PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 9, Pages 2495–2499
S 0002-9939(05)07752-X
Article electronically published on March 31, 2005

THE LAX CONJECTURE IS TRUE

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(Communicated by Jonathan M. Borwein)

Abstract. In 1958 Lax conjectured that hyperbolic polynomials in three variables are determinants of linear combinations of three symmetric matrices. This conjecture is equivalent to a recent observation of Helton and Vinnikov.

Consider a polynomial p on $\mathbf{R}^n$ of degree d (the maximum of the degrees of the monomials in the expansion of p). We call p *homogeneous* if $p(tw) = t^d p(w)$ for all real t and vectors $w \in \mathbf{R}^n$: equivalently, every monomial in the expansion of p has degree d. We denote the set of such polynomials by $\mathbf{H}^n(d)$. By identifying a polynomial with its vector of coefficients, we can consider $\mathbf{H}^n(d)$ as a normed vector space of dimension $\binom{n+d-1}{d}$.

A polynomial $p \in \mathbf{H}^n(d)$ is *hyperbolic* with respect to a vector $e \in \mathbf{R}^n$ if $p(e) \neq 0$ and, for all vectors $w \in \mathbf{R}^n$, the univariate polynomial $t \mapsto p(w - te)$ has all real roots. The corresponding *hyperbolicity cone* is the open convex cone (see [5])

$$\{w \in \mathbf{R}^n : p(w - te) = 0 \ \Rightarrow\ t > 0\}.$$

For example, the polynomial $w_1 w_2 \cdots w_n$ is hyperbolic with respect to the vector $(1, 1, \ldots, 1)$, since the polynomial $t \mapsto (w_1 - t)(w_2 - t) \cdots (w_n - t)$ has roots $w_1, w_2, \ldots, w_n$; hence the corresponding hyperbolicity cone is the open positive orthant.

Hyperbolic polynomials and their hyperbolicity cones originally appeared in the partial differential equations literature [4]. They have attracted attention more recently as fundamental objects in modern convex optimization [6, 1]. Three primary reasons drive this interest:

(i) the definition of "hyperbolic polynomial" is strikingly simple;
(ii) the class of hyperbolic polynomials, although not well-understood, is known to be rich — specifically, its interior in $\mathbf{H}^n(d)$ is nonempty;
(iii) optimization problems posed over hyperbolicity cones, with linear objective and constraint functions, are amenable to efficient interior point algorithms.

For more details on these reasons, see [6, 1].

In light of the interest of hyperbolic polynomials to optimization theorists, it is therefore natural to ask: how general is the class of hyperbolicity cones? In particular, do hyperbolicity cones provide a more general model for convex optimization

Received by the editors April 2, 2003.
2000 *Mathematics Subject Classification.* Primary 15A45, 90C25, 52A41.
Key words and phrases. Hyperbolic polynomial, Lax conjecture, hyperbolicity cone, semidefinite representable.
The research of the first author was supported by NSERC.

than "semidefinite programming" (the study of optimization problems with linear objectives and constraints and semidefinite matrix variables [9])?

We begin with some easy observations. A rich source of examples of hyperbolicity cones are *semidefinite slices*, by which we mean sets of the form

$$\Big\{w : \sum_{j=1}^{n} w_j G_j \in \mathbf{S}^d_{++}\Big\}, \tag{1}$$

for matrices $G_1, G_2, \ldots, G_n$ in the space $\mathbf{S}^d$ of all d-by-d real symmetric matrices, where $\mathbf{S}^d_{++}$ denotes the positive definite cone. Such cones are, in particular, "semidefinite representable" in the sense of [9].

Proposition 2. *Any nonempty semidefinite slice is a hyperbolicity cone.*

Proof. Suppose the semidefinite slice (1) contains the vector $\widehat{w}$. We claim the polynomial p on $\mathbf{R}^n$ defined by

$$p(w) = \det \sum_j w_j G_j \tag{3}$$

is hyperbolic with respect to $\widehat{w}$, with corresponding hyperbolicity cone described by (1). Clearly p is homogeneous of degree d, and $p(\widehat{w}) > 0$.

Define a matrix $\widehat{G} = \sum_j \widehat{w}_j G_j \in \mathbf{S}^d_{++}$, and notice, for any vector $w \in \mathbf{R}^n$ and scalar t, we have

$$\begin{aligned} p(w - t\widehat{w}) = \det \sum_j (w_j - t\widehat{w}_j) G_j &= \det \Big(\sum_j w_j G_j - t\widehat{G} \Big) \\ &= (\det \widehat{G}) \det \Big(\widehat{G}^{-1/2} \Big[\sum_j w_j G_j \Big] \widehat{G}^{-1/2} - tI \Big), \end{aligned}$$

where I denotes the identity matrix. Consequently, the univariate polynomial $t \mapsto p(w - t\widehat{w})$ has all real roots, namely the eigenvalues of the symmetric matrix $H = \widehat{G}^{-1/2}[\sum_j w_j G_j]\widehat{G}^{-1/2}$, so p is hyperbolic with respect to $\widehat{w}$. Furthermore, by definition, w lies in the corresponding hyperbolicity cone exactly when these roots (or equivalently, eigenvalues) are all strictly positive. But this property is equivalent to H being positive definite, which holds if and only if $\sum_j w_j G_j$ is positive definite, as required. $\square$

The class of semidefinite slices is quite broad. For example, any *homogeneous cone* (an open convex pointed cone whose automorphism group acts transitively) is a semidefinite slice [2] (see also [3]). In particular, therefore, any homogeneous cone is a hyperbolicity cone, a result first observed in [6].

What about the converse? When is a hyperbolicity cone a semidefinite slice? How general is the class of hyperbolic polynomials of the form (3)?

In considering a general hyperbolic polynomial p on $\mathbf{R}^n$ with respect to a vector e, we can suppose, after a change of variables, that $e = (1, 0, 0, \ldots, 0)$ and $p(e) = 1$. Consider the first nontrivial case, that of $n = 2$. By assumption, the polynomial $t \mapsto p(-t, 1)$ has all real roots, which we denote $g_1, g_2, \ldots, g_d$, so for some nonzero real k we have the identity

$$p(-t, 1) = k \prod_{j=1}^{d} (g_j - t).$$

By homogeneity, for any vector $(x, y) \in \mathbf{R}^2$ with $y \neq 0$, we deduce

$$p(x,y) = y^d p\Big(\frac{x}{y}, 1\Big) = y^d k \prod_{j=1}^{d} \Big(g_j + \frac{x}{y}\Big) = k \prod_{j=1}^{d} (g_j y + x).$$

By continuity and the fact that $p(1, 0) = 1$, we see that

$$p(x,y) = \prod_{j=1}^{d} (g_j y + x) = \det(xI + yG)$$

for all $(x, y) \in \mathbf{R}^2$, where G is the diagonal matrix with diagonal entries $g_1, g_2, \ldots, g_d$. Thus any such hyperbolic polynomial p does indeed have the form (3).

What about hyperbolic polynomials in more than two variables? The following conjecture [8] proposes that all hyperbolic polynomials in three variables are likewise easily described in terms of determinants of symmetric matrices.

Conjecture 4 (Lax, 1958)**.** *A polynomial p on $\mathbf{R}^3$ is hyperbolic of degree d with respect to the vector $e = (1, 0, 0)$ and satisfies $p(e) = 1$ if and only if there exist matrices $B, C \in \mathbf{S}^d$ such that p is given by*

$$p(x, y, z) = \det(xI + yB + zC). \tag{5}$$

An obvious consequence of this conjecture would be that, in $\mathbf{R}^3$, hyperbolicity cones and semidefinite slices comprise identical classes.

A polynomial on $\mathbf{R}^2$ is a *real zero polynomial* [7] if, for all vectors $(y, z) \in \mathbf{R}^2$, the univariate polynomial $t \mapsto q(ty, tz)$ has all real roots. Such polynomials are closely related to hyperbolic polynomials via the following elementary result.

Proposition 6. *If p is a hyperbolic polynomial of degree d on $\mathbf{R}^3$ with respect to the vector $e = (1, 0, 0)$, and $p(e) = 1$, then the polynomial on $\mathbf{R}^2$ defined by $q(y, z) = p(1, y, z)$ is a real zero polynomial of degree no more than d, and satisfying $q(0, 0) = 1$.*

Conversely, if q is a real zero polynomial of degree d on $\mathbf{R}^2$ satisfying $q(0, 0) = 1$, then the polynomial on $\mathbf{R}^3$ defined by

$$p(x, y, z) = x^d q\Big(\frac{y}{x}, \frac{z}{x}\Big) \ (x \neq 0) \tag{7}$$

(extended to $\mathbf{R}^3$ by continuity) is a hyperbolic polynomial of degree d on $\mathbf{R}^3$ with respect to e, and $p(e) = 1$.

Proof. To prove the first statement, note that for any point $(y, z) \in \mathbf{R}^2$ and complex μ, if $q(\mu(y, z)) = 0$, then $\mu \neq 0$ and $0 = p(1, \mu y, \mu z) = \mu^d p(\mu^{-1}, y, z)$, using the homogeneity of p. So, by the hyperbolic property, $-\mu^{-1}$ is real, and hence so is μ. The remaining claims are clear.

For the converse direction, since q has degree d, clearly p is well-defined and homogeneous of degree d and satisfies $p(e) = 1$. If $p(\mu, y, z) = 0$, then either $\mu = 0$ or $q(\mu^{-1}(y, z)) = 0$, in which case μ^{-1} and hence also μ must be real. □

(Notice, in the first claim of the proposition, that the polynomial q may have degree strictly less than d: consider, for example, the case $p(x, y, z) = x^d$.)

Helton and Vinnikov [7, p. 10] observe the following result, based heavily on [10].

Theorem 8. *A polynomial q on $\mathbf{R}^2$ is a real zero polynomial of degree d and satisfies $q(0,0) = 1$ if and only if there exist matrices $B, C \in \mathbf{S}^d$ such that q is given by*

$$q(y, z) = \det(I + yB + zC). \tag{9}$$

(Notice, as in the Lax conjecture, that the "if" direction is immediate.)

We claim that Theorem 8 is equivalent to the Lax conjecture. To see this, suppose p is a hyperbolic polynomial of degree d on $\mathbf{R}^3$ with respect to the vector $e = (1, 0, 0)$, and $p(e) = 1$. Then by Proposition 6, the polynomial on $\mathbf{R}^2$ defined by $q(y, z) = p(1, y, z)$ is a real zero polynomial of degree $d' \le d$, and satisfying $q(0, 0) = 1$. Hence by Theorem 8, equation (9) holds: we can assume $d' = d$ by replacing $B, C \in \mathbf{S}^{d'}$ with block diagonal matrices $\mathrm{Diag}(B, 0), \mathrm{Diag}(C, 0) \in \mathbf{S}^d$. Then, by homogeneity, for $x \neq 0$,

$$\begin{aligned} p(x, y, z) &= x^d p\Big(1, \frac{y}{x}, \frac{z}{x}\Big) = x^d q\Big(\frac{y}{x}, \frac{z}{x}\Big) \\ &= x^d \det\Big(I + \frac{y}{x}B + \frac{z}{x}C\Big) = \det(xI + yB + zC), \end{aligned}$$

as required. The converse direction in the Lax conjecture is immediate.

Conversely, let us assume the Lax conjecture, and suppose q is a real zero polynomial of degree d on $\mathbf{R}^2$ satisfying $q(0, 0) = 1$. (The converse direction in Theorem 8 is immediate.) Then by Proposition 6 the polynomial p defined by equation (7) is a hyperbolic polynomial of degree d on $\mathbf{R}^3$ with respect to e, and $p(e) = 1$. According to the Lax conjecture, equation (5) holds, so

$$q(y, z) = p(1, y, z) = \det(I + yB + zC),$$

as required. □

The exact analogue of the Lax conjecture fails in general for polynomials in $n > 3$ variables. To see this, note that the set of polynomials on $\mathbf{R}^n$ of the form $w \mapsto \det \sum_j w_j G_j$ (where $G_1, G_2, \ldots, G_n \in \mathbf{S}^d$) has dimension at most $n \cdot \binom{d+1}{2}$, being an algebraic image of a vector space of this dimension. If the degree d is large, this dimension is certainly smaller than the dimension of the set of hyperbolic polynomials: as we observed above, this latter set has nonempty interior in the space $\mathbf{H}^n(d)$ (by a result of Nuij [6, Thm. 2.1]), and so has dimension $\binom{n+d-1}{d}$.

More concretely, consider the polynomial defined by $p(w) = w_1^2 - \sum_2^n w_j^2$ for $w \in \mathbf{R}^n$. This polynomial is hyperbolic of degree $d = 2$ with respect to the vector $(1, 0, 0, \ldots, 0)$, and yet cannot be written in the form $\det \sum_j w_j G_j$ for matrices $G_1, G_2, \ldots, G_n \in \mathbf{S}^2$ if $n > 3$. To see this, choose any nonzero vector w satisfying $w_1 = 0$, and such that the first row of the matrix $\sum_j w_j G_j$ is zero.

The question of whether all hyperbolicity cones are semidefinite slices, or, more generally, are semidefinite representable, appears open.

Acknowledgement

We are very grateful to the Institute for Mathematics and its Applications at the University of Minnesota for their hospitality during our work on this topic.

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