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THE LEWY COUNTEREXAMPLE AND THE LOCAL EQUIVALENCE PROBLEM FOR G-STRUCTURES

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1. Let G be a Lie subgroup of $GL(n)$. Let M_1 and M_2 be differential manifolds of dimension n (in this paper all data will be assumed to be C^∞), and let $\mathcal{F}_i$, $i = 1, 2$, be the principal frame bundle on M_i. A sub-bundle, P_i, of $\mathcal{F}_i$ with structure group G is called a G-structure on M_i. The G-structure on M_1 is said to be equivalent to the G-structure on M_2 if there exists a diffeomorphism $f : M_1 \to M_2$ such that the induced diffeomorphism $f^* : \mathcal{F}_1 \to \mathcal{F}_2$ carries P_1 into P_2.

It is usually difficult to decide when two G-structures are equivalent; however the problem is a little simpler if we suppose that one of the structures, say P_1, is locally transitive, and look only at the local problem. Then the following is a necessary condition for the two structures to be locally equivalent:

* At every point $m_1 \in M_1$ and every point $m_2 \in M_2$ there exists a power series mapping ρ (in local coordinates with origins at m_1 and m_2) such that ρ formally effects a local equivalence between P_1 and P_2.

It might seem that (*) is not much of an improvement over the original problem; however, by techniques of homological algebra it can be converted into a much simpler statement about the vanishing of certain canonically defined tensors on P_2 (cf. [1], [2], [4]). The main problem therefore is to show that condition (*) is sufficient. This is known to be true in the following important cases:

1) G is of finite type.

2) The data are real analytic.

According to a recent result of Malgrange (unpublished) it is known to be true when G is elliptic. According to a result of the first author it is true when P_1 is flat. The purpose of this note is to show that condition (*) isn't always sufficient. In fact we will show that in certain cases the solution of the equivalence problem depends on the solution of a system of linear inhomogeneous partial differential equations resembling the Lewy counterexample [3]. These equations are determined, all the data in them are C^∞ and they have no solutions even in the weak (distribution) sense.

2. Let X_1, X_2 and X_3 be globally defined vector fields on R^3 satisfying the following commutation relations: $[X_1, X_2] = X_3, [X_1, X_3] = X_1, [X_2, X_3] = -X_2$. (Take for example the standard basis of so(3) and identify R^3 with a subset of SO(3) under the mapping exp: so(3) $\to$

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SO(3).) Let $X_i = \sum_{j=1}^{3} C_{ij} \frac{\partial}{\partial x_j}\cdot$ Let $(x_1, x_2, x_3, y_1, y_2)$ be coordinates on R^5 and consider on R^5 the moving frame:

$$X_1,\ X_2,\ X_3,\ \frac{\partial}{\partial y_1},\ \frac{\partial}{\partial y_2}.$$

We will define a G-structure on R^5 which has this moving frame as a global cross-section and has for a structure group the group of all 5×5 matrices of the form

$$\left(\begin{array}{ccc|c} & \mathrm{I} & & 0 \\ \hline a, & b, & c & \\ -b, & a, & d & \mathrm{I} \end{array}\right) \tag{2.1}$$

where the upper left hand block in (2.1) is the 3×3 identity matrix and the lower right hand block the 2×2 identity matrix. The G structure, which we will denote by P_1, is obtained by letting the matrices (2.1) act in all possible ways on the moving frame:

$$X_1,\ X_2,\ X_3,\ \frac{\partial}{\partial y_1},\ \frac{\partial}{\partial y_2}.$$

We will first of all determine the local diffeomorphisms of R^5 into itself which preserve P_1.

Let f be such a diffeomorphism, and let f have the form:

$$\begin{aligned} x_i' &= f_i(x,\ y), \qquad & i &= 1,\ 2,\ 3, \\ y_\alpha' &= \varphi_\alpha(x,\ y), \qquad & \alpha &= 1,\ 2. \end{aligned}$$

From the condition

$$f^* \frac{\partial}{\partial y_\alpha} = \frac{\partial}{\partial y_\alpha} \qquad \alpha = 1, 2,$$

we get

$$\frac{\partial \varphi_\alpha}{\partial y_\beta} = \delta_\beta^\alpha, \qquad \frac{\partial f_i}{\partial y_\beta} = 0.$$

Thus

$$\begin{aligned} x_i' &= f_i(x), \qquad & i &= 1,\ 2,\ 3, \\ y_\alpha' &= y_\alpha + \phi_\alpha(x), \qquad & \alpha &= 1,\ 2. \end{aligned} \tag{2.2}$$

Next applying f_* to X_i we get

$$
\begin{aligned}
f_* X_i &= f_* \left(\sum_{j=1}^{3} c_{ij} \frac{\partial}{\partial x_j} \right) \\
&= \sum_{j=1}^{3} c_{ij} \left(\sum_{k=1}^{3} \frac{\partial f_k}{\partial x_j} \frac{\partial}{\partial x_k} + \sum_{\alpha=1}^{2} \frac{\partial \phi_\alpha}{\partial x_j} \frac{\partial}{\partial y_\alpha} \right) \qquad (2.3) \\
&= \sum_{k=1}^{3} \mathcal{L}_{X_i} f_k \frac{\partial}{\partial x_k} + \sum_{\alpha=1}^{2} \mathcal{L}_{X_i} \phi_\alpha \frac{\partial}{\partial y_\alpha}.
\end{aligned}
$$

However, $f_* X_i$ must be the form

$$X_i + \sum_{\alpha=1}^{2} h_{\alpha i} \frac{\partial}{\partial y_\alpha},$$

where $h_{\alpha i}$ is a 2×3 matrix of the form

$$\begin{pmatrix} \mathrm{a}, & \mathrm{b}, & \mathrm{c} \\ -\mathrm{b}, & \mathrm{a}, & \mathrm{d} \end{pmatrix}.$$

From (2.1) we get a condition on f_1, f_2, f_3, namely, $f_1(x)$, $f_2(x)$, $f_3(x)$ must define a diffeomorphism of R^3 into itself preserving X_1, X_2, X_3.

We also get two conditions on ϕ_1, ϕ_2:

$$\mathcal{L}_{X_1} \phi_2 - \mathcal{L}_{X_2} \phi_1 = 0, \qquad \mathcal{L}_{X_1} \phi_2 + \mathcal{L}_{X_2} \phi_1 = 0.$$

These two equations can be more compactly written in complex form:

$$\mathcal{L}_{X_1 + \sqrt{-1} X_2} (\varphi_1 + \sqrt{-1} \varphi_2) = 0. \qquad (2.4)$$

Summing up what has been proved above:

Proposition 1. *The diffeomorphisms of P_1 consist of all mappings of R^5 into R^5 of the form:*

$$
\begin{aligned}
x_i' &= f_i(x), & i &= 1, 2, 3, \\
y_\alpha' &= y_\alpha + \phi_\alpha(x), & \alpha &= 1, 2,
\end{aligned}
$$

where $\tilde{f} = (f_1, f_2, f_3)$ belongs to the (local) Lie group on R^3 associated with X_1, X_2, X_3, and ϕ_1, ϕ_2 satisfy (2.4).

It is clear from Proposition 1 that the G-structure described above is transitive. In fact, it is frame transitive (the family of mappings induced on P_1 is transitive) and involutive (cf. [1] for definitions).

Now we consider another G-structure defined on R^5 as follows. Let

$$X_i' = X_i + \sum_{\alpha=1}^{2} g_{\alpha i}(x) \frac{\partial}{\partial y_\alpha}$$

where the $g_{\alpha i}$ are for the moment unspecified functions of x. Let P_2 be the G-structure obtained by applying all the matrices (2.1) to the moving frame

$$X_1', X_2', X_3', \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}.$$

We will consider what is involved in proving that P_1 and P_2 are locally equivalent. We must be able to find a local diffeomorphism f, of R^5 into R^5 with the property:

$$f_* X_i = X_i' + \sum_{\alpha=1}^{2} h_{\alpha i} \frac{\partial}{\partial y_\alpha}, \qquad i = 1,\ 2,\ 3,$$
$$f_* \frac{\partial}{\partial y_\alpha} = \frac{\partial}{\partial y_\alpha}, \qquad \alpha = 1,\ 2,$$

where $(h_{\alpha i})$ is of the form

$$\begin{pmatrix} \mathrm{a}, & -\mathrm{b}, & \mathrm{c} \\ \mathrm{b}, & \mathrm{a}, & \mathrm{d} \end{pmatrix}.$$

By an argument similar to that above we can show that f must have the following form in coordinates:

$$x_i' = f_i(x), \qquad i = 1,\ 2,\ 3,$$
$$y_\alpha' = y_\alpha + \phi_\alpha(x), \qquad \alpha = 1,\ 2,$$

where the conditions on f_1, f_2, f_3 are the same as in Proposition 1, but ϕ_1, ϕ_2 must satisfy the equation

$$(2.5) \qquad (\mathcal{L}_{X_1 + \sqrt{-1}\, X_2})(\phi_1 + \sqrt{-1}\, \phi_2) = g_1 + \sqrt{-1}\, g_2,$$

where

$$g_1 = g_{11}(\cdots, f_i(x), \cdots) - g_{22}(\cdots, f_i(x), \cdots),$$
$$g_2 = g_{12}(\cdots, f_i(x), \cdots) + g_{21}(\cdots, f_i(x), \cdots).$$

Lewy has shown that one can always choose the right hand side of (2.5) such that even locally it is impossible to find C^1 functions ϕ_1 and ϕ_2 which satisfy (2.5). On the other hand this equation is always formally solvable; so the condition (*) of §1 is certainly satisfied by P_1 and P_2. We can therefore conclude that (*) does not always guarantee that the two structures are locally equivalent.

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