The Annals of Probability
1976, Vol. 4, No. 4, 587–599

THE LAW OF LARGE NUMBERS AND THE CENTRAL LIMIT THEOREM IN BANACH SPACES

BY J. HOFFMANN-JØRGENSEN AND G. PISIER

Matematisk Institut, Aarhus and L'École Polytechnique, Palaiseau

Let $X_1, X_2, \cdots$ be independent random variables with values in a Banach space E. It is then shown that Chung's version of the strong law of large numbers holds, if and only if E is of type p. If the X_n's are identically distributed, then it is shown that the central limit theorem is valid, if and only if E is of type 2. Similar results are obtained for vectorvalued martingales.

1. Introduction. Let E be a Banach space, $(\Omega, \mathscr{F}, P)$ a probability space and $\{X_n\}$ a sequence of E-valued random variables, so that $\mathbb{E}X_n = 0$ for all $n \geqq 1$, where $\mathbb{E}$ denotes the expectation (that is, the Bochner integral with respect to P). We shall then study the law of large numbers, that is when do we have

$$\frac{1}{n}(X_1 + \cdots + X_n) \to_{n\to\infty} 0 \tag{1.1}$$

and the central limit theorem, that is when do we have

$$\frac{1}{(n)^{\frac{1}{2}}}(X_1 + \cdots + X_n) \to_{n\to\infty} \text{a Gaussian measure.} \tag{1.2}$$

In Section 2 we shall treat the law of large numbers. This problem has been treated by Fortet and Mourier in [7], [8] and [15] and by W. A. Woyczynski in [18]. We shall show that if $\{X_n\}$ satisfies (for some $1 \leqq p \leqq 2$)

$$\textstyle\sum_{n=1}^{\infty} n^{-p}\mathbb{E}||X_n||^p < \infty\,, \tag{1.3}$$

then a.s. convergence holds in (1.1) for all independent sequences $\{X_n\}$, if and only if E is of type p. And a.s. convergence in (1.1) holds for all martingale differences, (X_n), satisfying (1.3), if and only if E is isomorphic to a uniformly p-smooth space.

In Section 3 we treat the central limit theorem. This problem has been treated by Fortet and Mourier in [7], [8] and [15], Dudley and Strassen in [5] and [6], Giné in [9], and by Marcus and Jain in [14]. We shall here show that (1.2) holds under the classical hypothesis (independent, identically distributed, mean 0, finite second moment) if and only if E is of type 2.

Let (ε_n) be a *Bernoulli sequence*, that is (ε_n) are independent real random

Received April 24, 1975; revised September 17, 1975.
AMS 1970 *subject classifications.* Primary 60F05, 60B10; Secondary 46E15.
Key words and phrases. Central limit theorem, law of large numbers, Banach space valued random variables, martingales, Banach space type, modulus of uniform smoothness.

variables with $P(\varepsilon_n = \pm 1) = \frac{1}{2}$ for all n. Then we define

$$C(E) = \{(x_n) \in E^\infty \mid \sum_{j=1}^\infty \varepsilon_j x_j \text{ converges in probability}\}\,.$$

From Theorem 4 page 17 in [12] we know that we may substitute "converges in probability" with "converges in $L^p(E)$" for all $p \in \mathbb{R}_+$. Let $\|\cdot\|_p$ denote the usual metric in $L^p(E)$ for $p \in \mathbb{R}_+$ (for this and further notation see [10]), and put

$$|x|_p = \|\sum_{j=1}^\infty \varepsilon_j x_j\|_p \qquad \text{for} \quad x = (x_j) \in C(E)\,.$$

Then a standard argument shows that $(C(E), |\cdot|_p)$ is a Fréchet space for $p \in \mathbb{R}_+$. And since all the $|\cdot|_p$-topologies are stronger than the product topology on $C(E)$, we find from the closed graph theorem, that all the $|\cdot|_p$-topologies ($p \in \mathbb{R}_+$) are equivalent, that is:

LEMMA 1.1. *Let (ε_n) be a Bernoulli sequence, and define the set L by*

$$L = \{\sum_{j=1}^\infty \varepsilon_j x_j \mid (x_j) \in C(E)\}$$

then $L \subseteqq L^p(E)$ for all $p \in \mathbb{R}_+$, and all the L^p-topologies, for $p \in \mathbb{R}_+$, coincides on L.

Now let $p \in [1, 2]$, then E is said to be of *type p* if we have $l^p(E) \subseteqq C(E)$. Since the injection from $l^p(E)$ into $C(E)$ has a closed graph in case that E is of type p, we find that E is of type p, if and only if

(1.4) $\exists A \in \mathbb{R}_+$, so that $\mathbb{E}\|\sum_{j=1}^\infty \varepsilon_j x_j\|^p \leqq A \sum_{j=1}^\infty \|x_j\|^p$ for all $(x_j) \in C(E)$.

Finally we note that it follows easily from Theorem 2, page 11 in [12], that we have:

LEMMA 1.2. *If (X_n) are independent E-valued random variables so that the series $\sum X_n$ converges in probability, then it converges* a.s.

2. The strong law of large numbers. Let $\{X_n\}$ be a sequence of E-valued random variables satisfying

$$\mathbb{E}X_n = 0 \qquad \forall\, n \geqq 1\,. \tag{2.1}$$

Then $\{X_n\}$ is said to satisfy *the strong law of large numbers* if

$$\frac{1}{n} \sum_{j=1}^n X_j \to 0 \qquad \text{a.s.}$$

We shall here mainly consider Chung's condition (see [3])

$$\sum_{n=1}^\infty n^{-p}\mathbb{E}\|X_n\|^p < \infty \tag{2.2}$$

where p is some fixed positive number.

THEOREM 2.1. *Let $1 \leqq p \leqq 2$, then the following four statements are equivalent*:

(2.1.1) *E is of type p.*

(2.1.2) $\exists\, C > 0$ *so that* $\mathbb{E}\|\sum_{j=1}^n X_j\|^p \leqq C \sum_{j=1}^n \mathbb{E}\|X_j\|^p$ *for all independent*, $X_1, \cdots, X_n$, *with mean* 0 *and finite* pth *moment.*

(2.1.3) *The strong law of large numbers holds for all independent sequences*, $\{X_n\}$, *satisfying* (2.1) *and* (2.2).

(2.1.4) *If* $\sum_{j=1}^\infty j^{-p}\|x_j\|^p < \infty$ *and* (ε_j) *is a Bernoulli sequence, then* $n^{-1}\sum_{j=1}^n \varepsilon_j x_j \to 0$ *in probability.*

PROOF. Suppose that (2.1.1) holds, then there exists a constant B, so that (see (1.4))

$$\mathbb{E}\|\sum_{j=1}^n \varepsilon_j x_j\|^p \leqq B \sum_{j=1}^n \|x_j\|^p \qquad \forall\, x_1, \cdots, x_n \in E \quad \forall\, n\,.$$

So by Fubini's theorem we find

$$\mathbb{E}\|\sum_{j=1}^n \varepsilon_j X_j\|^p \leqq B \sum_{j=1}^n \mathbb{E}\|X_j\|^p$$

where (ε_j) is a Bernoulli sequence independent of $X_1, \cdots, X_n$, and $X_1, \cdots, X_n$ are independent random vectors with mean 0 and finite pth moment. Now from Theorem 4.3 in [10] it follows that

$$\mathbb{E}\|\sum_{j=1}^n X_j\|^p \leqq 8^p \mathbb{E}\|\sum_{j=1}^n \varepsilon_j X_j\|^p\,.$$

So (2.1.2) holds with $C = 8^p B$.

Suppose that (2.1.2) holds, and that (X_j) satisfies (2.1) and (2.2). Then the series $\sum j^{-1}X_j$ converges in $L^p(E)$ by (2.1.2), so by Lemma 1.2 it converges a.s. Now we notice that Kronecker's lemma is valid in any Banach space (with the same proof as in the real case), so applying Kronecker's lemma to $(X_j(w))$, for those w for which the series $\sum j^{-1}X_j(w)$ converges, we find that $n^{-1}(X_1 + \cdots + X_n) \to 0$ a.s., when $n \to \infty$. And so (2.1.3) holds.

(2.1.3) obviously implies (2.1.4). Now suppose that (2.1.4) holds, and let (x_j) be a sequence in E so that

$$\sum_{j=1}^\infty j^{-p}\|x_j\|^p < \infty\,.$$

Then from (2.1.4) and Lemma 1.1 we have that

$$\frac{1}{n}\sum_{j=1}^n \varepsilon_j x_j \to 0 \qquad \text{in} \quad L^p(E)\,.$$

Now let

$$F = \left\{(x_j) \in E^\infty \,|\, \sup_n \frac{1}{n}\{\mathbb{E}\|\sum_{j=1}^n \varepsilon_j x_j\|^p\}^{1/p} < \infty\right\}.$$

Then F is a linear space and we may define the norm

$$|x| = \sup_n \frac{1}{n}\{\mathbb{E}\|\sum_{j=1}^n \varepsilon_j x_j\|^p\}^{1/p}$$

for $x = (x_j) \in F$, and a standard argument shows that $(F, |\cdot|)$ is a Banach space. Now let

$$F_0 = \{(x_j) \in E^\infty \mid \sum_{j=1}^\infty j^{-p}||x_j||^p < \infty\}$$
$$|x|_0 = \{\sum_{j=1}^\infty j^{-p}||x_j||^p\}^{1/p}$$

then $(F_0, |\cdot|_0)$ is a Banach space, and by assumption we have $F_0 \subseteqq F$. The injection $F_0 \to F$ is clearly a linear operator with closed graph, and so it is continuous. That is, there exists $C > 0$ so that

$$\mathbb{E}||\sum_{j=1}^n \varepsilon_j x_j||^p \leqq Cn^p \sum_{j=1}^n j^{-p}||x_j||^p$$

for all $x_1, \cdots, x_n \in E$ and all $n > 1$. Now let $x_1, \cdots, x_n \in E$ and define

$$\begin{aligned} y_j &= 0 && \text{for } 1 \leqq j \leqq N \\ &= x_{j-N} && \text{for } N < j \leqq N + n \end{aligned}$$

where N is some integer. Then

$$\begin{aligned} \mathbb{E}||\sum_{j=1}^n \varepsilon_j x_j||^p &= \mathbb{E}||\sum_{j=1}^{N+n} \varepsilon_j y_j|| \leqq C(N + n)^p \sum_{j=N+1}^{N+n} \frac{||x_{j-N}||^p}{j^p} \\ &\leqq C\left(\frac{N+n}{N+1}\right)^p \sum_{j=1}^n ||x_j||^p \end{aligned}$$

for all $N \geqq 1$, and so

$$\mathbb{E}||\sum_{j=1}^n \varepsilon_j x_j||^p \leqq C \sum_{j=1}^n ||x_j||^p \qquad \forall x_1, \cdots, x_n \in E \quad \forall n \geqq 1 .$$

Hence E is of type p and the theorem is proved.

REMARKS. A. Beck has shown that if E is B-convex (for definition see [1]), then the strong law of large numbers holds for all independent sequences, $\{X_n\}$, satisfying (2.1) and

(2.3) $$\sup_n \mathbb{E}||X_n||^2 < \infty .$$

It was shown in [16] that E is B-convex if and only if E is of type p for some $p > 1$. Hence the result of Beck is included in Theorem 2.1, since (2.3) obviously implies (2.2) for all $p > 1$.

W. A. Woyczynski has in [18], shown that if E is a G_α-space (see [18] for a definition) with $\alpha = p - 1$, then the strong law of large numbers holds for all independent sequences, $\{X_j\}$, satisfying (2.1) and (2.2). So by Theorem 2.1 we find that every G_α-space is of type $(1 + \alpha)$. However, the G_α-spaces are in general much smoother than type $(1 + \alpha)$-spaces, since a fairly easy argument shows, that E is a G_α-space $(0 < \alpha \leqq 1)$, if and only if E is *uniformly* $(1 + \alpha)$-*smooth*, that is the modulus of smoothness, ρ, satisfies

(2.4) $$\rho(t) = O(t^{1+\alpha}) \qquad \text{as } t \to 0$$

where the modulus of smoothness is defined by

$$\rho(t) = \sup\{\tfrac{1}{2}(||x + y|| + ||x - y|| - 2) \mid ||x|| = 1 , \ ||y|| = t\} .$$

So G_α-spaces are in particular superreflexive (see: Theorem 4, page 169 in [4]). However, R. C. James has given an example of a B-convex space E, which is not even reflexive, that is E is of type $(1+\alpha)$ for some $\alpha > 0$, but E is not a G_α-space [11].

Let $\{X_1, X_2, \cdots\}$ be a finite of infinite sequence of random variables taking values in E, then $\{X_n\}$ is said to be a *martingale difference* if there exist σ-algebras $\mathscr{F}_0 \subseteqq \mathscr{F}_1 \subseteqq \cdots \subseteqq \mathscr{F}$, such that

$$X_n \text{ is } \mathscr{F}_n \text{ measurable } \forall n \tag{2.5}$$

$$\mathbb{E}(X_n \mid \mathscr{F}_{n-1}) = 0 \qquad \forall n . \tag{2.6}$$

If $\Omega = \{-1, 1\}^{\mathrm{N}}$, $\mathscr{F} =$ the Borel σ-algebra, $P =$ the normalized Haar measure, $\mathscr{F}_n =$ the σ-algebra spanned by the first n-coordinates for $n \geqq 1$ and $\mathscr{F}_0 = \{\phi, \Omega\}$, then $\{X_n\}$ is called a *Walsh–Paley martingale difference*. If $\{X_n\}$ is a Walsh-Paley martingale difference then (2.5) and (2.6) may be stated as follows:

$$X_n(\varepsilon) = X_n(\varepsilon_1, \cdots, \varepsilon_n) \text{ is only a function of the } n \text{ first coordinates} \tag{2.5$'$}$$

$$X_n(\varepsilon_1, \cdots, \varepsilon_{n-1}, 1) = -X_n(\varepsilon_1, \cdots, \varepsilon_{n-1}, -1) \qquad \forall \varepsilon \quad \forall n . \tag{2.6$'$}$$

THEOREM 2.2. *Let $1 \leqq p \leqq 2$, then the following four statements are equivalent*:

(2.2.1) E *is isomorphic to a uniformly p-smooth space*:

(2.2.2) $\exists C > 0$ *so that* $\mathbb{E}\|\sum_{j=1}^n X_j\|^p \leqq C \sum_{j=1}^n \mathbb{E}\|X_j\|^p$ *for all martingale differences*, $X_1, \cdots, X_n$, *with finite* pth *moment*;

(2.2.3) *The strong law of large numbers holds for martingale differences*, $\{X_n\}$, *satisfying* (2.2);

(2.2.4) *The strong law of large numbers holds for Walsh–Paley martingale differences*, $\{X_n\}$, *for which* $\sum_n n^{-p}\|X_n\|^p \in L^\infty(\Omega, \mathscr{F}, P)$.

SKETCH OF A PROOF. The equivalence of (2.2.1) and (2.2.2) is proved in [17]. The implication "(2.2.2) $\Rightarrow$ (2.2.3)" can be proved as in the proof of Theorem 2.1. It is evident that (2.2.3) implies (2.2.4). Finally the implication "(2.2.4) $\Rightarrow$ (2.2.2)" can be proved by standard martingale technique, and by using the method in Burkholder's paper [2]. (cf. the proofs of sublemmas 3.2 and 3.3 in [17]).

THEOREM 2.3. *Let $\{X_j\}$ be independent E-valued random variables with mean* 0, *and let $1 \leqq p \leqq 2$. If E is of type p, and $\{X_j\}$ satisfies*

(2.3.1) $\forall \varepsilon > 0 \quad \exists (a_j)$ *a sequence of positive numbers, so that* $\sum_{j=1}^\infty (a_j/j)^p < \infty$ *and* $\mathbb{E}(\|X_j\| 1_{\{\|X_j\| \geqq a_j\}}) \leqq \varepsilon \quad \forall j \geqq 1$;

then we have

$$\lim_{n\to\infty} \mathbb{E}\left\{\frac{1}{n} \|\textstyle\sum_{j=1}^n X_j\|\right\} = 0 .$$

REMARK. Even when $E = \mathbb{R}$, it is well known that (2.3.1) does not imply the strong law of large numbers.

PROOF. Let $\varepsilon > 0$ be given and choose (a_j) according to (2.3.1). Now we define

$$\begin{aligned} X_j' &= X_j \qquad \text{if} \quad \|X_j\| < a_j \\ &= 0 \qquad \text{if} \quad \|X_j\| \geqq a_j \\ X_j'' &= X_j' - \mathbb{E}X_j' . \end{aligned}$$

Then $\|X_j''\| \leqq 2a_j$, and so

$$\sum_{j=1}^{\infty} j^{-p} \mathbb{E}\|X_j''\|^p \leqq 2^p \sum_{j=1}^{\infty} \left(\frac{a_j}{j}\right)^p < \infty .$$

Hence by Theorem 2.1 and its proof we find that

$$\frac{1}{n} \sum_{j=1}^{n} X_j'' \to 0 \qquad \text{a.s.} \quad \text{and in} \quad L^1(E)$$

since $\mathbb{E}X_j = 0$ we find

$$\mathbb{E}X_j' = -\int_{\|X_j\| \geqq a_j} X_j \, dP$$

and so

$$\|\mathbb{E}X_j'\| \leqq \int_{\|X_j\| \geqq a_j} \|X_j\| \, dP \leqq \varepsilon .$$

Moreover we have

$$\mathbb{E}\|X_j - X_j'\| = \int_{\|X_j\| \geqq a_j} \|X_j\| \, dP \leqq \varepsilon$$

and so

$$\mathbb{E}\|\sum_{j=1}^{n} X_j\| \leqq \mathbb{E}\|\sum_{j=1}^{n} X_j''\| + \sum_{j=1}^{n} \|\mathbb{E}X_j'\| + \sum_{j=1}^{n} \mathbb{E}\|X_j - X_j'\| .$$

Hence we find

$$\limsup_{n\to\infty} \frac{1}{n} \mathbb{E}\|\sum_{j=1}^{n} X_j\| \leqq 2\varepsilon$$

and since $\varepsilon > 0$ is arbitrary we have proved the theorem.

THEOREM 2.4. *Let $\{X_j\}$ be independent E-valued random variables with mean 0, and satisfying*

(2.4.1) $$\forall \varepsilon > 0 \quad \exists K \quad \textit{compact} \quad \subseteq E \quad \textit{so that}$$
$$\mathbb{E}(\|X_j\| 1_{\{X_j \notin K\}}) \leqq \varepsilon \quad \forall j \geqq 1 ,$$

then we have

$$\lim \mathbb{E}\left\{\frac{1}{n} \|\sum_{j=1}^{n} X_j\|\right\} = 0 .$$

PROOF. Let $\varepsilon > 0$ be given and choose a compact set, K, according to (2.4.1), we may of course assume that K is convex. Now let

$$\begin{aligned} X_j' &= X_j \qquad \text{if} \quad X_j \in K \\ &= 0 \qquad \text{if} \quad X_j \notin K \\ X_j'' &= X_j' - \mathbb{E}X_j' . \end{aligned}$$

Then $X_j''(\omega) \in C$ for all $j \geqq 1$ and all ω, where $C = K - K$. Hence we have

$$\frac{1}{n} \sum_{j=1}^{n} X_j''(\omega) \in C \qquad \forall n \geqq 1 \quad \forall \omega .$$

Moreover we have

$$\frac{1}{n} \langle x', \sum_{j=1}^{n} X_j''(\omega)\rangle \to 0 \qquad \text{a.s.} \quad \forall x' \in E'$$

by the real valued strong law of large numbers. But this implies that

$$\frac{1}{n} \sum_{j=1}^{n} X_j'' \to 0 \qquad \text{a.s. and in} \quad L^1(E)$$

(for the L^1-convergence see Corollary 3.4 in [10]). Now the rest of the proof is identical to the last part of the proof of Theorem 2.3.

3. The central limit theorem. We shall now turn to the central limit theorem in Banach spaces. We shall need some additional notations. Let $(E, ||\cdot||)$ be a Banach space, and let E' be its dual space. If μ is a Radon probability on E, then its *characteristic functional*, $\hat{\mu}$, is defined by

$$\hat{\mu}(x') = \int_E e^{i\langle x', x\rangle} \mu(dx) \qquad \text{for } x' \in E' .$$

If μ has mean zero and finite 2nd moment, then its *covariance functional*, R, is defined by

$$R(x', y') = \int_E \langle x', x\rangle\langle y', x\rangle \mu(dx) \qquad \text{for } x', y' \in E' .$$

We shall say that a Radon probability, μ, on E is *pregaussian*, if μ has mean 0, finite second moment, and for some Radon measure, γ, on E we have

$$\hat{\gamma}(x') = \exp(-\tfrac{1}{2}R(x', x')) \qquad \forall x' \in E' \tag{3.1}$$

where R is the covariance functional of μ.

It is easily seen, that if γ satisfies (3.1), then γ is a Gaussian measure on E with mean 0 and the same covariance functional as μ.

Now let μ be a probability on E, and let $X_1, X_2, \cdots$ be independent random variables with distribution law μ. Then we put

$$Z_n = \frac{1}{(n)^{\frac{1}{2}}} \sum_{j=1}^{n} X_j$$

$$\mu_n = \text{the distribution law of} \quad Z_n .$$

We shall then say that μ *belongs to the domain of normal attraction* if there exists a Gaussian Radon measure, γ, on E, so that $\mu_n \to \gamma$ $||\cdot||$-weakly.

(If (T, τ) is a Hausdorff topological space, and ν_n and ν are Radon probabilities on (T, τ), then we say that $\nu_n \to \nu$ *τ-weakly*, if

$$\int_T f(t)\nu_n(dt) \to \int_T f(t)\nu(dt) \qquad \forall f \in C(T)$$

where $C(T)$ is the space of all continuous, bounded real functions on T.)

Fortet and Mourier (see [7], [8] and [15]) have proved that, if E is a separable, reflexive G_1-space admitting a Schauder basis, then any Radon probability, μ, with mean 0 and finite 2nd moment belongs to the domain of normal attraction. R. M. Dudley raised the problem of characterizing those Banach spaces for which this result holds. We shall in this section prove that this class of Banach spaces coincides with the type 2 spaces.

Suppose that μ is a Radon probability with mean 0 and finite 2nd moment. From the two inequalities

$$|e^{it} - 1 - it + \tfrac{1}{2}t^2| \leqq |t|^2 \qquad \forall\, t \in \mathbb{R},$$
$$|e^{it} - 1 - it + \tfrac{1}{2}t^2| \leqq |t|^3 \qquad \forall\, t \in \mathbb{R},$$

we find that

$$(3.2) \qquad |\hat{\mu}(x') - 1 + \tfrac{1}{2}R(x', x')| \leqq a^3\|x'\|^3 + \int_{\{\|x\|\geqq a\}} \|x'\|^2\|x\|^2\mu(dx)$$

for all $a > 0$ and all $x' \in E'$, where R is the covariance functional of μ. Let μ_n be defined as above, then it follows easily from (3.2) (see also [15]), that we have

$$(3.3) \qquad \lim_{n\to\infty} \hat{\mu}_n(x') = \exp(-\tfrac{1}{2}R(x', x')) \qquad \forall\, x' \in E'.$$

So if μ belongs to domain of normal attraction then μ is pregaussian, and the $\|\cdot\|$-weak limit of $\{\mu_n\}$, must satisfy (3.1). As a consequence of Lemma 5 in [13] we have:

LEMMA 3.1. *Let (T, τ) be a Hausdorff topological space and F a subset of $C(T)$, which separates points of T and satisfies*

$$(3.1.1) \qquad \text{If } f, g \in F \text{ then } fg \in F.$$

Let $\{\mu_n\}$ and μ be Radon probabilities on T, so that

$$(3.1.2) \qquad \lim_{n\to\infty} \int_T f(t)\mu_n(dt) = \int_T f(t)\mu(dt) \qquad \forall\, f \in F;$$

then $\mu_n \to \mu$ $\sigma(T, F)$-weakly, where $\sigma(T, F)$ is the weakest topology on T making all functions $f \in F$ continuous.

COROLLARY 3.2. *If μ is a pregaussian probability on E, and γ is the Gauss measure defined by* (3.1), *then $\mu_n \to \gamma$ $\sigma(E, E')$-weakly, where μ_n is the distribution law of*

$$Z_n = \frac{1}{(n)^{\frac{1}{2}}} \sum_{j=1}^n X_j$$

and $X_1, X_2, \cdots$ are independent random variables with distribution law μ.

PROOF. Let $F = \{e^{i\langle x', \cdot\rangle} \mid x' \in E'\}$, then F separates point in E and satisfies (3.1.1). Moreover, evidently we have that $\sigma(E, F) = \sigma(E, E')$, so the corollary is an immediate consequence of (3.3) and Lemma 3.1.

DEFINITION. Let (S, Σ, μ) be a positive measure space and $\Sigma_0 = \{A \in \Sigma \mid \mu(A) < \infty\}$; then a *second-order additive process* with *variance* μ, is a stochastic process

$X = \{X(A) \mid A \in \Sigma_0\}$, so that

(3.4) $X(A)$ has mean 0 and variance $\mu(A)$ for all $A \in \Sigma_0$.

(3.5) If $A_1, \cdots, A_n$ are disjoint sets in Σ_0, then $X(A_1), \cdots, X(A_n)$ are independent and

$$X(\bigcup_{j=1}^n A_j) = \sum_{j=1}^n X(A_j) .$$

If X in addition to (3.4) and (3.5) satisfies:

(3.6) $X(A)$ has a Gaussian distribution for all $A \in \Sigma_0$,

then X is called a *white noise* with *variance* μ.

PROPOSITION 3.3. *Let (S, Σ, μ) be a positive measure space and X a second order additive process with variance μ. If E is of type 2, then there exists a unique linear map*

$$f \to \int f\, dX$$

from $L^2(S, \Sigma, \mu, E)$ into $L^2(\Omega, \mathscr{F}, P, E)$ with the following properties

(3.3.1) $$\int f\, dX = \sum_{j=1}^n x_j X(A_j) \qquad \text{if } f = \sum_{j=1}^n x_j 1_{A_j}$$

and there exists a constant C, so that

(3.3.2) $$\mathbb{E}\|\int f\, dX\|^2 \leqq C \int \|f\|^2\, d\mu \qquad \forall f \in L^2(S, \Sigma, \mu, E) .$$

If X is a white noise, then we have

(3.3.3) $$\mathbb{E}\{\exp(i\langle x', \int f\, dX\rangle)\} = \exp(-\tfrac{1}{2} \int_S \langle x', f(s)\rangle^2 \mu(ds)) \qquad \forall f \in L^2(S, \Sigma, \mu, E) .$$

PROOF. Let $\mathscr{S}$ be the class of Σ_0-simple functions, then we may define

$$\int f\, dX$$

for $f \in \mathscr{S}$ by (3.3.1). Using the fact that $X(\cdot)$ is a $L^2(\Omega, \mathscr{F}, P)$-valued measure on Σ_0, it follows that $\int f\, dX$ is a well-defined linear map from $\mathscr{S}$ into $L^2(\Omega, \mathscr{F}, P, E)$. If $f \in \mathscr{S}$, then we may write f on the form $f = \sum_j x_j 1_{A_j}$, where $A_1, \cdots, A_n$ are disjoint sets from Σ_0. Now, since $X(A_1), \cdots, X(A_n)$ are independent random variables with mean 0 and variances $\mu(A_1), \cdots, \mu(A_n)$, we find from (2.1.2) that

$$\mathbb{E}\|\int f\, dX\|^2 \leqq C \sum_{j=1}^n \|x_j\|^2 \mu(A_j) = C \int \|f\|^2\, d\mu .$$

Hence the linear map

$$f \to \int f\, dX$$

is a continuous linear map: $\mathscr{S} \to L^2(\Omega, \mathscr{F}, P, E)$, and as such it admits a unique extension to $L^2(S, \Sigma, \mu, E)$.

Now, suppose that X is a white noise, and $f = \sum_1^n x_j 1_{A_j}$, where $A_1, \cdots, A_n$

are disjoint sets from Σ_0. Then

$$\begin{aligned}\mathbb{E}\exp(i\langle x', \textstyle\int f\,dX\rangle) &= \mathbb{E}\textstyle\prod_{j=1}^n \exp(i\langle x', x_j\rangle X(A_j)) \\ &= \exp(-\tfrac{1}{2}\textstyle\sum_{j=1}^n \langle x', x_j\rangle^2\mu(A_j)) \\ &= \exp(-\tfrac{1}{2}\textstyle\int_S \langle x', f(s)\rangle^2\mu(ds)) .\end{aligned}$$

So (3.3.3) follows by continuity.

PROPOSITION 3.4. *Let $\{X_t \mid t \geqq 0\}$ be a square integrable right continuous* (real) *martingale. If E is isomorphic to a uniformly 2-smooth space, then there exists a unique linear map*

$$f \to \int_0^\infty f\,dX_t$$

from $L^2(\mathbb{R}_+, \sigma, E)$ into $L^2(\Omega, \mathscr{F}, P, E)$, where $\sigma(t) = E(X(t))^2$, satisfying

$$\int_0^\infty f\,dX_t = \sum_{j=1}^n x_j(X(t_j) - X(t_{j-1})) \tag{3.4.1}$$
$$\text{if} \quad f = \sum_1^n x_j 1_{]t_{j-1},t_j]} \quad \text{and} \quad t_0 \leqq t_1 \leqq \cdots \leqq t_n$$

and there exists a constant C, so that

$$\mathbb{E}\|\int_0^\infty f\,dX_t\|^2 \leqq C\int_0^\infty \|f(t)\|^2\,d\sigma(t) \qquad \forall f \in L^2(\mathbb{R}_+, \sigma, E) . \tag{3.4.2}$$

PROOF. The proof is mutatis mutandis as the proof of Proposition 3.3.

THEOREM 3.5. *E is of type 2, if and only if every Radon probability, μ, on E with mean 0 and finite variance is pregaussian.*

PROOF. Suppose that E is of type 2, and let μ be a Radon probability on E with mean 0 and finite variance. Let W be a white noise with variance μ, then

$$X = \int_E xW(dx)$$

is an E-valued random variable by Proposition 3.3 since $f(x) = x \in L^2(E, \mathbb{B}(E), \mu, E)$. Moreover we have

$$\mathbb{E}\{e^{i\langle x', X\rangle}\} = \exp(-\tfrac{1}{2}\textstyle\int_E \langle x', x\rangle^2\mu(dx)) .$$

So the distribution law of X is a Radom probability on E statisfying (3.1). Hence μ is pregaussian.

Now suppose that every Radon probability with mean 0 and finite second moment is pregaussian. Let x_j be a sequence in E, so that $x_j \neq 0$ for all j, and

$$\sum_{j=1}^\infty \|x_j\|^2 = 1 .$$

Let $\lambda_j = \|x_j\|^2$ and $y_j = \|x_j\|^{-1}x_j$. Then the probability measure

$$\mu = \sum_{j=1}^\infty \lambda_j(\tfrac{1}{2}\delta_{y_j} + \tfrac{1}{2}\delta_{-y_j})$$

is a Radon measure on E having mean 0, second moment equal to 1, and

$$\int_E \langle x', x\rangle^2\mu(ds) = \sum_{j=1}^\infty \langle x', x_j\rangle^2 .$$

Hence there exists a Radon measure, γ, on E so that

$$\hat{\gamma}(x') = \exp\{-\tfrac{1}{2}\textstyle\sum_{j=1}^\infty \langle x', x_j\rangle^2\} .$$

Now, let $\eta_1, \eta_2, \cdots$ be independent (realvalued) normally distributed random variables with mean 0 and variance 1. Let

$$X_n = \sum_{j=1}^n \eta_j x_j .$$

Then

$$\mathbb{E}e^{i\langle x', x_n\rangle} = \exp\{-\tfrac{1}{2}\sum_{j=1}^n \langle x', x_j\rangle^2\} \to_{n\to\infty} \gamma(x') \qquad \forall\, x' \in E' .$$

Hence the law of X_n tends $\sigma(E, E')$-weakly to γ by Lemma 3.1, so by Theorem 6.2 in [10] (we may of course assume that E is separable) we have

$$\sum_1^\infty \eta_j x_j$$

converges a.s. From Theorem 3.1 in [10] it follows that this series converges in $L^2(E)$. Hence by Theorem 4.3 in [10] we find that $\sum \varepsilon_j x_j$ converges in $L^2(E)$, and so E is of type 2.

REMARK. From the proof it actually follows that if any discrete symmetric probability measure on E, with mean 0 and finite second moment, is pregaussian, then E is of type 2 and any Radon probability, with mean 0 and finite second moment, is pregaussian.

THEOREM 3.6. *Let E be of type 2, and μ a Radon probability on E with mean 0 and finite second moment, then μ is in the domain of normal attraction.*

PROOF. Let μ be a Radon probability on E, with mean 0 and finite second moment. Let Σ be the σ-algebra of μ-measurable sets and W a white noise with mean 0 and variance μ, defined on the probability space $(\Omega, \mathscr{F}, P)$. Let T be the linear map from $L^2(E, \Sigma, \mu, E)$ into $L^2(\Omega, \mathscr{F}, P, E)$ defined in Proposition 3.3.

Let $f(x) = x$, then $f \in L^2(E, \Sigma, \mu, E)$, and $U = Tf$ is a Gaussian E-valued random variable. Now we can find Σ-simple functions, f_p, so that

$$\int_E f_p(x)\mu(dx) = 0 \qquad \text{and} \qquad \|f - f_p\|_2 \leqq 2^{-p-1} .$$

Let $X_1, X_2, \cdots$ be independent random vectors with distribution law μ. Then we define

$$Z_n = n^{-\frac{1}{2}} \sum_{j=1}^n X_j \qquad Z_{np} = n^{-\frac{1}{2}} \sum_{j=1}^n f_p(X_j) \qquad U_p = Tf_p .$$

Then by the central limit theorem in finite dimensional space we have (here $\mathscr{L}(X)$ denotes the distribution law of X)

(i) $$\mathscr{L}(Z_{np}) \to_{n\to\infty} \mathscr{L}(U_p) \quad \|\cdot\|\text{-weakly} \qquad \forall\, p \geqq 1 .$$

Moreover, if C is the constant from (2.1.2), then

$$\begin{aligned}\mathbb{E}\|Z_{np} - Z_n\| &\leqq \mathbb{E}(\|Z_{np} - Z_n\|^2)^{\frac{1}{2}} \\ &\leqq (Cn^{-1} \textstyle\sum_{j=1}^n \mathbb{E}\|X_j - f_p(X_j)\|^2)^{\frac{1}{2}} \\ &= (C)^{\frac{1}{2}}\|f - f_p\|_2 \leqq (C)^{\frac{1}{2}}2^{-p-1} .\end{aligned}$$

So we find

(ii) $$\mathbb{E}\|Z_{np} - Z_n\| \leqq (C)^{\frac{1}{2}}2^{-p-1} \qquad \forall\, n, p \geqq 1 .$$

From Proposition 3.3 we have

(iii) $$\mathbb{E}\|U_p - U\| \leqq (C)^{\frac{1}{2}}2^{-p-1} \qquad \forall p \geqq 1 .$$

Now let L denote the set of all bounded Lipschitzian functions, $\varphi : E \to \mathbb{R}$. And let $\varphi \in L$. Then there exists a constant $A > 0$ so that

$$|\varphi(x)| \leqq A \qquad \forall x \in E$$
$$|\varphi(x) - \varphi(y)| \leqq A\|x - y\| \qquad \forall x, y \in E .$$

From (ii) and (iii) we find

$$\begin{aligned}|\mathbb{E}\varphi(Z_n) - \mathbb{E}\varphi(U)| &\leqq \mathbb{E}|\varphi(Z_n) - \varphi(Z_{np})| + |\mathbb{E}\varphi(Z_{np}) - \mathbb{E}\varphi(U_p)| \\ &\quad + \mathbb{E}|\varphi(U_p) - \varphi(U)| \\ &\leqq A(C)^{\frac{1}{2}}2^{-p} + |\mathbb{E}\varphi(Z_{np}) - \mathbb{E}\varphi(U_p)| .\end{aligned}$$

So from (i) we have

(iv) $$\lim_{n\to\infty} \mathbb{E}\varphi(Z_n) = \mathbb{E}\varphi(U) \qquad \forall \varphi \in L .$$

Now L is a function algebra contained in $C(E, \|\cdot\|)$, which separates points. Hence from Lemma 3.1 we find that

$$\mathscr{L}(Z_n) \to \mathscr{L}(U) \quad \sigma(E, L)\text{-weakly}.$$

But the $\sigma(E, L)$-topology is easily seen to be equal to the $\|\cdot\|$-topology, and so μ is in the domain of normal attraction.

REMARK. The counterexample in [5] shows that a Radon measure may be pregaussian without being in the domain of normal attraction. It still remains an open problem to characterize those spaces, E, in which all pregaussian measures are in the domain of normal attraction.

Often the preceding results have straightforward extensions to the case of operators of type p.

REFERENCES

[1] BECK, A. (1961). On the strong law of large numbers. Ergodic Theory, *Proc. Inst. Symp. New Orleans* (F. B. Wright, ed.) 21–53.

[2] BURKHOLDER, D. (1964). Maximal inequalities a necessary condition for almost everywhere convergence. *Z. Wahrscheinlichkeitstheorie und Verw. Gebeite* **3** 75–78.

[3] CHUNG, K. L. (1947). Note on some laws of large numbers. *Amer. J. Math.* **69** 189–192.

[4] DAY, M. M. (1973). *Normed Linear Spaces*, 3rd. ed. Springer-Verlag, New York.

[5] DUDLEY, R. M. (1974). The metric entropy and the central limit theorem in $C(S)$. *Ann. Inst. Fourier* **24** 49–60.

[6] DUDLEY, R. M., and STRASSEN, V. (1969). The central limit theorem and ε-entropy. Lecture Notes in Math. **89** 224–231. Springer-Verlag, New York.

[7] FORTET, M. R. and MOURIER, E. (1955–56). Les fonctions aléatoires comme éléments aléatoires dans les espaces de Banach. *Studia. Math.* **15** 62–79.

[8] FORTET, M. R. and MOURIER, E. (1965). Resultats complementaires sur les éléments aléatoires prenanant leurs valeurs dans un espace de Banach. *Bull. Sci. Math.* **78** 14–30.

[9] GINÉ, E. (1974). On the central limit theorem for sample continuous processes. *Ann. Probability* **2** 629–641.

[10] HOFFMANN-JØRGENSEN, J. (1964). Sums of independent Banach space valued random variables. *Studia. Math.* **52** 159–186.

[11] JAMES, R. C. (1974). A nonreflexive space is uniformly nonoctahedral. *Israel J. Math.* **18** 145–155.

[12] KAHANE, J.-P. (1968). *Some random series of functions.* Heath, Lexington, Mass.

[13] LECAM, L. (1957). *Convergence in distribution of stochastic processes.* Univ. of California Publ. Statist. **2** 207–236.

[14] MARCUS, M. B. and JAIN, N. (1975). Central limit theorems for $C(S)$-valued random variables. *J. Functional Analysis* **19** 216–231.

[15] MOURIER, E. (1952–53). Eléments aléatoires dans un espace de Banach. *Ann. Inst. H. Poincaré* **13** 159–244.

[16] PISIER, G. (1973–74). Sur les espaces qui ne contiennent pas de l_n^1 uniformément. *Sem. Maurey-Schwartz*, Ecole Polytechnique, Paris, p. VII. 1–VII. 19.

[17] PISIER, G. Martingales with values in uniformly convex convex spaces. *Israel J. Math.* (To appear).

[18] WOYCZYNSKI, W. A. (1973). Random series and law of large numbers in some Banach spaces. *Teor. Verojatnost. I Primenen* **18** 371–377.

MATEMATISK INSTITUT
UNIVERSITETSPARKEN
NY MUNKEGADE
8000 AARHUS, DENMARK

CENTRE DE MATHÉMATIQUES
DE L'ÉCOLE POLYTECHNIQUE
PLATEAU DE PALAISEAU
91120-PALAISEAU, FRANCE