The Lie Algebra Structure of Nonlinear Evolution Equations Admitting Infinite Dimensional Abelian Symmetry Groups

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Hereditary operators in Lie algebras are investigated. These are operators which are characterized by a special algebraic equation and their main property is that they generate abelian subalgebras of the given Lie algebra. These abelian subalgebras are infinite dimensional if the hereditary operator is not cyclic. As a consequence hereditary operators generate on a systematic level nonlinear dynamical systems which possess infinite dimensional abelian groups of symmetry transformations. We show that hereditary operators can be understood as special Lie algebra deformations with a linear interpolation property. In order to construct new hereditary operators out of given ones we study the permanence properties of these operators; this study of permanence properties leads in a natural way to a notion of compatibility. For local hereditary operators it is shown that eigenvector decompositions are time invariant (such an eigenvector decomposition is known[1] to characterize pure multisoliton solutions). Apart from the well-known equations (KdV, sine-Gordon, etc.), we give—as examples—many new nonlinear equations with infinite dimensional groups of symmetry transformations.

§ 1. Introduction

A detailed analysis of the celebrated Korteweg-de Vries equation reveals that this nonlinear evolution equation possesses an infinite dimensional *abelian* group of symmetry transformations. This group of symmetry transformations is given by the resolvents of the so-called generalized KdV equations. And this striking property is shared by many other nonlinear evolution equations; Only to name a few: Burgers equation, sine-Gordon equation, Zakharov-Shabat equations, Gardner equation etc. Furthermore one discovers that for these equations (except Burgers equation) the structure of this abelian symmetry group is intimately connected with the existence (and description) of multisoliton solutions, and in addition connected to the existence of infinitely many conservation laws (via Noether's theorem or rather a suitable generalization thereof).

The phenomena related to this observation have in recent years been one of the most active areas of research in applied mathematics and theoretical physics. Looking into the problems encountered in this research with a somewhat more systematic interest, one realizes that very often it is highly desirable to construct for a given element K of a Lie algebra its annihilator $K^{\perp}$ (or at least a large abelian subalgebra containing K). If one can do this, then, of course, integration of the equations for the infinitesimal generators of a Lie group yields a family of commuting flows.

In this paper we investigate mappings Φ having the property that (under additional assumptions) the annihilator $K^{\perp}$ of a given element K of a Lie algebra is mapped on itself. In a canonical way mappings of this kind are given by hereditary operators Φ on a Lie algebra L. These operators have been considered before.[1] They are characterized by a certain algebraic equation which implies that Φ is a selfmap in $K^{\perp}$.

Other authors[4),5)] have also considered special hereditary operators in connection with Hamiltonian systems. The operators considered in these papers are special insofar as they always possess a symplectic-implectic factorization[2),3)] (nevertheless these special cases cover the most important evolution equations).

We investigate the permanence properties of hereditary operators. Unfortunately it turns out that the set of these operators does not have a nice mathematical structure. Nevertheless we can give certain methods to construct new ones out of given ones (theorem 3.2). In special cases this method has been applied before[2),3)] without discovering its Lie algebra aspects.

In the last part of the paper we apply the methods developed so far to construct out of simple hereditary operators complicated new ones, thus generating on a systematic level many new classes of nonlinear evolution equations (of integro-differential type) having the property that they possess infinite dimensional abelian symmetry groups. These classes contain the well-known equations but also many new ones not yet considered in the literature.

In order to make the paper more coherent we have moved some information about hereditary operators to the Appendix. In the first part of the Appendix we clarify the interrelation between hereditary operators and special deformations of Lie products on a given vector space L. We call these deformations linear deformations of compatible Lie products. They are the tangential structure of what we call compatible deformations. To be more precise: Two Lie products $[\,,\,]_0$ and $[\,,\,]$ in a vector space L are said to be compatible if their sum $[\,,\,]_1 \underset{\text{def}}{=} [\,,\,]_0 + [\,,\,]$ is again a Lie product; and an isomorphism $\Phi:(L, [\,,\,]_0) \to (L, [\,,\,])$ is said to be a linear deformation if $(I+\Phi):(L, [\,,\,]_1) \to (L, [\,,\,])$ is again a Lie-algebra homomorphism. This property then yields a linear interpolation property for a continuous family of Lie products.

To the second part of the Appendix we have moved some of the tedious calculations which arise in the study of the permanence properties.

§ 2. Hereditary operators

Consider a vector space L (over $\boldsymbol{R}$ or $\boldsymbol{C}$) and let a Lie product $[\,,\,]$ be given in L. A linear map $\Phi: L \to L$ is called a *hereditary operator* if

$$\Phi^2[a, b] + [\Phi(a), \Phi(b)] - \Phi\{[a, \Phi(b)] + [\Phi(a), b]\} = 0 \tag{2·1}$$

for all $a, b \in L$. A simple calculation shows that Lie algebra isomorphisms transfer hereditary operators into hereditary operators.

In order to see what hereditary operators can do for the construction of annihilators or abelian subalgebras, we define that a linear map $\Phi: L \to L$ is said to *commute* with $a \in L$ if

$$\Phi[a, b] = [a, \Phi(b)] \qquad \text{for all } b \in L\,. \tag{2·2}$$

Now, if Φ is hereditary and commutes with a, then for this special a, two terms of (2·1) cancel and we get:

$$[\Phi(a), \Phi(b)] - \Phi[\Phi(a), b] = 0 \tag{2·3}$$

for all $b \in L$. Hence, if a hereditary Φ commutes with an element $a \in L$, then it commutes with $\Phi(a)$ (immediate consequence of (2·3)). If Φ is in addition injective, then Φ commutes with $\Phi(a)$ if and only if it commutes with a. To see this, we observe that if Φ commutes with $\Phi(a)$ then we get from (2·1)

$$\Phi^2[a, b] - \Phi[a, \Phi(b)] = \Phi\{\Phi[a, b] - [a, \Phi(b)]\} = 0\,.$$

Since Φ is injective, this gives

$$\Phi[a, b] - [a, \Phi(b)] = 0\,,$$

and Φ must commute with a.

Let us list some of the consequences of these observations:

2.1 **Consequences**: *Let Φ commute with a.*

(i) *Φ maps $a^\perp$ (the annihilator of a) into $a^\perp$, hence $\{\Phi^n(a) | n \in N_0\} \subset a^\perp$.*

(ii) *If Φ is in addition hereditary, then the linear hull of $\{\Phi^n(a) | n \in N_0\}$ is an abelian subalgebra of $(L, [\,,\,])$.*

(iii) *If Φ is invertible and hereditary, then the linear hull of $\{\Phi^n(a) | n \in \boldsymbol{Z}\}$ is an abelian subalgebra of $(L, [\,,\,])$.*

The assertion (i) is completely trivial. In case (ii) we know from the preceding argument that Φ commutes with all the $\Phi^n(a)$. Hence

$$[\Phi^n(a), \Phi^m(a)] = \Phi^m[\Phi^n(a), a] = \Phi^{n+m}[a, a] = 0\,,$$

and (iii) follows by almost the same argument.

§ 3. Permanence properties

In order to construct as many hereditary operators as possible it is desirable to study the structural properties of the set of these operators. But unfortunately these operators are neither a vector space nor a semigroup. But some structural properties are coming out of the notion of compatibility. Two hereditary operators Φ, Ψ are called *compatible* if $\Phi+\Psi$ is again hereditary. One easily sees that two operators Φ and Ψ are compatible if and only if, for all $a, b\in L$, we have

$$\begin{aligned} B_{\Phi,\Psi}(a, b) = & \Psi\{[\Phi(a), b]+[a, \Phi(b)]\}+\Phi\{[\Psi(a), b]+[a, \Psi(b)]\} \\ & -\Phi\Psi[a, b]-\Psi\Phi[a, b]-[\Psi(a), \Phi(b)]-[\Phi(a), \Psi(b)] \\ = & 0 . \end{aligned} \tag{3·1}$$

To show this one abbreviates

$$A_{\Phi}(a, b)=\Phi^2[a, b]+[\Phi(a), \Phi(b)]-\Phi\{[a, \Phi(b)]+[\Phi(a), b]\} \tag{3·2}$$

and one obtains in a straightforward way:

$$A_{\Phi+\Psi}(a, b)-A_{\Phi}(a, b)-A_{\Psi}(a, b)=-B_{\Phi,\Psi}(a, b).$$

Since $B_{\Phi,\Psi}$ is linear in the variables Φ and Ψ, this shows that whenever Ψ_1 and Ψ_2 are compatible hereditary operators such that Ψ_1 and Ψ_2 are compatible with Φ then, $\lambda_1\Psi_1+\lambda_2\Psi_2$ is again compatible with Φ (for arbitrary scalars λ_1, λ_2).

We need a technical lemma. The proof consists of a straightforward (but cumbersome) calculation and can be found in the Appendix.

3.1 *Lemma* *Let Φ be invertible.*

(i) *We have*

$$\Phi^{-1}A_{\Phi}(\Phi^{-1}(a), \Phi^{-1}(b))=\Phi A_{\Phi^{-1}}(a, b). \tag{3·3}$$

(ii) *If Φ and Ψ are hereditary, then we obtain*

$$A_{\Psi\Phi^{-1}}(a, b)=\Psi\Phi^{-1}B_{\Phi,\Psi}(\Phi^{-1}(a), \Phi^{-1}(b)). \tag{3·4}$$

(iii) *If Φ and Ψ are commuting hereditary operators, then we have*

$$A_{\Psi\Phi}(a, b)=-\Psi\Phi B_{\Psi,\Phi}(a, b). \tag{3·5}$$

These identities in fact yield the permanence properties for hereditary operators which are listed in the following:

3.2 THEOREM *Let Φ and Ψ be hereditary operators.*

(i) *If Φ is invertible, then Φ^{-1} is again hereditary.*

(ii) *If Φ and Ψ are compatible and if Φ is invertible, then $\Psi\Phi^{-1}$ is hereditary.*

(iii) *Let Φ be invertible and Ψ be injective. Then $\Psi\Phi^{-1}$ is hereditary if and only if $\Psi+\Phi$ is hereditary, i.e., they are compatible.*

(iv) *If Ψ and Φ are commuting and compatible, then $\Phi\Psi$ is hereditary.*

Proof (i) is a direct consequence of (3·3). Assertions (ii) and (iii) follow immediately from (3·4) and (iv) is a consequence of (3·5).

As a corollary we get that all polynomials $\varphi(\Phi)$ in a hereditary operator Φ are again hereditary. This fact even goes over to meromorphic functions in Φ, if they make sense.

§ 4. Application to (nonlinear) evolution equations

General remarks

Let S be a topological vector space and $C^\infty(S, S)$ the space of infinitely many times differentiable functions $S \rightarrow S$. Differentiable always means Hadamard-differentiable.[6] This is assumed in order to ensure that derivatives are linear maps and that the chain-rule holds.

We are interested in evolution equations of the form

$$u(t)_t = K(u(t)), \qquad u(t) \in S,\ K \in C^\infty(S, S). \tag{4·1}$$

For simplicity we assume that the initial value problem for (4·1) is *very well posed*. This means that for every $u_0 \in S$ there is a unique solution $u(t)$, $t \in R$, with $u(t=0) = u_0$ such that $u(t)$ is differentiable with respect to u_0. It is useful to consider the *resolvent map* $R_K(t)$ given by

$$u_0 \xrightarrow{R_K(t)} u(t).$$

Because of translation invariance with respect to t we have

$$R_K(t) \circ R_K(\tau) = R_K(t+\tau), \qquad R_K(0) = 1\,. \tag{4·2}$$

Thus $R_K(t)$ defines a differentiable one-parameter abelian group of transformations in S. From (4·1) we obtain

$$\frac{d}{dt} R_K(t) = K \circ R_K(t). \tag{4·3}$$

Hence, K is the infinitesimal generator of that group. Furthermore $R_K(t)$ is differentiable (differentiability with respect to the initial value). So, for every $u \in S$ there is a linear map $L_K(t, u): S \rightarrow S$ given by

$$L_K(t, u)v = \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} R_K(t)(u + \varepsilon v) \tag{4·4}$$

being the resolvent of the linearization (perturbation equation, tangential equation) of (4·1):

$$v(t)_t = \frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} K(u(t)+\varepsilon v(t)),\ u(t)=R_K(t)(u). \tag{4·5}$$

The family $L_K(t, u)$ has a group structure (coming out of (4·2)) which is given by the formula:

$$L_K(t, R_K(\tau)(u)) \circ L_K(\tau, u) = L_K(t+\tau, u). \tag{4·6}$$

Furthermore

$$L_K(t, u)K(u) = K(R_K(t)(u)) \tag{4·7}$$

since u_t is a solution of (4·5).

Roughly speaking, Lie algebras are important for evolution equations because the tangential equation (4·5) can be written in commutator form. In order to see this we denote the constant function $S \ni s \to v$ for $v \in S$ by $1_v \in C^\infty(S, S)$. If we define a Lie product in $C^\infty(S, S)$ by

$$[G, H](s) \underset{\text{def}}{=} \frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} \{G(s+\varepsilon H(s)) - H(s+\varepsilon G(s))\}, \tag{4·8}$$

where $s \in S$ and $G, H \in C^\infty(S, S)$, then the tangential equation (4·5) can be written as

$$v(t)_t = [K, 1_{v(t)}](u(t)), \qquad u(t) = R_K(t)(u). \tag{4·9}$$

An immediate consequence of that formula is that, *if $R_G(t)$ is a second resolvent group (with infinitesimal generator G) then the transformations $R_G(t)$ and $R_K(\tau)$ commute if and only if $[K, G]=0$.*

That means if $[K, G]=0$, then G can be understood as the infinitesimal generator of a flow commuting with (4·1). Therefore G is then called the *infinitesimal generator of a symmetry* of (4·1). This notion is also adopted if G is not the generator of a resolvent group (or in other words if the initial value problem for $u_t = G(u)$ is not very well posed). Because even in this case G yields important information about invariant manifolds, namely:

Let $[K, G]=0$, then

$$Ker(G) = \{s \in S \mid G(s) = 0\} \tag{4·10}$$

is a submanifold of S which is invariant under the flow given by (4·1).

To prove that we show that, for $u \in Ker(G)$, we have

$$\frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} G(u+\varepsilon u_t) = 0\,.$$

But this quantity is equal to $[G, K](u) + \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} K(u+\varepsilon G(u)) = [G, K](u)$.

Now, let us assume that Φ is a hereditary operator in $C^\infty(S, S)$ commuting with K. Denote $K_n = \Phi^n(K)$, then we know that

$$\{K_n | n \in N_0\}$$

(or even $\{K_n | n \in \boldsymbol{Z}\}$ if Φ invertible) is an abelian subalgebra of $C^\infty(S, S)$. If the initial value problem is very well posed for the following equations

$$u_t = K_n(u), \tag{4·11}$$

then the products of the resolvents $R_{K_m}(t)$, $m \in N_0$ or $\boldsymbol{Z}$, are an abelian symmetry group (often of infinite dimension) for all these equations. But the importance of Φ for the investigation of (4·1) does not stop at this point. There is another way of describing, in terms of Φ, submanifolds of S which are invariant under the flow given by (4·1). For suitable examples these submanifolds correspond to the so-called soliton-solutions.[1),7)] Let us call the operator Φ *local* if there are operators $\Phi(u): S \to S$, depending on $u \in S$, such that

$$(\Phi K)(u) = \Phi(u) K(u)$$

for all $u \in S$ and all $K \in C^\infty(S, S)$. An element $w \in S$ is said to be an eigenvector of (Φ, u) with eigenvalue λ if

$$(\Phi 1_w)(u) = \lambda w\,.$$

If Φ is local, this is equivalent to $\Phi(u) w = \lambda w$.

4.1. THEOREM *Let $\Phi: C^\infty(S, S) \to C^\infty(S, S)$ be a local hereditary operator which commutes with K. Then for arbitrary scalars $\alpha_1, \cdots \alpha_n, \lambda_1, \cdots, \lambda_n$, the set*

$$\{s \in S | K(s) = \sum_{k=1}^{m} \alpha_k w_k,\ w_k \text{ eigenvectors of } (\Phi, s) \text{ with eigenvalues } \lambda_k\}$$

is invariant under the flow given by Eq. (4·1).

Proof Define that $w(t)$ is $L_K(t, u) w$. Then using (4·9) and the fact that Φ commutes with K we get for $u(t) = R_K(t)(u)$:

$$\frac{d}{dt}\{(\Phi 1_{w(t)}(u(t)) - \lambda w(t)\} = \frac{d}{dt}\{\Phi(u(t)) w(t) - \lambda w(t)\}$$
$$= \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} K(u(t) + \varepsilon(\Phi 1_{w(t)}(u(t)) - \varepsilon \lambda w(t)).$$

Hence, $w(t)$ is an eigenvector of $(\Phi, u(t))$ with eigenvalue λ if and only if w is eigenvector of (Φ, u) with eigenvalue λ. Combining this with (4·7) we have the proof.

4.2. *Examples*

We are looking for (nonlinear) integro-differential equations admitting an abelian group of symmetries of infinite dimension. We start with a rather simple hereditary operator of the form

$$\Phi_{\alpha,\beta}=\alpha\Phi_1+\beta\Phi_2$$

having the property that $\Phi_{\alpha,\beta}$ is hereditary for all scalars α and β. Then we choose the solution space S for the evolution equations in such a way that, say $\alpha_0\Phi_1+\beta_0\Phi_2$ is invertible. This operator is certainly compatible with $\Phi_{\alpha,\beta}$ since $\Phi_{\alpha_0,\beta_0}+\Phi_{\alpha,\beta}=\Phi_{\alpha_0+\alpha,\beta_0+\beta}$ is again hereditary by assumption. Therefore, by application of theorem 3.2, we are led to the conclusion that

$$\Phi=(\alpha\Phi_1+\beta\Phi_2)(\alpha_0\Phi_1+\beta_0\Phi_2)^{-1}$$

is again hereditary. At this point we should remark that in general for a given concrete integro-differential operator the proof of hereditariness is a very cumbersome calculation. Guessing those integro-differential operators which are hereditary (most of them are not) is even more hopeless.

All the operators, which we are dealing with, will be of such a form that they commute (with respect to the Lie algebra $C^\infty(S, S)$) with K_0, where $K_0(u)=u_x$ (derivative of u). Hence,

$$K_n=\Phi^n K_0,\ n\in N_0(\text{or } n\in \boldsymbol{Z} \text{ if } \Phi \text{ is invertible})$$

is an abelian Lie algebra (in general of infinite dimension). Or, in other words, the evolution equations

$$u_t=K_n(u)$$

describe commuting flows and each of these flows has an infinite dimensional abelian group of symmetry transformations. The operator Φ is recursion operator for these equations in the sense of Olver.[8]

Another remark seems to be appropriate at this point: For a given complicated evolution equation it seems absolutely hopeless to guess whether or not this equation has infinitely many symmetries (hidden symmetries). Therefore our procedure for generating these equations on a systematic basis seems to be some progress in the right direction.

First we make some remarks about the notation we are adopting. Let $\mathscr{S}$ denote the predual of the tempered distributions on $\boldsymbol{R}$, i.e., $\mathscr{S}$ is the space of infinitely many times differentiable functions $\boldsymbol{R}\to\boldsymbol{C}$ such that all, including 0-th, derivatives vanish rapidly (faster than any polynomial) at $\pm\infty$. By $\mathscr{S}^-$ we denote the space of infinitely many times differentiable functions where the derivatives are only required to vanish rapidly at $-\infty$ and to be of at most polynomial growth at $+\infty$. Let $\delta>0$, then we mean the following spaces by $\mathscr{S}_\delta$, $\mathscr{S}_\delta^-$:

$$\mathscr{S}_\delta=\{\varphi\in\mathscr{S}|\varphi(x)\exp(-\delta x)\in\mathscr{S}\},$$

$$\mathscr{S}_\delta^- = \{\varphi \in \mathscr{S}^- | \varphi(x)\exp(-\delta x) \in \mathscr{S}^-\}.$$

The differential operator $\varphi \to \varphi_x$ is denoted by D and $D^{-1} : \mathscr{S}^- \to \mathscr{S}^-$ is its inverse, i. e.,

$$(D^{-1}\varphi)(x) = \int_{-\infty}^{x} \varphi(\xi)d\xi \,, \quad \varphi \in \mathscr{S}^- .$$

Now, let S be any of these function spaces under consideration. We only deal with local operators $\Phi : C^\infty(S, S) \to C^\infty(S, S)$, i.e., we start with linear operators $\Phi(u) : S \to S$ (depending C^∞ on $u \in S$) and define an operator $\Phi : C^\infty(S, S) \to C^\infty(S, S)$ by

$$(\Phi K)(u) = \Phi(u)K(u). \tag{4·12}$$

Such an operator Φ is hereditary if and only if

$$\frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} \{\Phi(u+\varepsilon\Phi(u)v)w - \Phi(u)\Phi(u+\varepsilon v)w\}$$

is, for all $u, v, w \in S$, symmetric in the variables v and w. This formula was discovered in an earlier paper.[1] We know[1] that the operator $\Phi : C^\infty(S, S) \to C^\infty(S, S)$ (where $S = \mathscr{S}, \mathscr{S}^-, \mathscr{S}_\delta$ or $\mathscr{S}_\delta^-$) given by the operator-valued function

$$\Phi_1(u) = \gamma I + \beta D + \alpha D u D^{-1} \tag{4·13}$$

is hereditary for arbitrary scalars α, β, γ. The proof for that fact was only given for the case $S = \mathscr{S}$. But since the proof only depends on the algebraic properties of D, it goes over unchanged to the present situation. Hence,

$$\Phi_2(u) = \Phi_1(u)D^{-1} = \gamma D^{-1} + \beta I + \alpha D u D^{-2} \tag{4·14}$$

is hereditary for $S = \mathscr{S}$. But this operator leaves $C^\infty(S, S)$ invariant if S is replaced by any of the subspaces $\mathscr{S}^-, \mathscr{S}_\delta, \mathscr{S}_\delta^-$. Therefore (4·14) defines a hereditary operator for all these spaces. Looking at the formula

$$\varphi(x) = (D - \varepsilon I)\left\{\exp(\varepsilon x)\int_{-\infty}^{x} \varphi(\xi)\exp(-\varepsilon\xi)d\xi\right\},$$

one discovers that $(D - \varepsilon I)$ is, for $0 \le \varepsilon \le \delta$, invertible in $\mathscr{S}_\delta$ as well as in $\mathscr{S}_\delta^-$. Obviously $(D - \varepsilon I)$ is among the operators given by (4·13), therefore

$$\Phi_3(u) = \Phi_1(u)(D - \varepsilon I)^{-1}, \qquad 0 \le \varepsilon \le \delta \tag{4·15}$$

is hereditary in $\mathscr{S}_\delta$ as well as in $\mathscr{S}_\delta^-$.

A very simple calculation[1] shows that all these operators Φ_1, Φ_2, Φ_3 commute (in the sense of formula (2·2)) with the function $K_0 \in C^\infty(S, S)$ given by $K_0(u)$

$=u_x$. This has as an immediate consequence that for $\Phi=\Phi_1$, Φ_2 or Φ_3 the functions

$$K_n=\Phi^n K_0,\ n\in N_0\quad (\text{or } n\in \boldsymbol{Z} \text{ if that makes sense})$$

are forming an infinite dimensional abelian Lie algebra. Or, in other words, the evolution equations

$$u_t=\Phi^n(u)u_x \tag{4·16}$$

are describing commuting flows. Among these equations one finds Burger's equation ($u_t=\Phi_1(u)u_x$ for $\beta=\alpha=1$, $\gamma=0$) but also many others not yet discovered in the literature. Among these

$$u_t=\Phi_2(u)u_x=\gamma u+\beta u_x+\alpha(uD^{-1}u)_x,\quad u\in\mathscr{S}^-$$
$$u_t=\Phi_3(u)u_x,\qquad u\in\mathscr{S}_\delta^-.$$

Using the substitutions $v=D^{-1}u$ or $v=(D-\varepsilon I)^{-1}u$, one can rewrite these equations as:

$$v_{xt}=\gamma v_x+\beta v_{xx}+\alpha(vv_x)_x,\quad v\in\mathscr{S}^-, \tag{4·17}$$
$$(v_x-\varepsilon v)_t=\gamma v_x+\beta v_{xx}+\alpha(vv_x)_x-2\alpha\varepsilon vv_x,\quad v\in\mathscr{S}_\delta^-. \tag{4·18}$$

It certainly seems impossible to guess that all these equations possess infinite dimensional abelian symmetry groups.

One obtains other classes of equations by considering other hereditary operators. The operator $\Phi: C^\infty(S,S)\to C^\infty(S,S)$ given by the operator-valued function

$$\Phi_4(u)=\rho I+\gamma D^2+\beta(2u+u_xD^{-1})+\alpha(u^2+u_xD^{-1}u) \tag{4·19}$$

is, for arbitrary scalars α,β,γ,ρ, hereditary in $\mathscr{S}$, $\mathscr{S}^-$, $\mathscr{S}_\delta$ as well as in $\mathscr{S}_\delta^-$ (see Ref. 1) formula H4). Application of theorem 3.2 yields a new class of hereditary operators

$$\Phi_5(u)=\rho D^{-2}+\gamma I+\beta(2u+u_xD^{-1})D^{-2}+\alpha(u^2+u_xD^{-1}u)D^{-2} \tag{4·20}$$

in $\mathscr{S}^-$ (and in $\mathscr{S}$ if $\rho=0$). One easily sees that for $0<\varepsilon\le\delta$ the operator $(D^2-\varepsilon^2 I)$ is invertible in $\mathscr{S}_\delta$ and $\mathscr{S}_\delta^-$. Hence

$$\Phi_6(u)=\Phi_4(u)(D^2-\varepsilon I)^{-1} \tag{4·21}$$

defines a new class of hereditary operators in $\mathscr{S}_\delta$ and $\mathscr{S}_\delta^-$. Again all these hereditary operators commute with $K_0(u)=u_x$. Therefore all the equations

$$u_t=\Phi^n(u)u_x,\ n\in N_0(\text{or } n\in\boldsymbol{Z} \text{ if that makes sense}) \tag{4·22}$$

describe commuting flows, for Φ equal to either Φ_4, Φ_5 or Φ_6. Among these equations one finds the KdV, the modified KdV, the Gardner equation and the sine-Gordon equation (see Ref. 1) for details). But apart from these well-known equations many other evolution equations not yet discovered as soliton equations belong to that class.

If the linear structure in the function spaces under consideration is restricted to the reals (i.e., scalars $=\boldsymbol{R}$), then

$$\Phi_7(u)=\gamma I+i\beta D+i\alpha u D^{-1}\operatorname{Re}(\bar{u}\cdot),\quad \alpha,\beta,\gamma\in\boldsymbol{R} \tag{4·23}$$

is again a class of hereditary operators (see Ref. 1), formula H6). Here $\operatorname{Re}(\bar{u}\cdot)$ stands for the real-linear operator given by

$$v\to\frac{1}{2}(\bar{u}v+u\bar{v}),\quad \bar{u},\ \bar{v}\ \text{complex conjugate.}$$

Then, running again through the (now almost boring) factorization procedure one obtains new classes of hereditary operators. All these then commute with $K_0(u)=u_x$. Hence the equations $u_t=\Phi^n(u)u_x$ do have infinite dimensional abelian symmetry groups. Apart from the nonlinear Schrödinger equation one finds among these equations many new ones.

§ 5. Concluding remarks

First of all, we like to emphasize that our approach yields much more nonlinear evolution equations with infinite dimensional abelian symmetry groups than the previous sections might indicate.

For example

$$\begin{aligned} v_{xt}=\alpha_1 v_{xx}+\alpha_2\sin(v)+\alpha_3(v_{xx}\cos(v)-v_x^2\sin(v)) \\ +\alpha_4\Big(2v_x\sin(v)+v_{xx}\int_{-\infty}^{x}\sin(v(\xi))d\xi\Big) \end{aligned} \tag{5·1}$$

admits an infinite dimensional (rather complicated) abelian symmetry group for arbitrary scalars $\alpha_1,\cdots,\alpha_4$. This is easily seen: Consider a special case of (4·19), namely

$$\Psi(u)=(D^2+4u_xD^{-1}u+4u^2).$$

This operator is injective and especially we obtain[1)]

$$\frac{1}{2}\sin\Big(2\int_{-\infty}^{x}u(\xi)d\xi\Big)=\Psi(u)^{-1}u_x\,. \tag{5·2}$$

If we like to check this relation, we should keep in mind that we are dealing with functions vanishing at $-\infty$, hence

$$D^{-1}\left(2u(x)\sin\left(2\int_{-\infty}^{x}u(\xi)d\xi\right)\right)=-\cos\left(2\int_{-\infty}^{x}u(\xi)d\xi\right)+1\,.$$

Now, consider the equation $u_t=(\Phi(u)+\varepsilon\Psi(u))\,\Psi^{-1}(u)u_x$, where $\Phi(u)$ is given by (4·19). Insertion of (5·2) and substitution

$$v(x)=2\int_{-\infty}^{x}u(\xi)d\xi$$

then yields (5·1) (after renaming the scalar parameters). Another equation one easily obtains out of a similar factorization of (4·19) is the following modified BBM equation:

$$u_t-u_{xxt}+u_x-3(uu_x+(uu_x)_{xx}-\gamma u_{xxx})+u_xu_{xx}=0\,. \tag{5·3}$$

Secondly, let us briefly emphasize the importance of theorem 4.1 in connection with multisoliton solutions. Let $\Phi(u)$ be one of the hereditary operators of the last sections and consider $u_t=\Phi(u)u_x$. We start by discussing the meaning of the decomposition given in theorem 4.1 for $m=1$. We then have the conclusion that u_t is an eigenvector of $\Phi(u)$ with eigenvalue λ. Comparing that information with the evolution equation we get (if the kernel of Φ is empty) $u_t=\lambda u_x$. Hence, $u(t)$ must be a travelling wave solution with speed λ. Now, in case that the solution space under consideration is either $\mathscr{S}$ or $\mathscr{S}_s$, multisoliton solutions are solutions which decompose asymptotically ($t=\pm\infty$) into travelling wave solutions, with rapidly vanishing overlap. Thus, if $\Phi(u)$ is a local operator (or even semilocal), these solutions are solutions belonging to the manifold described in theorem 4.1. And the eigenvalues are the asymptotic speeds of the travelling wave solutions. By the way, the same manifold describes Novikov's generalized multisoliton solutions.[9),10)]

Those equations we obtained from (4·13) have in general no multisoliton solutions whereas those obtained otherwise do have (for suitable α, β, γ) multisoliton solutions.

Another interesting observation is that the equations obtained out of (4·19) and (4·23) do have infinitely many conservation laws. This is not true for those equations obtained from (4·13). Because, in contrast to the second case, in the first case $\Phi(u)D$ is a symplectic operator (or rather its inverse is symplectic). Symplectic operators are Lie algebra homomorphisms relating the Lie algebra of gradients of covector fields with the Lie algebra given by (4·8) (for details see Refs. 2) and 3) and especially a forthcoming paper[11)] where the Lie algebra aspects of bi-Hamiltonian systems in the sense of Magri[12)] are extensively treated).

A theory very similar to the theory presented in this paper can be built up for dynamical systems with infinite dimensional *non-abelian* symmetry groups.[11)]

Finally, we like to remark that the transfer of hereditariness by Lie algebra isomorphisms corresponds to the well-known Bäcklund transformations. Concrete examples for that—again without mention of the Lie algebra aspects—are given in a forthcoming paper.[13]

Appendix

Linear deformations

Consider in a vector space L(over $\boldsymbol{R}$ or $\boldsymbol{C}$) two Lie-products $[\,,]$ and $[\,,]_0$. They are called *compatible* if $[\,,]_1$ defined by

$$[a, b]_1 = [a, b]_0 + [a, b]$$

is again a Lie product. In this case an isomorphism-into $\Phi:(L, [\,,]_0)\to(L, [\,,])$ is called a *linear deformation* of $(L[\,,])$ if $(I+\Phi):(L, [\,,]_1)\to(L, [\,,])$ is a homomorphism.

It is very easy to see that, if $[\,,]_0$ and $[\,,]$ are compatible, then $[a, b]_{\alpha,\beta}=\alpha[a, b]_0+\beta[a, b]$ defines a Lie-product for all scalars α, β. Furthermore, if Φ is such a linear deformation, then $(\beta I+\alpha\Phi):(L, [\,,]_{\alpha,\beta}\to(L, [\,,])$ is a homomorphism.

It is quite easy to characterize all linear deformations:

6.1. THEOREM *Let $\Phi:L\to L$ be linear, then the following are equivalent*:
(i) *Φ is a linear deformation* (*with respect to* $(L, [\,,])$.
(ii) *Φ is injective and hereditary.*

Proof

(i)⇒(ii): That Φ has to be injective is an immediate consequence of the definition. For $a, b\in L$ we must have $(I+\Phi)([a, b]+[a, b]_0)=[(I+\Phi)(a), (I+\Phi)(b)]$. Since $\Phi:(L, [\,,]_0)\to(L, [\,,])$ is an isomorphism, we can replace $[a, b]_0$ by $\Phi^{-1}[\Phi(a), \Phi(b)]$. This replacement yields:

$$[a, b]+\Phi[a, b]+[\Phi(a), \Phi(b)]+\Phi^{-1}[\Phi(a), \Phi(b)]$$
$$=[a, b]+[\Phi(a), b]+[a, \Phi(b)]+[\Phi(a), \Phi(b)].$$

The terms of order 0 and 2 in Φ cancel and the application of Φ to the first order terms yields Eq. (2·1).

(ii)⇒(i): First, we remark that (2·1) implies that $\Phi(L)$ is a subalgebra of L. Hence the definition

$$[a, b]_0=\Phi^{-1}[\Phi(a), \Phi(b)]$$

makes sense. Obviously $[\,,]_0$ is a Lie-product and $\Phi:(L, [\,,]_0)\to(L, [\,,])$ is an isomorphism-into. A straightforward calculation gives:

$$(I+\Phi)[a, b]_1=(I+\Phi)\{[a, b]+[a, b]_0\}$$
$$=[a, b]+[\Phi(a), \Phi(b)]+\Phi^{-1}[\Phi(a), \Phi(b)]+\Phi[a, b].$$

Insertion of (2·1) into the last term makes that equal to:

$$=[a, b]+[\Phi(a), \Phi(b)]+[a, \Phi(b)]+[\Phi(a), b]$$
$$=[(I+\Phi)(a), (I+\Phi)(b)].$$

So, $(I+\Phi):(L, [\ ,\ ]_1)\rightarrow(L,[\ ,\ ])$ is a homomorphism if $[\ ,\ ]_1$ is a Lie-product. To see this one easily calculates with the help of (2·1):

$$[a,[b, c]_1]_1=[a,[b, c]]+[a,[b, c]_0]_0-\Phi[a,[b, c]]$$
$$+\{[a,[b, \Phi(c)]+[a,[\Phi(b), c]]+[\Phi(a), [b, c]]\}$$

which clearly implies the Jacobi identity.

The crucial property of linear deformations is that, for all $\lambda\in\boldsymbol{R}$, the operator $I+\lambda\Phi$ has to be a homomorphism from the product $[\ ,\ ]+\lambda[\ ,\ ]_0$ to $[\ ,\ ]$. Here the linear dependence in λ is rather special. And, of course, there is a more general structure having linear deformations as tangential structure. We briefly describe that structure (without proofs).

In order to do that we assume that L is a topological vector space (if no topology is explicitly given we take the finest locally convex one). Let a family of Lie-products $[\ ,\ ]_\lambda$ be given, say for $0\leq\lambda\leq1$ and assume that the familiy is *differentiable isomorphic*, i. e., there are continuous linear bijections $\theta(\lambda):L\rightarrow L$ with

$$\theta(\lambda)[a, b]_\lambda=[\theta(\lambda)a, \theta(\lambda)b]_0 \qquad \forall\, a, b\in L \tag{A·1}$$

such that the $\theta(\lambda)$ are differentiable in λ. This family of Lie-products is said to be a *compatible deformation* of $[\ ,\ ]_0$ if

$$\frac{d}{d\lambda}[\ ,\ ]_\lambda \text{ is, for all } 0\leq\lambda\leq1, \text{ again a Lie-product} \tag{A·2}$$

and

$$\frac{d}{d\lambda}\theta(\lambda) \text{ is a homomorphism from } \left(L, \frac{d}{d\lambda}[\quad]\right) \text{ to } (L,[\quad]_0). \tag{A·3}$$

6.2. THEOREM *Let* $\theta(\lambda):(L, [\ ,\ ]_\lambda)\rightarrow(L, [\ ,\ ])$ *be a family of bijective isomorphisms. Then the following are equivalent*:

(i) $[\ ,\ ]_\lambda, 0\leq\lambda\leq1$ *is a compatible deformation of the product* $[\ ,\ ]$.

(ii) *For every* $0\leq\lambda\leq1$ *we have*

(a) $[\ ,\ ]_\lambda$ *and* $\frac{d}{d\lambda}[\ ,\ ]_\lambda$ *are compatible Lie-products;*

(b) $\left(\frac{d}{d\lambda}\theta(\lambda)\right)\theta(\lambda)^{-1}$ *is hereditary with respect to* $[\ ,\ ]_0$.

The proof of lemma 3.1

Let us give here the details of the calculations leading to the identities collected in lemma 3.1:

(i) is a simple verification.

(ii) Put $\tilde{a}=\Phi^{-1}(a), \tilde{b}=\Phi^{-1}(b)$. A straightforward calculation yields:

$$\begin{aligned}\Psi\Phi^{-1}([a,\ \Psi\Phi^{-1}(b)]+[\Psi\Phi^{-1}(a), b]) &= \Psi\Phi^{-1}([\Phi(\tilde{a}),\ \Psi(\tilde{b})]+[\Psi(\tilde{a}), \Phi(\tilde{b})])\\ &= \Psi\Phi^{-1}(-B_{\Phi,\Psi}(\tilde{a},\tilde{b})+\Psi([\Phi(\tilde{a}),\tilde{b}]+[\tilde{a}, \Phi(\tilde{b})])\\ &\qquad +\Phi([\Psi(\tilde{a}), \tilde{b}]+[\tilde{a},\ \Psi(\tilde{b})])-\Phi\Psi[\tilde{a},\ \tilde{b}]-\Psi\Phi[\tilde{a},\tilde{b}]\\ &= -\Psi\Phi^{-1}B_{\Phi,\Psi}(\tilde{a},\tilde{b})+\Psi\{[\Psi(\tilde{a}),\tilde{b}]+[\tilde{a},\ \Psi(\tilde{b})]-\Psi[\tilde{a},\tilde{b}]\}\\ &\qquad +\Psi\Phi^{-1}\Psi\{[\Phi(\tilde{a}),\tilde{b}]+[\tilde{a}, \Phi(\tilde{b})]-\Phi[\tilde{a},\tilde{b}]\}\\ &= -\Psi\Phi^{-1}B_{\Phi,\Psi}(\tilde{a},\tilde{b})-A_\Psi(\tilde{a},\tilde{b})+[\Psi(\tilde{a}),\ \Psi(\tilde{b})]\\ &\qquad -(\Psi\Phi^{-1})^2A_\Phi(\tilde{a},\tilde{b})+(\Psi\Phi^{-1})^2[\Phi(\tilde{a}), \Phi(\tilde{b})].\end{aligned}$$

Hence:

$$\begin{aligned}\Psi\Phi^{-1}B_{\Phi,\Psi}(\Phi^{-1}(a), \Phi^{-1}(b)) &= A_{\Psi\Phi^{-1}}(a, b)-A_\Psi(\Phi^{-1}(a), \Phi^{-1}(b))\\ &\qquad -(\Psi\Phi^{-1})^2A_\Phi(\Phi^{-1}(a), \Phi^{-1}(b)),\end{aligned}$$

and this gives (3·4) since Ψ and Φ are assumed to be hereditary.

(iii) For commuting hereditary operators Φ, Ψ we calculate

$$\begin{aligned}[\Psi\Phi(a),\ \Psi\Phi(b)] &= -\Psi^2[\Phi(a), \Phi(b)]+\Psi\{[\Psi\Phi(a), \Phi(b)]+[\Phi(a),\ \Psi\Phi(b)]\}\\ &= \Psi^2\Phi^2[a, b]-\Psi^2\Phi\{[a, \Phi(b)]+[\Phi(a), b]\}\\ &\quad -\Psi\Phi^2\{[\Psi(a), b]+[a,\ \Psi(b)]\}\\ &\quad +\Psi\Phi\{[\Phi\Psi(a), b]+[\Psi(a), \Phi(b)]\\ &\quad +[\Phi(a),\ \Psi(b)]+[a,\ \Psi\Phi(b)]\}.\end{aligned}$$

Insertion of this into

$$\begin{aligned}A_{\Psi\Phi}(a, b) &= (\Psi\Phi)^2[a, b]-\Phi\Psi\{[a,\ \Psi\Phi(b)]+[\Psi\Phi(a), b]\}\\ &\qquad +[\Psi\Phi(a),\ \Psi\Phi(b)]\end{aligned}$$

yields:

$$\begin{aligned}A_{\Psi\Phi}(a, b) &= 2(\Psi\Phi)^2[a, b]-(\Psi\Phi)\Psi\{[a, \Phi(b)]+[\Phi(a), b]\}\\ &\qquad -(\Psi\Phi)\Phi\{[\Psi(a), b]+[a,\ \Psi(b)]\}\end{aligned}$$

$$+ \Psi\Phi\{[\Psi(a), \Phi(b)]+[\Phi(a), \Psi(b)]\}$$
$$= - \Psi\Phi B_{\Psi,\Phi}(a, b).$$

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