
(ii) it is intutionsitic fuzzy symmetric, i.e.,$R^{-1} = R$.
(iii) it is intutionsitic fuzzy transitive, i.e., $R \circ R \subset R$.

We will denote the set of all IFERs on X as IFE(X).

Definition 2.8[18].*We define two IFRs on a set X, $\triangle$ and $\bigtriangledown$ as follows, respectively : for any $x, y \in X$,*

$$\triangle(x,y) = \begin{cases} (1,0) & \text{if } x = y, \\ (0,1) & \text{if } x \neq y, \end{cases}$$

and

$$\bigtriangledown(x,y) = (1,0).$$

It is clear that $\triangle$, $\bigtriangledown \in IFE(X)$.

Let R be an intuitionistic fuzzy equivalence relation on a set X and let $a \in X$. We define a complex mapping $Ra : X \to I \times I$ as follows : for each

$x \in X$

$$Ra(x) = R(a, x).$$

Then clearly $Ra \in \mathrm{IFS}(X)$. The intuitionistic fuzzy set Ra in X is called an *intuitionistic fuzzy equivalence class* of R containing $a \in X$. The set $\{Ra : a \in X\}$ is called the *intuitionistic fuzzy quotient set* of R by X as denoted by X/R.

Result 2.A[18, Theorem 2.15].*Let R be an intuitionistic fuzzy equivalence relation on a set X. Then the followings hold :*

(1) $Ra = Rb$ if and only if $R(a, b) = (1, 0)$ for any $a, b \in X$.

(2) $R(a, b) = (0, 1)$ if and only if $Ra \cap Rb = 0_\sim$ for any $a, b \in X$.

(3) $\bigcup_{a \in X} Ra = 1_\sim$.

(4) There exists the surjection $p : X \to X/R$ defined by $p(x) = Rx$ for each $x \in X$.

Definition 2.9[18]. *Let X be a set, let $R \in IFR(X)$ and let $(\lambda, \mu) \in [0, 1) \times (0, 1]$ such that $\lambda + \mu \leq 1$. We define a complex mapping $R_{(\lambda,\mu)} : X \times X \to I \times I$ as follows : for each $y \in X$,*

$$R_{(\lambda,\mu)}(x, y) = \begin{cases} (1, 0) & \textit{if} \;\; \mu_R(x, y) > \lambda \;\; \textit{and} \;\; \nu_R(x, y) < \mu, \\ (0, 1) & \textit{if} \;\; \mu_R(x, y) \leq \lambda \;\; \textit{and} \;\; \nu_R(x, y) \geq \mu. \end{cases}$$

Result 2.B[18, Proposition 2.19].*Let $P, Q \in IFR(X)$. Then*

(1) $P = Q$ if and only if $P_{(\lambda,\mu)} = Q_{(\lambda,\mu)}$ for each $(\lambda, \mu) \in [0, 1) \times (0, 1]$ with $\lambda + \mu \leq 1$.

(2) For each $(\lambda, \mu) \in [0, 1) \times (0, 1]$ with $\lambda + \mu \leq 1$,

$$(P \cap Q)_{(\lambda,\mu)} = P_{(\lambda,\mu)} \cap Q_{(\lambda,\mu)}, (P \cup Q)_{(\lambda,\mu)} = P_{(\lambda,\mu)} \cup Q_{(\lambda,\mu)},$$
$$(P \circ Q)_{(\lambda,\mu)} = P_{(\lambda,\mu)} \circ Q_{(\lambda,\mu)}, (P \vee Q)_{(\lambda,\mu)} = P_{(\lambda,\mu)} \vee Q_{(\lambda,\mu)}.$$

Definition 2.10[18].*Let X be a set, let $R \in$ IFR(X) and let $\{R_\alpha\}_{\alpha \in \Gamma}$ be the family of all the IFERs on X containing R. Then $\bigcap_{\alpha \in \Gamma} R_\alpha$ is called the IFER generated by R and denoted by R^e.*

It is easily seen that R^e is the smallest intuitionistic fuzzy equivalence relation containing R.

Definition 2.11[18]. *Let X be a set and let $R \in IFR(X)$. Then the intutionsitic fuzzy transitive closure of R, denoted by R^∞, is defined as follows :*

$$R^\infty = \bigcup_{n \in \mathbb{N}} R^n, \quad where \quad R^n = R \circ R \circ \cdots \circ R(n\ factors).$$

Result 2.C [18, Proposition 3.7]. *Let X be a set and let $R, Q \in IFE(X)$. We define $R \vee Q$ as follows: $R \vee Q = (R \cup Q)^\infty$, i.e., $R \vee Q = \bigcup_{n \in \mathbb{N}} (R \cup Q)^n$. Then $R \vee Q \in IFE(X)$.*

Result 2.D[18, Proposition 3.8]. *Let P and Q be any intutionsitic fuzzy equivalence relations on a set X. It $R \circ Q \in IFE(X)$, then $R \circ Q = R \vee Q$, where $R \vee Q$ denotes the least upper bound for $\{P, Q\}$ with respect to the inclusion.*

Result 2.E[18, Proposition 3.9]. *Let X be a set. If $R, Q \in IFE(X)$, then $R \vee Q = (R \circ Q)^\infty$.*

Result 2.F[18, Corollary 3.9]. *Let X be a set. If $R, Q \in IFE(X)$ such that $R \circ Q = Q \circ R$, then $R \vee Q = R \circ Q$.*

3. The lattice of intuitionistic fuzzy congruences on a semigroup

Definition 3.1[19]. *An IFR R on a groupoid S is said to be:*

(1) intuitionistic fuzzy left compatible if $\mu_R(x, y) \leq \mu_R(zx, zy)$ and $\nu_R(x, y) \geq \nu_R(zx, zy)$, for any $x, y, z \in S$.

(2) intuitionistic fuzzy right compatible if $\mu_R(x, y) \leq \mu_R(xz, yz)$ and $\nu_R(x, y) \geq \nu_R(xz, yz)$, for any $x, y, z \in S$.

(3) intuitionistic fuzzy compatible if $\mu_R(x, y) \wedge \mu_R(z, t) \leq \mu_R(xz, yt)$ and $\nu_R(x, y) \vee \nu_R(z, t) \geq \nu_R(xz, yt)$, for any $x, y, z, t \in S$.

Definition 3.2[19]. *An IFER R on a groupoid S is called an:*

(1) intuitionistic fuzzy left congruence (in short, $IFLC$) if it is intuitionistic fuzzy left compatible.

(2) intuitionistic fuzzy right congruence (in short, $IFRC$) if it is intuitionistic fuzzy right compatible.

(3) intuitionistic fuzzy congruence (in short, IFC) if it is intuitionistic fuzzy compatible.

We will denote the set of all IFCs [resp. IFLCs and IFRCs] on a groupoid S as IFC(S) [resp. IFLC(S) and IFRC(S)]. Then it is clear that $\triangle, \bigtriangledown \in$ IFC(S).

Result 3.A[19, Lemma 2.14].*Let R and Q be intuitionistic fuzzy compatible relations on a groupoid S. Then $Q \circ R$ is also an intuitionistic fuzzy compatible relation on S.*

Result 3.B[19, Theorem 2.15].*Let R and Q be intuitionistic fuzzy congruences on a groupoid S. Then the following conditions are equivalent :*

(1) $Q \circ R \in$ IFC(S).

(2) $Q \circ R \in$ IFE(S).

(3) $Q \circ R$ is intuitionistic fuzzy symmetric.

(4) $Q \circ R = R \circ Q$.

Result 3.C[19, Proposition 2.16].*Let S be a semigroup and let $Q, R \in$ IFC(S). If $R \circ Q = Q \circ R$, then $R \circ Q \in$ IFC(S).*

Let R be an intuitionistic fuzzy congruence on a semigroup S and let $a \in S$. The intuitionistic fuzzy set Ra in S is called an *intuitionistic fuzzy congruence class of R containing $a \in S$* and we will denote the set of all intuitionistic fuzzy congruence classes of R as S/R.

Result 3.D[21, Proposition 2.4].*Let S be a regular semigroup and let $R \in$ IFC(S). If Ra is an idempotent element of S/R, then there exists an idempotent $e \in S$ such that $Re = Ra$.*

For a semigroup S, it is clear that IFC(S) is a partially ordered set by the inclusion relation " $\subset$ ". Moreover, for any $P, Q \in$ IFC(S), $P \cap Q$ is the greatest lower bound of P and Q in (IFC(S),$\subset$) but $P \cup Q \notin$ IFC(S) in general(See Example 2.11 in [18]).

Lemma 3.3. *Let S be a semigroup and let $P, Q \in IFC(S)$. We define $P \vee Q$ as follows: $P \vee Q = \widehat{P \cup Q}$, i.e., $P \vee Q = \bigcup_{n \in \mathbb{N}} (P \cup Q)^n$. Then $P \vee Q \in IFC(S)$.*

proof. By Result 1.C, it is clear that $P \vee Q \in$ IFE(S). Let $x, y, t \in S$. Since P and Q are intuitionistic fuzzy left compatible,

$$\begin{aligned}\mu_{P \vee Q}(x,y) &= \bigvee_{n \in \mathbb{N}} [\mu_P(x,y) \vee \mu_Q(x,y)]^n \\ &\leq [\mu_P(tx,ty) \vee \mu_Q(tx,ty)]^n = \mu_{P \vee Q}(tx,ty)\end{aligned}$$

and

$$\begin{aligned}\nu_{P \wedge Q}(x,y) &= \bigwedge_{n \in \mathbb{N}} [\nu_P(x,y) \wedge \nu_Q(x,y)]^n \\ &\geq [\nu_P(tx,ty) \wedge \nu_Q(tx,ty)]^n = \nu_{P \wedge Q}(tx,ty).\end{aligned}$$

Thus $P \vee Q$ is intuitionistic fuzzy left compatible. Similarly, it can be easily seen that $P \vee Q$ is intuitionistic fuzzy right compatible. Hence $P \vee Q \in$ IFC(S). ■

The following is the immediate result of Result 1.D.

Theorem 3.4. *Let P and Q be any intuitionistic fuzzy congruence on a semigroup S. If $P \circ Q$ is an intuitionistic fuzzy congruence on S, then $P \circ Q = P \vee Q$ where $P \vee Q$ denotes the least upper bound for $\{P, Q\}$ with respect to the inclusion.*

The following is the immediate result of Result 1.E and Result 2.A. Moreover, this gives another description for $P \vee Q$ of two IFCs P and Q.

Proposition 3.5. *Let S be a semigroup. If $P, Q \in IFC(S)$, then $P \vee Q = (P \circ Q)^\infty$.*

The following is the immediate result of Result 2.F and Proposition 3.5.

Corollary 3.5. *Let S be a semigroup. If $P, Q \in IFC(S)$ such that $P \circ Q = Q \circ P$, then $P \vee Q = P \circ Q$.*

For a semigroup S, we define two binary operations $\vee$ and $\wedge$ on IFC(S) as follows : for any $P, Q \in$ IFC(S),

$$P \vee Q = \widehat{P \cup Q} \quad \text{and} \quad P \wedge Q = P \cap Q.$$

Then we obtain the following result from Definition 2.8, Lemma 3.3 and Theorem 3.4.

Theorem 3.6. *Let S be a semigroup. Then (IFC(S),$\wedge, \vee$) is a complete lattice with $\triangle$ and $\bigtriangledown$ as the least and greatest elements of IFC(S).*

Proposition 3.7. *Let P and Q be any intuitionistic fuzzy congruences on a group G. Then $R \circ Q = Q \circ P$. Hence, by Result 3.C and Corollary 3.5, $P \circ Q = P \vee Q$.*

Proof. Let $x, y \in G$. Then

$$\begin{aligned}
\mu_{P\circ Q}(x,y) &= \bigvee_{z\in G}[\mu_Q(x,z) \wedge \mu_{P(z,y)}] \\
&= \bigvee_{z\in G}[(\mu_Q(y,y) \wedge \mu_Q(z^{-1},z^{-1}) \wedge \mu_Q(x,z)) \\
&\quad \wedge(\mu_P(z,y) \wedge \mu_P(z^{-1},z^{-1}) \wedge \mu_P(x,x)] \\
&\leq \bigvee_{z\in S}[\mu_Q(yz^{-1}x,y) \wedge \mu_P(x,yz^{-1}x)] \\
&\leq \bigvee_{yz^{-1}x\in G}[\mu_P(x,yz^{-1}x) \wedge \mu_Q(yz^{-1}x,y)] \\
&= \mu_{Q\circ P}(x,y)
\end{aligned}$$

and

$$\begin{aligned}
\nu_{P\circ Q}(x,y) &= \bigwedge_{z\in G}[\nu_Q(x,z) \vee \nu_{P(z,y)}] \\
&= \bigwedge_{z\in G}[(\nu_Q(y,y) \vee \nu_Q(z^{-1},z^{-1}) \vee \nu_Q(x,z)) \\
&\quad \vee(\nu_P(z,y) \vee \nu_P(z^{-1},z^{-1}) \vee \nu_P(x,x)] \\
&\geq \bigwedge_{z\in G}[\nu_Q(yz^{-1}x,y) \vee \nu_P(x,yz^{-1}x)]
\end{aligned}$$

$$\begin{aligned} &\geq \bigwedge_{yz^{-1}x \in G} [\nu_P(x, yz^{-1}x) \vee \nu_Q(yz^{-1}x, y)] \\ &= \nu_{Q \circ P}(x, y) \end{aligned}$$

Thus $P \circ Q \subset Q \circ P$. Simlarly, we have $Q \circ P \subset P \circ Q$. Hence $P \circ Q = Q \circ P$. ■

Definition 3.8[4].*A lattice* $(L, \wedge, \vee)$ *is said to be modular if for any* $x, y, z \in L$ *with* $x \leq z$,

$$(x \vee y) \wedge z = x \vee (y \wedge z).$$

In any lattice L, it is well-known [4, Lemma I.5] that for any $x, y, z \in L$, if $x \leq z$ [resp. $x \geq z$], then $x \vee (y \wedge z) \leq (x \vee y) \wedge z$ [resp. $x \wedge (y \vee z) \geq (x \wedge y) \vee z$].

The inequality is called the *modular inequality*.

Theorem 3.9.*Let* S *be a semigroup and let* $\mathcal{A}$ *be any sublattice of (IFC(*S*),*$\wedge, \vee$*) such that* $P \circ Q = Q \circ R$ *for any* $P, Q \in \mathcal{A}$. *Then* $\mathcal{A}$ *is a modular lattice.*

Proof. Let $R, Q, P \in \mathcal{A}$ such that $R \subset P$. Let $x, y \in S$. Then

$$\begin{aligned} \mu_{(R \vee Q) \wedge P}(x, y) &= \mu_{(R \circ Q) \cap P}(x, y) && \text{(By Corollary 2.5)} \\ &= (\bigvee_{z \in S} [\mu_Q(x, z) \wedge \mu_R(z, y)]) \wedge \mu_P(x, y) \\ &= \bigvee_{z \in S} [\mu_Q(x, z) \wedge \mu_R(z, y) \wedge \mu_R(z, y) \wedge \mu_P(x, y)] \\ &\leq \bigvee_{z \in S} [\mu_Q(x, z) \wedge \mu_R(z, y) \wedge \mu_P(z, y) \wedge \mu_P(x, y)] && \text{(Since } R \subset P) \\ &\leq \bigvee_{z \in S} [\mu_Q(x, z) \wedge \mu_R(z, y) \wedge \mu_P(x, z)] && \text{(Since } P \in \text{ IFC}(S)) \\ &= \mu_{R \circ (Q \cap P)}(x, y) \\ &= \mu_{R \vee (Q \wedge P)}(x, y) && \text{(By Corollary 2.3)} \end{aligned}$$

and

$$\begin{aligned} \nu_{(R \vee Q) \wedge P}(x, y) &= \nu_{(R \circ Q) \cap P}(x, y) \\ &= (\bigwedge_{z \in S} [\nu_Q(x, z) \vee \nu_R(z, y)]) \vee \nu_P(x, y) \end{aligned}$$

$$
\begin{aligned}
&= \bigwedge_{z\in S}[\nu_Q(x,z)\vee\nu_R(z,y)\vee\nu_R(z,y)\vee\nu_P(x,y)]\\
&\geq \bigwedge_{z\in S}[\nu_Q(x,z)\vee\nu_R(z,y)\vee\nu_P(z,y)\vee\nu_P(x,y)]\\
&\geq \bigwedge_{z\in S}[\nu_Q(x,z)\vee\nu_R(z,y)\vee\nu_P(x,z)]\\
&= \nu_{R\circ(Q\cap P)}(x,y)=\nu_{R\vee(Q\wedge P)}(x,y).
\end{aligned}
$$

Thus $(R\vee Q)\wedge P\subset R\vee(Q\wedge P)$. It is clear that $R\vee(Q\wedge P)\subset(R\vee Q)\wedge P$ from the modular inequality. So $(R\vee Q)\wedge P=P\vee(Q\wedge P)$. Hence $\mathcal{A}$ is modular. ■

The following is the immediate result of Proposition 3.7 and Theorem 3.9.

Corollary 3.7 *If G is a group, then (IFC(G), $\wedge,\vee$) is a modular lattice.*

4. The lattice of intuitionistic fuzzy congruences on a regular semigroup.

For a semigroup S, S^1 denotes the monoid defined as follows :

$$
S^1=\begin{cases} S & \text{if } s \text{ has the ideuetity } 1,\\ S\cup\{1\} & \text{otherwise.}\end{cases}
$$

Definition 4.1 [13]. *The equivalence relations $\mathcal{L},\mathcal{R},\mathcal{H}$ and $\mathcal{D}$ on a semigroup S are defined as follows, respectively :*

(1) $\mathcal{L}=\{(a,b)\in S\times S: S^1a=S^1b\}$.
(2) $\mathcal{R}=\{(a,b)\in S\times S: aS^1=bS^1\}$.
(3) $\mathcal{H}=\mathcal{L}\cap\mathcal{R}$.
(4) $\mathcal{D}=\mathcal{L}\vee\mathcal{R}$.

The $\mathcal{L}-,\mathcal{R}-,\mathcal{H}-$ and $\mathcal{D}-$ classes of S containg the element a will, as usual, be denoted by La, Ra, Ha and Da, respectively. The set of all $\mathcal{L}-$ classes [resp.$\mathcal{R}-$ classes] of S can be partially ordered as follows : for any $a,b\in S$,

$$
La\leq Lb \quad \text{if and only if} \quad S^1a\subset S^1b
$$

and

$$Ra \leq Rb \quad \text{if and only if} \quad aS^1 \subset bS^1.$$

Definition 4.2[20]. *Let R be an intuitionistic fuzzy relation on a semigroup S. We define a complex mapping $R^\circ = (\mu_{R^\circ}, \nu_{R^\circ}) : S \times S \to I \times I$ as follows : for any $x, y \in S$,*

$$R^\circ(x,y) = (\bigwedge_{s,t \in S^1} \mu_R(sxt, syt), \bigvee_{s,y \in S^1} \nu_R(sxt, syt)).$$

It is clear that $R^\circ \in IFR(S)$.

Result 4.A[20, Proposition 3.3]. *Let S be a semigroup and let $R, Q \in IFR(S)$. Then :*

(1) $R^\circ \subset R$.
(2) $(R^\circ)^{-1} = (R^{-1})^\circ$.
(3) If $R \subset Q$, then $R^\circ \subset Q^\circ$.
(4) $(R^\circ)^\circ = R^\circ$.
(5) $(R \cap Q)^\circ = R^\circ \cap Q^\circ$.
(6) $R = R^\circ$ if and only if R is intuitionistic fuzzy left and right compatible.

Result 4.B[20, Theorem 3.4]. *Let S be a semigroup and let $R \in IFE(S)$. Then R° is the largest intuitionistic fuzzy congruence on S contained in R.*

Result 4.C[20, Theorem 3.6]. *Let S be a regular semigroup. If $P, Q \in \sum(\chi_{\mathcal{H}}, \chi_{\mathcal{H}^c})$, then $P \circ Q = Q \circ P$, where $\sum(\chi_{\mathcal{H}}, \chi_{\mathcal{H}^c}) = \{T \in IFC(S) : T \subset (\chi_{\mathcal{H}}, \chi_{\mathcal{H}^c})\}$.*

Definition 4.3[20]. *Let S be a regular semigroup and let $R \in IFC(S)$. Then R is called an intuitionistic fuzzy idempotent separating congruence (in short, IFISC) if $Re \neq R_f$ whenever $e \neq f$, i.e., $Re = R_f$ implies $e = f$ for any $e, f \in E_S$.*

We will denote the set of all IFISCs on S by IFISC(S).

Result 4.D[20, Theorem 4.7]. *Let S be a regular semigroup and let $T \in IFC(S)$. Then $T \in IFISC(S)$ if and only if $T \in \sum(\chi_{\mathcal{H}}, \chi_{\mathcal{H}^c})$.*

Proposition 4.4. *Let S be a semigroup and let $R \in IFE(S)$ and let $\sum(R) = \{T \in IFC(S) : T \subset R\}$. Then $\sum(R)$ is a sublattice of $IFC(S)$ with the greatest* *element R° and the least element $\triangle$.*

proof. It is clear that $\triangle \in$ IFC(S) and $\triangle \subset R$. Thus $\triangle \in \sum(R)$. So $\sum(R) \neq \phi$. Let $P, Q \in \sum(R)$. Then, by Result 3.A(1) and Result 3.B, $P \wedge Q \subset R$ and $P \vee Q \subset R^\circ \subset R$. Thus $P \wedge Q, P \vee Q \in \sum(R)$. Hence $\sum(R)$ is a sublattice of IFC(S) with the greatest element R° and the least element $\triangle$. ■

Proposition 4.5. *Let S be a regular semigroup. Then $\sum(\chi_{\mathcal{H}}, \chi_{\mathcal{H}^c})$ is a modular sublattice of IFC(S) with the greatest element $(\chi_{\mathcal{H}}, \chi_{\mathcal{H}^c})^\circ$ and the least element $\triangle$.*

Proof. By Proposition 4.4, $\sum(\chi_{\mathcal{H}}, \chi_{\mathcal{H}^c})$ is a sublattice of IFC(S) with the greatest element $(\chi_{\mathcal{H}}, \chi_{\mathcal{H}^c})^\circ$ and the least element $\triangle$. Hence, by Result 4.C and Theorem 3.9, $\sum(\chi_{\mathcal{H}}, \chi_{\mathcal{H}^c})$ is a modular sublattice of IFC(S). ■

The following is the immediate result of Proposition 4.5 and Result 4.D.

Theorem 4.6. *Let S be a regular semigroup. Then IFISC(S) is a modular sublattice of IFC(S) with the greatest element $(\chi_{\mathcal{H}}, \chi_{\mathcal{H}^c})^\circ$ and the least element $\triangle$.*

Lemma 4.7. *Let S be a semigroup and let $P, Q \in$ IFC(S) such that $Q \subset P$. We define a complex mapping $P/Q = (\mu_{P/Q}, \nu_{P/Q}) : S/Q \times S/Q \to I \times I$ as follows : for any $x, y \in S$,*

$$P/Q(Qx, Qy) = (\mu_P(x, y), \nu_P(x, y)).$$

Then P/Q is an intuitionistic fuzzy congruence on S/Q.

Proof. It is clear that $P/Q \in$ IFR(S/Q) from the definition of P/Q. Let $x \in S$. Then $P/Q(Qx, Qx) = (\mu_P(x, x), \nu_P(x, x)) = (1, 0)$. Thus P/Q is intuitionistic fuzzy reflexive. It is clear that P/Q is intuitionistic fuzzy symmetric from the definition of P/Q. Now let $x, y \in S$. Then

$$\mu_{P/Q}(Qx, Qy) = \mu_P(x, y) \geq \mu_{P \circ P}(x, y) = \bigvee_{z \in S} [\mu_P(x, z) \wedge \mu_P(z, y)]$$

$$\begin{aligned} &= \bigvee_{z\in S}[\mu_{P/Q}(Qx, Qz) \wedge \mu_{P/Q}(Qz, Qy)] \\ &= \mu_{P/Q\circ P/Q}(Qx, Qy) \end{aligned}$$

and

$$\begin{aligned} \nu_{P/Q}(Qx, Qy) &= \nu_P(x, y) \leq \nu_{P\circ P}(x, y) = \bigwedge_{z\in S}[\nu_P(x, z) \vee \nu_P(z, y)] \\ &= \bigwedge_{z\in S}[\nu_{P/Q}(Qx, Qz) \vee \nu_{P/Q}(Qz, Qy] \\ &= \nu_{P/Q\circ P/Q}(Qx, Qy). \end{aligned}$$

Thus $P/Q \circ P/Q \subset P/Q$, i.e., P/Q is intuitionistic fuzzy transitive. So $P/Q \in$ IFE(S/Q).

Let $x, y, z, t \in S$. Then

$$\begin{aligned} \mu_{P/Q}(Qx * Qz, Qy * Qt) &= \mu_{P/Q}(Qxz, Qyt) = \mu_P(xz, yt) \\ &\geq \mu_P(x, y) \wedge \mu_P(z, t) \\ &= \mu_{P/Q}(Qx, Qy) \wedge \mu_{P/Q}(Qz, Qt) \end{aligned}$$

and

$$\begin{aligned} \nu_{P/Q}(Qx * Qz, Qy * Qt) &= \nu_{P/Q}(Qxz, Qyt) = \nu_P(xz, yt) \\ &\leq \nu_P(x, y) \vee \nu_P(z, t) \\ &= \nu_{P/Q}(Qx, Qy) \vee \nu_{P/Q}(Qz, Qt). \end{aligned}$$

Thus P/Q is intuitionistic fuzzy compatible. Hence $P/Q \in$ IFC(S/Q). ■

Lemma 4.8. *Let S be a semigroup, let $T \in IFC(S)$ and let $IFC_T(S) = \{P \in IFC(S) : T \subset P\}$. Then there exists an order preserving bijection $\Phi : IFC_T(S) \to IFC(S/T)$.*

Proof. We define a mapping Φ : IFC$_T(S) \to$ IFC(S/T) as follows : for each $P \in$ IFC$_T(S)$,

$$\Phi(P) = P/T.$$

Then, by Lemma 4.7, Φ is well-defined. Let $P, Q \in$ IFC$_T(S)$ such that $P \subset Q$ and let $x, y \in S$. Then

$$\begin{aligned} \mu_{\Phi(P)}(Tx, Ty) &= \mu_{P/T}(Tx, Ty) = \mu_P(x, y) \\ &\leq \mu_Q(x, y) = \mu_{Q/T}(Tx, Ty) = \mu_{\Phi(Q)}(Tx, Ty) \end{aligned}$$

and

$$\begin{aligned}\nu_{\Phi(P)}(Tx,Ty) &= \nu_{P/T}(Tx,Ty)=\nu_P(x,y)\\ &\geq \nu_Q(x,y)=\nu_{Q/T}(Tx,Ty)=\nu_{\Phi(Q)}(Tx,Ty).\end{aligned}$$

Thus $\Phi(P)\subset\Phi(Q)$. So Φ is an order preserving mapping. It is clear that Φ is surjective. For any $P,Q\in \mathrm{IFC}_T(S)$, suppose $\Phi(P)=\Phi(Q)$ and let $x,y\in S$. Then $P/T(Tx,Ty)=Q/T(Tx,Ty)$. Thus $P(x,y)=Q(x,y)$. So Φ is injective. Hence Φ is an order preserving bijection. ■

The following result is straigh forward to verify.

Theorem 4.9. *Let S be a semigroup and let $T\in$ IFC(S). If $P,Q\in IFC_T(S)$, then $(P\wedge Q)/T=P/T\wedge Q/T$ and $(P\vee Q)/T=P/T\vee Q/T$. Hence $IFC_T(S)$ and IFC(S/T) are lattice isomorphic.*

Lemma 4.10. *Let S be a semigroup and let $\mathcal{C}\subset$ IFC(S) such that $T=\bigcap\mathcal{C}\in\mathcal{C}$. If $\mathcal{C}/T=\{P/T : P\in\mathcal{C}\}$ is a sublattice [resp. a sublattice of commuting intuitionistic fuzzy congruences] of IFC(S/T), then $\mathcal{C}$ is a sublattice [resp. a sublattice of commuting intuitionistic fuzzy congruences] of IFC(S).*

Proof. Suppose $\mathcal{C}/T$ is a sublattice of IFC(S/T). Let $P,Q\in\mathcal{C}$. Since $\mathcal{C}/T$ is a sublattice of IFC(S/T), $P/T\wedge Q/T, P/T\vee Q/T\in\mathcal{C}/T$. Since $P,Q\in \mathrm{IFC}_T(S)$, by Theorem 4.9, $P/T\wedge Q/T=(P\wedge Q)/T$ and $P/T\vee Q/T=(P\vee Q)/T$. Let $\Phi:$ $\mathrm{IFC}_T(S)\to \mathrm{IFC}(S/T)$ be the order preserving bijection defined in Lemma 4.8. Then $\Phi\mid_{\mathcal{C}}:\mathcal{C}\to\mathcal{C}/T$ is an order preserving bijection. Thus $P\wedge Q, P\vee Q\in\mathcal{C}$. Hence $\mathcal{C}$ is a sublattice of IFC(S).

Suppose $\mathcal{C}/T$ is a sublattice of commuting intuitionistic fuzzy congruences of IFC(S/T). Let $P,Q\in\mathcal{C}$ and let $x,y\in S$. Then

$$\begin{aligned}(P\circ Q)(x,y) &= (\bigvee_{z\in S}[\mu_Q(x,z)\wedge\mu_P(z,y)],\bigwedge_{z\in S}[\nu_Q(x,z)\vee\nu_P(z,y)])\\ &= (\bigvee_{z\in S}[\mu_{Q/T}(Tx,Tz)\wedge\mu_{P/T}(Tz,Ty)],\bigwedge_{z\in S}[\nu_{Q/T}(Tx,Tz)\vee\nu_{P/T}(Tz,Ty)])\\ &= (P/T\circ Q/T)(Tx,Ty)=(Q/T\circ P/T)(Tx,Ty)\\ &= (\bigvee_{z\in S}[\mu_{P/T}(Tx,Tz)\wedge\mu_{Q/T}(Tz,Ty)],\bigwedge_{z\in S}[\nu_{P/T}(Tx,Tz)\vee\nu_{Q/T}(Tz,Ty)])\\ &= (\bigvee_{z\in S}[\mu_P(x,z)\wedge\mu_Q(z,y)],\bigwedge_{z\in S}[\nu_P(x,z)\vee\nu_Q(z,y)])\end{aligned}$$

$$= \quad (Q \circ P)(x, y).$$

Thus $P \circ Q = Q \circ P$. Hence $\mathcal{C}$ is a sublattice of commuting intuitionistic fuzzy congruences of IFC(S). ■

Remark 4.11. *From Lemma 4.9 it is immediate that if $\mathcal{C}$ is a sublattice [resp.a sublattice of commuting intuitionistic fuzzy congruences] of IFC(S), then $\mathcal{C}$ /T is a sublattice [resp.a sublattice of commuting intuitionistic fuzzy congruences] of IFC(S/T).*

The following is the immediate result.

Proposition 4.12. *Let S be a semigroup and let $IFCo(S) = \{R \in IFC(S) : R(x, y) \in \{(0, 1), (1, 0)\} \quad \text{for any } x, y \in S\}$. Then IFCo($S$) is a sublattice of IFC(S).*

The following is the immediate result of Proposition 4.12 and Result 2.B.

Proposition 4.13. *Let S be a semigroup. Then $R \in IFC(S)$ if and only if $R_{(\lambda,\mu)} \in IFCo(S)$ for each $(\lambda, \mu) \in [0, 1) \times (0, 1]$ with $\lambda + \mu \leq 1$.*

Lemma 4.14. *Let S be a regular semigroup and let $Ro = \{(P, Q) \in IFCo(S) \times IFCo(S) : P(e, f) = Q(e, f) \text{ for any } e, f \in E_S\}$. Then*

(1) Ro is an equivalence relation on IFCo(S).

(2) Each Ro$-$ class is a sublattice of IFCo(S) of commuting intuitionistic fuzzy congruences.

Proof. The proof of (1) is clear.

(2) Let $\mathcal{A}$ be an $Ro-$ class, let $T = \bigcap_{P\in\mathcal{A}} P$, let $Q \in\mathcal{A}$ and let $e, f \in E_S$. Then $Q(e, f) = P(e, f)$ for each $P \in\mathcal{A}$ and $T(e, f) = (\bigwedge_{P\in\mathcal{A}}\mu_P(e, f), \bigvee_{P\in\mathcal{A}} \nu_P(e, f)) = P(e, f)$. Thus $T \in\mathcal{A}$. So $\mathcal{A}$ has the least element T.

Suppose there exist idempotents f_1 and f_2 in S/T such that $\mu_{Q/T}(f_1, f_2) > 0$ and $\nu_{Q/T}(f_1, f_2) < 1$. By Result 3.D, there exist idempotents e_1, e_2 in S such that $f_1 = Te_1$ and $f_2 = Te_2$. Then

$$\mu_Q(e_1, e_2) = \mu_{Q/T}(Te_1, Te_2) = \mu_{Q/T}(f_1, f_2) > 0$$

and

$$\nu_Q(e_1, e_2) = \nu_{Q/T}(Te_1, Te_2) = \nu_{Q/T}(f_1, f_2) < 1.$$

Since $Q(e_1, e_2) = T(e_1, e_2)$, $\mu_T(e_1, e_2) > 0$ and $\nu_T(e_1, e_2) < 1$. Since $T \in$ IFCo(S), $T(e_1, e_2) = (1, 0)$. By Result 2.A(1), $f_1 = Te_1 = Te_2 = f_2$. So Q/T is intuitionistic fuzzy idempotent separating.

Now, for each $P \in$ IFCo(S/T), we define a complex mapping $P' = (\mu_{P'}, \nu_{P'})$: $S \times S \to I \times I$ as follows : for any $x, y \in S$,

$$P'(x, y) = P(Tx, Ty).$$

Then clearly $P' \in$ IFCo(S) and $T \subset P'$. Suppose P is intuitionistic fuzzy idempotent separating and $\mu_{P'}(e, f) > 0, \nu_{P'}(e, f) < 1$ for any $e, f \in E_S$. Then $\mu_P(Te, T_f) = \mu_{P'}(e, f) > 0$ and $\nu_P(Te, T_f) < 1$. Since P is intuitionistic fuzzy idempotent separating, $Te = T_f$. Thus $T(e, f) = (1, 0)$. Since $T \subset P', 1 = \mu_T(e, f) \leq \mu_{P'}(e, f)$ and $0 = \nu_T(e, f) \geq \nu_{P'}(e, f)$. Thus $T(e, f) = P'(e, f)$ for any $e, f \in E_S$. So $P' \in\mathcal{A}$ and thus $P'/T = P$. Hence $\mathcal{A}/T = \{Q/T : Q \in \mathcal{A}\}$ is just the subset of IFCo(S/T) of idempotent separating intuitionistic fuzzy congruences, i.e., $\mathcal{A}/T =$ IFCo$(S/T)\cap$ IFISC(S/T). By Proposition 4.12 and Theorem 4.6, $\mathcal{A}/T$ is a sublattice of IFC(S/T). Furthermore, by Result 4.C and Result 4.D, $\mathcal{A}/T$ is a sublattice of IFC(S/T) of commuting intuitionistic fuzzy congruences. By Lemma 4.10, $\mathcal{A}$ is a sublattice of commuting intuitionistic fuzzy congruences. But $\mathcal{A}\subset$ IFCo(S). Hence $\mathcal{A}$ is a sublattice of IFCo(S) of commuting intuitionistic fuzzy congruences. This complete the proof. ∎

Theorem 4.15. *Let S be a regular semigroup and let $R = \{(P, Q) \in IFC(S) \times IFC(S) : P(e, f) = Q(e, f)$ for any $e, f \in E_S\}$. Then*

(1) R is an equivalence relation on IFC(S).

(2) Each $R-$ class is a modular sublattice of IFC(S).

Proof. The proof of (1) is clear.

(2) Let $\mathcal{A}$ be an $R-$ class, let $T = \bigcap \mathcal{A}$, and let $P \in\mathcal{A}$. Let $e, f \in E_S$. Then clearly $P(e, f) = Q(e, f)$ for each $Q \in\mathcal{A}$. Thus $P(e, f) = T(e, f)$. So $T \in\mathcal{A}$ and thus T is the least element of $\mathcal{A}$. Let $P, Q \in\mathcal{A}$ and let $e, f \in E_S$. Then clearly $(P \cap Q)(e, f) = T(e, f)$, i.e., $P \cap Q = T$. Since $T \in\mathcal{A}$, $P \cap Q \in\mathcal{A}$ for any $P, Q \in\mathcal{A}$. Now let $P, Q \in\mathcal{A}$, let $e, f \in E_S$ and let $(\lambda, \mu) \in [0, 1) \times (0, 1]$ with $\lambda + \mu \leq 1$. Then $T(e, f) = P(e, f) = Q(e, f)$. Thus, by Result 2.B(1),

$T_{(\lambda,\mu)}(e,f) = P_{(\lambda,\mu)}(e,f) = Q_{(\lambda,\mu)}(e,f)$. So there exists an $Ro-$ class $\mathcal{A}o$ such that $T_{(\lambda,\mu)}, P_{(\lambda,\mu)}, Q_{(\lambda,\mu)} \in \mathcal{A}o$. By Result 2.B(2) and Lemma 4.14, $(P \vee Q)_{(\lambda,\mu)} = P_{(\lambda,\mu)} \vee Q_{(\lambda,\mu)} \in \mathcal{A}o$. Then $(P \vee Q)_{(\lambda,\mu)}(e,f) = (P_{(\lambda,\mu)} \vee Q_{(\lambda,\mu)})(e,f) = T_{(\lambda,\mu)}(e,f)$. Thus $(P \vee Q)(e,f) = T(e,f)$. So $P \vee Q \in \mathcal{A}$. Hence $\mathcal{A}$ is a sublattice of IFC(S). Also, by Result 2.B(2) and Lemma 4.14, $(P \circ Q)_{(\lambda,\mu)} = P_{(\lambda,\mu)} \circ Q_{(\lambda,\mu)} = Q_{(\lambda,\mu)} \circ P_{(\lambda,\mu)} = (Q \circ P)_{(\lambda,\mu)}$. Then $P \circ Q = Q \circ P$. Hence, by Result 2.B(1) and Theorem 3.9, $\mathcal{A}$ is a modular sublattice of IFC(S). ■

Corollary 4.15. *Let S be a regular semigroup. Then*

(1) IFC(S) is a disjoint union of some modular sublattices of IFC(S).

(2) If S is a group, then IFC(S) is a modular lattice.

Proof. (1) It is clear form Theorem 4.15.

(2) Suppose S is a group. Then $E_S = \{e\}$, where e is the identity of S. Let $P, Q \in$ IFC(S). Then $P(e,e) = Q(e,e) = (1,0)$. Thus $R =$ IFC(S) $\times$ IFC(S) and each $R-$class is IFC(S). Hence, by Theorem 4.15, IFC(S) is a modular lattice. ■

5. Relationship between intuitionistic fuzzy normal subgroups and intuitionistic fuzzy congruences

Definition 5.1[14]. *Let $(X, \cdot)$ be a groupoid and let $A, B \in$ IFS(X). Then the intuitionistic fuzzy product of A and B, $A \circ B$ is defined as follows : for any $x \in X$*

$$(A \circ B)(x) = \begin{cases} (\bigvee_{yz=x}[\mu_A(y) \wedge \mu_B(z)], \bigwedge_{yz=x}[\nu_A(y) \vee \nu_B(z)]), \\ (0,1) \qquad \text{if } x \text{ is not expressible as } x = yz. \end{cases}$$

Definition 5.2[14]. *Let $(X, \cdot)$ be a groupoid and let $A \in$ IFS(X). Then A is called an intuitionistic fuzzy subgroupoid (in short, IFGP) of X if for any $x, y \in X$,*

$$\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y) \quad \text{and} \quad \nu_A(xy) \leq \nu_A(x) \vee \nu_A(y).$$

We will denote the set of all IFGPs of a groupoid X as IFGP(X). Then it is clear that $0_\sim$ and $1_\sim \in$ IFGP(X).

Definition 5.3[15].*Let G be a group and let $A \in$ IFGP(G). Then A is called an intuitionistic fuzzy subgroup (in short, IFG) of G if $A(x^{-1}) \geq A(x)$, i.e., $\mu_A(x^{-1}) \geq \mu_A(x)$ and $\nu_A(x^{-1}) \leq \nu_A(x)$, for each $x \in G$.*

We will denote the set of all IFGs of G as IFG(G).

Result 5.A[15, Proposition 3.4].*Let A be an IFG of a group G. Then $A \circ A = A$.*

Result 5.B[15, Proposition 3.5].*Let A and B be any two IFGs of a group G. Then the following conditions are equivalent :*

(1) $A \circ B \in$ IFG(G).

(2) $A \circ B = B \circ A$.

Definition 5.4[15].*Let G be a group and let $A \in$ IFG(G). Then A is said to be normal if $A(xy) = A(yx)$ for any $x, y \in G$.*

We will denote the family of all intuitionistic fuzzy normal subgroups of a group G as IFNG(G). In particular, we will denote the set $\{N \in \text{IFNG}(G) : N(e) = (1, 0)\}$ as IFN(G).

Result 5.C[15, Proposition 4.4].*Let G be a group and let $A, B \in$ IFNG(G). Then $A \circ B \in$ IFNG(G).*

Result 5.D[19, Proposition 3.18].*Let G be a group and let $R \in$ IFC(G). We define the complex mapping $A_R = (\mu_{A_R}, \nu_{R_A}) : G \to I \times I$ as follows : for each $a \in G$,*

$$A_R(a) = R(a, e) = Re(a).$$

Then $A_R \in$ IFN(G).

Definition 5.5[16].*Let G be a group, let $A \in$ IFG(G) and let $x \in G$. We define two complex mappings*

$$Ax = (\mu_{Ax}, \nu_{Ax}) : G \to I \times I$$

and

$$xA = (\mu_{xA}, \nu_{xA}) : G \to I \times I$$

as follows, respectively : for each $g \in G$*,*

$$Ax(g) = A(gx^{-1}) \quad \textit{and} \quad xA(g) = A(x^{-1}g).$$

Then Ax *[resp.*xA*] is called the intuitionistic fuzzy right [resp.left] coset of* G *determined by* x *and* A*.*

It is clear that if $A \in$ IFNG(G), then the intuitionistic fuzzy left coset and the intuitionistic fuzzy right coset of A on G coincide and in this case, we call *intuitionistic fuzzy coset* instead of intuitionistic fuzzy left coset or intuitionistic fuzzy right coset.

We denote as $C(G)$ the set of all congruences on a group G. As $C(G)$ a complate description of the congruences on a group in terms of its normal subgroups can be seen in many books, for example, in A.Rosenfeld [30] and J.M.Howie [13]. There can read as follows : There exists a lattice isomorphism of $N(G)$ onto $C(G)$. In this section, we shall obtain the similar result using intuitionistic fuzzy sets, where $N(G)$ denotes the set of all normal subgroups of G.

Lemma 5.6. *Let* G *be a group and let* $A \in$ *IFN*(G)*. We define the complex mapping* $R_A = (\mu_{R_A}, \nu_{R_A}) : G \times G \to I \times I$ *as follows : for each* $(a, b) \in G \times G$*,*

$$R_A(a, b) = A(ab^{-1}).$$

Then $R_A \in$ *IFC*(G)*.*

Proof. From the definition of R_A, it is clear that $R_A \in$ IFR(G). Moreover, R_A is intuitionistic fuzzy reflexive and intuitionistic fuzzy symmetric. Let $a, b \in G$. Then

$$\begin{aligned}
\mu_{R_A \circ R_A}(a, b) &= \bigvee_{t \in G} [\mu_{R_A}(a, t) \wedge \mu_{R_A}(t, b)] \\
&= \bigvee_{t \in G} [\mu_A(at^{-1}) \wedge \mu_A(tb^{-1})] \\
&\leq \bigvee_{t \in G} \mu_A((at^{-1})(tb^{-1})) \qquad \text{(Since } A \in \text{ IFG}(G)) \\
&= \mu_A(ab^{-1}) = \mu_{R_A}(a, b)
\end{aligned}$$

and

$$\begin{aligned}\nu_{R_A\circ R_A}(a,b) &= \bigwedge_{t\in G}[\nu_{R_A}(a,t)\vee\nu_{R_A}(t,b)] = \bigwedge_{t\in G}[\nu_A(at^{-1})\vee\nu_A(tb^{-1})] \\ &\geq \bigwedge_{t\in G}\nu_A((at^{-1})(tb^{-1})) = \nu_A(ab^{-1}) = \nu_{R_A}(a,b).\end{aligned}$$

Thus $R_A\circ R_A\subset R_A$. So R_A is intuitionistic fuzzy transitive. Hence $R_A\in$ IFE(G).

We can easily see that R_A is intuitionistic fuzzy compatible. Therefore $R_A\in$ IFC(G). ■

Proposition 5.7. *Let G be a group and let $A,B\in IFG(G)$. Then*

$$R_B\circ R_A = R_{A\circ B}$$

Proof. Let $(a,b)\in G$. Then

$$\begin{aligned}(R_B\circ R_A)(a,b) &= (\mu_{R_B\circ R_A}(a,b),\nu_{R_B\circ R_A}(a,b)) \\ &= (\bigvee_{z\in G}[\mu_{R_A}(a,z)\wedge\mu_{R_B}(z,b)], \bigwedge_{z\in G}[\nu_{R_A}(a,z)\vee\nu_{R_B}(z,b)]) \\ &= (\bigvee_{z\in G}[\mu_A(az^{-1})\wedge\mu_B(zb^{-1})], \bigwedge_{z\in G}[\nu_A(az^{-1})\vee\nu_B(zb^{-1})]) \\ &= (\bigvee_{az^{-1}=x,zb^{-1}=y}[\mu_A(x)\wedge\mu_B(y)], \bigwedge_{az^{-1}=x,zb^{-1}=y}[\nu_A(x)\vee\nu_B(y)]) \\ &= (\bigvee_{ab^{-1}=xy}[\mu_A(x)\wedge\mu_B(y)], \bigwedge_{ab^{-1}=xy}[\nu_A(x)\vee\nu_B(y)]) \\ &= (\mu_{A\circ B}(ab^{-1}),\nu_{A\circ B}(ab^{-1})) \\ &= (\mu_{R_{A\circ B}}(a,b),\nu_{R_{A\circ B}}(a,b)) = R_{A\circ B}(a,b).\end{aligned}$$

Hence $R_B\circ R_A = R_{A\circ B}$. ■

Theorem 5.8. *Let G be a group. Then (IFC(G), $\circ$) is a semilattice (i.e., a commutative idempotent semigroup).*

Proof. Let $H,K\in$ IFC(G) and let $(a,b)\in G\times G$. Then

$$(K\circ H)(a,b) = (\mu_{K\circ H}(a,b),\nu_{K\circ H}(a,b))$$

$$
\begin{aligned}
&= (\bigvee_{z\in G}[\mu_H(a,z)\wedge\mu_K(z,b)], \bigwedge_{z\in G}[\nu_H(a,z)\vee\nu_K(z,b)]) \\
&= (\bigvee_{z\in G}[\mu_H(az^{-1},e)\wedge\mu_K(e,z^{-1}b)], \bigwedge_{z\in G}[\nu_H(az^{-1},e)\vee\nu_K(e,z^{-1}b)]) \\
&\quad \text{(By Lemma 3.1)} \\
&= (\bigvee_{z\in G}[\mu_K(e,z^{-1}b)\wedge\mu_H(az^{-1},e)], \bigwedge_{z\in G}[\nu_K(e,z^{-1}b)\vee\nu_H(az^{-1},e)]) \\
&= (\bigvee_{z\in G}[\mu_K(a,az^{-1}b)\wedge\mu_H(az^{-1}b,b)], \bigwedge_{z\in G}[\nu_K(a,az^{-1}b)\vee\nu_H(az^{-1}b,b)]) \\
&\quad \text{(By Lemma 3.1)} \\
&= (\bigvee_{t\in G}[\mu_K(a,t)\wedge\mu_H(t,b)], \bigwedge_{t\in G}[\nu_K(a,t)\vee\nu_H(t,b)]) \qquad (t=az^{-1}b) \\
&= (\mu_{H\circ K}(a,b),\nu_{H\circ K}(a,b)) = (H\circ K)(a,b).
\end{aligned}
$$

Thus $K\circ H = H\circ K$. So, by Result 3.B, $H\circ K\in$ IFC(G). On the other had, we can easily see that $R\circ R = R$ for each $R\in$ IFC(G). Hence (IFC(G), $\circ$) is a semilattice. ■

The following result follows from Results 5.A, 5.B and 5.C.

Proposition 5.9. *Let G be a group. Then (IFN(G),$\circ$) is a semilattice.*

Theorem 5.10. *Let G be a group. Then there exists a bijection α : IFC(G) $\rightarrow$ IFN(G) such that $\alpha(R\circ S)=\alpha(R)\circ\alpha(S)$ and $\alpha(R\wedge S)=\alpha(R)\cap\alpha(S)$ for any $R,S\in$ IFC(G). Hence α : (IFC(G),$\wedge,\circ$) $\rightarrow$ (IFN(G),$\cap,\circ$) is a lattice isomorphism.*

Proof. We define two mappings α : IFC(G)$\rightarrow$IFN(G) and β : IFN(G) $\rightarrow$ IFC(G) respectively, as follows :

$$\alpha(R) = Re \quad \text{for each} \quad R\in \text{ IFC(G)}$$

and

$$\beta(N)(a,b) = N(ab^{-1}) \quad \text{for each} \quad N\in \text{ IFN(G) and any} \quad a,b\in G.$$

By Result 5.D and Lemma 5.6, α and β are well-defined.

We show that $\alpha\circ\beta = id_{IFN(G)}$ and $\beta\circ\alpha = id_{IFC(G)}$. Let $R\in$ IFC(G) and let $a,b\in G$. Then

$$[(\beta\circ\alpha)(R)](a,b) \;=\; [\beta(\alpha(R))](a,b) = \beta(Re)(a,b)$$

$$\begin{aligned} &= R_e(ab^{-1}) = R(e, ab^{-1}) \\ &= R(b,a) \quad \text{(Since } R \text{ is intuitionistic fuzzy right compatible)} \\ &= R(a,b). \quad \text{(Since } R \text{ is intuitionistic fuzzy symmetric)} \end{aligned}$$

Thus $(\beta \circ \alpha)(R) = R$. So $\beta \circ \alpha = id_{IFC(G)}$. Now let $N \in$ IFN(G) and let $a \in G$. Then

$$\begin{aligned} [(\alpha \circ \beta)(N)](a) &= [\alpha(\beta(N))](a) = (\beta(N))_e(a) = \beta(N)(e,a) \\ &= N(ea^{-1}) = N(a^{-1}) = N(a). \end{aligned}$$

Thus $(\alpha \circ \beta)(N) = N$. So $\alpha \circ \beta = id_{IFN(G)}$. Hence α is bijective.

Now, we show that $\alpha(R \circ S) = \alpha(R) \circ \alpha(S)$ and $\alpha(R \wedge S) = \alpha(R) \cap \alpha(S)$ for any $R, S \in$ IFC(G). Let $R, S \in$ IFC(G) and let $a \in G$. Then

$$[\alpha(R \circ S)](a) = (R \circ S)e(a) = (R \circ S)(e,a).$$

Thus

$$\begin{aligned} \mu_{R\circ S}(e,a) &= \bigvee_{z\in G} [\mu_S(e,z) \wedge \mu_R(z,a)] = \bigvee_{z\in G} [\mu_S(e,z) \wedge \mu_R(e, az^{-1})] \\ &\qquad \text{(Since } R \text{ is intuitionistic fuzzy right compatible)} \\ &= \bigvee_{z\in G} [\mu_{Se}(z) \wedge \mu_{Re}(az^{-1})] = \bigvee_{z\in G} [\mu_{Re}(az^{-1}) \wedge \mu_{Se}(z)] \\ &= \bigvee_{a=bz} [\mu_{Re}(b) \wedge \mu_{Se}(z)] = \mu_{Re\circ Se}(a) = \mu_{\alpha(R)\circ\alpha(S)}(a) \end{aligned}$$

and

$$\begin{aligned} \nu_{R\circ S}(e,a) &= \bigwedge_{z\in G} [\nu_S(e,z) \vee \nu_R(z,a)] = \bigwedge_{z\in G} [\nu_S(e,z) \vee \nu_R(e, az^{-1})] \\ &= \bigwedge_{z\in G} [\nu_R(e, az^{-1}) \vee \nu_S(e,z)] = \bigwedge_{a=bz} [\nu_{Re}(b) \vee \nu_{Se}(z)] \\ &= \nu_{Re\circ Se}(a) = \nu_{\alpha(R)\circ\alpha(S)}(a). \end{aligned}$$

So $\alpha(R \circ S) = \alpha(R) \circ \alpha(S)$. On the other hand,

$$\begin{aligned} \mu_{\alpha(R\wedge S)}(a) &= \mu_{(R\cap S)e}(a) = \mu_{R\cap S}(e,a) = \mu_R(e,a) \wedge \mu_S(e,a) \\ &= \mu_{Re}(a) \wedge \mu_{Se}(a) = \mu_{Re\cap Se}(a) = \mu_{\alpha(R)\cap\alpha(S)}(a) \end{aligned}$$

and

$$\begin{aligned} \nu_{\alpha(R\wedge S)}(a) &= \nu_{(R\cap S)_e}(a) = \nu_{R\cap S}(e,a) = \nu_R(e,a) \vee \nu_S(e,a) \\ &= \nu_{Re}(a) \vee \nu_{Se}(a) = \nu_{Re\cap Se}(a) = \nu_{\alpha(R)\cap\alpha(S)}(a). \end{aligned}$$

So $\alpha(R \wedge S) = \alpha(R) \cap \alpha(S)$. Hence α is a lattice isomorphism. This completes the proof. ■

The following is the immediate result of Corollary 3.9 and Theorem 5.10.

Corollary 5.10. *(IFN(G),* $\cap, \circ$*) is a modular lattice.*

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