THE LAW OF THE ITERATED LOGARITHM FOR A GAP SEQUENCE WITH INFINITE GAPS

SHIGERU TAKAHASHI

(Received June 4, 1963)

1. In the present note let $f(t)$ satisfy the following conditions

$$f(t+1) = f(t), \qquad \int_0^1 f(t)\,dt = 0 \text{ and } \qquad \int_0^1 f^2(t)\,dt < +\infty,$$

and let $\{n_k\}$ be a lacunary sequence of positive integers, that is,

$$n_{k+1}/n_k > q > 1. \tag{1. 1}$$

Then the sequence of functions $\{f(n_k t)\}$, although themselves not independent, exhibits the properties of independent random variables (c. f. [2]). In [1] S.Izumi proved that under certain smoothness condition of $f(t)$, $\{f(2^k t)\}$ obeys the law of the iterated logarithm. Further M. Weiss proved that this law holds for lacunary trigonometric series.

THEOREM of WEISS ([4]). *Let $\{n_k\}$ satisfy* (1. 1) *and $\{a_k\}$ be an arbitrary sequence of real numbers for which*

$$B_N = \left(\frac{1}{2}\sum_{k=1}^{N} a_k^2\right)^{1/2} \to +\infty \text{ and } a_N = o(\sqrt{B_N^2/\log\log B_N}), \quad \text{as } N \to +\infty.$$

Then we have, for almost all t,

$$\overline{\lim_{N\to\infty}} \frac{1}{\sqrt{2B_N^2 \log\log B_N}} \sum_{k=1}^{N} a_k \cos 2\pi n_k(t + \alpha_k) = 1.$$

However, there exist a sequence $\{n_k\}$ satisfying (1. 1) and a trigonometric polynomial $f(t)$ such that $\{f(n_k t)\}$ does not obey the law of the iterated logarithm. In [3] it is shown that if $\{n_k\}$ satisfies (1. 1) and $f(t)$ is a function of Lip α, $0 < \alpha \leqq 1$, then there exists a constant C such that

$$\overline{\lim_{N\to\infty}} \frac{1}{\sqrt{N \log\log N}} \left|\sum_{k=1}^{N} f(n_k t)\right| \leqq C, \qquad \text{a. e. in } t.$$

The purpose of this note is to prove the following

THEOREM. *Let $f(t)$ be a function of* Lip $\alpha, 0 < \alpha \leqq 1$, *and $\{n_k\}$ satisfy*

$$n_{k+1}/n_k \to +\infty, \qquad \textit{as } k \to +\infty. \tag{1. 2}$$

Then we have, for almost all t,

$$\overline{\lim_{N\to\infty}} \frac{1}{\sqrt{2N \log\log N}} \sum_{k=1}^{N} f(n_k t) = \|f\|.^{*)}$$

2. From now on let $f(t)$ and $\{n_k\}$ satisfy the conditions of the theorem. Further, without loss of generality we may assume that the Fourier series of $f(t)$ contains cosine terms only and

$$(2.\ 1) \qquad n_{k+1}/n_k > 3, \qquad \text{for } k \geqq 1.$$

These assumptions are introduced solely for the purpose of shortening the formulas. Let us put

$$(2.\ 2) \qquad f(t) \sim \sum_{k=1}^{\infty} a_k \cos 2\pi kt \quad \text{and} \quad S_N(t) = \sum_{k=1}^{N} a_k \cos 2\pi kt.$$

Since $f(t)$ is a function of Lip α, we have for some constant A

$$(2.\ 3) \qquad |f(t) - S_N(t)| < AN^{-\alpha} \log N,$$

$$(2.\ 3') \qquad \sum_{k=N}^{\infty} a_k^2 < A^2 N^{-2\alpha},$$

$$(2.\ 3'') \qquad |f(t)| < A \quad \text{and} \quad |S_N(t)| < A.$$

LEMMA 1. *If a positive number* λ *satisfies the condition*

$$(2.\ 4) \qquad \lambda\sqrt{M} < \log M,$$

then there exists an integer M_0, *not depending on* N, *such that*

$$\int_0^1 \exp\left\{\lambda \sum_{k=N+1}^{N+M} f(n_k t)\right\} dt < 2 \exp\{2(\lambda \|f\|)^2 M\}, \qquad \textit{for } M > M_0.$$

PROOF. We define m, L and $U_l(t)$ as follows;

$$(2.\ 5) \qquad m^6 \leqq M < (m+1)^6,$$

$$(2.\ 5') \qquad m(2L+2) \leqq M < m(2L+4),$$

and

$$(2.\ 5'') \qquad U_l(t) = \sum_{k=lm+1}^{(l+1)m} S_{M^{1/\alpha}}(n_{N+k} t).$$

Then we can easily see that

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*) We put $\|f\| = \left\{\int_0^1 f^2(t)\,dt\right\}^{1/2}$.

$$\lambda\left|\sum_{k=N+1}^{N+M} f(n_k t) - \sum_{l=0}^{(2L+1)} U_l(t)\right|$$

$$\leqq \lambda\left|\sum_{k=(2L+2)m+N+1}^{N+M} f(n_k t)\right| + \lambda \sum_{k=1}^{(2L+2)m}\left|f(n_{N+K}t) - S_{M^{1/\alpha}}(n_{N+K}t)\right|$$

$$\leqq A\lambda(2m + \log M^{1/\alpha}) = O(M^{-1/3}\log M) = o(1), \qquad \text{as } M \to +\infty.$$

Hence if $M > M_0$ for some M_0, it is seen that

$$\int_0^1 \exp\left\{\lambda \sum_{k=N+1}^{N+M} f(n_k t)\right\} dt < \sqrt{2}\int_0^1 \exp\left\{\lambda \sum_{l=0}^{2L+1} U_l(t)\right\} dt \tag{2. 6}$$

$$\leqq \sqrt{2}\left[\int_0^1 \exp\left\{2\lambda \sum_{l=0}^{L} U_{2l}(t)\right\} dt\right]^{1/2}\left[\int_0^1 \exp\left\{2\lambda \sum_{l=0}^{L} U_{2l+1}(t)\right\} dt\right]^{1/2}.$$

From (2. 3$''$) and (2. 4) we obtain

$$\lambda \operatorname*{Max}_{l\leqq L}|U_{2l}(t)| < \lambda A m = O(M^{-1/3}\log M) = o(1),$$

$$\lambda^3 \sum_{l=0}^{L} |U_{2l}(t)|^3 < \lambda^3 A^3 m^3 L = O(M^{-1/6}\log^3 M) = o(1), \quad \text{as } M\to\infty.$$

By the above relations and the inequality $e^z \leqq (1 + z + z^2/2)e^{|z|^3}$ for $|z| < \frac{1}{2}$, we have, for $M > M_0$,

$$\exp\left\{2\lambda \sum_{l=0}^{L} U_{2l}(t)\right\} < \sqrt{2}\prod_{l=0}^{L}\{1 + 2\lambda\, U_{2l}(t) + 2\lambda^2 U_{2l}^2(t)\}. \tag{2. 7}$$

Let us put

$$W_l(t) = \sum_{k=lm+2}^{(l+1)m}\ \sum_{j=lm+1}^{k-1}\ \sum_{(r,s)} a_r a_s \cos 2\pi(n_{k+N}s - n_{j+N}r)t, \tag{2. 8}$$

and

$$V_l(t) = 2\lambda U_l(t) + \lambda^2\left\{2U_l^2(t) - m\sum_{k=1}^{M^{1/\alpha}} a_k^2 - 2W_l(t)\right\}, \tag{2. 8$'$}$$

where $\sum_{(r,s)}$ denotes the summation over all (r, s) which belong to

$$\{(r, s);\ |n_{k+N}s - n_{j+N}r| \leqq n_{lm+N},\ 0 < s, r \leqq M^{1/\alpha}\}. \tag{2. 8$''$}$$

Then $V_l(t)$ is a sum of cosine terms whose frequencies are in the interval $[n_{lm+N} + 1, 2M^{1/\alpha}n_{(l+1)m+N}]$. By (2. 1) and (2. 5) we can find an integer M_0 such that $M > M_0$ implies

$$\frac{n_{2lm+N}}{2M^{1/\alpha}n_{(2j+1)m+N}} > \frac{3^{m(2l-2j-1)}}{2M^{1/\alpha}} > 3^{l-j}, \qquad \text{for } l > j.$$

Hence if $u_l \in [n_{lm+N}+1,\ 2M^{1/\alpha}n_{(l+1)m+N}]$ and $1 \leqq l_1 < l_2 < \cdots < l_s < l$, then we have

$$u_{2l} - \sum_{j=1}^{s} u_{2l_j} > u_{2l} - \sum_{j=1}^{l-1} u_{2j} > n_{2lm+N} - 2M^{1/\alpha}\sum_{j=1}^{l-1} n_{(2j+1)m+N}$$

$$> n_{2lm+N}\left(1 - \sum_{j=1}^{l-1} 3^{j-l}\right) > 2^{-1}\ n_{2lm+N} > 0, \qquad \text{for } M > M_0.$$

If u_l's are integers, then the above relation implies

$$\int_0^1 \cos 2\pi\, u_{2l}t \prod_{j=1}^{s} \cos 2\pi\, u_{2l_j}t\ dt = 0, \qquad \text{for } M > M_0.$$

Therefore, we have

$$\int_0^1 V_{2l}(t) \prod_{j=1}^{s} V_{2l_j}(t)dt = 0, \qquad \text{for } M > M_0. \tag{2. 9}$$

On the other hand if $k > j > lm$, then (2. 1) implies that the (r,s)-set in (2.8″) is contained in $\{(r,s);\ |sn_{k+N}/n_{j+N} - r| < 3^{-1},\ 1 \leqq s\}$. Therefore we have, by (2. 3′) and (2. 1),

$$\sum_{(r,s)} |a_l a_s| \leqq \left\{\sum_{s=1}^{\infty} a_s^2\right\}^{1/2} \left\{\sum_{r \geqq [n_{k+N}/n_{j+N}]} a_r^2\right\}^{1/2} \leqq \sqrt{2}\,\|f\| A\left(\frac{2n_{j+N}}{n_{k+N}}\right)^{\alpha}$$

$$\leqq 2^{1/2+\alpha}\ \|f\|\, A\left(\frac{n_{k-1+N}}{n_{k+N}}\right)^{\alpha} 3^{\alpha(j-k+1)}.$$

Therefore if we put

$$B_l = \sup_t |W_l(t)|, \tag{2.10}$$

then we have

$$B_l \leqq 2^{1/2+\alpha}\|f\|\, A\ \operatorname*{Max}_{k>lm}(n_{k-1}/n_k)^{\alpha} \sum_{k=lm+2}^{(l+1)m}\ \sum_{j=lm+1}^{k-1} 3^{\alpha(j-k+1)} \tag{2.10′}$$

$$\leqq Bm\ \operatorname*{Max}_{k>lm}(n_{k-1}/n_k)^{\alpha}, \qquad \text{for some constant } B > 0.$$

Since $m\sum_{k=1}^{M^{1/\alpha}} a_k^2 \leqq 2m\|f\|^2$, we obtain from (2. 8′) and (2.10)

$$\{1 + 2\lambda\ U_l(t) + 2\lambda^2 U_l^2(t)\} \leqq \{1 + V_l(t) + 2\lambda^2 m(\|f\|^2 + B_l)\}.$$

By (2. 7), (2. 9) and the above relation we have, for $M > M_0$,

$$\int_0^1 \exp\left\{2\lambda\sum_{l=0}^{L} U_{2l}(t)\right\}dt < \sqrt{2}\int_0^1 \prod_{l=0}^{L}\{1 + V_{2l}(t) + 2\lambda^2 m(\|f\|^2 + B_{2l})\}dt$$

$$=\sqrt{2}\prod_{l=0}^{L}\{1+2\lambda^2 m(\|f\|^2+B_{2l})\}\leqq\sqrt{2}\exp\left\{\sum_{l=0}^{L}2\lambda^2 m(\|f\|^2+B_{2l})\right\},$$

and in the same way

$$\int_0^1\exp\left\{2\lambda\sum_{l=0}^{L}U_{2l+1}(t)\right\}dt<\sqrt{2}\exp\left\{\sum_{l=0}^{L}2\lambda^2 m(\|f\|^2+B_{2l+1})\right\}.$$

From the above relations and (2. 6) we can see that for $M>M_0$

$$\int_0^1\exp\left\{\lambda\sum_{k=N+1}^{N+M}f(n_k t)\right\}dt<2\exp\left\{\sum_{l=0}^{2L+1}\lambda^2 m(\|f\|^2+B_l)\right\}.$$

On the other hand (2. 5′), (1. 2) and (2.10′) imply that if $M>M_0$ for some M_0, then

$$m\sum_{l=0}^{2L+1}(\|f\|^2+B_l)$$
$$\leqq M\|f\|^2+Bm\sum_{l=0}^{2L+1}\operatorname*{Max}_{k>lm}(n_{k-1}/n_k)^\alpha<2M\|f\|^2,$$

The last two relations prove the lemma.

3. LEMMA 2. *If a positive number $\psi(M)$ satisfies*

(3. 1) $$\psi(M)<(2\|f\|\ \log M)^2,$$

then for $M>M_0$, M_0 is the same as in Lemma 1, *we have*

$$\left|\left\{t;\ \sum_{k=N+1}^{N+M}f(n_k t)\geqq 2\|f\|\sqrt{M\psi(M)}\right\}\right|\leqq 2e^{-\psi(M)/2}.\ ^{*)}$$

PROOF. If we put $\lambda=(2\|f\|)^{-1}\psi^{1/2}(M)M^{-1/2}$, then λ satisfies the condition (2. 4). Hence by Lemma 1 and Tchebyschev's inequality we have

$$\left|\left\{t;\ \sum_{k=N+1}^{N+M}f(n_k t)\geqq 2\|f\|\sqrt{M\psi(M)}\right\}\right|$$
$$\leqq 2\exp\left\{2(\lambda\|f\|^2)M-2\lambda\|f\|\sqrt{M\psi(M)}\right\}=2e^{-\psi(M)/2}.$$

LEMMA 3. *We have, for almost all t*

$$\overline{\lim_{m\to\infty}}\frac{1}{\sqrt{2^{m+1}\log m}}\sum_{k=1}^{2^m}f(n_k t)\leqq 2\|f\|.$$

PROOF. If $m>m_0$ for some m_0, then $2^m>M_0$ and $\psi(2^m)=2(1+\varepsilon)\log m$

*) We consider t's within the interval [0, 1].

satisfies (3. 1) for any fixed $\varepsilon>0$. Therefore we have, by Lemma 2,

$$\sum_{m>m_0}\left|\left\{t:\sum_{k=1}^{2^m}f(n_kt)\geqq 2\|f\|\sqrt{2^{m+1}(1+\varepsilon)\log m}\right\}\right|\leqq 2\sum_{m>m_0}m^{-(1+\varepsilon)}<+\infty.$$

Since ε is arbitrary, Borel-Cantelli's lemma completes the proof.

LEMMA 4 . *We have, for almost all* t,

$$\overline{\lim_{m\to\infty}}\max_{N<2^m}\frac{1}{\sqrt{2^{m+1}\log m}}\sum_{k=2^m+1}^{2^m+N}f(n_kt)\leqq 6\|f\|.$$

PROOF. Let m be a positive integer such that

$$2^{[m/2]}>M_0,\qquad \frac{1}{2}\left\{1+\frac{1}{(\log m)}\right\}<\frac{9}{16} \tag{3. 2}$$

and, for any fixed $\varepsilon>0$,

$$2(m-l)+2(1+\varepsilon)\log m<(2\|f\|\log 2^l)^2,\ m>l\geqq[m/2]. \tag{3. 3}$$

Further let $X_l(t)$ be the positive part of the function

$$X_l(t)=\max_N\left[\sum_{k=N+1}^{N+2^l}f(n_kt);N\in\{r2^l+2^m,\ r=0,1,\cdots,2^{m-l}-1\}\right]. \tag{3. 4}$$

Then we have, by (2. 3″),

$$\max_{N<2^m}\sum_{k=2^m+1}^{2^m+N}f(n_kt)\leqq\sum_{l=0}^{m-1}X_l(t)<A2^{[m/2]}+\sum_{l=[m/2]}^{m-1}X_l(t). \tag{3. 5}$$

Putting

$$\psi(2^l)=2(m-l)+2(1+\varepsilon)\log m,\qquad\text{for } m>l\geqq[m/2],$$

then (3. 2), (3. 3) and Lemma 2 imply, for any positive integer N,

$$\left|\left\{t;\sum_{k=N+1}^{N+2l}f(n_kt)\geqq 2\|f\|\sqrt{2^l\psi(2^l)}\right\}\right|\leqq 2e^{-(m-l)}m^{-(1+\varepsilon)}. \tag{3. 6}$$

Therefore if we put

$$E_l=\{t;\ X_l(t)\geqq 2\|f\|\sqrt{2^l\psi(2^l)}\},$$

then we have, by (3. 4) and (3. 6),

$$|E_l|\leqq 2^{(m-l+1)}m^{-(1+\varepsilon)}e^{-(m-l)},$$

and

$$\sum_{m>m_0}\sum_{l=[m/2]}^{m-1}|E_l|<+\infty. \tag{3. 7}$$

Further if $t \bar{\in} \bigcup_{l=[m/2]}^{m-1} E_l$, then we have

$$\sum_{l=[m/2]}^{m-1} X_l(t) \leqq 2\|f\| \sum_{l=[m/2]}^{m-1} \sqrt{2^l\psi(2^l)}\ .$$

Since (3. 2) implies

$$\sqrt{\frac{2^l\psi(2^l)}{2^{l+1}\psi(2^{l+1})}} < \sqrt{\frac{1}{2}\left(1+\frac{1}{\log m}\right)} < \frac{3}{4}, \qquad \text{for } l < m,$$

we have

$$\sum_{l=[m/2]}^{m-1} \sqrt{2^l\psi(2^l)} < 4\sqrt{2^{m-1}\psi(2^{m-1})} < 3\sqrt{2^{m+1}\{1+(1+\varepsilon)\log m\}}.$$

From the last two relations and (3. 5) it is seen that

$$\operatorname*{Max}_{N<2^m} \sum_{k=2^m+1}^{2^m+N} f(n_kt) < A2^{[m/2]} + 6\|f\|\sqrt{2^{m+1}\{1+(1+\varepsilon)\log m\}}, \quad \text{for } t \bar{\in} \bigcup_{l=[m/2]}^{m-1} E_l.$$

The above relation and (3. 7) complete the proof.

From Lemma 3 and Lemma 4 we have, for almost all t,

$$\text{(3. 8)} \qquad \overline{\lim_{N\to\infty}} \frac{1}{\sqrt{2N\log\log N}} \sum_{k=1}^{N} f(n_kt) \leqq 8\|f\|.$$

4. Since (3. 8) can be proved under the conditions (1. 2), (2. 3), (2. 3′) and (2. 3″), we can also prove that for any fixed $M > 0$ and almost all t

$$\text{(4. 1)} \qquad \overline{\lim_{N\to\infty}} \frac{1}{\sqrt{2N\log\log N}} \sum_{k=1}^{N} \{f(n_kt) - S_M(n_kt)\} \leqq 8\|f(t) - S_M(t)\|.$$

Considering $-\{f(t) - S_M(t)\}$, we have for almost all t

$$\text{(4. 2)} \qquad \underline{\lim_{N\to\infty}} \frac{1}{\sqrt{2N\log\log N}} \sum_{k=1}^{N} \{f(n_kt) - S_M(n_kt)\}$$

$$= -\overline{\lim_{n\to\infty}} \frac{-1}{\sqrt{2N\log\log N}} \sum_{k=1}^{N} \{f(n_kt) - S_M(n_kt)\}$$

$$\geqq -8\|f(t) - S_M(t)\|.$$

On the other hand from (1. 2) we can take $N_0 = N_0(M)$ such that

$$n_{k+1}/Mn_k > (M+1)/M, \qquad \text{for } k \geqq N_0.$$

Hence $\sum_{k\geqq N_0} S_M(n_kt)$ is a lacunary trigonometric series and it is easily seen that if

$a_m \neq 0$ for some $m \leqq M$, then this series satisfies the conditions of the theorem of Weiss stated in §1. Therefore we obtain, for almost all t,

$$\overline{\lim_{N\to\infty}} \frac{1}{\sqrt{2N \log\log N}} \sum_{k=1}^{N} S_M(n_k t) = \| S_M(t) \|. \tag{4. 3}$$

From (4. 1), (4. 2) and (4. 3) we have, for almost all t,

$$\left| \overline{\lim_{N\to\infty}} \frac{1}{\sqrt{2N \log\log N}} \sum_{k=1}^{N} f(n_k t) - \| S_M(t) \| \right| \leqq 8 \| f(t) - S_M(t) \|. \tag{4. 4}$$

Since $\| f(t) - S_M(t) \| \to 0$ as $M \to +\infty$, (4. 4) proves the theorem.

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DEPARTMENT OF MATHEMATICS, KANAZAWA UNIVERSITY.