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THE LEFSCHETZ NUMBER FOR EQUIVARIANT MAPS

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1. Introduction and results

Let G be a compact Lie group. A G-ENR (Euclidean Neighborhood Retract) is a G-space which is a G-retract of some G-invariant open subspace in a Euclidean G-space. In this paper we will consider the Lefschetz number

$$\lambda(f) = \sum_i (-1)^i \operatorname{trace} f_{*,i}\colon H_i(X; Z)/\mathrm{Tor} \to H_i(X; Z)/\mathrm{Tor}$$

of a self G-map $f\colon X \to X$ of a compact G-ENR X. f restricts to the self map $f^G\colon X^G \to X^G$ of the G-fixed point set X^G of X. Then we will show

Theorem 1. *Let $f\colon X \to X$ be a self G-map of a compact G-ENR X.*

(i) *If X has only one isotropy type (H), then $\lambda(f) \equiv 0 \bmod \chi(G/H)$ where $\chi(\ )$ denotes the Euler characteristic.*

(ii) *If the G-action on X is semifree, then $\lambda(f) \equiv \lambda(f^G) \bmod \chi(G)$.*

(iii) *If G is finite and of prime power order p^k, then $\lambda(f) \equiv \lambda(f^G) \bmod p$.*

(iv) *If G is connected and abelian (i.e., torus), then $\lambda(f) = \lambda(f^G)$.*

In section 4 we will prove this theorem by using the fixed point index defined by Dold [2]. (i) of the theorem is a special case of Dold [3; (8.18)]. If G is finite and the G-action is free, related results are in Nakaoka [9] and Gottlieb [5]. As a corollary of the theorem we obtain

Corollary 2. (i) *If the G-action on X is semifree and $\lambda(f) \not\equiv 0 \bmod \chi(G)$, then f has a fixed point in X^G.*

(ii) *If G is of prime power order p^k and $\lambda(f) \not\equiv 0 \bmod p$, then f has a fixed point in X^G.*

Proof. In either case it follows $\lambda(f^G) \neq 0$ and by the Lefschetz fixed point theorem there exists a fixed point of $f^G\colon X^G \to X^G$. q.e.d.

If G is a compact monogenic Lie group (i.e., finite cyclic group, torus and product of these) and $f \in G$ is its generator, then we may regard f as a self G-map of a G-ENR X. In this case we can show, as in the proof of Theorem 1, that $\lambda(f) = \chi(X^G)$, although this has already appeared in the literature, tom

Dieck [1; (5.3.11)] and Huang [6; Corollary 1] for G a finite cyclic group, Kobayashi [7; p. 63] for X a Riemannian manifold. As applications of this we will show the following two results.

Proposition 3. *If X is a compact G-ENR and G is monogenic, then*

$$|\chi(X^G)| \leq \sum_i \operatorname{rank} H_i(X; Z) .$$

In connection with this we note that if G is finite and of prime power order p^k, Floyd [4] shows

$$|\chi(X^G)| \leq \sum_i \dim H_i(X^G; Z_p) \leq \sum_i \dim H_i(X; Z_p) .$$

Proposition 4. *Let G be of order 2 and f be its generator. Let M be a $2n$-dimensional closed smooth G-manifold and orientable over Z. If f is orientation preserving, then*

$$\chi(M^G) \equiv \operatorname{trace} f_{*,n} \qquad \bmod 2 .$$

If f is orientation reversing, then

$$\chi(M^G) = \operatorname{trace} f_{*,n} = 0 .$$

Here $f_{*,n}$ is the automorphism of $H_n(M;Z)$ induced from f.

These two propositions will be proved in section 5.

2. A lemma

If M is a G-space and $x \in M$, then $G(x)$ denotes the orbit of x and G_x the isotropy subgroup at x. The conjugacy class (G_x) of an isotropy subgroup G_x is called an isotropy type. For a subgroup H of G let $M_{(H)} = \{x \in M \mid (G_x) = (H)\}$. If N is a G-invariant subspace of M and $h: N \to M$ a G-map, then the fixed point set $\operatorname{Fix}(h)$ of h is a union of orbits. If N and M are smooth G-manifolds, then for any fixed orbit $G(x) \subset \operatorname{Fix}(h)$ we may take G-invariant tubular neighborhoods T and T' of $G(x)$ such that $T \subset T'$ and $h(T) \subset T'$. We decompose T into $T = T_t \oplus T_n$, where $T_t = T \cap N_{(H)}$, $H = G_x$, is the component tangent to $N_{(H)}$, and T_n the component normal to $N_{(H)}$. Similarly we decompose T' into $T' = T'_t \oplus T'_n$. Then we see $h(T_t) \subset T'_t$. We may regard T and T' as G-vector bundles over $G(x) \approx G/H$.

Lemma 5. *Let M be a smooth G-manifold and N a G-invariant codimension 0 submanifold of M with finite isotropy types. If $f: N \to M$ is a G-map with $\operatorname{Fix}(f)$ compact, then there exists a G-map $h: N \to M$ such that*

(i) *h is G-homotopic to f relative to the outside of some G-invariant compact neighborhood of $\operatorname{Fix}(f)$,*

(ii) *$\operatorname{Fix}(h)$ consists of a finite number of orbits,*

(iii) *if* $f(N_{(H)}) \cap M_{(H)} = \phi$ *then* $h(N_{(H)}) \cap M_{(H)} = \phi$ *and hence* $\mathrm{Fix}(h) \cap N_{(H)} = \phi$,

(iv) *for any fixed orbit* $G(x) \subset \mathrm{Fix}(h)$ *if* $T = T_t \oplus T_n$ *and* $T' = T'_t \oplus T'_n$ *are* G*-invariant tubular neighborhoods of* $G(x)$ *as above, then* $h|T: T \to T'$ *is fibre preserving and decomposes into* $h|T = (h|T_t) \oplus 0$ *where* $0: T_n \to T'_n$ *maps any vector to* 0.

Proof. (I) *The case in which the* G*-action on* N *is free.* $N \times M$ is a G-manifold with diagonal G-action, and its action is also free. Thus the orbit spaces N/G and $N \times_G M$ are smooth manifolds. Define a G-map $\tilde{f}: N \to N \times M$ by $\tilde{f}(x) = (x, f(x))$ for $x \in N$. Passing to the orbit spaces, $\tilde{f}$ induces a map $\tilde{f}/G: N/G \to N \times_G M$. By the transversality theorem we obtain a smooth map $h_1: N/G \to N \times_G M$ such that

(i) h_1 is transverse to Δ/G, where Δ is the diagonal set in $N \times M$, and

(ii) h_1 is close enough and homotopic to $\tilde{f}/G$ relative to $N - V/G$, where V is some G-invariant compact neighborhood of $\mathrm{Fix}(h)$.

By the dimension reason $h_1^{-1}(\Delta/G)$ is a finite set, in particular it is empty if $\dim G > 0$. If $f(N) \cap M_{(1)} = \phi$ where $M_{(1)}$ is the points of M with the identity isotropy subgroup, then $\mathrm{Fix}(f) = \phi$, $\tilde{f}/G(N/G) \cap \Delta/G = \phi$ and hence we may take $h_1 = \tilde{f}/G$. By the equivariant covering homotopy property we may lift the homotopy of (ii) and obtain a G-map $h_2: N \to N \times M$ G-homotopic to $\tilde{f}$ relative to the outside of some G-invariant compact neighborhood of $\mathrm{Fix}(f)$. $h_2^{-1}(\Delta)$ consists of a finite number of orbits. Let $p_1: N \times M \to N$ and $p_2: N \times M \to M$ be the projections. $p_1 h_2: N \to N$ is a diffeomorphism since it is close enough to $p_1 \tilde{f} =$ identity. Let $h_3 = h_2(p_1 h_2)^{-1}: N \to N \times M$ and $h = p_2 h_3: N \to M$. then $h_3(x) = (x, h(x))$ and $\mathrm{Fix}(h) = h_3^{-1}(\Delta) \approx h_2^{-1}(\Delta)$. h is a desired G-map.

(II) *The general case.* Let $\{(H_1), (H_2), \cdots, (H_a)\}$ be the set of isotropy types on N ordered in such a way that if H_i is conjugate to a subgroup of H_j then $j \leq i$. Consider the following assertion A(i) for $0 \leq i \leq a$:

A(i). *There exist a* G*-map* $h_i: N \to M$ *and a* G*-invariant neighborhood* U_i *of* $X_i = N_{(H_1)} \cup \cdots \cup N_{(H_i)}$ *such that*

(i) h_i *is* G*-homotopic to* f *relative to the outside of some* G*-invariant compact neighborhood of* $\mathrm{Fix}(f|X_i)$,

(ii) $\mathrm{Fix}(h_i) \cap (U_i - X_i) = \phi$,

(iii) $h_i|U_i: U_i \to M$ *satisfies the conditions* (ii), (iii) *and* (iv) *of the lemma.*

If $i=0$, then $X_i = \phi$ and hence we may take $U_i = \phi$, $h_i = f$. Thus A(0) is valid. A(a) is equivalent to the lemma since $X_a = N$. Thus, to prove the lemma it suffices to prove that A(i) implies A($i+1$).

Now suppose A(i). As in the author [8; Lemma 3.1] there exists a G-invariant codimension 0 submanifold P (with boundary) of N such that $X_i \subset \mathrm{Int}\, P \subset P \subset \mathrm{Int}\, U_i$. Let $Q = N - \mathrm{Int}\, P$ and $K = H_{i+1}$. Consider an $N(K)$-map

$h_i|Q^K$: $Q^K \to M^K$, where $N(K)$ is the normalizer of K in G. $h_i|Q^K$ may also be considered as an $N(K)/K$-map. Since K is the maximal isotropy subgroup on Q, then the action of $N(K)/K$ on Q^K is free. Thus we may apply the preceding argument (I) to the $N(K)/K$-map $h_i|Q^K$, and obtain a resulting $N(K)/K$-map $Q^K \to M^K$. By G-equivariancy it extends to a G-map f_1: $Q_{(K)}=G(Q^K) \to M$, which satisfies the conditions (i)~(iv) of the lemma. To be precise for the condition (i) it says that f_1 is G-homotopic to $h_i|Q_{(K)}$ relative to the outside of some G-invariant compact neighborhood (in $Q_{(K)}$) of $\mathrm{Fix}(h_i|Q_{(K)})$. Moreover its G-homotopy may be so taken as to be relative to a neighborhood of $\partial Q_{(K)}$, since h_i has no fixed point in a neighborhood of $\partial Q_{(K)}$. Let T be a G-invariant tubular neighborhood of $Q_{(K)}$ in Q and π: $T \to Q_{(K)}$ be the projection. Then we may extend f_1 to a G-map f_2: $T \to M$ such that

(i) for some two neighborhoods $U \subset U'$ (U' compact) of $\mathrm{Fix}(f_1)$ in $Q_{(K)}$, $f_2=f_1 \circ \pi$ on $T|U$ and $f_2=h_i$ on $T|Q_{(K)}-U'$,

(ii) $\mathrm{Fix}(f_2) \cap (T-Q_{(K)})=\phi$.

From $h_i|Q$ and f_2, as in the author [8; Lemma 3.2], we obtain a G-map f_3: $Q \to M$ such that

(i) $f_3=h_i$ on a neighborhood A of ∂Q, $f_3=f_2$ on a neighborhood of $Q_{(K)}$, $f_3=h_i=f$ on the outside of a G-invariant compact neighborhood B of $\mathrm{Fix}(f_1)$ ($=\mathrm{Fix}(f_2)$),

(ii) f_3 is G-homotopic to $h_i|Q$ relative to $A \cup (Q-B)$.

Define h_{i+1}: $N \to M$ as $h_{i+1}=h_i$ on P and $h_{i+1}=f_3$ on Q. Then h_{i+1} is a G-map required in A($i+1$). q.e.d.

3. Fixed point index

We first recall the definition of the fixed point index from Dold [2]. Let $F \subset N \subset R^n \subset R^n \cup \{\infty\} = S^n$, where F is compact and N is open. The *fundamental class* $o_F \in H_n(N, N-F; Z)$ is the image of 1 under the composite homomorphism

$$Z = H_n(S^n; Z) \to H_n(S^n, S^n-F; Z) \cong H_n(N, N-F; Z)\,.$$

Let h: $N \to R^n$ be a map with $\mathrm{Fix}(h)$ compact. Define the map $1-h$: $(N, N-F) \to (R^n, R^n-0)$ by $(1-h)(x)=x-h(x)$ for $x \in N$. Then the *fixed point index* $\mathrm{ind}(h)$ of h is defined as $\mathrm{ind}(h)=(1-h)_* o_F \in H_n(R^n, R^n-0; Z)=Z$. Dold uses the symbol I_h for the index, but we use the symbol $\mathrm{ind}(h)$ to facilitate the printing.

Let R^n be a Euclidean G-space, N be a G-invariant open subspace of R^n, and h: $N \to R^n$ be a G-map satisfying the conditions (ii) and (iv) of Lemma 5. Let $\mathrm{Fix}(h)=G(x_1) \cup G(x_2) \cup \cdots \cup G(x_a)$ with $G_{x_i}=H_i$ $(1 \leq i \leq a)$. If T_i is a small G-invariant open tubular neighborhood of $G(x_i)$ in N, then by the additivity of the index [2; (1.5)] it follows that

$$\mathrm{ind}(h) = \sum_{i=1}^{a} \mathrm{ind}(h \mid T_i)\,.$$

For a while let $x=x_i$, $H=H_i$, $T=T_i$. We may consider that a fibre in T over $g(x)\in G(x)$ is a subspace in R^n which is a parallel translation to $g(x)$ of (a small open disc in) a linear subspace through the origin. Let $\pi\colon T\to G(x)\subset R^n$ be the projection, and T' be the other G-invariant open tubular neighborhood of $G(x)$ as in (iv) of Lemma 5. Define $1-h+\pi\colon T\to T'$ as $(1-h+\pi)(v)=v-h(v)+\pi(v)$. This map is fibre preserving, and the following diagram is commutative for any $g\in G$.

$$\begin{array}{ccc} H_n(T,\ T-G(x)) & \xrightarrow{j_*} & H_n(T,\ T-g(x)) = Z \\ {\scriptstyle (1-h+\pi)^*}\downarrow & & \downarrow{\scriptstyle (1-h+\pi)_*} \\ H_n(T',\ T'-G(x)) & \xrightarrow[j_*]{} & H_n(T',\ T'-g(x)) = Z\,, \end{array}$$

where $j\colon (T,\ T-G(x))\to(T,\ T-g(x))$ is the inclusion. Let $o=o_{G(x)}\in H_n(T,\ T-G(x))$ be the fundamental class. Let $\alpha_g=j_*(1-h+\pi)_*o\in Z$. By the commutativity of the diagram, $\alpha_g=(1-h+\pi)_*j_*o$ and $j_*o=1$ in $H_n(T,\ T-g(x))=Z$. Since $1-h+\pi$ is G-equivariant, α_g are all equal for every $g\in G$. So, if α is its the same value, then we see that $(1-h+\pi)_*o=\alpha\cdot o$ in $H_n(T',\ T'-G(x))$.

$$(1-h)_*\colon H_n(T,\ T-G(x)) \to H_n(R^n,\ R^n-0)$$

factors as

$$H_n(T,\ T-G(x)) \xrightarrow{(1-h+\pi)_*} H_n(T',\ T'-G(x)) \xrightarrow{(1-\pi)_*} H_n(R^n,\ R^n-0)\,.$$

Thus we see that in $H_n(R^n,\ R^n-0)$

$$(1-h)_*o = (1-\pi)_*(1-h+\pi)_*o = \alpha\cdot(1-\pi)_*o\,,$$

and hence $\mathrm{ind}(h\mid T)=\alpha\cdot\mathrm{ind}(\pi)$. Since $\mathrm{ind}(\pi)=\chi(G/H)$ by [2; (4.1)], it follows that $\mathrm{ind}(h\mid T_i)$ is a multiple of $\chi(G/H_i)$ for $i=1, 2, \cdots, a$.

Let $\mathrm{Fix}(h)\cap N^G=\{x_1, x_2, \cdots, x_b\}$ $(1\le b\le a)$. For $1\le i\le b$ the tubular neighborhood T_i is a disc with x_i as its center. As before T_i decomposes into the direct sum $T_i=T_{i,t}\oplus T_{i,n}$ where $T_{i,t}=T_i^G$ is the component tangent to N^G and $T_{i,n}$ is the component normal to N^G. Then, from the condition (iv) of Lemma 5 we see that h on T_i decomposes into $h(u, v)=(h(u), 0)$. Thus, by [2; (1.4), (1.6)], $\mathrm{ind}(h\mid T_i)=\mathrm{ind}(h\mid T_i^G)$ and hence

$$\sum_{i=1}^{b} \mathrm{ind}(h\mid T_i) = \mathrm{ind}(h^G)\,.$$

From the above argument it follows the following.

(i) If $\mathrm{Fix}(h)\subset N^G\cup N_{(H)}$, then $\mathrm{ind}(h)\equiv\mathrm{ind}(h^G) \bmod \chi(G/H)$.

(ii) If G is finite and of prime power order p^k, then $\mathrm{ind}(h)\equiv\mathrm{ind}(h^G) \bmod p$. For $\chi(G/H)\equiv 0 \bmod p$ for any proper subgroup H of G.

(iii) If G is connected and abelian, then $\mathrm{ind}(h)=\mathrm{ind}(h^G)$. For $\chi(G/H)=0$ for any proper subgroup H of G.

4. Proof of Theorem 1

Let $f\colon X\to X$ be as in the theorem. Let N be a G-invariant open subspace in a Euclidean G-space R^n, and $i\colon X\to N$, $r\colon N\to X$ be G-maps such that $ri=$identity. We easily see that X^H is also an ENR for any $H<G$. We apply Lemma 5 to the G-map $ifr\colon N\to R^n$ and obtain a G-map $h\colon N\dot{\to}R^n$ satisfying the conditions (i)$\sim$(iv). By [2; (1.7), (4.1)] it follows that $\lambda(f^H)=\mathrm{ind}((ifr)^H)=\mathrm{ind}(h^H)$. From this and (ii), (iii) in the preceding section, (iii) and (iv) of the theorem immediately follow. If X has only one isotropy type (H), then $(ifr)(N_{(K)})\subset R^n_{(H)}$ for any $K<G$. Thus, from (iii) of Lemma 5 it follows $\mathrm{Fix}(h)\subset N_{(H)}$ and from (i) in the preceding section it follows $\lambda(f)=\mathrm{ind}(h)\equiv 0 \bmod \chi(G/H)$. This proves (i) of the theorem. If the G-action on X is semifree, from (iii) of the lemma it follows $\mathrm{Fix}(h)\subset N^G\cup N_{(1)}$. Thus (ii) of the theorem follows from (i) in the preceding section.

5. Proof of Proposition 3 and 4

Let X be a compact G-ENR and N, i, r be as in section 4. If G is a compact monogenic Lie group, we regard a generator f of G as a G-map $f\colon X\to X$. Then $\mathrm{Fix}(f)=X^G$. Let $h\colon N\to R^n$ be a G-map obtained by Lemma 5 from the G-map $ifr\colon N\to R^n$. In this case we may construct h satisfying the additional condition $\mathrm{Fix}(h)\subset N^G$. This is ensured by the fact $\mathrm{Fix}(ifr)=i(X^G)\subset N^G$. Then we see that $\lambda(f)=\mathrm{ind}(h)=\mathrm{ind}(h^G)=\lambda(f^G)$. Since f^G is the identity map of X^G, it follows $\lambda(f)=\chi(X^G)$. As noticed in Introduction this has already appeared in the literature. Using this result, Proposition 3 and 4 are proved as follows.

(1) Proof of Proposition 3. Let $f_{*,i}\colon H_i(X; C)\to H_i(X; C)$ be the automorphism induced from f, where C is the complex numbers. Let $z_1, z_2, \cdots, z_r \in C$ be the eigenvalues of $f_{*,i}$ where $r=\dim H_i(X; C)=\mathrm{rank}\, H_i(X; Z)$. Since the $\chi(G)$ times composition of $f_{*,i}$ is the identity, then $z_j^{\chi(G)}=1$ and thus $|z_j|=1$ for $1\le j\le r$. We see that

$$|\mathrm{trace}\, f_{*,i}| = |\textstyle\sum_{j=1}^r z_j| \le \sum_{j=1}^r |z_j| = \mathrm{rank}\, H_i(X; Z)\,,$$

and

$$|\chi(X^G)| = |\lambda(f)| \le \textstyle\sum_i \mathrm{rank}\, H_i(X; Z)\,. \qquad \text{q.e.d.}$$

(2) Proof of Proposition 4. Let G, f and M be as in the proposition. Note that a smooth G-manifold with a finite number of isotropy types is a G-ENR. Let $z\in H_{2n}(M; Z)$ be the fundamental class defined from an orientation of M for which f is either orientation preserving or reversing. Consider the following commutative diagram.

$$\begin{array}{ccc} H_i(M;Z) & \xrightarrow{\ \cap z\ } & H^{2n-i}(M;Z) \\ {\scriptstyle f_{*,i}}\downarrow & & \uparrow{\scriptstyle f^{*,2n-1}} \\ H_i(M;Z) & \xrightarrow{\ \cap f_{*,2n}(z)\ } & H^{2n-1}(M;Z)\,, \end{array}$$

where $\cap$ denotes the cap product and the horizontal homomorphisms are the isomorphisms of the Poincaré duality. Note that the inverse of the isomorphism $f^{*,2n-i}$ is itself since f is an involution. It follows that if f is orientation preserving then $\operatorname{trace} f_{*,i}=\operatorname{trace} f^{*,2n-i}$, and if f is orientation reversing then $\operatorname{trace} f_{*,i}=-\operatorname{trace} f^{*,2n-i}$. From the universal coefficient theorem it follows that $\operatorname{trace} f^{*,i}=\operatorname{trace} f_{*,i}$. It thus follows that if f is orientation preserving then $\lambda(f)\equiv\operatorname{trace} f_{*,n} \bmod 2$, and if f is orientation reversing then $\lambda(f)=\operatorname{trace} f_{*,n}=0$. Thus $\lambda(f)=\chi(M^G)$ implies the proposition. q.e.d.

In case M is odd dimensional in Proposition 4, we then easily see that $\lambda(f)=\chi(M^G)=0$ if f is orientation preserving, and $\lambda(f)=\chi(M^G)\equiv 0 \bmod 2$ if f is orientation reversing.

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