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THE LEFT REGULAR *-REPRESENTATION OF AN INVERSE SEMIGROUP

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ABSTRACT. The left regular *-representation of the semigroup algebra of an inverse semigroup is faithful. Clifford semigroups with a particular type of semilattice are shown to have the weak containment property if and only if each subgroup is amenable.

An *inverse semigroup* is a semigroup in which for each element s there exists a unique element, which we denote s^*, such that $ss^*s = s$ and $s^*ss^* = s^*$. From this it follows that the idempotents commute, products of idempotents are idempotents, and $(st)^* = t^*s^*$ [**5**, pp. 130–131]. For an inverse semigroup S, E_S will denote the set of idempotents of S. E_S is a lower semilattice under the operation $e \wedge f = ef$. A *Clifford semigroup* is an inverse semigroup T whose idempotents are central; then $T = \bigcup\{G_e \colon e \in E_T\}$ where G_e is the greatest subgroup of T containing e.

We identify $s \in S$ with the function in $l^1(S)$ which is 1 at s and 0 elsewhere. Then $l^1(S)$ is a Banach algebra with multiplication the continuous bilinear extension of the semigroup multiplication, called the (l^1-)*semigroup algebra* of S. The involution * on S extends to a unique continuous involution * on $l^1(S)$ by conjugate linearity. Then $l^1(S)$ is a Banach star algebra.

Barnes [**1**] constructed the *left regular *-representation* of $l^1(S)$ (on $l^2(S)$), which we will denote by L_S, by

$$L_S(a)b = \begin{cases} ab & \text{if } a^*ab = b, \text{ i.e. if } a^*a \geq bb^*, \\ 0 & \text{otherwise.} \end{cases}$$

for the elements of S, and extending to $l^1(S) \times l^2(S)$ by continuity and linearity. (The left regular representation of $c_{00}(S)$ on $c_{00}(S)$ is defined for $a, b \in S$ by $\lambda(a)b = ab$ and extended by linearity. It need not be continuous if the range space is given the l^2-norm, and is a *-representation only if S is a group, in which case it is the same as Barnes's construction.) He proved that if E_S was a lattice, and not just a semilattice, under the semilattice order, than this representation was faithful. By imbedding S in a symmetric inverse semigroup [**5**, pp. 133–135], and hence $l^1(S)$ in the latter's l^1-algebra, he proved that $l^1(S)$ was an A^*-algebra for S an inverse semigroup. We prove here that the left regular *-representation is always faithful.

Let A be a Banach *-algebra. If S and T are *-representations on Hilbert spaces, one says that S *weakly contains* T if there is a *-homomorphism U such that

$$\begin{array}{ccc} A & \xrightarrow{S} & \overline{S(A)} \\ & \searrow^{T} & \downarrow U \\ & & \overline{T(A)} \end{array}$$

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commutes, where the closures are in the norm topologies. This is equivalent to $||Tx|| \leq ||Sx||$ for all $x \in A$. See [**3**, p. 60] for further references. Thus a faithful *-representation of a Banach *-algebra weakly contains all other *-representations if and only if it induces a faithful *-representation of the enveloping C^*-algebra.

An inverse semigroup is said to have the *weak containment property* if its left regular *-representation weakly contains all other *-representations. For a group, this is equivalent to amenability [**3**]. Paterson [**6**] proved that a Clifford semigroup has the weak containment property if all its subgroups are amenable, and proved this condition to be necessary if the semilattice was of a certain type. We extend this necessity to a wider class of semilattice.

Let E be a semilattice. For $e \in E$ define ψ_e by

$$\psi_e(f) = \begin{cases} 1 & \text{if } f \geq e, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 1. *The ψ_e separate $l^1(E)$, where E is a lower semilattice.*

PROOF. Let $x \in l^1(E)$. Let $\phi \in l^\infty(E)$ with $\phi(e) = \prod_{i=1}^n \psi_{u(i)}(e)$ for all $e \in E$. (Such a function will be called a product of ψ_e's.) Then I claim $\psi_e(x) = 0$ for all $e \in E$ implies that $\phi(x) = 0$.

Let $F = \{e \in E\colon e \geq u(r) \text{ for } 1 \leq r \leq n\}$. Direct F downwards, i.e. by $\geq$. Then for $f \in E, \prod_{i=1}^n \psi_{u(i)}(f) = \lim_{e\in F} \psi_e(f)$. The ψ_e are uniformly bounded, so $\phi(x) = \lim_{e\in F} \psi_e(x) = 0$.

The final part of the proof is an obvious simplification of [**4**, Theorem 3.4].

Suppose $\psi_e(x) = 0$ for all $e \in E$.

Let $x = \sum_{r=1}^\infty a_r e_r$ with $a_r \in \mathbf{C}, a_1 \neq 0$ and the e_r distinct, then $\lim_{e\downarrow} \psi_e(x) = \sum_{r=1}^\infty a_r = 0$. For $r \geq 2$, there exists $f(r) \in E$ such that $\psi_{f(r)}(e_1) \neq \psi_{f(r)}(e_r)$. Define $\phi_n \in l^\infty(E)$ by

$$\phi_n(e) = \prod_{r=2}^n \frac{\psi_{f(r)}(e) - \psi_{f(r)}(e_r)}{\psi_{f(r)}(e_1) - \psi_{f(r)}(e_r)}.$$

Then $\phi_n(e_1) = 1, \phi_n(e_r) = 0$ for $2 \leq r \leq n$ and $||\phi_n||_\infty = 1$. Then $\phi_n(x) = 0$ as ϕ_n is a sum of multiples of ψ_n's and their products, so

$$|a_1| \leq \sum_{r=n+1}^\infty |a_r|.$$

Thus $a_1 = 0$, a contradiction.

For $s \in S$, define $\pi_s \in l^\infty(S)$ by $\pi_s(t) = 1$ if $t = s$, 0 otherwise.

THEOREM 2. *Let S be an inverse semigroup. Then the left regular *-representation is faithful.*

PROOF. We shall treat $l^1(S)$ as a subspace of $l^2(S)$.

Let $x \in l^1(S)$ and suppose $L_S(x) = 0$. Then $L_S(x)e = 0$ for all $e \in E$. x can be written as $\sum_{s\in S} \alpha_s s$ with $\alpha_s \in \mathbf{C}$.

For $e \in E$ let x_e be $\sum\{\alpha_s s\colon s^*s = e\}$. Then $x_e \in l^1(S)$ and $x = \sum_{e\in E} x_e$. For $f \in E$,

$$L_S(x_e)f = \begin{cases} x_e f & \text{if } e \geq f, \\ 0 & \text{otherwise.} \end{cases}$$

Let $u \in E$. Let $F = \{e \in E: e \geq u\}$. Then F is a semilattice. Then for $f \in F$, $L_S(x)f = \sum_{e \geq f} x_e f$, so $0 = \sum_{e \geq f}(x_e f)u = \sum_{e \geq f} x_e u$. Let $w_s = \sum\{\pi_s(x_f u)f: f \in F\}$. Now

$$\|w_s\| \leq \sum_{f \in F} |\pi_s(x_f u)| \leq \sum_{f \in F} \|x_f u\| \leq \sum_{f \in F} \|x_f\| \leq \|x\|,$$

so $w_s \in l^1(F)$.

But for all $f \in F$,

$$\sum_{e \geq f} w_s(e) = \sum_{e \geq f} \pi_s(x_e u) = \pi_s\left(\sum_{e \geq f} x_e u\right) = 0.$$

So by Lemma 1, $w_s = 0$.

Now $x_u = x_u u$, so $\pi_s(x_u) = \pi_s(x_u u) = w_s(u) = 0$. Therefore $x_u = 0$. But u was arbitrary, so $x = 0$.

W. D. Munn (personal communication) has proved the corresponding algebraic result (namely that L_S faithfully represents the semigroup ring on $c_{00}(S)$) but his proof proceeds by finite induction on the support of ring elements.

THEOREM 3. *Let S be a Clifford semigroup with the weak containment property. Then every subgroup of S is amenable if for all $e \in E_S$, there exists a finite subset F of E_S such that whenever $u < e$, there exists $f \in F$ such that $u \leq f < e$. (This is equivalent to saying that for all $e \in E_S$ there is a minimal idempotent u of $l^1(E_S)$ such that $eu = e$ and $fu = 0$ for all $f < e$.)*

PROOF. Pick f in E_S. Define *-endomorphism F of $l^1(S)$ by

$$Ps = \begin{cases} s & \text{if } s \in G_e \text{ with } e \geq f, \\ 0 & \text{if } s \in G_e \text{ with } e \not\geq f. \end{cases}$$

Then $L_S P$ is a *-representation of $l^1(S)$, so for $x \in l^1(S)$, $\|L_S P(x)\| \leq \|L_S(x)\|$.

Let F be a finite set such that $h < f$ for all $h \in F$, and whenever $f > u \in E_S$, $u \leq h$ for some $h \in F$. Then for $x \in l^1(G_f)$,

$$\|L_S P(x)\| = \|L_S P(x \Pi_F (1 - h))\| = \|L_{G_f}(x)\|,$$

so as $x = Px$,

$$\|L_S(x)\| = \|L_S P(x)\| = \|L_{G_f}(x)\|.$$

Let T be a *-representation of $l^1(G_f)$ on a Hilbert space $\mathcal{H}$. Let $\tilde{T}: l^1(s) \to BL(\mathcal{H})$ be the *-homomorphism with

$$\tilde{T}(s) = \begin{cases} T(fs) & \text{if } s \in G_e \text{ with } e \geq f, \\ 0 & \text{if } s \in G_e \text{ with } e \not\geq f. \end{cases}$$

Then $\tilde{T}$ is a *-representation of $l^1(S)$ on $\mathcal{H}$ extending T, so for $x \in l^1(G_f)$, $\|\tilde{T}x\| \leq \|L_S(x)\|$ by [**2**].

Thus $\|Tx\| \leq \|L_{G_f}(x)\|$ for each *-representation T of $l^1(G_f)$. By [**2**, Proposition 2.7.1], $l^1(G_f)$ has the weak containment property (see formulation in [**3**]) and so by [**3** or **7**], G_f is amenable.

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