THE LEE-YANG AND PÓLYA-SCHUR PROGRAMS. II. THEORY OF STABLE POLYNOMIALS AND APPLICATIONS

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ABSTRACT. In the first part of this series we characterized all linear operators on spaces of multivariate polynomials preserving the property of being non-vanishing in products of open circular domains. For such sets this completes the multivariate generalization of the classification program initiated by Pólya-Schur for univariate real polynomials. We build on these classification theorems to develop here a theory of multivariate stable polynomials. Applications and examples show that this theory provides a natural framework for dealing in a uniform way with Lee-Yang type problems in statistical mechanics, combinatorics, and geometric function theory in one or several variables. In particular, we answer a question of Hinkkanen on multivariate apolarity.

CONTENTS

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Introduction

In two seminal papers from 1952 [30, 61] Lee and Yang proposed the program of analyzing phase transitions in terms of zeros of the partition function and proved a celebrated theorem locating the zeros of the partition function of the ferromagnetic Ising model on the imaginary axis in the complex magnetic plane. This theorem has since been proved and generalized in many ways by e.g. Asano, Fisher, Newman, Ruelle, Lieb-Sokal, Biskup *et al*, etc; see §8 and references therein. Nevertheless, the Lee-Yang theorem seems to have retained an aura of mystique. In his 1988 Gibbs lecture [48] Ruelle proclaimed: "I have called this beautiful result a failure because, while it has important applications in physics, it remains at this time isolated in mathematics." Ruelle's statement was apparently motivated by the fact that the Lee-Yang theorem also inspired speculations about possible statistical mechanics models underlying the zeros of Riemann or Selberg zeta functions and the Weil conjectures [25, 36, 48] but "the miracle has not happened" [48].

Recently, Lee-Yang like problems and techniques have appeared in various mathematical contexts such as combinatorics, complex analysis, matrix theory and probability theory [1, 6, 7, 8, 9, 10, 13, 14, 16, 21, 24, 25, 47, 56, 59]. The past decade has also been marked by important developments on other aspects of phase transitions, conformal invariance, percolation theory [27, 29, 55]. However, as Hinkkanen noted in [25], the power in the ideas behind the Lee-Yang theorem has not yet been fully exploited: "It seems that the theory of polynomials, linear in each variable, that do not have zeros in a given multidisk or a more general set, has a long way to go, and has so far unnoticed connections to various other concepts in mathematics."

In this paper we show that the Lee-Yang theorem and the mathematics around it are intimately connected with the dynamics of zero loci of multivariate polynomials under linear transformations and Problems 1–2 below. As we point out in §8, such connections have been implicitly noted in essentially all known proofs and extensions of the Lee-Yang theorem. For instance, Lieb and Sokal [32] reduced the at the time best Lee-Yang theorem, due to Newman [38], to the following statement: if $P, Q \in \mathbb{C}[z_1, \dots, z_n]$ are polynomials which are non-vanishing when all variables are in the open right half-plane, then the polynomial $P(\partial/\partial z_1, \dots, \partial/\partial z_n)Q(z_1, \dots, z_n)$ also has this property unless it is identically zero (Theorem 8.3). Thus, to better understand Lee-Yang type theorems one is naturally led to consider the problems of describing linear operators on polynomial spaces that preserve the property of being non-vanishing when the variables are in prescribed subsets of $\mathbb{C}^n$.

Let us formulate these problems explicitly as in [5]. Given an integer $n \geq 1$ and $\Omega \subset \mathbb{C}^n$ we say that $f \in \mathbb{C}[z_1, \dots, z_n]$ is Ω-*stable* if $f(z_1, \dots, z_n) \neq 0$ whenever $(z_1, \dots, z_n) \in \Omega$. A $\mathbb{K}$-linear operator $T : V \to \mathbb{K}[z_1, \dots, z_n]$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and V is a subspace of $\mathbb{K}[z_1, \dots, z_n]$, is said to *preserve* Ω-*stability* if for any Ω-stable polynomial $f \in V$ the polynomial $T(f)$ is either Ω-stable or $T(f) \equiv 0$. For $\kappa = (\kappa_1, \dots, \kappa_n) \in \mathbb{N}^n$ let $\mathbb{K}_\kappa[z_1, \dots, z_n] = \{f \in \mathbb{K}[z_1, \dots, z_n] : \deg_{z_i}(f) \leq \kappa_i, 1 \leq i \leq n\}$, where $\deg_{z_i}(f)$ is the degree of f in z_i. By slight abuse of terminology, if $\Psi \subset \mathbb{C}$ and $\Omega = \Psi^n$ then Ω-stable polynomials will also be referred to as Ψ-stable.

Problem 1. *Characterize all linear operators* $T : \mathbb{K}_\kappa[z_1, \dots, z_n] \to \mathbb{K}[z_1, \dots, z_n]$ *that preserve* Ω*-stability for a given set* $\Omega \subset \mathbb{C}^n$ *and* $\kappa \in \mathbb{N}^n$.

Problem 2. *Characterize all linear operators* $T : \mathbb{K}[z_1, \dots, z_n] \to \mathbb{K}[z_1, \dots, z_n]$ *that preserve* Ω*-stability, where* Ω *are prescribed subsets of* $\mathbb{C}^n$.

In physics [54, 56] it is useful to distinguish between hard-core pair interactions (subject to constraints, e.g. the maximum degree of a graph) and soft-core pair interactions (essentially constraint-free). By analogy with this dichotomy, one may say that results pertaining to Problem 1 are "hard" or "algebraic" (bounded degree) while those for Problem 2 are "soft" or "transcendental" (unbounded degree), cf. [5].

For $n = 1$, $\mathbb{K} = \mathbb{R}$, and $\Omega = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ Problems 1–2 amount to classifying linear operators that preserve the set of real polynomials with all real zeros. This question has a distinguished history that goes back to Hermite, Laguerre, Hurwitz and Pólya-Schur, see [9] and references therein. In particular, in [41] Pólya and Schur characterized all diagonal operators with this property, which led to a rich subsequent literature on this subject [6, 8, 17, 18, 19, 20, 21, 26, 31, 35, 40, 43, 52]. However, it was not until very recently that full solutions to this question – and, more generally, to Problems 1–2 for $n = 1$ and any open circular domain Ω – were obtained in [6]. Quite recently, Problems 1–2 were solved in [5] whenever $\Omega = \Omega_1 \times \cdots \times \Omega_n$ and the Ω_i's are open circular domains. For such sets these results complete the multivariate generalization of the classification program initiated by Pólya-Schur [41]. They also go beyond e.g. [6, 8] and have interesting consequences, as we will now see.

In Part A we build on the classification theorems of [5] to develop a self-contained theory of multivariate stable polynomials. To begin with, in §1 we study operators on multi-affine polynomials inspired by natural time evolutions (symmetric exclusion processes) for interacting particle systems [34]. We give a new simple proof of [10, Theorem 4.20] (see also [33]) stating that these operators preserve stability and extend it to all circular domains. In §2 we use these symmetrization procedures to give a new proof of the Grace-Walsh-Szegö coincidence theorem that unlike most proofs known so far avoids (univariate) apolarity theory. In §3 we establish a "master composition theorem" that extends to several variables all the classical Hadamard-Schur convolution results due to Schur-Maló-Szegö, Walsh, Cohn-Egerváry-Szegö, de Bruijn, etc [17, 35, 43]. This also generalizes the multivariate composition theorems based on the Weyl product [8] as well as Hinkkanen's theorem [25] and provides a unifying framework for results of this type. In §4 we obtain "hard" multivariate generalizations of Pólya-Schur's classification of multiplier sequences that extend the "soft" theorems of [8].

As noted in [44], the concept of apolarity has a rich pedigree going all the way back to Apollonius and was much studied in invariant theory, umbral calculus, and algebraic geometry [28, 53]. In [44] Rota adds: "Grace's [apolarity] theorem is an instance of what might be called a sturdy theorem. For almost one hundred years it has resisted all attempts at generalization. Almost all known results about the distribution of zeros of [univariate] polynomials in the complex plane are corollaries of Grace's theorem." In §5 we establish Grace type theorems for multivariate polynomials and provide an answer to a question of Hinkkanen [25, §5]. In §6 we prove "hard" Lieb-Sokal lemmas that sharpen the "soft" ones in [32] (whose importance in the Lee-Yang program is explained in §8.1.)

In Part B we study statistical mechanical and combinatorial applications of the theory of stable polynomials developed in Part A and [5]. We show that the key steps in existing proofs and extensions of the Lee-Yang and Heilmann-Lieb theorems as well as various other theorems on graph polynomials follow in a simple and unified manner from this theory. These results are due to Asano [2], Ruelle [45, 46, 51], Newman [37, 38], Lieb-Sokal [32], Hinkkanen [25], Choe et al [16], Wagner [59].

A. Theory of Multivariate C-Stable Polynomials

Let us first fix some of the notation that we will use throughout. Recall that the *support* of a polynomial $f(z) = \sum_{\alpha\in\mathbb{N}^n} a(\alpha)z^\alpha \in \mathbb{C}[z_1,\dots,z_n]$ is the set $\mathrm{supp}(f) = \{\alpha \in \mathbb{N}^n : a(\alpha) \neq 0\}$, where $z = (z_1,\dots,z_n) \in \mathbb{C}^n$, $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$, and $z^\alpha = z_1^{\alpha_1}\cdots z_n^{\alpha_n}$. Set $[n] = \{1,\dots,n\}$ and $(1^n) = (1,\dots,1) \in \mathbb{N}^n$. We say that f is of degree at most $\kappa = (\kappa_1,\dots,\kappa_n) \in \mathbb{N}^n$ if $f \in \mathbb{C}_\kappa[z_1,\dots,z_n]$ (cf. the introduction) and of degree κ if $\deg_{z_i}(f) = \kappa_i$, $i \in [n]$. Polynomials in $\mathbb{C}_{(1^n)}[z_1,\dots,z_n]$ are called *multi-affine*.

We employ the usual partial order on $\mathbb{N}^n$: if $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$ and $\beta = (\beta_1,\dots,\beta_n) \in \mathbb{N}^n$ then $\alpha \le \beta$ if $\alpha_i \le \beta_i$ for all $i \in [n]$. Let $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $\alpha! = \alpha_1!\cdots\alpha_n!$, and

$$\binom{\beta}{\alpha} = \begin{cases} \frac{\beta!}{\alpha!(\beta-\alpha)!} \text{ if } \alpha \le \beta, \\ 0 \text{ otherwise.} \end{cases}$$

The open unit disk is denoted by $\mathbb{D}$ and open half-planes bordering on the origin by $\mathbb{H}_\theta = \{z \in \mathbb{C} : \mathrm{Im}(e^{i\theta}z) > 0\}$, where $\theta \in [0, 2\pi)$. Note that $\mathbb{H}_0$ is the open upper half-plane while $\mathbb{H}_{\frac{\pi}{2}}$ is the open right half-plane. $\mathbb{H}_0$-stable polynomials are referred to as *stable* polynomials and those with all real coefficients are called *real stable*, cf. [5]–[10]. We denote the sets of stable, respectively real stable polynomials in n variables by $\mathcal{H}_n(\mathbb{C})$, respectively $\mathcal{H}_n(\mathbb{R})$. Polynomials in $\mathbb{C}[z_1,\dots,z_n]$ which are $\mathbb{H}_{\frac{\pi}{2}}$-stable are said to be *weakly Hurwitz stable.* In [16] these are termed polynomials with the *half-plane property.* The notions of $\mathbb{H}_\theta$-stability are equivalent modulo rotations for complex polynomials but this is not so for real polynomials. However, for real polynomials with non-negative coefficients [10, Theorem 4.5] yields the following hierarchy of half-plane properties: if such a polynomial is $\mathbb{H}_0$-stable then it is $\mathbb{H}_\theta$-stable for any $\theta \in [0, \pi]$.

1. Symmetrization Procedures

The symmetric group on n elements, $\mathfrak{S}_n$, acts on $\mathbb{C}[z_1,\dots,z_n]$ by permuting the variables: $\sigma(f)(z_1,\dots,z_n) = f(z_{\sigma(1)},\dots,z_{\sigma(n)})$, $\sigma \in \mathfrak{S}_n$, $f \in \mathbb{C}[z_1,\dots,z_n]$. Define the *symmetrization operator* $\mathrm{Sym} : \mathbb{C}[z_1,\dots,z_n] \to \mathbb{C}[z_1,\dots,z_n]$ by

$$\mathrm{Sym}(f) = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} \sigma(f).$$

Clearly, Sym is a linear operator whose image consists of *symmetric polynomials*, that is, polynomials invariant under the action of $\mathfrak{S}_n$.

Recall that a *circular domain* in $\mathbb{C}$ is any open or closed disk, exterior of a disk, or half-plane. The Grace-Walsh-Szegö coincidence theorem is an important and very useful result on the geometry of polynomials, see, e.g., [16, 22, 35, 43, 60].

Theorem 1.1 (Grace-Walsh-Szegö)**.** *Let $f \in \mathbb{C}_{(1^n)}[z_1,\dots,z_n]$ be a symmetric polynomial, C be an open or closed circular domain, and $\xi_1,\dots,\xi_n \in C$. Suppose further that either the total degree of f equals n or that C is convex (or both). Then there exists at least one point $\xi \in C$ such that*

$$f(\xi_1,\dots,\xi_n) = f(\xi,\dots,\xi). \tag{1.1}$$

This theorem was essential for proving the sufficiency part of the characterization of linear operators preserving C-stability in [5]. We will see here that Theorem 1.1 is actually a consequence of stronger (asymmetric) symmetrization procedures on stable polynomials which were used in [10] to prove correlation inequalities for symmetric exclusion processes. More precisely, we will deduce Theorem 1.1 from the following result.

Theorem 1.2. *Let C be an open or closed circular domain.*

(a) *If C is convex then the symmetrization operator* Sym *preserves C-stability on multi-affine polynomials, i.e.,* $\mathrm{Sym} : \mathbb{C}_{(1^n)}[z_1, \ldots, z_n] \to \mathbb{C}_{(1^n)}[z_1, \ldots, z_n]$ *preserves C-stability.*
(b) *If C is non-convex and $f \in \mathbb{C}_{(1^n)}[z_1, \ldots, z_n]$ is C-stable and such that all variables are active in f (i.e., $\partial f/\partial z_i \neq 0$, $i \in [n]$) then* $\mathrm{Sym}(f)$ *is C-stable.*

Remark 1.1. If C is non-convex and $f \in \mathbb{C}_{(1^n)}[z_1, \ldots, z_n]$ is C-stable then the condition in Theorem 1.2 (b) that all variables are active is actually equivalent to the requirement that f has total degree n, i.e., $\partial^n f/\partial z_1 \cdots \partial z_n \neq 0$.

Remark 1.2. One can easily construct examples showing that Sym does not preserve C-stability when acting on arbitrary (not multi-affine) polynomials.

It is not difficult to prove Theorem 1.2 *assuming* the Grace-Walsh-Szegö theorem. However, in §2 we will prove the latter *via* Theorem 1.2. This will make the theory developed here and in [5] self-contained.

We will derive Theorem 1.2 from the next result which was first proved in [10, Theorem 4.20]. From a physical viewpoint [34], Proposition 1.3 implies that stability is preserved by the natural time evolution of symmetric exclusion processes.

Proposition 1.3. *Let $f \in \mathbb{C}_{(1^n)}[z_1, \ldots, z_n]$, C be an open or closed circular domain, $0 \leq p \leq 1$, and $\tau = (ij) \in \mathfrak{S}_n$ be a transposition.*

(a) *If C is convex and f is C-stable then so is $pf + (1-p)\tau(f)$.*
(b) *If C is non-convex and f is C-stable and depending on both z_i and z_j then $pf + (1-p)\tau(f)$ is also C-stable.*

Our proof of Proposition 1.3 relies on the maximum principle for harmonic functions which we use to prove the following lemma. Another recent elementary proof of Proposition 1.3 was independently given in [33]. Let $\overline{\mathbb{H}}_0$ be the closed upper half-plane.

Lemma 1.4. *Let $f(z, w) = a + bz + cw + dzw \in \mathbb{C}[z, w]$ and define*

$$V_1(f)(x) = \mathrm{Im}(a\bar{c}) + \mathrm{Im}(a\bar{d} + b\bar{c})x + \mathrm{Im}(b\bar{d})x^2,$$
$$V_2(f)(x) = \mathrm{Im}(a\bar{b}) + \mathrm{Im}(a\bar{d} + c\bar{b})x + \mathrm{Im}(c\bar{d})x^2.$$

Suppose that $d \neq 0$.

(1) *If f is $\overline{\mathbb{H}}_0$-stable then* $\mathrm{Im}(b/d) > 0$ *or* $\mathrm{Im}(c/d) > 0$.
(2) *If f is $\overline{\mathbb{H}}_0$-stable and* $\mathrm{Im}(c/d) > 0$ *then $V_1(f)(x) > 0$ and $V_2(f)(x) \geq 0$ for all $x \in \mathbb{R}$.*
(3) *If f is $\overline{\mathbb{H}}_0$-stable and* $\mathrm{Im}(b/d) > 0$ *then $V_2(f)(x) > 0$ and $V_1(f)(x) \geq 0$ for all $x \in \mathbb{R}$.*
(4) *If* $\mathrm{Im}(b/d) > 0$ *and* $\mathrm{Im}(c/d) > 0$ *then f is $\overline{\mathbb{H}}_0$-stable if and only if for some (and then any) $i \in \{1, 2\}$ one has $V_i(f)(x) > 0$ for all $x \in \mathbb{R}$.*

Proof. Since the partial derivative of a $\mathbb{H}_0$-stable polynomial is $\mathbb{H}_0$-stable or identically zero (if in doubt apply Theorem 3.1) we have $\mathrm{Im}(c/d) \geq 0$ and $\mathrm{Im}(b/d) \geq 0$ if f is $\overline{\mathbb{H}}_0$-stable. If $\mathrm{Im}(c/d) = \mathrm{Im}(b/d) = 0$ then the polynomial $d^{-1}f(z-c/d, w-b/d) = wz + (ad-bc)/d^2$ is $\overline{\mathbb{H}}_0$-stable. This is a contradiction since $\{wz : w, z \in \overline{\mathbb{H}}_0\} = \mathbb{C}$. Hence $\mathrm{Im}(c/d) > 0$ or $\mathrm{Im}(b/d) > 0$ if f is $\overline{\mathbb{H}}_0$-stable.

Assume that $\mathrm{Im}(b/d) > 0$ and $\mathrm{Im}(c/d) > 0$. Solving for w in $f(z, w) = 0$ we see that f is $\overline{\mathbb{H}}_0$-stable if and only if

$$\mathrm{Im}(z) \geq 0 \quad \Longrightarrow \quad \rho(z) := \mathrm{Im}\left(\frac{a+bz}{c+dz}\right) > 0. \tag{1.2}$$

Now, ρ is a harmonic function in the half-plane $\{z \in \mathbb{C} : \mathrm{Im}(z) > -\mathrm{Im}(c/d)\}$ which contains $\overline{\mathbb{H}}_0$. Let $K_r = \{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0, |z| \leq r\}$. By the maximum principle the minimum of ρ on K_r is attained on the boundary of K_r. For real x we have

$$\rho(x) = \frac{\mathrm{Im}(a\bar{c}) + \mathrm{Im}(a\bar{d} + b\bar{c})x + \mathrm{Im}(b\bar{d})x^2}{|c+dx|^2}.$$

Moreover, $\rho(z) \to \mathrm{Im}(b/d) > 0$ as $z \to \infty$. Since the same arguments apply if one instead solves for z in $f(z, w) = 0$, this verifies (4).

If just $\mathrm{Im}(c/d) > 0$ we will still have $\rho(x) > 0$ for all $x \in \mathbb{R}$ and if only $\mathrm{Im}(b/d) > 0$ then $\rho(x) > 0$ for all $x \in \mathbb{R} \setminus \{-c/d\}$. By symmetry in z and w this verifies (2) and (3) of the lemma. □

Recall the multivariate version of Hurwitz' theorem on the "continuity of zeros", cf. [16, Footnote 3, p. 96].

Theorem 1.5 (Hurwitz' theorem)**.** *Let D be a domain (open connected set) in $\mathbb{C}^n$ and suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of non-vanishing analytic functions on D that converge to f uniformly on compact subsets of D. Then f is either non-vanishing on D or else identically zero.*

To deal with discs and exteriors of discs we also need Lemmas 1.6 and 1.8 below – which were proved in [5, Lemmas 6.1 and 6.2] – and Corollary 1.7.

Lemma 1.6. *Let $\{C_i\}_{i=1}^n$ be a family of circular domains, $f \in \mathbb{C}[z_1, \ldots, z_n]$ be of degree $\kappa \in \mathbb{N}^n$, and $J \subseteq [n]$ a (possibly empty) set such that C_j is the exterior of a disk whenever $j \in J$. Denote by g be the polynomial in the variables z_j, $j \in J$, obtained by setting $z_i = c_i \in C_i$ arbitrarily for $i \notin J$. If f is $C_1 \times \cdots \times C_n$-stable then* $\mathrm{supp}(g)$ *has a unique maximal element γ with respect to the standard partial order on $\mathbb{N}^J$. Moreover, γ is the same for all choices of $c_i \in C_i$, $i \notin J$.*

An immediate consequence of Lemma 1.6 is the following.

Corollary 1.7. *Let $\kappa \in \mathbb{N}^n$ and $I_\kappa : \mathbb{C}_\kappa[z_1, \ldots, z_n] \to \mathbb{C}_\kappa[z_1, \ldots, z_n]$ be the linear operator defined by $I_\kappa(z^\alpha) = z^{\kappa-\alpha}$, $\alpha \leq \kappa$. Then I_κ restricts to a bijection between the set of $\mathbb{D}$-stable ($\overline{\mathbb{D}}$-stable) polynomials in $\mathbb{C}_\kappa[z_1, \ldots, z_n]$ and the set of $\mathbb{C} \setminus \overline{\mathbb{D}}$-stable ($\mathbb{C} \setminus \mathbb{D}$-stable) polynomials in $\mathbb{C}_\kappa[z_1, \ldots, z_n]$ of degree κ.*

If $\{C_i\}_{i=1}^n$ is a family of circular domains and $\kappa = (\kappa_1, \ldots, \kappa_n) \in \mathbb{N}^n$ we let $\mathcal{N}_\kappa(C_1, \ldots, C_n)$ be the set of $C_1 \times \cdots \times C_n$-stable polynomials in $\mathbb{C}_\kappa[z_1, \ldots, z_n]$ that have degree κ_j in z_j whenever C_j is non-convex. Note that if all C_j are convex then $\mathcal{N}_\kappa(C_1, \ldots, C_n)$ consists of all $C_1 \times \cdots \times C_n$-stable polynomials in $\mathbb{C}_\kappa[z_1, \ldots, z_n]$.

Recall that a *Möbius transformation* is a bijective conformal map of the extended complex plane $\widehat{\mathbb{C}}$ given by

$$\phi(\zeta) = \frac{a\zeta + b}{c\zeta + d}, \quad a, b, c, d \in \mathbb{C}, ad - bc = 1. \tag{1.3}$$

Note that one usually has the weaker requirement $ad - bc \neq 0$ but this is equivalent to (1.3) which proves to be more convenient.

Lemma 1.8. *Suppose that $C_1, \dots, C_n, D_1, \dots, D_n$ are open circular domains and $\kappa = (\kappa_1, \dots, \kappa_n) \in \mathbb{N}^n$. Then there are Möbius transformations*

$$\zeta \mapsto \phi_i(\zeta) = \frac{a_i\zeta + b_i}{c_i\zeta + d_i}, \quad i \in [n],$$

as in (1.3) *such that the (invertible) linear transformation $\Phi_\kappa : \mathbb{C}_\kappa[z_1, \dots, z_n] \to \mathbb{C}_\kappa[z_1, \dots, z_n]$ defined by*

$$\Phi_\kappa(f)(z_1, \dots, z_n) = (c_1z_1 + d_1)^{\kappa_1} \cdots (c_nz_n + d_n)^{\kappa_n} f(\phi_1(z_1), \dots, \phi_n(z_n))$$

restricts to a bijection between $\mathcal{N}_\kappa(C_1, \dots, C_n)$ and $\mathcal{N}_\kappa(D_1, \dots, D_n)$.

Proof of Proposition 1.3. Clearly, it is enough to prove the proposition for closed circular domains. Suppose first that C is the closed upper half-plane $\overline{\mathbb{H}}_0$ and let $f \in \mathbb{C}_{(1^n)}[z_1, \dots, z_n]$ be $\overline{\mathbb{H}}_0$-stable. Assuming, as we may, that $i = 1$ and $j = 2$, we need to prove that

$$pf(\xi_1, \xi_2, \dots, \xi_n) + (1 - p)f(\xi_2, \xi_1, \dots, \xi_n) \neq 0$$

whenever $\xi_1, \dots, \xi_n \in \overline{\mathbb{H}}_0$. By fixing $\xi_3, \dots, \xi_n$ arbitrarily in $\overline{\mathbb{H}}_0$ and considering the multi-affine polynomial in variables z_1, z_2 given by

$$(z_1, z_2) \mapsto g(z_1, z_2) := f(z_1, z_2, \xi_3, \dots, \xi_n)$$

we see that the problem reduces to proving that for any $p \in (0, 1)$ the polynomial $pf(z, w) + (1 - p)f(w, z)$ is $\overline{\mathbb{H}}_0$-stable provided that $f(z, w)$ is $\overline{\mathbb{H}}_0$-stable. This is easy to check if $d = 0$ so we may assume that $d \neq 0$. Now, if $\{i, j\} = \{1, 2\}$ then

$$V_i\Big(pf(z, w) + (1 - p)f(w, z)\Big) = pV_i\Big(f(z, w)\Big) + (1 - p)V_j\Big(f(z, w)\Big)$$

which proves the proposition for $C = \overline{\mathbb{H}}_0$ by Lemma 1.4 (since $p \in (0, 1)$ implies that we will be in case (4) of Lemma 1.4).

Let C be a closed disk and suppose that f is C-stable. Then by compactness f is $\tilde{C}$-stable for some open disk $\tilde{C} \supset C$. The result now follows by applying Lemma 1.8 (with $\kappa = (1^n)$, $C_\ell = \tilde{C}$, $D_\ell = D$, $\ell \in [n]$, where D is an arbitrarily fixed open half-plane) and using the fact the partial symmetrization operator commutes with the operator Φ_κ defined in Lemma 1.8.

The case of the closed exterior of a disk follows from the disk case considered above and Corollary 1.7 for $\kappa = (1^n)$ (cf. Remark 1.1). □

In the theory of interacting particle systems [34] it is well known that the symmetrization of a polynomial f can be achieved by applying $f \mapsto (f + \tau(f))/2$ infinitely many times with different transpositions τ. For the sake of completeness, we will give a proof of this fact in the Appendix.

Proof of Theorem 1.2. Consider first the case when C is an open circular domain and recall Remark 1.1. Since $\mathrm{Sym}(f)$ is obtained by applying $f \mapsto (f + \tau(f))/2$ infinitely many times with different transpositions τ (see Lemma 9.2 below) the result follows from Hurwitz' Theorem 1.5 and Proposition 1.3.

If C is closed write

$$f(z, \dots, z) = B \prod_{j=1}^{d} (z - c_j),$$

where $c_j \notin C$ for $j \in [d]$ and $B \neq 0$. Clearly, the polynomial $F(z_1, \dots, z_n)$ defined by

$$F(z_1, \dots, z_n) = B \prod_{j=1}^{d} (z_j - c_j)$$

is D-stable, where D is a suitable open circular domain containing C but none of the c_j's. Then by the above $\mathrm{Sym}(F)$ is D-stable and since $\mathrm{Sym}(F) = \mathrm{Sym}(f)$ the theorem follows. □

2. The Grace-Walsh-Szegö Coincidence Theorem

Using Theorem 1.2 we can give a new proof of the Grace-Walsh-Szegö coincidence theorem that does not rely on apolarity theory as do most known proofs so far, see [22, 35, 43, 57, 60] and §5. We actually prove a more general version of this result which holds for families of circular domains.

Theorem 2.1. *Suppose $f(z_{11}, \dots, z_{1\kappa_1}, \dots, z_{n1}, \dots, z_{n\kappa_n})$ is a multi-affine polynomial in $|\kappa|$ complex variables which is symmetric in $\{z_{ij} : j \in [\kappa_i]\}$ for all $i \in [n]$, where $\kappa = (\kappa_1, \dots, \kappa_n) \in \mathbb{N}^n$ with $\kappa_i \geq 1$, $i \in [n]$. Let further C_i, $i \in [n]$, be circular domains and $\xi_{ij} \in C_i$, $j \in [\kappa_i]$, $i \in [n]$. Then there exist $\xi_i \in C_i$, $i \in [n]$, such that*

$$f(\xi_{11}, \dots, \xi_{1\kappa_1}, \dots, \xi_{n1}, \dots, \xi_{n\kappa_n}) = f(\xi_1, \dots, \xi_1, \dots, \xi_n \dots, \xi_n)$$

provided that f has total degree κ_i in $\{z_{ij} : j \in [\kappa_i]\}$ whenever C_i is non-convex.

Proof. Clearly, it is enough to prove that if the polynomial in n variables

$$g(z_1, \dots, z_n) := f(z_1, \dots, z_1, \dots, z_n \dots, z_n)$$

is $C_1 \times \cdots \times C_n$-stable then $f(z_{11}, \dots, z_{1\kappa_1}, \dots, z_{n1}, \dots, z_{n\kappa_n})$ is $C_1^{\kappa_1} \times \cdots \times C_n^{\kappa_n}$-stable, which we will now do by considering one variable at a time. By assumption g has degree κ_i in the variable z_i whenever C_i is non-convex (symmetry prevents cancellation). Fix $\zeta_j \in C_j$, $j \in [n-1]$. The polynomial $h(z_n) := g(\zeta_1, \dots, \zeta_{n-1}, z_n)$ is C_n-stable and we may therefore write

$$h(z_n) = B \prod_{j=1}^{d} (z_n - \alpha_j),$$

where $B \neq 0$ and $\alpha_j \notin C_i$ for $j \in [d]$, so the polynomial

$$H(w_1, \dots, w_{\kappa_n}) := B \prod_{j=1}^{d} (w_j - \alpha_j)$$

is also C_n-stable. Now, if C_n is non-convex then by Lemma 1.6 one has $d = \kappa_n$ and by Theorem 1.2 the symmetrization operator Sym acting on κ_n variables maps $\mathbb{H}_0$ to a C_n-stable polynomial. Since the numbers ζ_i, $i \in [n-1]$, were arbitrary this means that the polarization of g that splits the variable z_n symmetrically into κ_n new variables, i.e., the linear operator

$$g(z_1,\dots,z_n)\mapsto f(z_1,\dots,z_1,\dots,z_{n-1},\dots,z_{n-1},z_{n1},\dots,z_{n\kappa_n})$$

preserves the stability in question. By polarizing one variable at a time we conclude that $f(z_{11},\dots,z_{1\kappa_1},\dots,z_{n1},\dots,z_{n\kappa_n})$ is $C_1^{\kappa_1}\times\cdots\times C_n^{\kappa_n}$-stable. □

Remark 2.1. In [47] Ruelle produced a proof of the Grace-Walsh-Szegö coincidence theorem (Theorem 1.1) using similar ideas.

Remark 2.2. A yet more general version of Theorem 2.1 was actually given by Walsh in [60, Theorem 1] without assuming any degree conditions, the only requirement in [60] being that C_i, $i\in[n]$, should be just (closed) circular domains. However, in such generality Walsh's aforementioned result fails already for $n=1$.

3. Master Composition Theorems

Composition (or convolution) theorems such as the Schur-Maló-Szegö theorems ([17, Theorem 2.4], [43, Theorem 3.4.1d]), the Cohn-Egerváry-Szegö theorem ([43, Theorem 3.4.1d]), Walsh's theorems ([43, Theorems 3.4.2c and 5.3.1]) or de Bruijn's theorems [11, 12] play an important role in the analytic theory of univariate complex polynomials and allow to locate their zeros in certain circular domains [35, 43].

Using results of [5] we establish "master composition theorems" that provide a unifying framework for *multivariate* generalizations of the classical theorems mentioned above. Let us first recall two of the classification theorems from [5].

Theorem 3.1. *Let $\kappa\in\mathbb{N}^n$, $T:\mathbb{C}_\kappa[z_1,\dots,z_n]\to\mathbb{C}[z_1,\dots,z_n]$ be a linear operator, and $C=\mathbb{H}_\theta$ for some $0\le\theta<2\pi$. Then T preserves C-stability if and only if*

(a) *T has range of dimension at most one and is of the form $T(f)=\alpha(f)P$, where α is a linear functional on $\mathbb{C}_\kappa[z_1,\dots,z_n]$ and P is a C-stable polynomial, or*

(b) *The polynomial (in $2n$ variables)*

$$T[(z+w)^\kappa]:=\sum_{\alpha\le\kappa}\binom{\kappa}{\alpha}T(z^\alpha)w^{\kappa-\alpha} \tag{3.1}$$

is C-stable.

Remark 3.1. Theorem 3.1 trivially implies the following (well-known) multivariate Gauss-Lucas theorem: if $f\in\mathbb{C}_\kappa[z_1,\dots,z_n]$ is $\mathbb{H}_\theta$-stable for some $0\le\theta<2\pi$ then $\partial f/\partial z_i$ is $\mathbb{H}_\theta$-stable or identically zero for any $i\in[n]$. More generally, the $(n+1)$-variable polynomial $f+z_{n+1}\partial f/\partial z_i$ is $\mathbb{H}_\theta$-stable.

Theorem 3.2. *Let $\kappa\in\mathbb{N}^n$, $T:\mathbb{C}_\kappa[z_1,\dots,z_n]\to\mathbb{C}[z_1,\dots,z_n]$ be a linear operator, and $C=\mathbb{D}$ or $\mathbb{H}_{\frac{\pi}{2}}$. Then T preserves C-stability if and only if*

(a) *T has range of dimension at most one and is of the form $T(f)=\alpha(f)P$, where α is a linear functional on $\mathbb{C}_\kappa[z_1,\dots,z_n]$ and P is a C-stable polynomial, or*

(b) *The polynomial (in $2n$ variables)*

$$T[(1+zw)^\kappa]:=\sum_{\alpha\le\kappa}\binom{\kappa}{\alpha}T(z^\alpha)w^{\alpha} \tag{3.2}$$

is C-stable.

The polynomials in (3.1) and (3.2) are called the *algebraic symbols* of T with respect to the circular domains under consideration (for $\mathbb{H}_{\frac{\pi}{2}}$ it is often more convenient – but equivalent – to choose (3.2) rather than (3.1), cf. [5, Remark 6.1]).

The main result of this section is as follows.

Theorem 3.3. *Let* $\kappa = (\kappa_1, \dots, \kappa_n) \in \mathbb{N}^n$, $f(u, v) \in \mathbb{C}[u_1, \dots, u_n, v_1, \dots, v_n]$, *and* $g(z, w) \in \mathbb{C}[z_1, \dots, z_n, w_1, \dots, w_n]$. *Suppose that* $\deg_{u_i}(f) \le \kappa_i$ *and* $\deg_{z_i}(g) \le \kappa_i$ *for all* $i \in [n]$.

(a) *If* f *and* g *are* $\mathbb{H}_\theta$*-stable for some* $0 \le \theta < 2\pi$, *then the polynomial (in* $4n$ *variables)*
$$\sum_{\alpha \le \kappa} \frac{\partial^\alpha f}{\partial u^\alpha}(u, v) \cdot \frac{\partial^{\kappa-\alpha} g}{\partial z^{\kappa-\alpha}}(z, w)$$
is $\mathbb{H}_\theta$*-stable or identically zero.*

(b) *If* f *and* g *are* $\mathbb{H}_0$*-stable, then the polynomial*
$$\sum_{\alpha \le \kappa} (-1)^\alpha \frac{(\kappa - \alpha)!}{\alpha!} \cdot \frac{\partial^\alpha f}{\partial u^\alpha}(u, v) \cdot \frac{\partial^\alpha g}{\partial z^\alpha}(z, w)$$
is $\mathbb{H}_0$*-stable or identically zero.*

(c) *If* f *and* g *are* $\mathbb{D}$*-stable, then the polynomial*
$$\sum_{\alpha \le \kappa} \frac{(\kappa - \alpha)!}{\alpha!} \cdot \frac{\partial^\alpha f}{\partial u^\alpha}(0, v) \cdot \frac{\partial^\alpha g}{\partial z^\alpha}(z, w)$$
is $\mathbb{D}$*-stable or identically zero.*

Proof. We only prove (a) since the proofs of (b) and (c) are almost identical. Let $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{N}^n$ be fixed. Define a $\mathbb{C}[u_1, \dots, w_n]$-valued linear operator T on the space of all polynomials h in $4n$ variables $u_1, \dots, w_n$ satisfying $\deg_{u_j}(h) \le \kappa_j$, $\deg_{v_j}(h) \le \gamma_j$, $\deg_{z_j}(h) \le \gamma_j$, and $\deg_{z_j}(h) \le \gamma_j$ for all $j \in [n]$ by setting
$$T[h(u, v, z, w)] = \sum_{\alpha \le \kappa} \frac{\partial^\alpha h}{\partial u^\alpha}(u, v, z, w) \cdot \frac{\partial^{\kappa-\alpha} g}{\partial z^{\kappa-\alpha}}(z, w).$$
Let $\tilde{u} = (\tilde{u}_1, \dots, \tilde{u}_n)$, $\tilde{v} = (\tilde{v}_1, \dots, \tilde{v}_n)$, $\tilde{z} = (\tilde{z}_1, \dots, \tilde{z}_n)$ and $\tilde{w} = (\tilde{w}_1, \dots, \tilde{w}_n)$ be new sets of variables. The symbol of T with respect to $\mathbb{H}_\theta$, that is,
$$\begin{aligned} &T[(u + \tilde{u})^\kappa (v + \tilde{v})^\gamma (z + \tilde{z})^\gamma (w + \tilde{w})^\gamma] \\ &\quad = (v + \tilde{v})^\gamma (z + \tilde{z})^\gamma (w + \tilde{w})^\gamma \sum_{\alpha \le \kappa} \frac{\kappa!}{(\kappa - \alpha)!} (u + \tilde{u})^{\kappa-\alpha} \frac{\partial^{\kappa-\alpha} g}{\partial z^{\kappa-\alpha}}(z, w) \\ &\quad = \kappa! (v + \tilde{v})^\gamma (z + \tilde{z})^\gamma (w + \tilde{w})^\gamma g(z + u + \tilde{u}, w) \end{aligned}$$
is clearly $\mathbb{H}_\theta$-stable which proves (a) by Theorem 3.1 since $\gamma \in \mathbb{N}^n$ was arbitrary. □

An important special case of the above theorem is particularly attractive, as is its proof.

Corollary 3.4. *Let* $\kappa \in \mathbb{N}^n$ *and* $f, g \in \mathbb{C}[z_1, \dots, z_n, w_1, \dots, w_n]$ *be of the form*
$$f(z, w) = \sum_{\alpha \le \kappa} \binom{\kappa}{\alpha} P_\alpha(w) z^\alpha, \quad g(z, w) = \sum_{\alpha \le \kappa} \binom{\kappa}{\alpha} Q_\alpha(z) w^\alpha,$$
where $z = (z_1, \dots, z_n)$, $w = (w_1, \dots, w_n)$.

(a) *If f and g are $\mathbb{H}_\theta$-stable for some $0 \le \theta < 2\pi$, then so is the polynomial*

$$\sum_{\alpha\le\kappa}\binom{\kappa}{\alpha}P_\alpha(w)Q_{\kappa-\alpha}(z) = \frac{1}{\kappa!}\sum_{\alpha\le\kappa}\frac{\partial^\alpha f}{\partial z^\alpha}(0,w)\cdot\frac{\partial^{\kappa-\alpha}g}{\partial w^{\kappa-\alpha}}(z,0)$$

unless it is identically zero.

(b) *If f and g are $\mathbb{H}_0$-stable, then so is the polynomial*

$$\sum_{\alpha\le\kappa}(-1)^\alpha\binom{\kappa}{\alpha}P_\alpha(w)Q_\alpha(z) = \frac{1}{\kappa!}\sum_{\alpha\le\kappa}(-1)^\alpha\frac{(\kappa-\alpha)!}{\alpha!}\cdot\frac{\partial^\alpha f}{\partial z^\alpha}(0,w)\cdot\frac{\partial^\alpha g}{\partial w^\alpha}(z,0)$$

unless it is identically zero.

(c) *If f and g are $\mathbb{D}$-stable, then so is the polynomial*

$$\sum_{\alpha\le\kappa}\binom{\kappa}{\alpha}P_\alpha(w)Q_\alpha(z) = \frac{1}{\kappa!}\sum_{\alpha\le\kappa}\frac{(\kappa-\alpha)!}{\alpha!}\cdot\frac{\partial^\alpha f}{\partial z^\alpha}(0,w)\cdot\frac{\partial^\alpha g}{\partial w^\alpha}(z,0)$$

unless it is identically zero.

Proof. Suppose that f, g are as in part (a) of the corollary. Let

$$T : \mathbb{C}_\beta[z_1,\dots,z_n] \to \mathbb{C}_\kappa[z_1,\dots,z_n] \text{ and } S : \mathbb{C}_\kappa[z_1,\dots,z_n] \to \mathbb{C}_\gamma[z_1,\dots,z_n]$$

be the linear operators whose algebraic symbols with respect to $\mathbb{H}_\theta$ (cf. (3.1)) are f, respectively g, with $\beta,\gamma \in \mathbb{N}^n$ appropriately chosen. By Theorem 3.1 both S and T preserve $\mathbb{H}_\theta$-stability, hence so does their (operator) composition ST whose symbol is precisely the polynomial in (a). Applying Theorem 3.1 again we conclude that this polynomial is $\mathbb{H}_\theta$-stable unless it is of the form $A(z)B(w)$ for some polynomials A and B. If this is the case and these polynomials are not identically zero then $A(z)$ must be $\mathbb{H}_\theta$-stable (being the polynomial P in part (a) of Theorem 3.1) and by exchanging the roles of f and g we get that $B(w)$, thus also $A(z)B(w)$, must be $\mathbb{H}_\theta$-stable. This proves (a). Parts (b) and (c) follow similarly. □

Example 1. Let us show how the classical (univariate) Schur-Maló-Szegö theorem can be easily derived from Corollary 3.4. If $\sum_{k=0}^n\binom{n}{k}a_kz^k$ and $\sum_{k=0}^n\binom{n}{k}b_kz^k$ are two polynomials with real zeros only and in addition all the zeros of the latter polynomial are non-positive then the bivariate polynomials

$$f(z,w) = \sum_{k=0}^n\binom{n}{k}a_kz^k \text{ and } g(z,w) = \sum_{k=0}^n\binom{n}{k}b_{n-k}z^{n-k}w^k$$

are stable (note that f actually depends only on z). Corollary 3.4 (a) implies that the Schur-Maló-Szegö composition of the two given polynomials, i.e., the univariate polynomial

$$\sum_{k=0}^n\binom{n}{k}a_kb_kz^k$$

is also stable (that is, real-rooted).

Example 2. Theorems 3.11 and 4.6 in [8] (the former actually follows from the latter) provide some multivariate extensions of the classical composition results mentioned above. In particular, [8, Theorem 4.6] shows that the Weyl product of polynomials (defined via the product formula in the Weyl algebra) preserves stability, that is, if $f(z,w)$ and $g(z,w)$ are stable polynomials in $z=(z_1,\dots,z_n)$ and $w=(w_1,\dots,w_n)$ then so is the polynomial

$$\sum_{\alpha\in\mathbb{N}^n}\frac{(-1)^\alpha}{\alpha!}\frac{\partial^\alpha f}{\partial z^\alpha}(z,w)\cdot\frac{\partial^\alpha g}{\partial w^\alpha}(z,w).$$

To see that this is in fact a consequence of Theorem 3.3 (b) let $\kappa_N=(N,\dots,N)\in\mathbb{N}^n$ and let f and g be stable polynomials as in the statement of Theorem 3.3. Then since stability is closed under scaling the variables with positive numbers the polynomial

$$H_N(u,v,z,w)=\sum_{\alpha\le\kappa}(-1)^\alpha\frac{(\kappa-\alpha)!}{\alpha!}\cdot\frac{\kappa_N^\alpha(\kappa_N-\alpha)!}{\kappa_N!}\cdot\frac{\partial^\alpha f}{\partial u^\alpha}(Nu,v)\cdot\frac{\partial^\alpha g}{\partial z^\alpha}(z,w)$$

is stable for large N. But then the polynomial

$$\lim_{N\to\infty}H_N(u/N,v,z,w)=\sum_{\alpha}\frac{(-1)^\alpha}{\alpha!}\frac{\partial^\alpha f}{\partial u^\alpha}(u,v)\cdot\frac{\partial^\alpha g}{\partial z^\alpha}(z,w)$$

is stable or identically zero, as claimed.

We conclude with one further consequence. For $t=(t_1,\dots,t_n)\in\mathbb{R}_+^n$ define the *t-deformed Weyl product* of f and g by

$$\sum_{\alpha\in\mathbb{N}^n}\frac{(-1)^\alpha t^\alpha}{\alpha!}\frac{\partial^\alpha f}{\partial u^\alpha}(u,v)\cdot\frac{\partial^\alpha g}{\partial w^\alpha}(z,w).$$

Using again the fact that stability is closed under scaling the variables with positive constants we deduce from above that the t-deformed Weyl product preserves stability. As a special (univariate) case, note that if $n=1$, $t<0$ and $f\in\mathbb{R}[z]\setminus\{0\}$, $g\in\mathbb{R}[w]\setminus\{0\}$ have all real zeros then by the above the univariate polynomial

$$\sum_{k\in\mathbb{N}}\frac{t^k}{k!}f^{(k)}(z)\cdot g^{(k)}(w)\big|_{w=z}=\sum_{k\in\mathbb{N}}\frac{t^k}{k!}f^{(k)}(z)\cdot g^{(k)}(z)$$

has all real zeros. We thus recover de Bruijn's [11, Theorem 2] and [12, Lemma 1].

4. Hard Pólya-Schur Theory: Bounded Degree Multiplier Sequences

Using the characterization of linear operators preserving real stability obtained in [5] we can establish "hard" (bounded degree) multivariate versions of Pólya-Schur's classification of *multiplier sequences* [41] that extend the "soft" (unbounded degree) theorems of [8].

A sequence of real numbers $\{\lambda(k)\}_{k\in\mathbb{N}}$ is called a multiplier sequence if the linear operator on univariate polynomials defined by $T(z^k)=\lambda(k)z^k$, $k\in\mathbb{N}$, preserves real-rootedness, that is, $T(f)\in\mathcal{H}_1(\mathbb{R})\cup\{0\}$ whenever $f\in\mathcal{H}_1(\mathbb{R})$. A *multivariate multiplier sequence* is then defined as a sequence $\{\lambda(\alpha)\}_{\alpha\in\mathbb{N}^n}$ of real numbers such that the linear operator $T:\mathbb{R}[z_1,\dots,z_n]\to\mathbb{R}[z_1,\dots,z_n]$ defined by $T(z^\alpha)=\lambda(\alpha)z^\alpha$, $\alpha\in\mathbb{N}^n$, preserves real stability, see [8]. These were characterized in [8] but here we will prove the corresponding "hard" theorems. Given $\kappa\in\mathbb{N}^n$ we say that a sequence $\{\lambda(\alpha)\}_{\alpha\le\kappa}$ of real numbers is a *κ-multiplier sequence* if the linear operator $T:\mathbb{R}_\kappa[z_1,\dots,z_n]\to\mathbb{R}_\kappa[z_1,\dots,z_n]$ defined by $T(z^\alpha)=\lambda(\alpha)z^\alpha$, $\alpha\le\kappa$, preserves real stability. This is the multivariate generalization of n-multiplier sequences [19].

Recall the following lemma from [5].

Lemma 4.1. *Let $f, g \in \mathbb{R}[z_1, \dots, z_n] \setminus \{0\}$, set $h = f + ig$ and suppose that f and g are not constant multiples of each other. The following are equivalent:*

(1) *h is stable;*
(2) *$|h(z)| > |h(\bar{z})|$ for all $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ with $\mathrm{Im}(z_j) > 0$, $j \in [n]$, where $\bar{z} = (\bar{z}_1, \dots, \bar{z}_n)$;*
(3) *$f + z_{n+1} g \in \mathcal{H}_{n+1}(\mathbb{C})$;*
(4) *f and g are stable and*
$$\mathrm{Im}\left(\frac{f(z)}{g(z)}\right) \geq 0$$
whenever $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ with $\mathrm{Im}(z_j) > 0$, $j \in [n]$.

A polynomial $f \in \mathbb{C}[z_1, \dots, z_n] \setminus \{0\}$ is said to have the *same-phase property* if there exists $\alpha \in \mathbb{R}$ such that all the non-zero coefficients of $e^{-i\alpha} f$ are positive. The next lemma was first proved in [16, Theorem 6.1]. We will use it in the proof of Lemma 4.3 below and for completeness we provide here a self-contained proof based on our results so far.

Lemma 4.2. *If $f \in \mathbb{C}[z_1, \dots, z_n] \setminus \{0\}$ is homogeneous of degree d and $\mathbb{H}_\theta$-stable for some $\theta \in [0, 2\pi)$ then f has the same-phase property.*

Proof. Note first that the assumptions of the lemma imply that f is $\mathbb{H}_{\theta'}$-stable for any $\theta' \in [0, 2\pi)$. Without loss of generality we may also assume that $\partial_j f \not\equiv 0$, where $\partial_j = \frac{\partial}{\partial z_j}$. We will now use induction on d. The statement is trivially true for $d = 0$ so suppose $d \geq 1$. Applying $\frac{\partial}{\partial t}$ to the identity $f(tz_1, \dots, tz_n) = t^d f(z_1, \dots, z_n)$ and setting $t = 1$ we get
$$f(z_1, \dots, z_n) = d^{-1} \sum_{j=1}^{n} z_j \partial_j f(z_1, \dots, z_n). \tag{4.1}$$
Each polynomial $\partial_j f$ is stable (e.g. by Theorem 3.1) and homogeneous of degree $d - 1$. By the induction hypothesis there exists $\alpha_j \in \mathbb{R}$ such that $e^{-i\alpha_j} \partial_j f$ has all non-negative coefficients. In view of (4.1) it is therefore enough to show that $\alpha_j \equiv \alpha_k \bmod 2\pi$, $j, k \in [n]$. For each $j \in [n]$ we get by Remark 3.1, Lemma 4.1 (3) $\Leftrightarrow$ (4) and homogeneity that
$$\mathrm{Im}\left(\frac{\partial_j f(z)}{f(z)}\right) \leq 0, \quad z = (z_1, \dots, z_n) \in \mathbb{H}_0^n, \tag{4.2}$$
and
$$\mathrm{Re}\left(\frac{\partial_j f(z)}{f(z)}\right) \geq 0, \quad z = (z_1, \dots, z_n) \in \mathbb{H}_{\frac{\pi}{2}}^n. \tag{4.3}$$
By continuity (4.2) also holds for all $z \in \mathbb{R}^n$ for which $f(z) \neq 0$. Using homogeneity we see that $\mathrm{Im}(\partial_j f(-z)/f(-z)) = -\mathrm{Im}(\partial_j f(z)/f(z))$ hence $\partial_j f(z)/f(z)$ is a real rational function. Since $e^{-i\alpha_j} \partial_j f(\zeta) \in \mathbb{R}$ we deduce that $e^{-i\alpha_j} f(\zeta) \in \mathbb{R}$ for $j \in [n]$ and $\zeta \in \mathbb{R}^n$. Now, from (4.3) with $z \in \mathbb{R}_+^n$ and the fact that $e^{-i\alpha_j} \partial_j f(z) > 0$ for all such z we conclude that $e^{-i\alpha_j} f(z) > 0$ whenever $z \in \mathbb{R}_+^n$, $j \in [n]$, and thus $\alpha_j \equiv \alpha_k \bmod 2\pi$, $j, k \in [n]$, as required. □

Lemma 4.3. *Let $f(z, w) = \sum_{\alpha \in \mathbb{N}^n} a(\alpha) z^\alpha w^\alpha \in \mathbb{C}[z_1, \dots, z_n, w_1, \dots, w_n]$. Then f is stable if and only if it can be written as*
$$f(z, w) = C f_1(z_1 w_1) \cdots f_n(z_n w_n),$$

where $C \in \mathbb{C}$ and $f_1(t), \ldots, f_n(t)$ are univariate real polynomials with real and non-negative zeros only.

Proof. The sufficiency part follows simply by noticing that if $\mu \leq 0$ then $\mu + zw$ is a stable polynomial in two variables.

Suppose that $f(z, w) = \sum_{\alpha \in \mathbb{N}^n} a(\alpha) z^\alpha w^\alpha$ is stable. We claim that its support $J := \text{supp}(f)$ has unique minimal and maximal elements with respect to the standard partial order on $\mathbb{N}^n$. Assume the contrary, let α, α' be two different minimal elements and let i, j be indices such that $\alpha_i > \alpha'_j$ and $\alpha_j < \alpha'_i$. By [13, Theorem 3.2] J is a jump system and since $\alpha'' = \alpha - e_i \notin J$, there is an index $k \neq i$ such that $\alpha'' + e_k \in J$. Let $g(z_i, w_i, z_k, w_k)$ be the polynomial

$$\frac{\partial^{\alpha''}}{\partial z^{\alpha''}} \frac{\partial^{\alpha''}}{\partial w^{\alpha''}} f \Big|_{z_\ell = w_\ell = 0,\, \ell \notin \{i,k\}}$$

and set

$$h(z_i, w_i, z_k, w_k) = \lim_{\lambda \to 0^+} \lambda^{-2} g(\lambda z_i, \lambda w_i, \lambda z_k, \lambda w_k).$$

By Hurwitz' theorem (Theorem 1.5) h is a stable polynomial. However, by construction h is of the form $A z_i w_i + B z_k w_k$ with $AB \neq 0$, which is a contradiction since polynomials of this type cannot be stable. This shows that J has a unique minimal element.

If $f(z, 1)$ has degree at most κ_i in the variable z_i, $i \in [n]$, we may consider the stable polynomial $z^\kappa w^\kappa f(-z^{-1}, -w^{-1})$, where $\kappa = (\kappa_1, \ldots, \kappa_n)$, $z^{-1} = (z_1^{-1}, \ldots, z_n^{-1})$ and similarly for w^{-1}. By the above the support of the latter polynomial has a unique minimal element, thus providing a unique maximal element for the support J of f.

Let now ξ, κ be the minimal, respectively maximal element of J and let $T : \mathbb{C}_\kappa[z_1, \ldots, z_n] \to \mathbb{C}_\kappa[z_1, \ldots, z_n]$ be the linear operator defined by

$$T[z^\alpha] = \lambda(\alpha) z^\alpha = (-1)^\alpha \binom{\kappa}{\alpha}^{-1} a(\alpha) z^\alpha, \quad 0 \leq \alpha \leq \kappa.$$

Let $\{e_j\}_{j=1}^n$ be the standard orthonormal basis of $\mathbb{R}^n$. We want to show that

$$\lambda(\alpha) = \lambda(\xi)^{-n+1} \lambda(\xi + (\alpha_1 - \xi_1)e_1) \cdots \lambda(\xi + (\alpha_n - \xi_n)e_n), \quad \xi \leq \alpha \leq \kappa. \tag{4.4}$$

This will then prove the necessity part since f will split into a product as in the statement of the lemma and the polynomials $f_j(t)$, $1 \leq j \leq n$, will have the desired properties since $f_j(z_j w_j)$ is necessarily stable.

To prove (4.4) note that the algebraic symbol of T is given by

$$G_T(z, w) = \sum_{\alpha \leq \kappa} (-1)^\alpha a(\alpha) z^\alpha w^{\kappa - \alpha} = w^\kappa f(z, -w^{-1}),$$

which is stable. By Theorem 3.1 T preserves stability. Since $G_T(z, w)$ is homogeneous we may assume that $(-1)^\alpha a(\alpha) \geq 0$ for all α in view of Lemma 4.2. Now, it is easy to check that

$$a + bz + cw + dzw \in \mathbb{R}[z, w] \text{ is stable } \iff bc \geq ad \tag{4.5}$$

(see, e.g., [13] or just adapt the arguments in the proof of Lemma 1.4) and of course

$$a + 2bz + cz^2 \in \mathbb{R}[z] \text{ is stable } \iff b^2 \geq ac. \tag{4.6}$$

Let $\gamma \in \mathbb{N}^n$ and $1 \leq i, j \leq n$ be such that $\gamma + e_i + e_j \leq \kappa$. Applying T to the polynomials $z^\gamma(1 + z_i)(1 + z_j)$ and $z^\gamma(1 - z_i)(1 + z_j)$ and keeping (4.5) and (4.6) in mind we see that $\lambda(\gamma)\lambda(\gamma+e_i+e_j) \geq \lambda(\gamma+e_i)\lambda(\gamma+e_j)$ and $\lambda(\gamma)\lambda(\gamma+e_i+e_j) \leq \lambda(\gamma+e_i)\lambda(\gamma+e_j)$ hence

$$\lambda(\gamma)\lambda(\gamma+e_i+e_j) = \lambda(\gamma+e_i)\lambda(\gamma+e_j) \text{ whenever } \gamma \in \mathbb{N}^n \text{ and } \gamma+e_i+e_j \leq \kappa. \quad (4.7)$$

From (4.7) and [13, Corollary 3.7] we deduce that $\lambda(\gamma) > 0$ for all $\xi \leq \gamma \leq \kappa$. The proposed formula (4.4) now follows by induction over $k := |\alpha| - |\xi|$. □

Recall the following theorem from [5].

Theorem 4.4. *Let $\kappa \in \mathbb{N}^n$ and $T : \mathbb{R}_\kappa[z_1, \dots, z_n] \to \mathbb{R}[z_1, \dots, z_n]$ be a linear operator. Then T preserves real stability if and only if either*

(a) *T has at most 2-dimensional range and is given by $T(f) = \alpha(f)P + \beta(f)Q$, where α, β are real linear forms on $\mathbb{R}_\kappa[z_1, \dots, z_n]$ and $P, Q \in \mathcal{H}_n(\mathbb{R})$ are such that $P + iQ \in \mathcal{H}_n(\mathbb{C})$, or*

(b) *Either $T[(z+w)^\kappa] \in \mathcal{H}_{2n}(\mathbb{R})$ or $T[(z-w)^\kappa] \in \mathcal{H}_{2n}(\mathbb{R})$.*

The "hard" multivariate version of Pólya-Schur's theorem [41] is as follows.

Theorem 4.5. *Let $\kappa \in \mathbb{N}^n$, let $\lambda := \{\lambda(\alpha)\}_{\alpha \leq \kappa}$ be a sequence of real numbers and $T : \mathbb{R}_\kappa[z_1, \dots, z_n] \to \mathbb{R}_\kappa[z_1, \dots, z_n]$ be the corresponding linear operator. The following are equivalent:*

(a) *λ is a κ-multiplier sequence;*

(b) *$\pm\lambda$ is the product of one-dimensional κ_i-multiplier sequences that are either all alternating in sign or all non-negative, i.e.,*

$$\pm\lambda(\alpha) = \lambda_1(\alpha_1) \cdots \lambda_n(\alpha_n), \quad \alpha \leq \kappa,$$

where λ_i is a κ_i-multiplier sequence, $i \in [n]$, and either all λ_i's are non-negative or all are alternating in sign;

(c) *Either $T[(z+w)^\kappa] \in \mathcal{H}_{2n}(\mathbb{R})$ or $T[(z-w)^\kappa] \in \mathcal{H}_{2n}(\mathbb{R})$;*

(d) *$T[(z+w)^\kappa]$ or $T[(z-w)^\kappa]$ can be written as*

$$f_1(z_1w_1) \cdots f_n(z_nw_n),$$

where $f_1(t), \dots, f_n(t)$ are univariate polynomials with real zeros only, and all these zeros have the same sign (collectively).

Proof. This is an immediate consequence of Theorem 4.4 and Lemma 4.3. □

Remark 4.1. Note that the κ-multiplier sequences with constant sign are precisely the sequences whose corresponding operators preserve stability.

Corollary 4.6. *Let $\kappa \in \mathbb{N}^n$, $\lambda := \{\lambda(\alpha)\}_{\alpha \leq \kappa}$ be a sequence of complex numbers and $T : \mathbb{C}_\kappa[z_1, \dots, z_n] \to \mathbb{C}_\kappa[z_1, \dots, z_n]$ be the corresponding linear operator. Then T preserves weak Hurwitz stability if and only if λ is (a constant complex multiple of) a non-negative κ-multiplier sequence.*

Proof. By Theorem 3.1 T preserves weak Hurwitz stability if and only if the polynomial

$$T[(z+w)^\kappa] = \sum_{\alpha \leq \kappa} \binom{\kappa}{\alpha} \lambda(\alpha) z^\alpha w^{\kappa-\alpha}$$

is weakly Hurwitz stable, which occurs exactly when

$$\sum_{\alpha \leq \kappa} (-1)^\alpha \binom{\kappa}{\alpha} \lambda(\alpha) z^\alpha w^\alpha$$

is stable. The assertion now follows from Lemma 4.3 and Remark 4.1. □

5. Multivariate Apolarity

The goal of this section is to develop a higher-dimensional apolarity theory and establish Grace type theorems for arbitrary multivariate polynomials.

Two univariate polynomials $f(z) = \sum_{k=0}^{n} \binom{n}{k} a_k z^k$ and $g(z) = \sum_{k=0}^{n} \binom{n}{k} b_k z^k$ of degree at most n are *apolar* if

$$\{f, g\}_n := \sum_{k=0}^{n} (-1)^k f^{(k)}(0) g^{(n-k)}(0) = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k b_{n-k} = 0.$$

Grace's classical apolarity theorem is as follows [22, 35, 43, 57].

Theorem 5.1 (Grace)**.** *Let f and g be apolar polynomials of degree $n \geq 1$. If f has all zeros in a circular domain C then g has at least one zero in C.*

Note that we may reformulate Grace's theorem as follows.

Theorem 5.2. *Let f and g be polynomials of degree $n \geq 1$ and C be a circular domain. If f is C-stable and g is $\mathbb{C} \setminus C$-stable then $\{f, g\}_n \neq 0$.*

For two polynomials $f, g \in \mathbb{C}[z_1, \dots, z_n]$ and $\kappa \in \mathbb{N}^n$ define

$$\{f, g\}_\kappa := \sum_{\alpha \leq \kappa} (-1)^\kappa f^{(\alpha)}(0) g^{(\kappa-\alpha)}(0)$$

and call f and g *apolar* if they both have degree at most κ and $\{f, g\}_\kappa = 0$.

Hinkkanen [25] wondered if Grace's theorem could be extended to several variables (he actually only considered multi-affine polynomials) but the precise form of such an extension remained uncertain. He also claimed that arguments due to Ruelle and Dyson [47, 50] could be extended to prove the following result.

Lemma 5.3. *Let A and B be closed subsets of $\mathbb{C}$ which do not contain the origin and let $f, g \in \mathbb{C}_{(1^2)}[z_1, z_2]$. If f is $A \times B$-stable and g is $(\mathbb{C} \setminus A) \times (\mathbb{C} \setminus B)$-stable then $\{f, g\}_{(1^2)} \neq 0$.*

Lemma 5.3 is false, as one can see by considering for instance $f(z_1, z_2) = z_1 + z_2$, $g(z_1, z_2) = 1$, and $A = B = \{\mathrm{Im}(z) \geq 1\}$. However, it holds under additional degree constraints (e.g. if both f ang g have total degree 2) which are tacitly assumed in [50, Footnote 7]. In [25] Hinkkanen also proposed two possible generalizations of Grace's theorem as the following questions.

Question 1 (Hinkkanen)**.** *Let A_i, $i \in [n]$, be closed subsets of $\mathbb{C}$ that do not contain the origin and $f, g \in \mathbb{C}_{(1^n)}[z_1, \dots, z_n]$. If f is $A_1 \times \cdots \times A_n$-stable and g is $(\mathbb{C} \setminus A_1) \times \cdots \times (\mathbb{C} \setminus A_n)$-stable then $\{f, g\}_{(1^n)} \neq 0$.*

Question 2 (Hinkkanen)**.** *Let C_i, $i \in [n]$, be closed circular domains and $f, g \in \mathbb{C}_{(1^n)}[z_1, \dots, z_n]$. If f is $C_1 \times \cdots \times C_n$-stable and g is $(\mathbb{C} \setminus C_1) \times \cdots \times (\mathbb{C} \setminus C_n)$-stable then $\{f, g\}_{(1^n)} \neq 0$.*

We will now see that these questions are not true in full generality, but if we strengthen the hypothesis slightly in the second question then it is true for arbitrary degree polynomials.

Note that if $f, g \in \mathbb{C}_\kappa[z_1, \dots, z_n]$ then

$$\{f, g\}_\kappa(z) := \sum_{\alpha \leq \kappa} (-1)^\alpha f^{(\alpha)}(z) g^{(\kappa-\alpha)}(z)$$

is a constant function so in this case $\{f,g\}_\kappa(z)=\{f,g\}_\kappa$ for $z\in\mathbb{C}^n$. Elementary computations also yield the following.

Lemma 5.4. *Let $\kappa=(\kappa_1,\dots,\kappa_n)\in\mathbb{N}^n$, $a_i,b_i,c_i,d_i\in\mathbb{C}$, $a_id_i-b_ic_i=1$, $i\in[n]$, $f,g\in\mathbb{C}_\kappa[z_1,\dots,z_n]$ and set*

$$F(z)=(c_1z_1+d_1)^{\kappa_1}\cdots(c_nz_n+d_n)^{\kappa_n}f\left(\frac{a_1z_1+b_1}{c_1z_1+d_1},\dots,\frac{a_nz_n+b_n}{c_nz_n+d_n}\right),$$
$$G(z)=(c_1z_1+d_1)^{\kappa_1}\cdots(c_nz_n+d_n)^{\kappa_n}g\left(\frac{a_1z_1+b_1}{c_1z_1+d_1},\dots,\frac{a_nz_n+b_n}{c_nz_n+d_n}\right).$$

Then $\{f,g\}_\kappa=\{F,G\}_\kappa$.

Remark 5.1. Note that Lemma 5.4 asserts that the functional $\{\cdot,\cdot\}_\kappa$ is invariant under the action of the group of Möbius transformations normalized as in (1.3). For $n=1$ this is quite well-known [43] and motivates the name "apolar invariant" for $\{\cdot,\cdot\}_\kappa$ which is classically used in invariant theory, umbral calculus, and the theory of algebraic curves [28, 44, 53].

Lemma 5.5. *Let $f,g\in\mathbb{C}[z_1,\dots,z_n]$ and suppose that g has degree $\kappa\in\mathbb{N}^n$. If f is $\mathbb{D}$-stable and g is $\mathbb{C}\setminus\mathbb{D}$-stable then $\{f,g\}_\kappa\neq 0$.*

Proof. Let
$$f(z)=\sum_\alpha a_\alpha z^\alpha \text{ and } g(z)=\sum_\alpha b_\alpha z^\alpha.$$
Then $h(z):=\sum_\alpha(-1)^\alpha b_{\kappa-\alpha}z^\alpha$ is $\overline{\mathbb{D}}$-stable by Corollary 1.7. By compactness there is $\epsilon>0$ such that $|h(z)|>\epsilon$ for $z\in\overline{\mathbb{D}}^n$. This means that there is $\delta>0$ such that $\sum_\alpha(-1)^\alpha b_{\kappa-\alpha}(1+\delta)^{|\alpha|}z^\alpha$ is $\mathbb{D}$-stable. Then by applying Corollary 3.4 (c) to the $\mathbb{D}$-stable polynomials
$$\sum_\alpha a_\alpha z^\alpha w^\alpha \text{ and } \sum_\alpha(-1)^\alpha b_{\kappa-\alpha}(1+\delta)^{|\alpha|}z^\alpha w^\alpha$$
we deduce that the polynomial
$$F(z,w):=\sum_\alpha\frac{(-1)^\alpha a_\alpha b_{\kappa-\alpha}(1+\delta)^{|\alpha|}}{\binom{\kappa}{\alpha}}z^\alpha w^\alpha$$
is $\mathbb{D}$-stable or identically zero. The assumptions on f and g guarantee that $a_0b_\kappa\neq 0$, so F is $\mathbb{D}$-stable and
$$\{f,g\}_\kappa=\kappa!F\big((1+\delta)^{-1/2},\dots,(1+\delta)^{-1/2}\big)\neq 0,$$
as claimed. □

We find it most natural to state two apolarity theorems: one for discs and exterior of discs (Theorem 5.6) and one for half-planes (Theorem 5.8).

Theorem 5.6. *Let C_i, $i\in[n]$, be open discs or exterior of discs and let $f,g\in\mathbb{C}_\kappa[z_1,\dots,z_n]$. Suppose that*

(i) *f is $C_1\times\cdots\times C_n$-stable and $\deg_{z_j}(f)=\kappa_j$ whenever C_j is the exterior of a disk, and*
(ii) *g is $(\mathbb{C}\setminus C_1)\times\cdots\times(\mathbb{C}\setminus C_n)$-stable and $\deg_{z_j}(f)=\kappa_j$ whenever C_j is a disk.*

Then $\{f,g\}_\kappa\neq 0$.

Proof. For $i \in [n]$ let Φ_i be a Möbius transformation $(a_i z + b_i)/(c_i z + d_i)$ as in (1.3) for which $\Phi_i(\mathbb{D}) = C_i$. By Lemma 1.8 the polynomial

$$F(z) := (c_1 z_1 + d_1)^{\kappa_1} \cdots (c_n z_n + d_n)^{\kappa_n} f(\Phi_1(z_1), \dots, \Phi_n(z_n))$$

is $\mathbb{D}$-stable. Note that $-d_i/c_i \notin \mathbb{C} \setminus \mathbb{D}$ since none of the C_i's is a half-plane. The polynomial

$$G(z) := (c_1 z_1 + d_1)^{\kappa_1} \cdots (c_n z_n + d_n)^{\kappa_n} g(\Phi_1(z_1), \dots, \Phi_n(z_n))$$

is therefore $\mathbb{C} \setminus \mathbb{D}$-stable. By Lemma 1.8 the degree of g is precisely κ so Lemma 5.5 applies and by Lemma 5.4 we get $\{f, g\}_\kappa = \{F, G\}_\kappa \neq 0$. □

The *homogeneous part* of a polynomial $f \in \mathbb{C}[z_1, \dots, z_n]$ is the polynomial f_H obtained by extracting the terms of maximum total degree, i.e.,

$$f_H(z_1, \dots, z_n) = \lim_{t \to \infty} t^{-d} f(tz_1, \dots, tz_n),$$

where $d = \max\{|\alpha| : \partial^\alpha f / \partial z^\alpha \neq 0\}$.

Lemma 5.7. *Suppose $f \in \mathbb{C}[z_1, \dots, z_n]$ is $\mathbb{H}_0$-stable and $i \in [n]$. Then*

$$\left(\frac{\partial f}{\partial z_i}\right)_H = \frac{\partial f_H}{\partial z_i}.$$

Proof. Suppose that f has total degree d. Clearly, it is enough to prove that either $\partial f / \partial z_i$ is identically zero or its total degree is $d - 1$.

Assume that $\partial f / \partial z_i \not\equiv 0$ and its total degree is $d' < d - 1$. By Remark 3.1 the polynomial $f + z_{n+1} \partial f / \partial z_i$ is $\mathbb{H}_0$-stable. Consider now the univariate polynomials

$$p(z) = f(z, \dots, z) = f_H(1, \dots, 1) z^d + \cdots,$$
$$q(z) = \frac{\partial f}{\partial z_i}(z, \dots, z) = \left(\frac{\partial f}{\partial z_i}\right)_H (1, \dots, 1) z^{d'} + \cdots,$$

These polynomials are of degree d and d' respectively, since the leading coefficients are non-zero by Lemma 4.2. Solving for z_{n+1} in $p(z) + z_{n+1} q(z) = 0$ we see that $\mathrm{Im}(p(z)/q(z)) \geq 0$ whenever $\mathrm{Im}(z) > 0$. This is a contradiction since $p(z)/q(z) = Cz^{d-d'} + o(z^{d-d'})$ when $z \to \infty$, where $C \neq 0$ and $d - d' \geq 2$. □

Theorem 5.8. *Let C_1 and C_2 be two open half-planes with non-empty intersection, $\kappa \in \mathbb{N}^n$, and $f, g \in \mathbb{C}_\kappa[z_1, \dots, z_n]$. If f is C_1-stable, g is C_2-stable, and $\kappa \leq \alpha + \beta$ for some $\alpha \in \mathrm{supp}(f), \beta \in \mathrm{supp}(g)$, then $\{f, g\}_\kappa \neq 0$.*

Proof. By an affine transformation of the variables we may assume that there is an $\epsilon > 0$ such that $f(z - i\varepsilon)$ and $g(-z + i\varepsilon)$ are $\mathbb{H}_0$-stable, where $\varepsilon = (\epsilon, \dots, \epsilon)$. Then so are the $2n$-variable polynomials $f(z + w - i\varepsilon)$ and $g(-z - w + i\varepsilon)$ and by Corollary 3.4 also the polynomial

$$F(z, w) = \sum_{\alpha \leq \kappa} (-1)^{\kappa - \alpha} f^{(\alpha)}(w - i\varepsilon) g^{(\kappa - \alpha)}(-z + i\varepsilon)$$

unless it is identically zero. If it is not identically zero then the conclusion of the theorem follows by setting $z = w = i\varepsilon$. To complete the proof we show that $F(z, w)$ is not identically zero. Let $G_\alpha(z, w) = (-1)^{\kappa - \alpha} f^{(\alpha)}(w - i\varepsilon) g^{(\kappa - \alpha)}(-z + i\varepsilon)$. This polynomial is $\mathbb{H}_0$-stable or identically zero (by Remark 3.1) and by Lemma 5.7 and Lemma 4.2 all non-zero coefficients in its homogeneous part have the same phase as those in the homogeneous part of $f(w - i\varepsilon)g(-z + i\varepsilon)$. By the assumptions on the supports of f and g there is an α such that $G_\alpha(z, w) \neq 0$ so $\lim_{t\to\infty} t^{-d-e+|\kappa|} F(tz, tw) \neq 0$, where d and e are the total degrees of f and g, respectively. In particular, $F(z, w)$ is not identically zero. □

6. Hard Lieb-Sokal Lemmas

In [32, Proposition 2.2] Lieb and Sokal proved that the operation that replaces one variable with differentiation with respect to another variable preserves weak Hurwitz stability. This result played a key role in the study of Laplace transforms of Lee-Yang measures and the extensions of Newman's strong Lee-Yang theorem obtained in [32], see §8.1. It was also an essential ingredient in proving the sufficiency part of the classification theorems of [5].

The Lieb-Sokal result is a "soft" (transcendental/unbounded degree) result since it amounts to saying that the linear operator on $\mathbb{C}[z_1, \ldots, z_n]$ acting on monomials as

$$z^\alpha \mapsto (-1)^{\alpha_1} \cdot \frac{\partial^{\alpha_1}(z_2^{\alpha_2} z_3^{\alpha_3} \cdots z_n^{\alpha_n})}{\partial z_2^{\alpha_1}}, \quad \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n, \tag{6.1}$$

(which one may schematically represent as "$z_1 \mapsto -\partial/\partial z_2$") preserves ($\mathbb{H}_0$-)stability, see Theorem 8.3 in §8.1.

By considering certain linear operators on finite-dimensional polynomial spaces we can establish "hard" versions of Lieb-Sokal's result.

Lemma 6.1. *Let $n, d \in \mathbb{N}$ with $n \geq 2$ and let $\kappa \in \mathbb{N}^n$ be such that $\kappa_1 = \kappa_2 = d$. Define a linear operator $T_d : \mathbb{C}_\kappa[z_1, \ldots, z_n] \to \mathbb{C}_\kappa[z_1, \ldots, z_n]$ by*

$$T_d(f) = \frac{1}{d!} \sum_{k=0}^{d} \frac{\partial^d f}{\partial z_1^k \partial z_2^{d-k}}.$$

Then T_d preserves $\mathbb{H}_\theta$-stability for any $0 \leq \theta < 2\pi$.

Proof. The symbol of T_d, i.e., the $2n$-variable polynomial $T_d[(z + w)^\kappa]$ is given by

$$\frac{1}{d!}(z_3 + w_3)^{\kappa_3} \cdots (z_n + w_n)^{\kappa_n} \sum_{k=0}^{d} \frac{d!}{(d-k)!} \frac{d!}{k!} (z_1 + w_1)^{d-k} (z_2 + w_2)^k$$
$$= (z_1 + z_2 + w_1 + w_2)^d (z_3 + w_3)^{\kappa_3} \cdots (z_n + w_n)^{\kappa_n}$$

which is $\mathbb{H}_\theta$-stable. The conclusion follows from Theorem 3.1. □

Remark 6.1. An interesting property of T_d is that $T_d(f)$ is actually a polynomial in the $n - 1$ variables $z_1 + z_2, z_3, \ldots, z_n$ for any $f \in \mathbb{C}_\kappa[z_1, \ldots, z_n]$. Indeed, let $F(z_1, z_2, \ldots, z_n) = T_d(f)(z_1, z_2, \ldots, z_n)$. It is straightforward to show that

$$\frac{\partial}{\partial t} F(z_1 + t, z_2 - t, z_3, \ldots, z_n) = 0, \quad t \in \mathbb{C},$$

and by letting $t = z_2$ we get $F(z_1 + t, z_2 - t, \ldots, z_n) = F(z_1 + z_2, 0, \ldots, z_n)$.

Using Lemma 6.1 and Remark 6.1 we deduce the following "hard" result that substantially improves (6.1) when the top degree is specified.

Corollary 6.2. *Let $n, d \in \mathbb{N}$ with $n \geq 2$ and let $\kappa \in \mathbb{N}^n$ be such that $\kappa_1 = \kappa_2 = d$. Define linear operators $S_d, R_d : \mathbb{C}_\kappa[z_1, \ldots, z_n] \to \mathbb{C}_\kappa[z_1, \ldots, z_n]$ by*

$$S_d\left[\sum_{k=0}^{d} z_1^k Q_k(z_2, \ldots, z_n)\right] = \frac{1}{d!}\sum_{k=0}^{d} k!\left(\frac{\partial}{\partial z_2}\right)^{d-k} Q_k(z_2, \ldots, z_n)$$

and

$$R_d\left[\sum_{k=0}^{d} z_1^k Q_k(z_2, \ldots, z_n)\right] = \frac{1}{d!}\sum_{k=0}^{d}(-1)^k (d-k)!\left(\frac{\partial}{\partial z_2}\right)^{k} Q_k(z_2, \ldots, z_n).$$

Then S_d and R_d preserve ($\mathbb{H}_0$-)stability up to degree κ.

The above "hard" results do indeed imply the "soft" ones. To see this fix $\beta \in \mathbb{N}^n$ and set $(\beta)_\alpha = \alpha!\binom{\beta}{\beta-\alpha}$ for $\alpha \in \mathbb{N}^n$. In [5, Lemma 8.2] it was shown that the linear operator on $\mathbb{C}[z_1, \ldots, z_n]$ defined by $z^\alpha \mapsto (\beta)_\alpha z^\alpha$, $\alpha \in \mathbb{N}^n$, preserves stability. In particular, if $\sum_{k=0}^{d} z_1^k Q_k(z_2)$ is stable then $\sum_{k=0}^{d} \frac{z_1^k}{(d-k)!} Q_k(z_2)$ is stable and this extends to n variables. Therefore, the "soft" Lieb-Sokal result (respectively, Theorem 8.3) follows from Lemma 6.1 (respectively, Corollary 6.2).

7. Transcendental Symbols and the Weyl Algebra

Define the *complex Laguerre-Pólya class* $\overline{\mathcal{H}}_n(\mathbb{C})$ as the class of entire functions in n variables that are limits, uniformly on compact sets, of polynomials in $\mathcal{H}_n(\mathbb{C})$, see, e.g., [31, Chap. IX]. The usual (real) *Laguerre-Pólya class* $\overline{\mathcal{H}}_n(\mathbb{R})$ consists of all functions in $\overline{\mathcal{H}}_n(\mathbb{C})$ with real coefficients.

If $T : \mathbb{K}[z_1, \ldots, z_n] \to \mathbb{K}[z_1, \ldots, z_n]$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, is a linear operator we define its *transcendental symbol*, $\overline{G}_T(z, w)$, to be the formal power series in $w_1, \ldots, w_n$ with polynomial coefficients in $\mathbb{K}[z_1, \ldots, z_n]$ given by

$$\overline{G}_T(z, w) := \sum_{\alpha \in \mathbb{N}^n} (-1)^\alpha T(z^\alpha)\frac{w^\alpha}{\alpha!}.$$

By abuse of notation we write $\overline{G}_T(z, w) = T[e^{-z\cdot w}]$, where $z\cdot w = z_1w_1 + \ldots + z_nw_n$. Let us recall from [5] the transcendental characterizations of complex, respectively real stability preservers.

Theorem 7.1. *Let $T : \mathbb{C}[z_1, \ldots, z_n] \to \mathbb{C}[z_1, \ldots, z_n]$ be a linear operator. Then T preserves ($\mathbb{H}_0$-)stability if and only if either*

(a) *T has range of dimension at most one and is given by $T(f) = \alpha(f)P$, where α is a linear form on $\mathbb{C}[z_1, \ldots, z_n]$ and $P \in \mathcal{H}_n(\mathbb{C})$, or*
(b) *$\overline{G}_T(z, w) \in \overline{\mathcal{H}}_{2n}(\mathbb{C})$.*

Remark 7.1. From Theorem 7.1 one can easily deduce a characterization of linear operators preserving Ω-stability for any open half-plane Ω. For instance, the analog of Theorem 7.1 (b) for the open right half-plane $\mathbb{H}_{\frac{\pi}{2}}$ is that the *transcendental symbol of T with respect to $\mathbb{H}_{\frac{\pi}{2}}$*, i.e., the formal power series

$$T[e^{z\cdot w}] := \sum_{\alpha \in \mathbb{N}^n} T(z^\alpha)\frac{w^\alpha}{\alpha!},$$

defines an entire function which is the limit, uniformly on compact sets, of weakly Hurwitz stable polynomials.

Theorem 7.2. *Let $T : \mathbb{R}[z_1, \dots, z_n] \to \mathbb{R}[z_1, \dots, z_n]$ be a linear operator. Then T preserves real stability if and only if either*

(a) *T has at most 2-dimensional range and is given by $T(f) = \alpha(f)P + \beta(f)Q$, where α, β are real linear forms on $\mathbb{R}[z_1, \dots, z_n]$ and $P, Q \in \mathcal{H}_n(\mathbb{R})$ are such that $P + iQ \in \mathcal{H}_n(\mathbb{C})$, or*
(b) *Either $\overline{G}_T(z, w) \in \overline{\mathcal{H}}_{2n}(\mathbb{R})$ or $\overline{G}_T(z, -w) \in \overline{\mathcal{H}}_{2n}(\mathbb{R})$.*

To illustrate the power of Theorems 7.1 and 7.2 we show that the main results of [8] for partial differential operators immediately follow from these two theorems. Recall that a (Weyl algebra) finite order linear partial differential operator with polynomial coefficients is an operator $T : \mathbb{K}[z_1, \dots, z_n] \to \mathbb{K}[z_1, \dots, z_n]$, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, of the form

$$T = \sum_{\alpha \le \beta} Q_\alpha(z) \frac{\partial^\alpha}{\partial z^\alpha}, \tag{7.1}$$

where $\beta \in \mathbb{N}^n$ and $Q_\alpha \in \mathbb{K}[z_1, \dots, z_n]$, $\alpha \le \beta$.

Theorem 7.3. *Let $T : \mathbb{K}[z_1, \dots, z_n] \to \mathbb{K}[z_1, \dots, z_n]$, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, be defined by (7.1) and set*

$$F(z, w) = \sum_{\alpha \le \beta} Q_\alpha(z) w^\alpha \in \mathbb{K}[z_1, \dots z_n, w_1, \dots, w_n].$$

Then

(a) *T preserves stability if and only if $F(z, -w)$ is stable;*
(b) *T preserves real stability if and only if $F(z, -w)$ is real stable.*

Proof. The (transcendental) symbol of T is given by

$$T[e^{-z \cdot w}] = e^{-z \cdot w} F(z, -w),$$

so (a) and (b) follow immediately from Theorems 7.1 and 7.2, respectively. □

Remark 7.2. Theorem 7.3 was first established in [8, Theorems 1.2–1.3] by different methods. An interesting consequence noted in [8, Theorem 1.11] is that if a Weyl algebra operator T preserves (real) stability then so does its Fischer-Fock dual T^*. As shown in [8], this duality result is a powerful multivariate generalization of the classical Hermite-Poulain-Jensen theorem and Pólya's curve theorem [17, 43].

B. Applications

We will now apply the theory developed in Part A to show that (the key steps in) existing proofs and generalizations of the Lee-Yang and Heilmann-Lieb theorems follow in a simple and unified way from the characterizations of Ω-stability preservers in terms of operator symbols obtained in [5]. These results are due to Asano [2], Ruelle [45, 46, 51], Newman [37, 38], Lieb-Sokal [32], Hinkkanen [25], Choe et al [16], Wagner [59]. For brevity's sake, we will only focus on the main arguments used in deriving them and in some cases we point out possible extensions.

8. Recovering Lee-Yang and Heilmann-Lieb Type Theorems

Let us first recall the original version of the Lee-Yang theorem for the *partition function* of the *ferromagnetic Ising model* (at inverse temperature 1). This function may be written as

$$Z(h_1,\ldots,h_n)=\sum_{\sigma\in\{-1,1\}^n}\mu(\sigma)e^{\sigma\cdot h},$$

where $\sigma\cdot h=\sum_{i=1}^n\sigma_ih_i$ and $\mu(\sigma)=e^{\sum_{i,j=1}^n J_{ij}\sigma_i\sigma_j}$.

Theorem 8.1 (Lee-Yang [30]). *If $J_{ij}\geq 0$ for all $i,j\in[n]$ then*

(a) *$Z(h_1,\ldots,h_n)\neq 0$ whenever $\mathrm{Re}(h_i)>0$, $1\leq i\leq n$;*
(b) *All zeros of $Z(h,\ldots,h)$ lie on the imaginary axis.*

Remark 8.1. In physical terms [3, 4, 30, 32, 56], the J_{ij} are ferromagnetic (≥ 0) coupling constants while the h_i are external (magnetic) fields sometimes also called fugacities. Theorem 8.1 (b) asserts that the zeros of the partition function of the ferromagnetic Ising model accumulate on the imaginary axis in the complex fugacity plane and a (first-order) phase transition occurs only at zero magnetic field.

Before we give a proof of the Lee-Yang theorem let us make a historical digression. In his work on the zeros of the Riemann zeta function Pólya was led to a simple yet useful result:

Lemma 8.2 (Pólya [39], Hilfssatz II). *Let $a>0$, $b\in\mathbb{R}$, and $G(z)$ be a real entire function of genus 0 or 1 with at least one real zero and only real zeros. Then the function*

$$G(z+ia)e^{ib}+G(z-ia)e^{-ib} \tag{8.1}$$

has only real zeros.

Hilfssatz II was subsequently employed by Kac [40, pp. 424–426] to settle a special case of Theorem 8.1 that proved to be inspirational for Lee and Yang's final proof [30] (cf. [5, Remark 4.2]). Recently, Lee-Yang type results and applications to Fourier transforms with all real zeros were obtained in [1, 14, 15] by iterating the process of Hilfssatz II. A simple proof of this result and multivariate extensions is as follows. Let R be the linear operator on formal power series in n variables with complex coefficients $f(z)=\sum_{\alpha\in\mathbb{N}^n}a(\alpha)z^\alpha$ defined by

$$R\left(\sum_{\alpha\in\mathbb{N}^n}a(\alpha)z^\alpha\right)=\sum_{\alpha\in\mathbb{N}^n}\mathrm{Re}(a(\alpha))z^\alpha=\frac{1}{2}\Big(f(z)+\overline{f(\bar{z})}\Big).$$

By Lemma 4.1 R maps the set of stable polynomials into the set of real stable polynomials and consequently also the complex Laguerre-Pólya class $\overline{\mathcal{H}}_n(\mathbb{C})$ (cf. §7) into the Laguerre-Pólya class $\overline{\mathcal{H}}_n(\mathbb{R})$. In the special case when $n=1$ and $G(z)$ is as in Lemma 8.2 it follows from Hadamard's factorization theorem that $G(z)\in\overline{\mathcal{H}}_1(\mathbb{R})$ hence $G(z+ia)e^{ib}\in\overline{\mathcal{H}}_1(\mathbb{C})$ and by the above

$$2R\big(G(z+ia)e^{ib}\big)=G(z+ia)e^{ib}+G(z-ia)e^{-ib}\in\overline{\mathcal{H}}_1(\mathbb{R}),$$

so the function in (8.1) has only real zeros. Note also that $e^{-iz}\in\overline{\mathcal{H}}_n(\mathbb{C})$ and thus $2R(e^{-iz})=\cos(z)\in\overline{\mathcal{H}}_n(\mathbb{R})$.

More general versions of Theorem 8.1 were obtained in e.g. [32] and [38], see §8.1. For simplicity of argument and exposition we will concentrate for the moment just on the original Lee-Yang theorem and give a short proof based on the ideas in [32] combined with Theorem 7.1.

Proof of Theorem 8.1. Note that (b) follows from (a) by symmetry in $\sigma \mapsto -\sigma$. To prove (a) define $\mathcal{M}$ to be the set of functions $\mu : \{-1,1\}^n \to \mathbb{C}$ whose Laplace transform

$$Z_\mu = \sum_{\sigma \in \{-1,1\}^n} \mu(\sigma) e^{\sigma \cdot h}$$

is the limit, uniformly on compact sets, of weakly Hurwitz stable polynomials (i.e., non-vanishing whenever all variables are in the open right half-plane $\mathbb{H}_{\frac{\pi}{2}}$).

Claim: Let $i, j \in [n]$ and $J_{ij} \geq 0$. If $\mu \in \mathcal{M}$ then $\tilde{\mu}_{ij} \in \mathcal{M}$, where

$$\tilde{\mu}_{ij}(\sigma) = \begin{cases} e^{J_{ij}} \mu(\sigma) \text{ if } \sigma_i = \sigma_j, \\ e^{-J_{ij}} \mu(\sigma) \text{ if } \sigma_i \neq \sigma_j. \end{cases}$$

Let us show that the claim implies the theorem. Indeed, if $\mu_0 : \{-1,1\}^n \to \mathbb{C}$ is such that $\mu(\sigma) = 1$ for all $\sigma \in \{-1,1\}^n$ then its Laplace transform Z_{μ_0} equals $(e^{h_1} + e^{-h_1}) \cdots (e^{h_n} + e^{-h_n})$. As noted above one has $\cos(z) \in \overline{\mathcal{H}}_n(\mathbb{R})$, which implies that $\mu_0 \in \mathcal{M}$ by a rotation of the variables. Then by successively applying to μ_0 the transformations defined above for all pairs $(i,j) \in [n] \times [n]$ one gets (a).

To prove the claim note that $Z_{\tilde{\mu}_{ij}} = T(Z_{\mu_0})$, where

$$T = \cosh(J_{ij}) + \sinh(J_{ij}) \frac{\partial^2}{\partial z_i \partial z_j}.$$

By Theorem 7.1 and Remark 7.1 the operator T preserves weak Hurwitz stability. Since T is a second order (linear) differential operator, by standard results in complex analysis we have that if $f_k \to f$ uniformly on compacts then $T(f_k) \to T(f)$ uniformly on compacts. This proves the claim. □

8.1. **Newman's Theorem and the Lieb-Sokal Approach.** In [38] Newman proved a strong Lee-Yang theorem stating that the Lee-Yang property holds for one-component ferromagnetic pair interactions if and only if it holds for zero pair interactions. This theorem was subsequently generalized in [32] by Lieb and Sokal who showed that one-component ferromagnetic pair interactions are "universal multipliers for Lee-Yang measures" and established a similar result for two-component ferromagnets. Lieb-Sokal's key observation was that it would suffice to show that a certain linear differential operator preserves the Lee-Yang property, which they proved by reducing the problem to the following statement about polynomials.

Theorem 8.3 (Lieb-Sokal). *Let $\{P_i(u)\}_{i=1}^m$ and $\{Q_i(v)\}_{i=1}^m$ be polynomials in n complex variables $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n)$, and define*

$$R(u,v) = \sum_{i=1}^m P_i(u) Q_i(v),$$

$$S(z) = \sum_{i=1}^m P_i(\partial/\partial z) Q_i(z),$$

where $z = (z_1, \ldots, z_n)$, $\partial/\partial z = (\partial/\partial z_1, \ldots, \partial/\partial z_n)$. If R is weakly Hurwitz stable (in $2n$ variables) then S is either weakly Hurwitz stable or identically zero.

Proof. Define a linear operator

$$T : \mathbb{C}[u_1, \dots, u_n, v_1, \dots, v_n] \to \mathbb{C}[u_1, \dots, u_n, v_1, \dots, v_n]$$

by letting

$$T(u^\alpha v^\beta) = \frac{\partial^\alpha}{\partial v_1^{\alpha_1} \cdots \partial v_n^{\alpha_n}} \big(v^\beta\big), \quad \alpha, \beta \in \mathbb{N}^n,$$

and extending linearly. Clearly, the theorem is equivalent to proving that T preserves weak Hurwitz stability. By Theorem 7.1 and Remark 7.1 this amounts to showing that the formal power series

$$\overline{G}_T(u, v, \xi, \eta) = \sum_{\alpha,\beta} T(u^\alpha v^\beta) \frac{\xi^\alpha \eta^\beta}{\alpha! \beta!}$$

(i.e., the transcendental symbol for $\mathbb{H}_{\frac{\pi}{2}}$) defines an entire function which is the limit, uniformly on compact sets, of weakly Hurwitz stable polynomials. An elementary computation then yields

$$\overline{G}_T(u, v, \xi, \eta) = \prod_{i=1}^n \big(e^{\eta_i v_i} e^{\eta_i \xi_i}\big),$$

which satisfies the above requirement since $e^{zw} = \lim_{n\to\infty}(1 + zw/n)^n$. □

8.2. **The Schur-Hadamard Product and Convolution.** The following version of Theorem 8.1 is usually referred to as the Lee-Yang "circle theorem", see, e.g., [25, 51].

Theorem 8.4. *Let $A = (a_{ij})$ be a Hermitian $n \times n$ matrix whose entries are in the closed unit disk $\overline{\mathbb{D}}$. Then the polynomial*

$$f(z_1, \dots, z_n) = \sum_{S\subseteq[n]} z^S \prod_{i\in S} \prod_{j\notin S} a_{ij}$$

is $\mathbb{D}$-stable. In particular, $f(z, \dots, z)$ has all its zeros on the unit circle.

Hinkkanen's proof [25] of Theorem 8.4 makes use of a composition theorem for the *Schur-Hadamard product* of multi-affine polynomials which is defined as follows: if $f(z) = \sum_{S\subseteq[n]} a(S) z^S$ and $g(z) = \sum_{S\subseteq[n]} b(S) z^S$ then

$$(f \bullet g)(z) = \sum_{S\subseteq[n]} a(S) b(S) z^S.$$

The next result is Hinkkanen's composition theorem [25, Theorem C].

Theorem 8.5. *Let $f, g \in \mathbb{C}_{(1^n)}[z_1, \dots, z_n]$. If f, g are $\mathbb{D}$-stable then so is $f \bullet g$ unless it is identically zero.*

Proof. Let g be a fixed $\mathbb{D}$-stable multi-affine polynomial in n variables and let T be the linear transformation on multi-affine polynomials in n variables given by $T(f) = f \bullet g$. Recall Theorem 3.2 (b). The symbol of T is

$$T\Big[(1 + zw)^{[n]}\Big] = g(z_1 w_1, \dots, z_n w_n),$$

which is clearly $\mathbb{D}$-stable (in $2n$ variables). Theorem 3.2 yields the result. □

Remark 8.2. The proof of Theorem 8.4 given in [25] is as follows. For $i, j \in [n]$ with $i < j$ (note the typo "$i \neq j$" in [25]) let

$$f_{ij}(z_1, \dots, z_n) = (1 + a_{ij} z_i + \overline{a_{ij}} z_j + z_i z_j) \prod_{k \in [n] \setminus \{i,j\}} (1 + z_k).$$

It is not hard to see that f_{ij} is $\mathbb{D}$-stable and by taking the Schur-Hadamard product of all these polynomials one gets

$$(f_{12} \bullet \cdots \bullet f_{(n-1)n})(z) = \sum_{S \subseteq [n]} z^S \prod_{i \in S} \prod_{j \notin S} a_{ij},$$

which is again $\mathbb{D}$-stable by Theorem 8.5.

Using Corollary 3.4 we can extend Hinkkanen's composition theorem to arbitrary (not necessarily multi-affine) $\mathbb{D}$-stable polynomials:

Theorem 8.6. *Let* $f(z) = \sum_{\alpha \le \kappa} \binom{\kappa}{\alpha} a(\alpha) z^\alpha$ *and* $g(z) = \sum_{\alpha \le \kappa} \binom{\kappa}{\alpha} b(\alpha) z^\alpha$ *be* $\mathbb{D}$-*stable polynomials. Then so is*

$$(f \bullet g)(z) := \sum_{\alpha \le \kappa} \binom{\kappa}{\alpha} a(\alpha) b(\alpha) z^\alpha$$

unless it is identically zero.

Proof. Apply Corollary 3.4 (c) to the $\mathbb{D}$-stable polynomials $f(z)$ and $g(zw)$, where $zw = (z_1 w_1, \dots, z_n w_n)$. □

Closely related to the Schur-Hadamard product is the *convolution operator* on multi-affine polynomials [16] defined as follows: if $f(z) = \sum_{S \subseteq [n]} a(S) z^S$ and $g(z) = \sum_{S \subseteq [n]} b(S) z^S$ then

$$(f \star g)(z) = \sum_{S, T \subseteq [n]} a(S) b(T) z^{S \Delta T},$$

where $S \Delta T = (S \cup T) \setminus (S \cap T)$. A corresponding composition result – this time for weak Hurwitz stability – is given in [16, Proposition 4.20].

Theorem 8.7. *Let* $f, g \in \mathbb{C}_{(1^n)}[z_1, \dots, z_n]$. *If* f, g *are weakly Hurwitz stable then so is* $f \star g$ *unless it is identically zero.*

Proof. Let g be a fixed $\mathbb{H}_{\frac{\pi}{2}}$-stable multi-affine polynomial in n variables and let T be the linear transformation on multi-affine polynomials in n variables given by $T(f) = f \star g$. The symbol of T (cf. Theorem 3.1) is just

$$i^n T\Big[(z+w)^{[n]}\Big] = (z+w)^{[n]} g\left(\frac{1 + z_1 w_1}{z_1 + w_1}, \dots, \frac{1 + z_n w_n}{z_n + w_n}\right). \tag{8.2}$$

Now if $u, v \in \mathbb{H}_{\frac{\pi}{2}}$ then also $u^{-1}, v^{-1}, u + v \in \mathbb{H}_{\frac{\pi}{2}}$ hence

$$\frac{1 + uv}{u + v} = (u+v)^{-1} + \left(u^{-1} + v^{-1}\right)^{-1} \in \mathbb{H}_{\frac{\pi}{2}},$$

so that the polynomial in (8.2) is $\mathbb{H}_{\frac{\pi}{2}}$-stable (in $2n$ variables). Theorem 3.1 again yields the desired conclusion. □

8.3. **Asano Contractions.** Many known proofs of the Lee-Yang theorem are based on so-called *Asano contractions* or variations thereof [2, 49, 50, 51]. Let

$$f(z_1,\dots,z_n) = a(z_3,\dots,z_n) + b(z_3,\dots,z_n)z_1 + c(z_3,\dots,z_n)z_2 + d(z_3,\dots,z_n)z_1z_2$$

be a polynomial in $n \geq 2$ variables which is multi-affine in z_1 and z_2. The Asano contraction of f is

$$A(f)(z_1,\dots,z_n) = a(z_3,\dots,z_n) + d(z_3,\dots,z_n)z_1.$$

Note that $A(f)$ does not depend on z_2. The key fact used in the aforementioned proofs is a property of Asano contractions that may be stated as follows.

Lemma 8.8. *Let $\kappa = (\kappa_1,\dots,\kappa_n) \in \mathbb{N}^n$ with $n \geq 2$ and $\kappa_1 = \kappa_2 = 1$. Then*

$$A : \mathbb{C}_\kappa[z_1,\dots,z_n] \to \mathbb{C}_\kappa[z_1,\dots,z_n]$$

is a linear operator that preserves $\mathbb{D}$-stability.

Proof. It is clear that A is linear. Its (algebraic) symbol is

$$A[(1+zw)^\kappa] = (1+zw)^{(\kappa_3,\dots,\kappa_n)}(1+z_1w_1w_2),$$

which is $\mathbb{D}$-stable, so the assertion follows from Theorem 3.2. □

8.4. **Multi-Affine Part and Folding mod 2.** Recall that a matching in a graph $G = (V,E)$ is a subset M of E such that no vertex of the graph (V,M) has degree exceeding one. The general version of the Heilmann-Lieb theorem on the *monomer-dimer model* is the following.

Theorem 8.9 (Heilmann-Lieb [23]). *Let $G = (V,E)$ be a loopless graph and define its matching polynomial with edge weights $\{\lambda_e\}_{e\in E}$ and vertex weights $\{z_i\}_{i\in V}$ as*

$$M_G(z,\lambda) = \sum_{\text{matchings } M} \prod_{e=ij\in M} \lambda_e z_i z_j.$$

If $\lambda_e \geq 0$, $e \in E$, then $M_G(z,\lambda)$ is a weakly Hurwitz stable polynomial (in z).

In [16] and [56, §5] it was shown that the Lee-Yang and Heilmann-Lieb theorems can actually be given a unified combinatorial formulation and proof. The idea is to form the "test" polynomial

$$F_G(z,\lambda) = \prod_{e=\{i,j\}\in E} (1+\lambda_e z_i z_j) \tag{8.3}$$

associated to a graph $G = (V,E)$, $|V| = n$, equipped with vertex weights $\{z_i\}_{i\in V}$ and non-negative edge weights $\{\lambda_e\}_{e\in E}$. This polynomial is weakly Hurwitz stable in the z_i's and by applying to it appropriate linear operators one gets precisely the polynomials occurring in the Lee-Yang theorem and the Heilmann-Lieb theorem, respectively. Thus one only has to check that the linear operators used in this process preserve weak Hurwitz stability. These operators are defined as follows.

The linear operator MAP : $\mathbb{C}[z_1,\dots,z_n] \to \mathbb{C}_{(1^n)}[z_1,\dots,z_n]$ extracts the multi-affine part of a polynomial, that is, if $f(z) = \sum_{\alpha\in\mathbb{N}^n} a(\alpha)z^\alpha$ then

$$\mathrm{MAP}(f)(z) = \sum_{\alpha:\, \alpha_i \leq 1,\, i\in[n]} a(\alpha)z^\alpha.$$

The transcendental symbol (see Remark 7.1) of MAP is

$$\sum_{\alpha:\, \alpha_i \le 1,\, i \in [n]} z^\alpha \frac{w^\alpha}{\alpha!} = (1 + zw)^{[n]}.$$

Clearly, this is a weakly Hurwitz stable polynomial. Theorem 7.1 and Remark 7.1 imply that MAP preserves weak Hurwitz stability. It is easy to see that

$$M_G(z, \lambda) = \text{MAP}\left[F_G(z, \lambda)\right], \tag{8.4}$$

which yields the Heilmann-Lieb theorem.

The linear operator MOD : $\mathbb{C}[z_1, \ldots, z_n] \to \mathbb{C}_{(1^n)}[z_1, \ldots, z_n]$ "folds mod 2" the powers in the Taylor expansion of a polynomial, i.e., if $f(z) = \sum_{\alpha \in \mathbb{N}^n} a(\alpha) z^\alpha$ then

$$\text{MOD}(f)(z) = \sum_{\alpha \in \mathbb{N}^n} a(\alpha) z^{\alpha \bmod 2},$$

where $\alpha \bmod 2 = (\alpha_1 \bmod 2, \ldots, \alpha_n \bmod 2)$. The algebraic symbol of MOD (up to degree κ) with respect to the open right-half plane $\mathbb{H}_{\frac{\pi}{2}}$ (cf. Theorem 3.2) is

$$\text{MOD}[(1 + zw)^\kappa] = 2^{-n}(1 + w)^\kappa (1 + z)^{[n]} \prod_{i=1}^{n} \left[1 + \left(\frac{1 - w_i}{1 + w_i}\right)^{\kappa_i} \frac{1 - z_i}{1 + z_i}\right].$$

If $\text{Re}(\zeta) > 0$ then $|(1 - \zeta)/(1 + \zeta)| < 1$ so the above symbol is weakly Hurwitz stable. By Theorem 3.2 we conclude that MOD preserves weak Hurwitz stability. MOD is employed in [16, §4.8] to prove the Asano contraction lemma (Lemma 8.8) and thereby the Lee-Yang theorem as well (cf. §8.3 and [56, §5]).

8.5. **Generalizations of the Heilmann-Lieb Theorem.** It is natural to study graph polynomials with more general degree constraints than those defining the matching polynomial and to establish Heilmann-Lieb type theorems for such polynomials. This has been pursued by e.g. Ruelle [45, 46] and Wagner [58, 59]. If $G = (V, E)$ is a graph with vertex set $V = \{v_1, \ldots, v_n\}$ we let $\deg G \in \mathbb{N}^n$ be the *degree vector* of G, i.e., the i-th coordinate of $\deg G$ is the degree of v_i in G. Let $\kappa = (\kappa_1, \ldots, \kappa_n) \in \mathbb{N}^n$ and suppose that $\deg G \le \kappa$. Then given degree weights $u : \mathbb{N}^\kappa \to \mathbb{C}$ and non-negative edge weights $\{\lambda_e\}_{e \in E}$ one may ask what are the non-vanishing properties of the polynomial

$$F_G(z, \lambda, u) = \sum_{H \subseteq E} \lambda^H u(\deg(V, H)) z^{\deg(V, H)}. \tag{8.5}$$

This question was considered by Wagner in [59]. When it comes to weak Hurwitz stability it is natural (and of course sufficient) to require that the linear "truncation" operator $T : \mathbb{C}_\kappa[z_1, \ldots, z_n] \to \mathbb{C}_\kappa[z_1, \ldots, z_n]$ defined by $T(z^\alpha) = u(\alpha) z^\alpha$, $u(\alpha) \in \mathbb{C}$, $\alpha \le \kappa$, preserves weak Hurwitz stability. Now these are precisely the multivariate multiplier sequences (up to degree κ) that were characterized in Corollary 4.6 of this paper as follows:

$$u(\alpha) = u_1(\alpha_1) \cdots u_n(\alpha_n), \quad \alpha \le \kappa,$$

where for each $i \in [n]$ the polynomial

$$\sum_{k=0}^{\kappa_i} \binom{\kappa_i}{k} u_i(k) z^k \tag{8.6}$$

has all real non-positive zeros. We thus recover the following generalization of the Heilmann-Lieb theorem due to Wagner [59], which extends a theorem of Ruelle [45]. By the necessity in Corollary 4.6 the theorem below is optimal.

Theorem 8.10 (Wagner [59]). *Let $G = (V, E)$ be a graph whose degree vector satisfies $\deg G \leq \kappa$ and let $F_G(z, \lambda, u)$ and u be as in (8.5) and (8.6), respectively. If $\{\lambda_e\}_{e\in E}$ are non-negative edge weights then*

(a) *$F_G(z, \lambda, u)$ is weakly Hurwitz stable considered as a polynomial in z;*
(b) *All zeros of the univariate polynomial*

$$\sum_{k=0}^{|E|}\left(\sum_{\substack{H\subseteq E\\ |H|=k}} u(\deg(V,H))\lambda^H\right)t^k$$

are real and non-positive.

Proof. The first statement follows from Corollary 4.6 and the fact that the test polynomial defined in (8.3) is weakly Hurwitz stable. If we set all the z_j's, $1 \leq j \leq n$, equal to $-it$ we obtain the univariate polynomial (in t)

$$\sum_{k=0}^{|E|}\left(\sum_{\substack{H\subseteq E\\ |H|=k}} u(\deg(V,H))\lambda^H\right)(-1)^k t^{2k}$$

which is then real stable. Clearly, this forces the polynomial in (b) to have all zeros real and non-positive. □

Remark 8.3. In [59] Wagner also proves non-vanishing properties in sectors, which cannot be obtained by our methods. However, Theorem 8.10 is slightly more general in that we consider max-degree at every vertex (not uniform max-degree).

9. Appendix

We give here a simple proof of the fact that Sym can be viewed as a (convergent) infinite product of operations as those in Proposition 1.3. For $\sigma \in \mathfrak{S}_n$ define an operator $T_\sigma : \mathbb{C}[z_1, \ldots, z_n] \to \mathbb{C}[z_1, \ldots, z_n]$ by

$$T_\sigma(f) = \frac{1}{|\langle\sigma\rangle|}\sum_{\tau\in\langle\sigma\rangle}\tau(f),$$

where $\langle\sigma\rangle$ is the subgroup of $\mathfrak{S}_n$ generated by σ. Given $\alpha, \beta \in \mathbb{N}^n$ we write $\alpha \sim \beta$ if α is a rearrangement of β. If $f(z) = \sum_{\alpha\in\mathbb{N}^n} a(\alpha)z^\alpha \in \mathbb{R}[z_1, \ldots, z_n]$ let the *symmetry index* of f be defined by $\mathfrak{s}(f) = \sum_{\alpha\sim\beta}|a(\alpha) - a(\beta)|$. For $f = g + ih \in \mathbb{C}[z_1, \ldots, z_n]$ with $g, h \in \mathbb{R}[z_1, \ldots, z_n]$ we define its symmetry index as $\mathfrak{s}(f) = \mathfrak{s}(g) + \mathfrak{s}(h)$. Hence $\mathfrak{s}(f) = 0$ if and only if f is symmetric.

Lemma 9.1. *Let $\sigma \in \mathfrak{S}_n$ and $f(z) = \sum_\alpha a(\alpha)z^\alpha \in \mathbb{C}[z_1, \ldots, z_n]$. Then*

$$\mathfrak{s}(T_\sigma(f)) \leq \mathfrak{s}(f) \tag{9.1}$$

with equality if and only if $a(\sigma(\alpha)) = a(\alpha)$ for all $\alpha \in \mathbb{N}^n$, i.e., $T_\sigma(f) = f$.

Proof. We may assume that $f \in \mathbb{R}[z_1, \dots, z_n]$. Since $\mathfrak{s}(\tau(f)) = \mathfrak{s}(f)$ for all $\tau \in \mathfrak{S}_n$ we have by the triangle inequality

$$\begin{aligned}
\mathfrak{s}(T_\sigma(f)) &= \sum_{\alpha\sim\beta} \frac{1}{|\langle\sigma\rangle|}\Big|\sum_{\tau\in\langle\sigma\rangle} a(\tau(\alpha)) - a(\tau(\beta))\Big| \\
&\leq \sum_{\alpha\sim\beta} \frac{1}{|\langle\sigma\rangle|}\sum_{\tau\in\langle\sigma\rangle} |a(\tau(\alpha)) - a(\tau(\beta))| \\
&= \frac{1}{|\langle\sigma\rangle|}\sum_{\tau\in\langle\sigma\rangle} \mathfrak{s}(\tau(f)) \\
&= \mathfrak{s}(f)
\end{aligned}$$

with equality if and only if the following condition holds:

$$\text{If } \alpha \sim \beta \text{ then } a(\tau(\alpha)) - a(\tau(\beta)) \text{ have the same sign for all } \tau \in \langle\sigma\rangle. \qquad \text{(A)}$$

Clearly, if $a(\sigma(\alpha)) = a(\alpha)$ for all $\alpha \in \mathbb{N}^n$ then equality holds in (9.1). On the other hand, if equality in (9.1) holds let $\beta = \sigma(\alpha)$ and assume that $a(\alpha) \geq a(\beta)$ (the case $a(\alpha) \leq a(\beta)$ follows similarly). Then by (A) we have

$$a(\alpha) \geq a(\sigma(\alpha)), \quad a(\sigma(\alpha)) \geq a(\sigma^2(\alpha)), \quad \dots, \quad a(\sigma^{k-1}(\alpha)) \geq a(\alpha),$$

where k is the order of σ. Hence $\alpha \mapsto a(\alpha)$ is constant on $\langle\sigma\rangle$-orbits, which completes the proof. □

For $f \in \mathbb{C}[z_1, \dots, z_n]$ let

$$\mathcal{A}(f) = \{T_{\tau_k} \cdots T_{\tau_1}(f) : \tau_1, \dots, \tau_k \in \mathfrak{S}_n \text{ are transpositions}\}$$

and denote by $\overline{\mathcal{A}}(f)$ the set of polynomials that are limits, uniformly on compact sets, of polynomials in $\mathcal{A}(f)$.

Lemma 9.2. *If* $f \in \mathbb{C}[z_1, \dots, z_n]$ *then* $\mathrm{Sym}(f) \in \overline{\mathcal{A}}(f)$.

Proof. We claim that the set $\mathfrak{s}\big(\overline{\mathcal{A}}(f)\big) := \{\mathfrak{s}(g) : g \in \overline{\mathcal{A}}(f)\}$ is closed. Suppose that $x_k \to x$ as $k \to \infty$, where $x_k = \mathfrak{s}(g_k)$ with $g_k \in \overline{\mathcal{A}}(f)$ for $k \in \mathbb{N}$. Let $|\cdot|_r$ be the supremum norm on the ball of radius r in $\mathbb{C}^n$. If $\sigma \in \mathfrak{S}_n$ we have by the triangle inequality and invariance under permutations that $|T_\sigma(g)|_r \leq |g|_r$. It follows that $|h|_r \leq |f|_r$ for all $h \in \overline{\mathcal{A}}(f)$. Hence, by Montel's theorem, $\{g_k\}_{k\in\mathbb{N}}$ forms a normal family so there is a subsequence converging uniformly on compacts to a polynomial $g \in \overline{\mathcal{A}}(f)$ with $\mathfrak{s}(g) = x$.

Hence $y := \inf \mathfrak{s}\big(\overline{\mathcal{A}}(f)\big)$ is achieved for some $g \in \overline{\mathcal{A}}(f)$. If $\tau(g) \neq g$ for some transposition $\tau \in \mathfrak{S}_n$ then by Lemma 9.1 we have $\mathfrak{s}(T_\tau(g)) < \mathfrak{s}(g)$. However, one clearly has $T_\tau(g) \in \overline{\mathcal{A}}(f)$, which contradicts the minimality of $\mathfrak{s}(g)$. We deduce that $\tau(g) = g$ for all transpositions $\tau \in \mathfrak{S}_n$ and thus $g = \mathrm{Sym}(f)$ and $\mathfrak{s}(g) = 0$. □

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