The Annals of Probability
1985, Vol. 13, No. 3, 825–859

THE LIL WHEN X IS IN THE DOMAIN OF ATTRACTION OF A GAUSSIAN LAW[1]

BY J. KUELBS

University of Wisconsin

If X takes values in a Banach space B and is in the domain of normal attraction of a Gaussian law on B with $EX = 0$, $E(\|X\|^2/L_2\|X\|) < \infty$, then it is known that X satisfies the compact law of the iterated logarithm as described in Goodman, Kuelbs and Zinn [9], Theorem 4.1. In this paper the analogous result is demonstrated when X is merely in the domain of attraction of a Gaussian law. The functional LIL is also obtained in this setting. These results refine Corollary 7 of Kuelbs and Zinn [22], as well as various functional LILs.

1. Introduction. The interplay between the law of the iterated logarithm (LIL) and the central limit theorem (CLT) with Gaussian limit has long been recognized, and the paper by Kesten [13] proved some fundamental results in this area as well as posed a number of interesting problems. In Kuelbs and Zinn [22] we examined these questions in the Banach space setting and provided partial answers to some of the problems posed by Kesten. Our work was also motivated by Klass [14, 15] and a recent paper of Pruitt [23]. Here we continue this line of investigation and prove the LIL and the functional LIL(FLIL) for B-valued random variables in the domain of attraction of a Gaussian random variable G.

To make things precise we need to establish some notation and discuss the CLT and LIL.

Throughout B is a real separable Banach space with topological dual B^* and norm $\|\cdot\|$. We also assume $X, X_1, X_2, \cdots$ are independent identically distributed B-valued random variables, and as usual $S_n = X_1 + \cdots + X_n$ for $n \geq 1$. We use Lx to denote the function $\max(1, \log_e x)$ and we write L_2x to denote $L(Lx)$. The law of X is denoted by $\mathscr{L}(X)$.

When saying X has CLT behavior with Gaussian limit G we always are assuming G is mean zero and $G \neq \delta_0$. Such behavior can be exhibited in three ways. That is, X could be in the domain of normal attraction or satisfy the classical CLT (we write $X \in$ CLT), X could be in the domain of attraction (we write $X \in \mathrm{DA}(G)$), or X could be in the domain of partial attraction (we write $X \in \mathrm{DPA}(G)$). These concepts are defined just as for real-valued X, and Araujo and Giné [6] or Kuelbs and Zinn [22] contains them explicitly.

Received September 1983; revised August 1984.
[1] Supported in part by NSF Grant MCS-8219742.
AMS 1980 *subject classifications.* Primary 60B05, 60B11, 60B12, 60F10, 60F15; secondary 28C20, 60B10.
Key words and phrases. Law of the iterated logarithm, cluster set, domain of attraction of a Gaussian random variable.

If (M, d) is a metric space and $A \subseteq M$ we define the distance from $x \in M$ to A by

$$d(x, A) = \inf_{y \in A} d(x, y).$$

If $\{x_n\}$ is a sequence of points in M, then $C(\{x_n\})$ denotes the cluster set of $\{x_n\}$. That is, $C(\{x_n\}) = \{x: \lim \inf_n d(x, x_n) = 0\}$, and we will use the notation $\{x_n\} \twoheadrightarrow A$ if both $\lim_n d(x_n, A) = 0$ and $C(\{x_n\}) = A$.

The classical normalizing constants in the LIL are

$$a_n = \sqrt{2nL_2 n}, \tag{1.1}$$

and we say X satisfies the classical compact LIL (we write $X \in$ CLIL) if there exists a nonrandom compact set $D \subseteq B$ such that

$$\{(S_n - nE(X))/a_n\} \twoheadrightarrow D \quad \text{w.p.1} \tag{1.2}$$

The set D is called the "limit set", and always turns out to be the unit ball of an appropriate Hilbert space determined by the covariance structure of X. (see (1.4) and the reference indicated following (1.4)). It is also well known that if $\{(S_n - \delta_n)a_n^{-1}\}$ is pointwise conditionally compact, w.p.1. for any shifts $\{\delta_n\}$, then $E(X)$ exists and (1.2) holds. Hence the use of $nE(X)$ in (1.2), rather than arbitrary shifts, does not limit the class of X satisfying the classical compact LIL. A similar situation occurs if X satisfies the classical CLT or, in our notation, if $X \in$ CLT. That is, if the sequence of laws $\{\mathscr{L}((S_n - \delta_n)/\sqrt{n})$ is tight, then EX exists and $\mathscr{L}((S_n - nEX)/\sqrt{n}) \to \mathscr{L}(G)$ where G is centered Gaussian, i.e., $X \in$ CLT.

The B-valued random variable X has LIL behavior with respect to the centerings $\{\delta_n\} \subseteq B$ if there exists a normalizing sequence $\gamma_n \uparrow \infty$ such that

$$0 < \lim \sup_n \| S_n - \delta_n \| / \gamma_n < \infty \quad \text{w.p.1.} \tag{1.3}$$

Of course, LIL behavior is more general than the CLIL.

A random variable X is weakly square integrable with weak moment zero (we write $X \in WM_0^2$) if $Ef(X) = 0$ for each $f \in B^*$ and the covariance function

$$T(f, g) = E(f(X)g(X)) \quad (f, g \in B^*) \tag{1.4}$$

exists and is finite. If $X \in WM_0^2$, the covariance structure of X determines a Hilbert space $H_{\mathscr{L}(X)} \subseteq B$, and we use $K_{\mathscr{L}(X)}$ to denote the unit ball of $H_{\mathscr{L}(X)}$. Details of the construction of $H_{\mathscr{L}(X)}$ can be found in Goodman, Kuelbs and Zinn [9].

For real-valued X it is well known that $E(X^2) < \infty$, $X \in$ CLT, and $X \in$ CLIL are equivalent. Further, in this case the limit set is $[-\sigma, \sigma]$ where $\sigma^2 = E(X - E(X))^2$. For X B-valued the following result is known; an independent proof is due to Heinkel [11].

THEOREM A (Goodman, Kuelbs, Zinn [9]; Heinkel [11]). *Let X be B-valued such that $X \in CLT$. Then $X \in CLIL$ iff $E(\| X \|^2 / L_2 \| X \|) < \infty$. Further, the limit* *set for the CLIL is $K_{\mathscr{L}(G)}$ where G is the mean zero Gaussian random variable in the domain of normal attraction of X when using normalization constants $\sqrt{n}$.*

In regard to the more general concept of LIL behavior, we have the following result and problem due to Kesten.

THEOREM B (Kesten [13]). *Let X be real valued. Then, X has LIL behavior with respect to the centering sequence $\delta_n = \text{med}(S_n)$, where $\text{med}(S_n)$ is any choice of median S_n, iff X is in the DPA(G) where G is a mean zero Gaussian random variable with variance one. That is, there exists a sequence $\gamma_n \uparrow \infty$ such that*

$$0 < \limsup_n | S_n - \text{med}(S_n) | / \gamma_n < \infty \quad w.p.1 \tag{1.5}$$

iff $X \in \text{DPA}(G)$ where G is $N(0, 1)$. Further, for every fixed $\varepsilon > 0$, $\{\gamma_n\}$ can be chosen so that

$$n^{-1/2+\varepsilon}\gamma_n \nearrow \infty. \tag{1.6}$$

PROBLEM (Kesten [13]). If (1.5) holds, find the accumulation points of

$$\{(S_n - \text{med}(S_n))/\gamma_n\}, \tag{1.7}$$

and of the polygonal functions $\{\eta_n/\gamma_n\}$ where

$$\eta_n(t) = \begin{cases} S_k - \text{med}(S_k) & t = k/n, \quad k = 0, 1, \cdots, n \\ \text{linearly interpolated elsewhere for} & 0 \le t \le 1. \end{cases} \tag{1.8}$$

If the polygonal functions $\{\eta_n\}$ defined in (1.8), with centering sequence $\{\text{med}(S_n)\}$ possibly modified, have a normalized sequence $\{\gamma_n\}$ such that $\{\eta_n/\gamma_n\}$ has a nondegenerate limit set of functions, we say X has functional LIL behavior.

In Kuelbs and Zinn [22] we partially solved Kesten's problem for B-valued X and certain sequences $\{\gamma_n\}$ provided the centerings $\{\text{med}(S_n)\}$ are replaced by truncated means. Before we state this result we need the following concepts.

If G is a mean zero B-valued Gaussian random variable with $\mu = \mathscr{L}(G)$, then μ induces a Brownian motion on B with transition measure $\mu_t(A) = \mu(A/\sqrt{t})$ defined for Borel sets A of B. For details regarding the construction and properties of μ-Brownian motion consider Kuelbs and LePage [21] or Kuelbs [16]. μ-Brownian motion induces a mean zero Gaussian measure W on the real separable Banach space C_B, the space of B-valued continuous functions on [0, 1] with norm

$$\| f \|_{\infty,B} = \sup_{0 \le t \le 1} \| f(t) \| \quad (f \in C_B). \tag{1.9}$$

For W we have the Hilbert space H_W with unit ball K_W which we denote by $\mathscr{H}$ and $\mathscr{K}$, respectively.

A result proved in Kuelbs and Zinn [22] is the following.

THEOREM C. *Let X be B valued and assume $X \in \text{DPA}(G)$ where G is a mean zero Gaussian variable. Let $K = K_{\mathscr{L}(G)}$ and $\mathscr{K}$ be as described above. Then, there* *exists a subsequence of integers* $\{n_k\}$ *and normalizing constants* $d_k \nearrow \infty$ *such that if*

$$\delta_n = nE(XI(\|X\| \le d_k)) \quad n \in (n_{k-1}, n_k] \tag{1.10}$$

and

$$\gamma_n = \sqrt{2Lk}d_k \quad n \in (n_{k-1}, n_k] \tag{1.11}$$

then

$$\{(S_n - \delta_n)/\gamma_n\} \twoheadrightarrow K \quad w.p.1. \tag{1.12}$$

Further, let $\{\eta_n\}$ *be defined as in* (1.8) *with* $\text{med}(S_k)$ *replaced by* $j\delta_n/n$ *when* $t = j/n$ *and* $n \in (n_{k-1}, n_k]$. *Then,* $X \in FLIL$ *with*

$$\{\eta_n/\gamma_n\} \twoheadrightarrow \mathscr{K} \quad w.p.1 \tag{1.13}$$

where the convergence in (1.13) *is in the norm on* C_B.

It is perhaps useful to point out that by using the argument sketched in Kuelbs [18] it also is the case that if $X \in$ CLT and $E(\|X\|^2/L_2\|X\|) < \infty$, then the polygonal functions $\{\eta_n\}$ defined in (1.8), with $\text{med}(S_k)$ now replaced by $E(S_k)$ and $\gamma_n = a_n$, satisfy

$$\{\eta_n/\gamma_n\} \twoheadrightarrow \mathscr{K} \quad \text{w.p.1}$$

where $\mathscr{K}$ is defined as above and convergence is in the norm on C_B.

Hence if $X \in$ CLT and $(E\|X\|^2/L_2\|X\|) < \infty$, then X satisfies a compact LIL and a FLIL with classical normalizing constants $\{a_n\}$. Further, if $X \in$ DPA(G), then X satisfies a compact LIL and FLIL with normalizing constants $\{\gamma_n\}$, but our proof of Theorem C deals only with the existence of $\{\gamma_n\}$ and not its regularity properties. In Kuelbs and Zinn [22] we also dealt with certain regularity properties of the normalizing sequence $\{\gamma_n\}$ when $X \in$ DA(G) and in situations related to results in Klass [14, 15]. Here we present a refinement of some of these results obtaining a compact LIL and FLIL with respect to a regular sequence $\{\gamma_n\}$ when $X \in$ DA(G).

NOTATION. We write $g(t) \sim f(t)$ as $t \to \infty$ if

$$\lim_{t\to\infty} g(t)/f(t) = 1,$$

and, if there are constants $0 < A < B < \infty$ such that

$$A < \liminf_{t\to\infty} g(t)/f(t) \le \limsup_{t\to\infty} g(t)/f(t) < B,$$

we write $f(t) \approx g(t)$.

2. Statement of results. If $X \in$ DA(G) where G is a centered Gaussian random variable, then it is well known that $E(X)$ exists, so we use this fact freely in the statement of our results. Our first theorem deals with the compact LIL for $X \in$ DA(G) with regularity conditions on the normalizing sequence $\{\gamma_n\}$. This result is an extension of Corollary 7 of Kuelbs and Zinn [22].

THEOREM 1. *Let X be in the* DA(G) *where G is a centered Gaussian random variable and assume $K = K_{\mathscr{L}(G)}$. Then, there exists a strictly increasing continuous function d: $[0, \infty) \to [0, \infty)$ such that*

$$d(t) \sim \sqrt{t}T(d(t)) \tag{2.1}$$

where T: $[0, \infty) \to [0, \infty)$ is nondecreasing, slowly varying, and if

$$\gamma_n = \sqrt{2nL_2n}T(d(n/L_2n)), \tag{2.2}$$

then

$$K \subseteq C(\{(S_n - nE(X))/\gamma_n\}) \quad w.p.1. \tag{2.3}$$

Further

$$\{(S_n - nE(X))/\gamma_n\} \twoheadrightarrow K \quad w.p.1 \tag{2.4}$$

iff

$$E(\alpha^{-1}d^{-1}\alpha(\|X\|)) < \infty \tag{2.5}$$

where $\alpha(t) = t/L_2t$.

REMARKS. (I) Some examples appear in Section 6 which show K may be a proper subset of the cluster set in (2.3).

(II) If $X \in$ CLT, then the proof of Theorem 1 will show $T(t) \sim 1$ is possible, and hence the integrability condition (2.5) is that $E(\|X\|^2/L_2\|X\|) < \infty$. Thus Theorem 1 reduces to Theorem A in this case.

In dealing with the functional LIL when $X \in$ DA(G), G centered Gaussian, recall the real separable Banach space C_B with norm as given in (1.9). Further, let $\{\alpha_k\} \subseteq B^*$ be such that $\{S\alpha_k: k \geq 1\}$ is a C.O.N.S. in $H_{\mathscr{L}(G)} \subseteq B$ where the mapping $S: B^* \to B$ is as in Lemma 2.1 of Goodman, Kuelbs and Zinn [9] with $\mu = \mathscr{L}(G)$. The polygonal functions of interest in this situation are as in (1.8) with med(S_k) replaced by $kE(X)$. The limit set involved in the FLIL for X is the set

$$\mathscr{K} = \Big\{f \in C_B: f(t) \in H_{\mathscr{L}(G)}, 0 \leq t \leq 1,$$

$$f(t) = \textstyle\sum_k \displaystyle\int_0^t \frac{d}{ds}\alpha_k(f(s))\, ds S\alpha_k, \text{ and } \textstyle\sum_k \displaystyle\int_0^1 \left[\frac{d}{ds}\alpha_k(f(s))\right]^2 ds \leq 1\Big\}. \tag{2.6}$$

The reader can examine Kuelbs and LePage [21] for details on $\mathscr{K}$, and the identification of $\mathscr{K}$ as the unit ball of the Hilbert space H_W where W is μ-Wiener measure induced on C_B by μ-Brownian motion. Hence $\mathscr{K}$ as given in (2.6) is the same $\mathscr{K}$ as used in Theorem C.

We now are able to state the FLIL in this setting.

THEOREM 2. *Let X be in the* DA(G) *where G is a centered Gaussian random vector and assume $\mathscr{K}$ is as in* (2.6). *Let d, T, $\{\gamma_n\}$ be as in Theorem* 1 *and assume* *the polygonal functions* $\{\eta_n\}$ *are as in* (1.8) *with* $\mathrm{med}(S_k)$ *replaced by* $kE(X)$. *Then,* $\{\eta_n\} \subseteq C_B$ *w.p.*1, *and*

$$\{\eta_n/\gamma_n\} \twoheadrightarrow \mathscr{K} \quad w.p.1, \tag{2.7}$$

where the convergence is in C_B, *iff*

$$E(\alpha^{-1}d^{-1}\alpha(\|X\|)) < \infty \tag{2.8}$$

where $\alpha(t) = t/L_2t$.

3. Some useful propositions and their proofs. The first proposition collects some facts about the CLT when X is in the DA(G), G centered Gaussian, and the second provides a useful upper bound covering certain instances when we have LIL behavior with respect to the centerings $\delta_n = 0$. This upper bound easily applies to X in the DA(G), G Gaussian, and partially solves a conjecture of Klass [15, page 154]. Additional remarks can be found after the statement of the second proposition. The proof of each will follow from a series of lemmas some of which prove useful at subsequent stages of the paper.

PROPOSITION 1. *Let* X *be in the* DA(G) *where* G *is a centered Gaussian random variable. Then,* EX *exists, and there exists a strictly increasing continuous function* d: $[0, \infty) \to [0, \infty)$ *such that*

$$d(t) \sim \sqrt{t}T(d(t)) \quad (t \to \infty) \tag{3.1}$$

where T: $[0, \infty) \to [0, \infty)$ *is nondecreasing, slowly varying, and*

$$\mathscr{L}((S_n - nE(X)/d(n)) \to \mathscr{L}(G). \tag{3.2}$$

Further, if

$$\gamma_n = \sqrt{2nL_2n}T(d(n/L_2n)), \tag{3.3}$$

then for every open convex set U *in* B *and* $t > 0$ *we have*

$$\liminf_n \frac{nT^2(d(n/L_2n))}{\gamma_n^2} \log P\left(\frac{S_n - nE(X)}{\gamma_n} \in U\right) \geq t^{-2}\log P(G \in tU). \tag{3.4}$$

PROOF. Since $X \in$ DA(G) where G is Gaussian, we have $0 < E\|X\|^p < \infty$ for all $p \in [0, 2)$ [6, page 150], and hence $E(X)$ exists. It suffices to prove the theorem for the case $EX = 0$ as the general case will follow by replacing X by $X - E(X)$. Further, $X \in$ DA(G) implies $h(X) \in$ DA($h(G)$) for all $h \in B^*$.

To define the function $d(t)$ we choose $f \in B^*$ such that $Ef^2(G) = 1$ (such an f exists as $G \neq \delta_0$). Letting $\eta = f(X)$, we see $0 < E|\eta| < \infty$, and we define the strictly increasing continuous function $g(t)$ by

$$g(t) = \begin{cases} t^2/\int_0^t E(|\eta| I(|\eta| > u))\,du & t > 0 \\ 0 & t = 0. \end{cases} \tag{3.5}$$

To see that $g(t)$ is strictly increasing, observe that under the circumstances one can directly verify that $g(t)/t$ is nondecreasing, $\lim_{t\to\infty} g(t)/t = \infty$, and $g(t)/t$ is nonzero for $t > 0$. Hence $g(t) = t(g(t)/t)$ is strictly increasing on $[0, \infty)$ and has a strictly increasing continuous inverse for $t > 0$. Letting $d(t)$ denote the inverse of $g(t)$ we then have $t = g(d(t))$ and hence $d(t)$ satisfies

$$d^2(t) = t \int_0^{d(t)} E(|\eta| I(|\eta| > u))\, du \quad (t \geq 0). \tag{3.6}$$

Since $g(t)/t \nearrow \infty$ and (3.6) holds $d(t)$ is such that as $t \to \infty$ we have

$$d(t)/g(d(t)) = d(t)/t \downarrow 0, \tag{3.7}$$

and

$$d^2(t)/t \nearrow E(f^2(X)) = E(\eta^2) \quad \text{(possibly infinite)}. \tag{3.8}$$

Further, since

$$E(\eta^2 I(|\eta| \leq r)) = -rE(|\eta| I(|\eta| > r)) + \int_0^r E(|\eta| I(|\eta| > u))\, du, \tag{3.9}$$

we have by setting $r = d(t)$ and combining (3.6) and (3.9) that

$$d^2(t) = tE(\eta^2 I(|\eta| \leq d(t))) + td(t)E(|\eta| I(|\eta| > d(t))). \tag{3.10}$$

Now let

$$T(t) = \sqrt{E(\eta^2 I(|\eta| \leq t))} \quad (0 \leq t < \infty), \tag{3.11}$$

and observe that (3.10) implies

$$d(t) = \sqrt{t}\,T(d(t)) \sqrt{1 + \frac{d(t)E(|\eta| I(|\eta| > d(t)))}{T^2(d(t))}}. \tag{3.12}$$

Since $\eta = f(X)$ is in DA($f(G)$), we have $E(\eta^2 I(|\eta| \leq r))$ slowly varying at ∞, and since $d(t) \nearrow \infty$ as $t \to \infty$, (3.12) and the "equivalence lemma" of Hahn and Klass [10], or Feller [8], implies

$$d(t) \sim \sqrt{t}\,T(d(t)) \quad (\text{as } t \to \infty) \tag{3.13}$$

Hence (3.1) is established with $T(t)$ as in (3.11) and $d(t)$ the inverse of $g(t)$.

To finish the proof of Proposition 1 we proceed with some lemmas.

LEMMA 3.1. *Let $X \in$ DA(G) with normalizing constants $\{\bar{d}_n\}$ and assume $EX = 0$. Let $h \in B^*$ be such that*

$$E(h^2(G)) = \sigma_h^2 > 0, \tag{3.14}$$

and suppose we define $d_h(\cdot)$ as the inverse of the function $g_h(\cdot)$ defined as in (3.5) *with η replaced by $h(X)$. Let T_h be the analogue of T for h. Then,*

$$\text{(3.15)} \quad \begin{aligned} &\text{(i)} \quad \lim_n d_h(n)/\bar{d}_n = \sigma_h, \quad \textit{and} \\ &\text{(ii)} \quad T_h(t) \sim \sigma_h T(t) \quad \textit{at} \quad t \to \infty. \end{aligned}$$

In particular, $\bar{d}_n \sim d(n)$ where $d(n)$ is defined by (3.13), *and $X \in DA(G)$ with normalizing constants $\{d(n)\}$.*

PROOF. Since $X \in \mathrm{DA}(G)$ with normalizing constants $\{\bar{d}_n\}$, we know from Corollary 2.12 of [2] that

$$\mathscr{L}\left(\frac{S_n - nE(XI(\|X\| < \bar{d}_n))}{\bar{d}_n}\right) \to \mathscr{L}(G). \tag{3.16}$$

Hence for all $h \in B^*$

$$\begin{aligned}\mathscr{L}&\left(\frac{h(S_n - nE(XI(\|X\| \le \bar{d}_n)))}{\bar{d}_n}\right)\\ &= \mathscr{L}\left(\frac{h(S_n) - nE(h(X)I(\|X\| \le \bar{d}_n))}{\bar{d}_n}\right) \to_{n\to\infty} \mathscr{L}(h(G)).\end{aligned} \tag{3.17}$$

To prove (3.15) recall from (3.11) and (3.13) that

$$d_h(t) \sim \sqrt{t}\sqrt{E(h^2(X)I(|h(X)| \le d_h(t)))} \tag{3.18}$$

and hence by [6, page 88], or the classical CLT if $Eh^2(X) < \infty$,

$$\mathscr{L}\left(\frac{h(S_n) - nE(h(X)I(|h(X)| \le d_h(n)))}{d_h(n)}\right) \to N(0, 1). \tag{3.19}$$

Using the algebraic identity

$$\begin{aligned}&\frac{h(S_n) - nE(h(X)I(|h(X)| < d_h(n)))}{d_h(n)}\\ &\quad= \frac{h(S_n) - nE(h(X)I(\|X\| \le \bar{d}_n))}{\bar{d}_n}\,\frac{\bar{d}_n}{d_h(n)}\\ &\qquad+ \frac{n[E(h(X)I(\|X\| \le \bar{d}_n)) - E(h(X)I(|h(X)| \le d_h(n)))]}{d_h(n)}\end{aligned} \tag{3.20}$$

we see by the convergence of types theorem [6, page 21] that

$$\lim_n(\bar{d}_n/d_h(n)) = \sigma_h^{-1}, \tag{3.21}$$

and the last term on the right-hand side of (3.20) tends to zero as n goes to infinity. Hence (3.15-i) holds, and (3.15-ii) follows since (3.15-i) implies

$$T_h(d_h(n)) \sim d_h(n)\sqrt{n} \sim \bar{d}_n\sigma_h/\sqrt{n}$$

and

$$T(d(n)) \sim d(n)/\sqrt{n} \sim \bar{d}_n/\sqrt{n}.$$

Hence

$$T_h(d_h(n)) \sim \sigma_h T(d(n)),$$

and, since $d(n) \sim \bar{d}_n$ (see the following remark) with T_h and T both slowly varying, (3.15-i) and the uniform convergence result of [24] together imply that

$$T_h(d_h(n)) \sim T_h(d(n) \cdot d_h(n)/d(n)) \sim T_h(\sigma_h d(n)) \sim T_h(d(n)).$$

Combining the above we have

$$T_h(d(n)) \sim \sigma_h T(d(n)).$$

Passing from the above equation to (3.15-ii) now follows as T_h and T are both increasing and slowly varying with $\lim_n d(n+1)/d(n) = \lim_n \bar{d}_{n+1}/\bar{d}_n = 1$. That is, if $t \in [d(n), d(n+1)]$, then by the uniform convergence result we have

$$\frac{T_h(t)}{T(t)} \le \frac{T_h(d(n+1))}{T(d(n))} \sim \frac{T_h(\bar{d}_{n+1})}{T(d(n))} \sim \frac{T_h(\sigma_n^{-1} d_h(n+1))}{T(d(n))}$$

$$\sim \frac{T_h(d_h(n+1))}{T(d(n))} \sim \frac{d_h(n+1)/\sqrt{n+1}}{d(n)/\sqrt{n}} \sim \frac{\bar{d}_{n+1}\sigma_h}{\bar{d}_n} \sim \sigma_h.$$

Similarly,

$$\frac{T_h(t)}{T(t)} \ge \frac{T_h(d(n))}{T(d(n+1))} \sim \frac{\bar{d}_n \sigma_h}{\bar{d}_{n+1}} \sim \sigma_h$$

so (3.15-ii) is proved.

REMARK. Since $Ef^2(G) = 1$ and $d(n) = d_f(n)$, (3.15-i) implies $\bar{d}_n \sim d(n)$, and hence $\{\mathscr{L}(S_n/d(n))\}$ is shift-convergent. Applying Corollary (2.12) of [2] again, we have $X \in \mathrm{DA}(G)$ with normalizing constants $\{d(n)\}$, and also that

$$\text{(3.22)} \qquad \lim_n \frac{\| nE(XI(\|X\| \le \bar{d}_n)) - nE(XI(\|X\| \le d(n))) \|}{d(n)} = 0.$$

Further, we see from Lemma 3.1 that if $h \in B^*$ with $E(h^2(G)) = \sigma_h^2 > 0$, then $Eh^2(X) = \infty$ for all such h or $Eh^2(X) < \infty$ for all such h.

LEMMA 3.2. *Let X be in the* DA(G) *where G is centered Gaussian and assume $EX = 0$. Let $d(t)$ be defined as above. Then* (3.2) *holds with* $E(X) = 0$.

PROOF. From Lemma 3.1 we have $\bar{d}_n \sim d(n)$, and since (3.16) and (3.22) hold we have (3.2) with $E(X) = 0$ provided

$$\text{(3.23)} \qquad \lim_n \| nE(XI(\|X\| \le d(n)))/d(n) \| = 0.$$

To prove (3.23) fix $\varepsilon > 0$. Let Π_N and Q_N be the mappings of Lemma 2.1 of [9] or [18] defined for the Hilbert space $H_{\mathscr{L}(G)}$. Let N be fixed so that

$$\text{(3.24)} \qquad E\|Q_N G\| < \varepsilon/4$$

(see, for example, [6, page 143] for (3.24)).

Let $X', X_1', X_2', \cdots; S_n'; G'$ be independent copies of $X, X_1, X_2, \cdots; S_n; G$, respectively, on some suitable probability space which also supports the X sequence. Since $d(n) \sim \bar{d}_n$ and Q_N is continuous and linear we have by [3], that

$$\text{(3.25)} \qquad \begin{aligned} \limsup_n E \left\| Q_N\left(\frac{S_n}{d(n)}\right) \right\| &\le \lim_n E \frac{\| Q_N(S_n - S_n') \|}{d(n)} \\ &= E\| Q_N(G - G') \| < \frac{\varepsilon}{2}, \end{aligned}$$

and

$$\lim_n E \frac{\| Q_N S_n - nE(Q_N XI(\| x \| < d(n))) \|}{d(n)} = E \| Q_N G \| < \frac{\varepsilon}{4}. \tag{3.26}$$

Combining (3.25) and (3.26) we easily have

$$\limsup_n \left\| \frac{nE(Q_N XI(\| X \| \le d(n)))}{d(n)} \right\| \le \varepsilon. \tag{3.27}$$

Since $\| x \| = \| (\Pi_N + Q_N)(x) \| \le \| \Pi_N x \| + \| Q_N x \|$ and $\varepsilon > 0$ is arbitrary (3.27) will yield (3.23) if we show (since $EX = E(\Pi_N X) = E(Q_N X) = 0$)

$$\lim_n \| nE(\Pi_N XI(\| X \| > d(n))) \| / d(n) = 0. \tag{3.28}$$

Since $\Pi_N(B) = \Pi_N H_{\mathscr{L}(G)}$ is a finite dimensional subspace of $H_{\mathscr{L}(G)}$ (of dimension N if $H_{\mathscr{L}(G)}$ has dimension greater than or equal to N), and all norms on finite dimensional subspaces are equivalent, to prove (3.28) it suffices to show

$$\lim_n \| (n/d(n)) E(\alpha_j(X) I(\| X \| > d(n))) \| = 0 \tag{3.29}$$

for all $j = 1, \cdots, N$ where $\{\alpha_j\} \subseteq B^*$ are related to the mappings Π_N as in Lemma 2.1 of [9] or [18].

To obtain (3.29) fix j and observe that $E\alpha_j^2(G) = 1$, and hence by Lemma 3.1 and the remark following its proof we have $d_{\alpha_j}(n) \sim d(n)$. Further, since $\alpha_j \in B^*$,

$$d_{\alpha_j}(n) \sim \sqrt{n} T_{\alpha_j}(d_{\alpha_j}(n)) \tag{3.30}$$

where

$$T_{\alpha_j}^2(t) = E(\alpha_j^2(X) I(| \alpha_j(X) | \le t)) \tag{3.31}$$

is slowly varying as $t \to \infty$. Hence for every constant c, $0 < c < 1$, we have

$$\begin{aligned}
&\limsup_n \left| \frac{n}{d(n)} E(\alpha_j(X) I(\| X \| > d(n))) \right| \\
&\le \limsup_n \left| \frac{n}{d_{\alpha_j}(n)} E(\alpha_j(X) I(| \alpha_j(X) | > c d_{\alpha_j}(n))) \right| \\
&\quad + \limsup_n \left| \frac{n}{d_{\alpha_j}(n)} E(\alpha_j(X) I(| \alpha_j(x) | \le c d_{\alpha_j}(n), \| X \| > d(n))) \right| \\
&\le \limsup_n \frac{\sqrt{n} T_{\alpha_j}(d_{\alpha_j}(n))}{T_{\alpha_j}^2(d_{\alpha_j}(n))} | E(\alpha_j(X) I(| \alpha_j(X) | > c d_{\alpha_j}(n))) | \\
&\quad + \limsup_n c n P(\| X \| > d(n)) \\
&\le \limsup_t \frac{t}{T_{\alpha_j}^2(t)} | E(\alpha_j(X) I(| \alpha_j(X) | > ct)) | = 0
\end{aligned} \tag{3.32}$$

by the "equivalence lemma" of Hahn and Klass [10] and since $X \in \mathrm{DA}(G)$ implies $\lim_n nP(\| X \| > d(n)) = 0$. Thus (3.29) is verified and since j was arbitrary the lemma is proved.

To prove (3.4) and finish the proof of Proposition 1, we establish the following lemma related to results in [1] and [4].

LEMMA 3.3. *Let X be in the* DA(G) *where G is a centered Gaussian random variable and assume $EX = 0$. Let $d(t)$ and $T(t)$ be defined as above and assume* (3.3). *Then* (3.4) *holds.*

PROOF. Let $\Phi(x) = T(d(x))$. Then we first observe that $\Phi(x)$ is slowly varying as $x \to \infty$. That is, if $s > 0$

$$\lim_{x\to\infty} \frac{\Phi(sx)}{\Phi(x)} = \lim_{x\to\infty} \frac{T(d(sx))}{T(d(x))} = \lim_{x\to\infty} \frac{T((d(sx)/d(x))d(x))}{T(d(x))} = 1 \tag{3.33}$$

by the uniform convergence result [24] since $d(x) \nearrow \infty$, T is slowly varying, and for $s > 0$ fixed we have

$$\min(1, s) \le \limsup_{x\to\infty}(d(sx)/d(x)) \le \max(s, 1). \tag{3.34}$$

To verify (3.34) note that the right (left) hand side is obvious if $s \le 1$ ($s > 1$). Hence we consider the right (left) hand side when $s > 1$ ($s \le 1$).

If $s > 1$, then, since g is strictly increasing with inverse d, we have $g(d(sx) = sx$. Further, letting $z(u) = E(|\eta| I(|\eta| > u))$ we have that

$$\begin{aligned} g(sd(x)) &= s^2d^2(x) \Big/ \int_0^{sd(x)} z(u)\,du \quad \text{(from (3.5))} \\ &= s^2x \int_0^{d(x)} z(u)\,du \Big/ \int_0^{sd(x)} z(u)\,du \quad \text{(from (3.6))} \\ &\ge sx, \end{aligned} \tag{3.35}$$

since

$$s\int_0^{d(x)} z(u) \ge \int_0^{sd(x)} z(u)\,du$$

for $s \ge 0$, $d(x) > 0$ as $z(u) \downarrow 0$. Hence for $s > 1$, $sd(x) \ge d(sx)$ and (3.34) holds in this case.

If $0 < s \le 1$, then

$$\begin{aligned} \frac{d^2(sx)}{d^2(x)} &= s\int_0^{d(sx)} z(u)\,du \Big/ \int_0^{d(x)} z(u)\,du \\ &\ge s\int_0^{d(sx)} z(u)\,du \Big/ \int_0^{(1/s)d(sx)} z(u)\,du \\ &\qquad \left(\text{since } \frac{d(x)}{d(sx)} = \frac{d(1/s\cdot sx)}{d(sx)} \le \frac{1}{s} \text{ by applying } g \text{ and using } \frac{1}{s} \ge 1\right) \\ &\ge s^2 \quad (\text{since } z(u) \downarrow 0). \end{aligned}$$

Hence (3.34) holds as claimed.

Applying (3.2) we have

$$\mathscr{L}(S_n/\sqrt{n}\Phi(n)) \to \mathscr{L}(G) \tag{3.36}$$

where $\Phi(x) = T(d(x))$ and Φ is slowly varying.

If γ_n is as in (3.3) and $b_n = \sqrt{n}\Phi(n)$, then the next step is to verify that

$$\lim_n \gamma_n/b_n = \infty. \tag{3.37}$$

To verify (3.37) we recall that Φ slowly varying implies for all sufficiently large x we have the representation, see [24, page 2],

$$\Phi(x) = \exp\left\{\eta(x) + \int_B^x \frac{\varepsilon(s)}{s}\,ds\right\} \tag{3.38}$$

where $\lim_{x\to\infty}\eta(x) = c$, $|c| < \infty$, and $\lim_{s\to\infty}\varepsilon(s) = 0$. Hence

$$\begin{aligned}\liminf_n \frac{\gamma_n}{b_n} &= \liminf_n \sqrt{2L_2 n}\,\frac{\Phi(n/L_2 n)}{\Phi(n)}\\ &= \liminf_n \sqrt{2L_2 n}\,\exp\left\{-\int_{n/L_2 n}^{n} \frac{\varepsilon(s)}{s}\,ds\right\}\\ &\geq \liminf_n (L_2 n)^{1/2}\exp\{-\sup_{n/L_2 n<s<n}|\varepsilon(s)|\log_3 n\}\\ &= \liminf_n (L_2 n)^{1/2-\delta_n} = \infty\end{aligned} \tag{3.39}$$

where $\delta_n = \sup_{n/L_2 n\leq s\leq n}|\varepsilon(s)| \to 0$ as $n \to \infty$.

Now let $p_n = [t^2 n/2L_2 n]$, $q_n = [n/p_n]$, and $r_n = \gamma_n/tq_n$ where $t > 0$. Then

$$\begin{aligned}&\text{(i)}\quad \lim_n n\Phi^2(n/L_2 n)q_n/\gamma_n^2 = t^{-2}\\ &\text{(ii)}\quad r_n \sim \sqrt{p_n}\Phi(p_n).\end{aligned} \tag{3.40}$$

Now $p_n q_n \sim n$, $p_n q_n \leq n$. Hence take U open and convex in B, and set $U_\varepsilon = \{y: d(y, U^c) > \varepsilon\}$ where U^c denotes the complement of U. Then U_ε is open and convex, and hence

$$P\left(\frac{S_{p_n}}{r_n} \in tU_\varepsilon\right)^{q_n} \leq P\left(\frac{S_{p_n q_n}}{r_n} \in tq_n U_\varepsilon\right) \leq P\left(\frac{S_{p_n q_n}}{\gamma_n} \in U_\varepsilon\right). \tag{3.41}$$

Since $S_n = S_{p_n q_n} + (S_n - S_{p_n q_n})$ we have

$$P(S_n/\gamma_n \in U) \geq P(S_{p_n q_n}/\gamma_n \in U_\varepsilon,\ \|S_n - S_{p_n q_n}\| < \varepsilon\gamma_n), \tag{3.42}$$

and since $p_n q_n \sim n$, $X \in \mathrm{DA}(G)$ with normalizing constants $\{b_n\}$ and $\gamma_n/b_n \to \infty$ we have

$$\lim_n P(\|S_n - S_{p_n q_n}\| < \varepsilon\gamma_n) = 1.$$

By the independence of $S_n - S_{p_nq_n}$ and $S_{p_nq_n}$ we thus have

$$\liminf_n \frac{n\Phi^2(n/L_2n)}{\gamma_n^2} \log P\left(\frac{S_n}{\gamma_n} \in U\right)$$

$$\text{(3.43)} \qquad \begin{aligned} &\geq \liminf_n \frac{n\Phi^2(n/L_2n)}{\gamma_n^2} \log P\left(\frac{S_{p_nq_n}}{\gamma_n} \in U_\varepsilon\right) \\ &\geq \liminf_n \frac{n\Phi^2(n/L_2n)}{\gamma_n^2} q_n \log P\left(\frac{S_{p_n}}{r_n} \in tU_\varepsilon\right) \\ &\geq t^{-2}\log P(G \in tU_\varepsilon) \end{aligned}$$

by (3.40-i) and (3.40-ii) since $X \in \mathrm{DA}(G)$ with normalization constants $b_n = \sqrt{n}\Phi(n)$.

Letting $\varepsilon \downarrow 0$ we have $U_\varepsilon \nearrow U$ so the lemma holds. Hence (3.4) holds and the proposition is proved.

The next proposition provides a useful upper bound result. The function $d_q(t)$ of the proposition is analogous to the function $d(t)$ of Proposition 1, but here it depends on a fairly general continuous seminorm $q(\cdot)$ on B.

PROPOSITION 2. *Let X be B-valued such that $EX = 0$ and assume $q(\cdot)$ is a seminorm on B such that $0 < Eq(X) < \infty$ and $q(x) \leq \|x\|$ for all $x \in B$. Set $\eta = q(X)$ and define $g(t)$ as in* (3.5). *Let $d_q(t)$ denote the inverse of $g(t)$, and let $\{\gamma_n\}$ be a nondecreasing sequence such that*

$$\text{(3.44)} \qquad \gamma_n \geq \alpha_n \equiv L_2nd_q(n/L_2n)$$

and

$$\text{(3.45)} \qquad \gamma_n = \sqrt{n}\beta(n)$$

where $\beta(n)$ is a positive sequence such that for some $x_0 \in [1, \infty)$ we have

$$\text{(3.46)} \qquad \inf_{x \geq x_0} \inf_{t \geq 1} \beta([tx])/\beta([x]) \geq c > 0.$$

If

$$\text{(3.47)} \qquad \begin{aligned} &\text{(i)} \quad q(S_n/\gamma_n) \to_{\text{prob}} 0, \\ &\text{(ii)} \quad \lim_n \gamma_n/d_q(n) = \infty, \quad \textit{and} \\ &\text{(iii)} \quad P(q(X_n) > M\gamma_n \text{ i.o.}) = 0 \quad \textit{for some} \quad M < \infty, \end{aligned}$$

then

$$\text{(3.48)} \qquad \limsup_n q(S_n/\gamma_n) \leq \sqrt{2} \quad \text{w.p.1.}$$

REMARKS. (I) For real-valued X a result related to this proposition is given in Klass [15, page 154]. See remark (V) below for some related facts. Proposition

2 also yields a portion of Klass's conjecture (Klass, [15, page 154]), but it is not the entire conjecture as condition (3.47-iii) is two-sided, and for real-valued X we are also assuming $\lim_n \alpha_n/d_q(n) = \infty$ (when $\gamma_n = \alpha_n$), but this is not always the case.

(II) If $\lim_n \alpha_n/d_q(n) = \infty$ and $q(\cdot)$ is a continuous, type 2 seminorm on B, i.e., $E(q^2(X_1 + \cdots + X_n)) \le A \sum_{j=1}^n E(q^2(X_j))$ for some $A < \infty$ and all independent mean zero B-valued $\{X_j\}$, then $\lim Eq(S_n)/\alpha_n = 0$ by the proof of Lemma 1 of Kuelbs and Zinn [22]. Hence (3.47-i) follows from (3.47-ii) when q is type 2.

(III) If $\gamma_n = \sqrt{n}\beta(n)$ where β is nondecreasing then (3.46) obviously holds with $c = 1$. This is the situation considered by Klass [14, 15].

(IV) It is of interest to check when $\lim_n \alpha_n/d_q(n) = \infty$. Since $\alpha_n = L_2 n d_q(n/L_2 n)$, and $d_q(n)$ satisfies

$$d_q(n) = \sqrt{n}\left(\int_0^{d_q(n)} \psi(u)\,du\right)^{1/2}$$

where $\psi(u) = E(q(X)I(q(X) > u))$, we have

$$\frac{\alpha_n}{d_q(n)} = \left(L_2 n \int_0^{d_q(n/L_2 n)} \psi(u)\,du \Big/ \int_0^{d_q(n)} \psi(u)\,du\right)^{1/2}.$$

Now for $s \ge 1$, $x > 0$, $d_q(sx) \le sdq(x)$ (since $g(d_q(sx)) = sx$ and $g(sd_q(x)) \ge sx$ by the argument used in Lemma 3.3), and hence

$$\begin{aligned}\int_0^{d_q(n)} \psi(u)\,du &\le \textstyle\sum_{j=1}^{[L_2 n]+1} \displaystyle\int_{(j-1)d_q(n/L_2 n)}^{jd_q(n/L_2 n)} \psi(u)\,du \\ &\le ([L_2 n] + 1) \int_0^{d_q(n/L_2 n)} \psi(u)\,du\end{aligned}$$

as $\psi(u) \downarrow 0$. Thus

$$\liminf_n \alpha_n/d_q(n) > \liminf_n (L_2 n/\textstyle\sum_{j=1}^{[L_2 n]+1} c_j(n))^{1/2}$$

where

$$c_j(n) = \int_{(j-1)d_q(n/L_2 n)}^{jd_q(n/L_2 n)} \psi(u)\,du \Big/ \int_0^{d_q(n/L_2 n)} \psi(u)\,du,$$

and hence $\liminf_n \alpha_n/d_q(n) = \infty$ if

$$\lim_{n\to\infty;\, j\to\infty;\, j \le [L_2 n]+1} c_j(n) = 0.$$

For example, if

$$\lim_{u\to\infty} \frac{E(q^2(X)I(q(X) \le u))}{\int_0^u \psi(s)\,ds} = 1, \tag{3.49}$$

then, since integration by parts gives

$$E(q^2(X)I(q(X) \le u)) = \int_0^u \psi(s)\,ds - u\psi(u),$$

we see $\lim_{u\to\infty} u\psi(u)/\int_0^u \psi(s)\,ds = 0$. Hence (3.49) and $\psi(u) \downarrow 0$ imply

$$\begin{aligned}
&\sup_{2\le j\le L_2 n} c_j(n) \\
&\quad\le \sup_{2\le j\le L_2 n} \int_{(j-1)d_q(n/L_2 n)}^{jd_q(n/L_2 n)} \psi\left((j-1)d_q\left(\frac{n}{L_2 n}\right)\right) du \Big/ \int_0^{d_q(n/L_2 n)} \psi(u)\,du \\
&\quad\le d_q\left(\frac{n}{L_2 n}\right)\psi\left(d_q\left(\frac{n}{L_2 n}\right)\right) \Big/ \int_0^{d_q(n/L_2 n)} \psi(u)\,du.
\end{aligned}$$

Thus, in this case, $\lim_n \sup_{2\le j\le L_2 n} c_j(n) = 0$ and $\lim_n \alpha_n/d_q(n) = \infty$.

(V) If $X \in \mathrm{DA}(G)$ where G is Gaussian, $EX = 0$, and q is a seminorm on B such that

$$|h_0(x)| \le q(x) \le \|x\| \quad (x \in B)$$

for some $h_0 \in B^*$, then (3.47-ii) holds. To verify this we show

$$\lim_{u\to\infty} u^2 P(q(X) > u)/E(q^2(X)I(q(X) \le u)) = 0, \tag{3.50}$$

and hence by the "equivalence lemma" of Hahn and Klass [10]

$$\lim_{u\to\infty} \int_0^u \psi(x)\,dx \Big/ E(q^2(X)I(q(X) \le u)) = 1$$

where $\psi(x)$ is as in (IV). Hence by the argument of (IV), we have (3.47-ii) as claimed. Thus it suffices to prove (3.50).

Since $X \in \mathrm{DA}(G)$, $EX = 0$, choose $f \in B^*$ such that $Ef^2(G) = 1$ and define $d(t)$ as in the proof of Proposition 1. Then, for $d(n) \le u \le d(n+1)$ we have

$$\begin{aligned}
&\frac{u^2 P(q(X) > u)}{E(q^2(X)I(q(X) \le u))} \\
&\qquad\le \frac{nd^2(n+1)P(q(X) > d(n))}{nE(q^2(X)I(q(X) \le d(n)))} \\
&\qquad\le \frac{nP(\|X\| > d(n))}{\dfrac{d^2(n)}{d^2(n+1)}\;\dfrac{nE(q^2(X)I(q(X) \le d(n)))}{d^2(n)}} \qquad (\text{since } q(x) \le \|x\|) \\
&\qquad\to 0 \quad \text{as} \quad n \to \infty
\end{aligned}$$

since

(i) $nP(\|X\| > d(n)) \to 0$
(ii) $\lim_n d(n+1)/d(n) = 1$, and
(iii) $\liminf_n (n/d^2(n))E(q^2(X)I(q(X) \le d(n))) > 0$.

The fact that $\lim nP(\|X\| > d(n)) = 0$ follows from Theorem 2.10-1 of [2], and the condition (ii) is easily checked by the properties of $d(t)$. To verify (iii) we recall $|h_0(x)| \le q(x) \le \|x\|$ and we apply Theorem 2.10-2 of [2] which implies

$$\liminf_n \frac{n}{d^2(n)} E(q^2(X)I(q(X) \le d(n)))$$

$$\ge \liminf_n \frac{n}{d^2(n)} E(h_0^2(X)I(q(X) \le d(n)))$$

$$\ge \liminf_n \frac{n}{d^2(n)} E(h_0^2(X)I(\|X\| \le d(n)))$$

$$= \liminf_n nE\left(h_0^2\left(\frac{X}{d(n)}\right) I\left(\left\| \frac{X}{d(n)} \right\| \le 1\right)\right) \ge E(h_0^2(G)) > 0$$

with $\delta = 1$, and we are using the fact that

$$\lim_n nh_0^2(E(XI(\|X\| \le d(n))))/d^2(n) = 0.$$

This last limit is obvious since $EX = 0$, and $d(n) \ge \sqrt{n}/2$ for all n sufficiently large.

PROOF OF PROPOSITION 2. The proof of this result will proceed via several lemmas.

LEMMA 3.4. *Under the assumptions of Proposition* 2 *we have*

$$\lim Eq(S_n/\gamma_n) = 0. \tag{3.51}$$

PROOF. Since

$$d_q(n) = \sqrt{n}\left(\int_0^{d_q(n)} \psi(u)\, du\right)^{1/2}$$

where $\psi(u) = E(q(X)I(q(X) > u)) \downarrow 0$, we have by integration by parts that

$$d_q^2(n) = n\{E(q^2(X)I(q(X) \le d_q(n))) + d_q(n)\psi(d_q(n))\} \tag{3.52}$$

and hence

$$d_q(n) \ge n\psi(d_q(n)). \tag{3.53}$$

Thus, if

$$\begin{aligned} Y_{j,n} &= X_j I(q(X_j) \le d_q(n)) \quad (1 \le j \le n) \\ Z_{j,n} &= X_j - Y_{j,n} \qquad\qquad\quad (1 \le j \le n), \end{aligned} \tag{3.54}$$

then

$$Eq(Z_{1,n} + \cdots + Z_{n,n})/\gamma_n \le n\psi(d_q(n))/\gamma_n \le d_q(n)/\gamma_n \to 0 \tag{3.55}$$

as $n \to \infty$ by (3.47-ii). Now $q(S_n/\gamma_n) = q(\sum_{j=1}^n Y_{j,n}/\gamma_n + \sum_{j=1}^n Z_{j,n}/\gamma_n)$ converges in probability to zero by (3.47-i) and since $q(\sum_{j=1}^n Z_{j,n}/\gamma_n) \to_{\text{prob}} 0$ by (3.55) we also have

$$q(\textstyle\sum_{j=1}^n Y_{j,n}/\gamma_n) \to_{\text{prob}} 0. \tag{3.56}$$

Further, since $ES_n = 0$ and (3.55) holds we have

(3.57) $$q(E(\textstyle\sum_{j=1}^n Y_{j,n})/\gamma_n) \to 0.$$

The next step is to show

(3.58) $$E(q(\textstyle\sum_{j=1}^n Y_{j,n} - E(\sum_{j=1}^n Y_{j,n}))/\gamma_n) \to 0$$

which completes the proof.

Let $\{Y'_{j,n}: 1 \le j \le n\}$ be an independent copy of $\{Y_{j,n}: 1 \le j \le n\}$. Let

$$T_n = \textstyle\sum_{j=1}^n (Y_{j,n} - E(Y_{j,n})) \quad \text{and} \quad T'_n = \sum_{j=1}^n (Y'_{j,n} - E(Y_{j,n})).$$

Then $q(T_n/\gamma_n) \to_{\text{prob}} 0$, $q(T'_n/\gamma_n) \to_{\text{prob}} 0$ and hence the technique of Hoffmann-Jorgenson (see, for example, Lemma 6.1(a) of [22]) implies

(3.59) $$E(q(T_n - T'_n)/\gamma_n) \le 24t_{0,n} + 6E(\max_{1\le j\le n} q(Y_{j,n} \leftarrow Y'_{j,n})/\gamma_n)$$

where $t_{0,n} = \inf\{t: P(q(T_n - T'_n) > t\gamma_n) \le 1/24\}$. Now $\lim_n t_{0,n} = 0$ since $q(T_n/\gamma_n) \to_{\text{prob}} 0$ and $q(T'_n/\gamma_n) \to_{\text{prob}} 0$, and since $\lim_n \gamma_n/d_q(n) = \infty$ we have

$$\lim_n E(\max_{1\le j\le n} q(Y_{j,n} - Y'_{j,n})/\gamma_n) \le \lim_n 2d_q(n)/\gamma_n = 0.$$

Thus

(3.60) $$\lim_n E(q(T_n - T'_n))/\gamma_n = 0,$$

and since $ET'_n = 0$ we have

(3.61) $$\lim_n E(q(T_n/\gamma_n)) = 0.$$

Since $S_n = T_n + E(\sum_{j=1}^n Y_{j,n}) + \sum_{j=1}^n Z_{j,n}$ the above gives

(3.62) $$\lim Eq(S_n/\gamma_n) = 0$$

as claimed.

The next lemma is a standard result and its proof is essentially that of Lemma 3.2.4 of Stout [25]. As a result we omit the details.

LEMMA 3.5. *If $\{\gamma_n\}$ is nondecreasing and* (3.45), (3.46) *holds, then we have*

$$P(q(X_n) > M\gamma_n \text{ i.o.}) = 0 \quad \text{for some} \quad M < \infty$$

iff

$$P(q(X_n) > M\gamma_n \text{ i.o.}) = 0 \quad \text{for all} \quad M \in (0, \infty).$$

To prove Proposition 2, we now let $n_k = [\rho^k]$, $k \ge 1$, where $[\cdot]$ denotes the greatest integer function and $\rho > 1$ is to be specified later. We define for $1 \le j \le n_{k+1}$ and $0 < \tau \le 1$

(3.63) $$\begin{aligned} u_j &= X_j I(q(X_j) \le d_q(\tau n_k/L_2 n_k)) \\ v_j &= X_j I(d_q(\tau n_k/L_2 n_k) < q(X_j) \le \gamma_{n_k}) \\ w_j &= X_j I(\gamma_{n_k} < q(X_j)), \end{aligned}$$

and set

$$U_n = \sum_{j=1}^n u_j \quad V_n = \sum_{j=1}^n v_j \quad W_n = \sum_{j=1}^n w_j. \tag{3.64}$$

Then $S_n = U_n + V_n + W_n$ with $ES_n = EU_n + EV_n + EW_n = 0$, and hence for every $\varepsilon > 0$

$$P(\max_{n_k < n \le n_{k+1}} q(S_n) > (\sqrt{2} + 3\varepsilon)\gamma_{n_k}) \le I_{1,k} + I_{2,k} + I_{3,k} \tag{3.65}$$

where

$$\begin{aligned} I_{1,k} &= P(\max_{n_k < n \le n_{k+1}} q(U_n - EU_n) > (\sqrt{2} + \varepsilon)\gamma_{n_k}) \\ I_{2,k} &= P(\max_{n_k < n \le n_{k+1}} q(V_n - EV_n) > \varepsilon\gamma_{n_k}) \\ I_{3,k} &= P(\max_{n_k < n \le n_{k+1}} q(W_n - EW_n) > \varepsilon\gamma_{n_k}). \end{aligned} \tag{3.66}$$

In view of (3.65) we have (3.48) if we show for each $\varepsilon > 0$ there exists $\rho > 1$ such that

$$\lim_{r\to\infty} \sum_{k \ge r} I_{j,k} = 0 \tag{3.67}$$

for $j = 1, 2, 3$.

LEMMA 3.6. *If $\varepsilon > 0$, then $\lim_{r\to\infty} \sum_{k\ge r} I_{3,k} = 0$.*

PROOF. First we observe that (3.53) and (3.47-ii) imply

$$\begin{aligned} \lim_k \max_{n_k < n \le n_{k+1}} Eq(W_n)/\gamma_{n_k} &\le \lim_k n_{k+1} E(q(X) I(q(X) > \gamma_{n_k}))/\gamma_{n_k} \\ &\le 2\rho \lim_k n_k E(q(X) I(q(X) > d(n_k)))/\gamma_{n_k} \\ &\le 2\rho \lim_k d(n_k)/\gamma_{n_k} = 0. \end{aligned} \tag{3.68}$$

Further, since $\gamma_n \nearrow$, and for $\varepsilon > 0$

$$P(\max_{n_k < n \le n_{k+1}} q(W_n) > \varepsilon\gamma_{n_k}/2) \le n_{k+1} P(q(X) > \gamma_{n_k}),$$

we have C such that

$$\begin{aligned} \lim_r \sum_{k\ge r} I_{3,k} &\le \lim_{r\to\infty} \sum_{k \ge r} C(n_k - n_{k-1}) P(q(X) > \gamma_{n_k}) \\ &\le C \lim_{r\to\infty} \sum_{n \ge n_{r-1}} P(q(X) > \gamma_n) = 0 \end{aligned}$$

where (3.47-iii) and Lemma 3.5 is used to obtain the limit. Hence the lemma is proved.

LEMMA 3.7. *If $\varepsilon > 0$, then $\lim_{r\to\infty} \sum_{k\ge r} I_{2,k} = 0$.*

PROOF. From (3.68) we have

$$\lim_k \max_{n_k \le n \le n_{k+1}} Eq(W_n - EW_n)/\gamma_{n_k} = 0, \tag{3.69}$$

and since (3.51) holds with $ES_n = 0$ we have by Lemma 3.3 of de Acosta and

Kuelbs [4] that

$$\text{(3.70)} \qquad \lim_k \max_{n_k < n \le n_{k+1}} Eq(U_n - EU_n)/\gamma_n = 0.$$

Hence

$$\text{(3.71)} \qquad \lim_k \max_{n_k < n \le n_{k+1}} Eq(V_n - EV_n)/\gamma_n = 0$$

since $V_n - EV_n = S_n - (U_n - EU_n) - (W_n - EW_n)$.

Hence for k sufficiently large (3.71) implies

$$\text{(3.72)} \quad I_{2,k} \le P(\max_{n_k < n \le n_{k+1}} q(V_n - EV_n) - Eq(V_{n_{k+1}} - EV_{n_{k+1}}) > \varepsilon\gamma_{n_k}/2)$$

Arguing as in Lemma 5.3 of Kuelbs and Zinn [22], starting with (3.72) replacing (5.20) in Lemma 5.3, we thus have the lemma proved.

An exponential moment which is useful in handling the limiting behavior of the $\{U_n: n \ge 1\}$ is given in the next proposition. The proof is essentially that given for Lemma 3.2 of [5], so it is omitted.

LEMMA 3.8. *Let $Y_1, Y_2, \cdots$ be independent B-valued random variables such that*

$$\text{(3.73)} \qquad q(Y_j) \le c_n \quad (1 \le j \le n)$$

for some measurable seminorm q. Let $T_n = \sum_{j=1}^n Y_j$. Then

$$\text{(3.74)} \quad E(\exp\{\lambda[q(T_n) - Eq(T_n)]\}) \le \exp\left\{\frac{\lambda^2}{2} \sum_{j=1}^n E(q^2(Y_j))\exp(2\lambda c_n)\right\}.$$

To finish the proof of Proposition 2 we need the following lemma.

LEMMA 3.9. *For $\varepsilon > 0$, $\lim_{r\to\infty} \sum_{k\ge r} I_{1,k} = 0$.*

PROOF. Since (3.70) holds and $\varepsilon > 0$, for all k sufficiently large

$$I_{1,k} \le P(\max_{n_k < n \le n_{k+1}} q(U_n - E(U_n)) - Eq(U_{n_{k+1}} - EU_{n_{k+1}}) > (\sqrt{2} + \varepsilon/2)\gamma_{n_k}).$$

Since $\{q(U_n - EU_n): n_k < n \le n_{k+1}\}$ is a submartingale and $e^{\gamma x}$ is increasing and convex for $\gamma > 0$, we have by the maximal inequality for submartingales that for k sufficiently large

$$I_{1,k} \le \exp\{-\gamma(\sqrt{2} + \varepsilon/2)\gamma_{n_k}\} \cdot E(\exp\{\gamma[q(U_{n_{k+1}} - EU_{n_{k+1}}) - E(q(U_{n_{k+1}} - EU_{n_{k+1}}))]\}).$$

Applying the previous lemma we thus have for k sufficiently large that

$$I_{1,k} \le \exp\left\{-\gamma\left(\sqrt{2} + \frac{\varepsilon}{2}\right)\gamma_{n_k} + \frac{\gamma^2}{2} \sum_{j=1}^{n_{k+1}} E(q^2(X_j)I(q(X_j) \le b_k))e^{4\gamma b_k}\right\}$$

where $b_k = d_q(\tau n_k/L_2 n_k)$. Setting $\gamma = \sqrt{2}/d_q(n_k/L_2 n_k)$ and checking that $d_q(\tau x) \le \tau^{1/2} d_q(x)$ for $0 < \tau \le 1$, we have for all k sufficiently large and any $\delta > 0$

$$I_{1,k} \le \exp\left\{-L_2 n_k\left[2 + \frac{\varepsilon}{\sqrt{2}} - \frac{n_{k+1}}{L_2 n_k d_q^2(n_k/L_2 n_k)} \cdot E\left(q^2(X) I\left(q(X) \le d_q\left(\frac{\tau n_k}{L_2 n_k}\right)\right)\right) e^{4\sqrt{2\tau}}\right]\right\}$$

$$\le \exp\{-L_2 n_k(2 + \varepsilon/\sqrt{2} - (\delta + \rho)e^{4\sqrt{2\tau}})\}$$

since $n_{k+1} \le (\rho + \delta)n_k$ eventually and for each $0 < \tau \le 1$ and all $x > 0$

$$xE(q^2(X)I(q(X) \le d_q(\tau x)))/d_q^2(x) \le 1.$$

Hence, since $\varepsilon > 0$ is fixed, we choose $\rho > 1$, $\delta > 0$, and $0 < \tau \le 1$ sufficiently small so that $2 + \varepsilon/2 - (\delta + \rho)e^{4\sqrt{2\tau}} > 1 + \varepsilon/8$. This gives $I_{1,k} \le \exp\{-(1 + \varepsilon/8)L_2 n_k\}$ for all k sufficiently large, and hence the lemma (and Proposition 2) is proved.

4. Proof of Theorem 1. Let d, T be as in Proposition 1 (see (3.6) and (3.11)). Let $\{\gamma_n\}$ be as defined by (2.2). Then, since $T \circ d$ is slowly varying by Lemma 3.3, it is easy to check that

$$\gamma_n \sim \sqrt{2}(\alpha^{-1} d\alpha)(n). \tag{4.1}$$

Hence, by Corollary 7 of [22] (see the proof of Corollary 7), we have that

$$P(\{(S_n - nE(X))\gamma_n^{-1}\} \text{ conditionally compact in } B) = 1 \tag{4.2}$$

iff the integrability condition (2.5) holds. Since $d^{-1} = g$ with g as in (3.5), and

$$\int_0^t E(|\eta| I(|\eta| > u))\, du \sim E(\eta^2 I(|\eta| \le t)) \tag{4.3}$$

as $t \to \infty$ (see the "equivalence lemma" of Hahn and Klass [10] and (3.9)) with $T^2(t) = E(\eta^2 I(|\eta| \le t))$ slowly varying we have the integrability condition (2.5) equivalent to

$$E(\|X\|^2/\tilde{T}^2(\|X\|/L_2\|X\|)L_2\|X\|) < \infty \tag{4.4}$$

where $\tilde{T}(t) = \max(T(t), 1)$.

Now assume the integrability condition (2.5) holds. Then (4.2) holds, and we show that under these conditions

$$P(C(\{(S_n - nE(X))/\gamma_n\}) \subseteq K) = 1. \tag{4.5}$$

Applying Lemma 1 of [19], we have a nonrandom set $A \subseteq B$ such that

$$A = C(\{(S_n - nE(X))/\gamma_n\}) \quad \text{w.p.1.} \tag{4.6}$$

Hence it suffices to show $A \subseteq K$ to obtain (4.5).

If $A \nsubseteq K$, there exists $a \in A - K$ and since K is a compact convex set containing the zero vector of B we have an $h \in B^*$ such that

$$0 < \alpha = \sup_{x \in K} h(x) < h(a). \tag{4.7}$$

The existence of $h \in B^*$ satisfying (4.7) follows from the Hahn-Banach theorem in the following way. Choose $\varepsilon > 0$ such that $a \notin (1+\varepsilon)K$. Choose $b \in (1+\varepsilon)K$ such that

$$\inf_{x \in (1+\varepsilon)K} \| x - a \| = \| b - a \|. \tag{4.8}$$

Let $L = \{b + t(a-b): 0 \le t \le 1\}$. Then $L \cap K = \varnothing$ since $x \in L \cap K$ implies $\| x - b \| = t \| a - b \| < \| b - a \|$ unless $t = 1$, and this contradicts (4.8) as $K \subseteq (1+\varepsilon)K$; if $t = 1$, then $x = a \notin (1+\varepsilon)K$. Thus $L \cap K$ is empty, and, since K, L are compact and convex sets, the Hahn-Banach theorem implies there is $h \in B^*$ such that

$$0 \le \sup_{x \in K} h(x) < \inf_{x \in L} h(x) = \min\{h(a), h(b)\}. \tag{4.9}$$

Further, $\sup_{x \in K} h(x) > 0$ as $\sup_{x \in K} h(x) \ge h(b/1+\varepsilon) > 0$, so (4.7) holds.

From (4.6) and (4.7) we now have

$$\limsup_n h((S_n - nE(X))/\gamma_n) \ge h(a) > \alpha \quad \text{w.p.1.} \tag{4.10}$$

Now the definition of K as being $K_{\mathscr{L}(G)}$ implies (see Lemma 2.1 of [18] for details) that

$$0 < \alpha = (E(h^2(G)))^{1/2} = \sigma_h, \tag{4.11}$$

and hence by Proposition 2 (with $q(x) = | h(x) |$ and X replaced by $X - E(X)$) we have

$$\limsup_n | h(S_n - nE(X))/\gamma_n | \le \sigma_h \quad \text{w.p.1.} \tag{4.12}$$

Hence (4.10) is contradicted, and, as a result, $A \subseteq K$ as claimed. Further, since (4.2) holds and $A \subseteq K$ we thus have

$$P(\lim_n d((S_n - nE(X))/\gamma_n, K) = 0) = 1 \tag{4.13}$$

when (2.5) holds.

To see that Proposition 2 applies to $X - E(X)$, note that by Lemma 3.1 (applied to $X - E(X)$) we have $d_q(t) = d_h(t)$ and hence

$$\alpha_n = L_2 n d_q(n/L_2 n) \sim \gamma_n \sigma_h / \sqrt{2} \tag{4.14}$$

where $\{\gamma_n\}$ is as in (2.2). Hence we have

$$\frac{\alpha_n}{\sqrt{n}} = \sqrt{L_2 n} \int_0^{d_q(n/L_2 n)} \psi(u)\, du$$

where $\psi(u) = E(| h(X) | I(| h(X) | > u))$, so $\alpha_n/\sqrt{n} \nearrow$ and (3.45) holds with $\gamma_n = \alpha_n$. The conditions in (3.47) also hold with $S_n = \sum_{j=1}^n (X_j - E(X))$ and $\gamma_n = \alpha_n$ as defined in (4.14). See remarks (IV) and (V) following the statement of Proposition 2 to verify (3.47-ii) with $\gamma_n = \alpha_n$. Since the integrability condition (2.5) implies (using Lemma 3.5 and γ_n as in (2.2)).

$$P(\| X_n \| > M\gamma_n \text{ i.o.}) = 0 \quad \text{for all} \quad M \in (0, \infty), \tag{4.15}$$

and (4.14) holds with $\sigma_h > 0$, we have (3.47-iii). That (3.47-i) holds with

$S_n = \sum_{j=1}^{n} (X_j - E(X))$ and $\gamma_n = \alpha_n$ follows since by Lemma 3.1 (see the proof of Proposition 1)

$$\mathscr{L}(S_n/d(n)) \to \mathscr{L}(G), \quad \sigma_h d(n) \sim d_q(n), \tag{4.16}$$

and by (3.47-ii) $\lim_n \alpha_n/d_q(n) = \infty$.

Thus, by Proposition 2 we have $\lim \sup_n h(S_n - nE(X))/\alpha_n \le \sqrt{2}$ w.p.1, and since (4.14) holds we thus have (4.12) as claimed and $A \subseteq K$.

Hence if the integrability condition (2.5) holds, then (4.13) and (4.5) hold. Conversely, if (4.13) holds, then from Lemma 3.5 it is easy to see that if $\{\gamma_n\}$ is as in (2.2), then (2.5) holds.

To finish the proof, it suffices to show that if $X \in \mathrm{DA}(G)$, then (2.3) holds.

To verify (2.3) choose b in K with $\| b \|_{\mathscr{L}(G)} < 1$. Then fix $\varepsilon > 0$ and let $U = \{x \in B: \| x - b \| < \varepsilon\}$. Then, by the Cameron-Martin formula as used in [4], we have

$$\lim \inf_{t \to \infty} t^{-2} \log P(G \in tU) \ge -\tfrac{1}{2} \| b \|^2_{\mathscr{L}(G)}, \tag{4.17}$$

and hence (3.4) implies

$$\lim \inf_n \frac{nT^2(d(n/L_2 n))}{\gamma_n^2} \log P(S_n - nE(X) \in \gamma_n U) \ge -\tfrac{1}{2} \| b \|^2_{\mathscr{L}(G)}. \tag{4.18}$$

Thus for each $\varepsilon > 0$ and any $\delta > 0$ there exists n_0 such that $n \ge n_0$ implies

$$P(S_n - nE(X) \in \gamma_n U) \ge \exp\{-(1 + \delta) \| b \|^2_{\mathscr{L}(G)} L_2 n\}. \tag{4.19}$$

To show $b \in A$ we apply Lemma 5 of [19] with $\{a_n\}$ replaced by $\{\gamma_n\}$ as in (2.2). The regularity properties of $\{\gamma_n\}$ make the necessary modifications of the proof of Lemma 4 in [19] easy to carry out, and hence Lemma 5 of [19] is applicable with $\{\gamma_n\}$ replacing $\{a_n\}$. Choosing $\delta > 0$ such that $(1 + \delta) \| b \|^2_{\mathscr{L}(G)} < 1$, (4.19) implies

$$\sum_n P(\| (S_n - nE(X))/\gamma_n - b \| < \varepsilon)/n = \infty, \tag{4.20}$$

and hence $b \in A$ as required. Thus (2.3) holds. Of course, (2.4) now holds iff (2.5) holds, so the theorem is proved.

5. Proof of Theorem 2. Let d, T be as in Theorem 1 (see Proposition 1 and (3.6), (3.11)), and let $\{\gamma_n\}$ be as defined by (2.2).

First assume integrability condition (2.8) holds. Fix $\varepsilon > 0$. For $N \ge 1$, $x \in B$ let

$$\Pi_N(x) = \sum_{r=1}^{N} \alpha_r(x) S\alpha_r \quad Q_N(x) = x - \Pi_N(x) \tag{5.1}$$

where $\{\alpha_r\} \subseteq B^*$ and $\{S\alpha_r: r \ge 1\}$ is a C.O.N.S. in $H_{\mathscr{L}(G)}$ as described in Lemma 2.1 of [18]. Then we have (see [18, page 243]) an N_0 such that for $N \ge N_0$

$$Q_N K \subseteq \{x: \| x \| \le \varepsilon\} \tag{5.2}$$

where $K = K_{\mathscr{L}(G)}$. Hence, by Theorem 1

$$\{(S_n - nE(X))/\gamma_n\} \twoheadrightarrow K \quad \text{w.p.1}, \tag{5.3}$$

and since Q_N is a continuous map from B into B we have from (5.2) and (5.3) that

(5.4) $$P(\omega: \limsup_n \| Q_N((S_n(\omega) - nE(X))/\gamma_n) \| \le \varepsilon) = 1.$$

Further, since the polygonal functions are as in (1.8) with med(S_k) replaced by $kE(X)$, then it is easy to see that (5.4) implies

(5.5) $$P(\omega: \limsup_n \| Q_n(\eta_n(\cdot, \omega)) \|_{\infty,B}/\gamma_n \le \varepsilon) = 1.$$

Hence from (5.5) and the fact that $\cup_N \Pi_N(\mathscr{K})$ is a dense subset of $\mathscr{K}$ it suffices to prove that for each integer N we have

(5.6) $$\{\Pi_N(\eta_n/\gamma_n)\} \twoheadrightarrow \Pi_N\mathscr{K} \quad \text{w.p.1.}$$

To prove (5.6) first note that N is fixed, the random variables $\{\Pi_N(X_n): n \ge 1\}$ take values in $\Pi_N B$, and our processes have values in $C_{\Pi_N B}[0, 1]$. Since all norms on finite dimensional spaces are equivalent, we will replace the B-norm on $\Pi_N B$ by the $H_{\mathscr{L}(G)}$ norm, i.e., the usual Euclidean norm with respect to the basis $\{S\alpha_r: 1 \le r \le N\}$, and for $f \in C_{\Pi_N B}[0, 1]$ we define

(5.7) $$\| f \|_\infty = \sup_{0 \le t \le 1} \| f(t) \|_2$$

where $\| x \|_2 = \| x \|_{H_{\mathscr{L}(G)}}$ for $x \in H_{\mathscr{L}(G)}$.

Hence it suffices to prove (5.6) when (5.7) replaces $\| f \|_{\infty,B}$. To do this we proceed via some lemmas.

To simplify notation we define

(5.8) $$\xi_n(t, \omega) = \Pi_N \eta_n(t, \omega) \quad (0 \le t \le 1, n \ge 1).$$

LEMMA 5.1. *$\{\xi_n/\gamma_n: n \ge 1\}$ is conditionally compact w.p.1.*

PROOF. Fix $\varepsilon > 0$ and $\beta > 1$. It suffices to choose an integer $q > 0$ such that if $n_k = [\beta^k]$, then

$$\limsup_k \sup_{n_k < n \le n_{k+1}} \sup_{|s-t| \le 2^{-q}} \| \xi_n(t) - \xi_n(s) \|_2 \gamma_n^{-1} \le 3\varepsilon \quad \text{w.p.1.}$$

Now

$$\begin{aligned}
&\sup_{n_k < n \le n_{k+1}} \sup_{|s-t| \le 2^{-q}} \| \xi_n(t) - \xi_n(s) \|_2 \gamma_n^{-1} \\
&\quad \le \sup_{n_k < n \le n_{k+1}} \sup_{|s-t| \le 2^{-q}} \| \xi_n(t) - \xi_n(s) \|_2 \gamma_{n_k}^{-1} \\
&\quad \le \sup_{|s-t| \le 2^{-q}} \| \xi_{n_{k+1}}(t) - \xi_{n_{k+1}}(s) \|_2 \gamma_{n_k}^{-1} \\
&\quad \le 3 \sup_{1 \le j \le 2^q} \sup_{j-1 \le 2^q t \le j} \| \xi_{n_{k+1}}(t) - \xi_{n_{k+1}}((j-1)2^{-q}) \|_2 \gamma_{n_k}^{-1},
\end{aligned}$$

so it suffices to show

(5.9) $$\limsup_k \sup_{j-1 \le 2^q t \le j} \| \xi_{n_{k+1}}(t) - \xi_{n_{k+1}}((j-1)2^{-q}) \|_2 \gamma_{n_k}^{-1} \le \varepsilon \quad \text{w.p.1.}$$

for $j = 1, 2, \cdots, 2^q$.

Let $Y = \Pi_N(X) - E\Pi_N(X)$, $Y_j = \Pi_N(X_j) - E\Pi_N(X_j)$, $j \ge 1$. Then the remarks following Proposition 2 apply. That is, since $Y \in \mathrm{DA}(\Pi_N G)$ it follows that

$$\lim_n E\| Y_1 + \cdots + Y_n \|_2 \gamma_n^{-1} = 0. \tag{5.10}$$

To obtain (5.10) we use the fact that $\| \cdot \|_2$ is an inner product norm (and hence type 2 and co-type 2); so by Lemma 1 of [22]

$$E\| Y_1 + \cdots + Y_n \|_2 \approx d_{\|Y\|_2}(n). \tag{5.11}$$

On the other hand, by [3]

$$\lim_n E(\| Y_1 + \cdots + Y_n \|_2/d(n)) = E\| G \|_2, \tag{5.12}$$

so

$$d(n) \approx d_{\|Y\|_2}(n)$$

and hence the remarks (IV) and (V) following Proposition 2 imply $\lim_n d(n)/\gamma_n = 0$ and thus (5.10).

Hence, if

$$c_k = [n_{k+1}2^{-q+1}], \quad e_k = [(j-1)n_{k+1}2^{-q}], \quad f_k = [jn_{k+1}2^{-q}] + 1,$$

then for all k sufficiently large we have

$$\begin{aligned} &P(\sup_{j-1 \le 2^q t \le j} \| \xi_{n_{k+1}}(t) - \xi_{n_{k+1}}(j-1)2^{-q}) \|_2 \gamma_{n_k}^{-1} > \varepsilon) \\ &\quad \le P(\max_{e_k \le \ell \le f_k} \| \textstyle\sum_{i=e_k}^{\ell} Y_i \|_2 \gamma_{n_k}^{-1} > \varepsilon) \\ &\quad \le P(\max_{1 \le \ell \le c_k} \| \textstyle\sum_{i=1}^{\ell} Y_i \|_2 \gamma_{n_k}^{-1} > \varepsilon) \le 2P(\| \textstyle\sum_{i=1}^{c_k} Y_i \|_2 \gamma_{n_k}^{-1} > \varepsilon) \end{aligned} \tag{5.13}$$

by Ottaviani's inequality as $\sum_{i=1}^n Y_i/\gamma_n \to_{\text{prob}} 0$ and $\gamma_{n_k} \sim \gamma_{n_{k+1}}$.

To prove (5.9) we now turn to the method of proof in Proposition 2. We define for $1 \le j \le n_{k+1}$

$$\begin{aligned} u_j' &= Y_j I\left(\| Y_j \|_2 \le d\left(\frac{n_k}{L_2 n_k}\right)\right) \quad v_j' = Y_j I\left(d\left(\frac{n_k}{L_2 n_k}\right) \le \| Y_j \|_2 \le \gamma_{n_k}\right) \\ w_j' &= Y_j I(\gamma_{n_k} \le \| Y_j \|_2), \end{aligned} \tag{5.14}$$

and set

$$U_n' = \textstyle\sum_{j=1}^n u_j' \quad V_n' = \sum_{j=1}^n v_j' \quad W_n' = \sum_{j=1}^n w_j'. \tag{5.15}$$

Then $T_n = \sum_{j=1}^n Y_j = U_n' + V_n' + W_n'$ with $E(T_n) = E(U_n') + E(V_n') + E(W_n') = 0$, and hence for every $\varepsilon > 0$

$$P(\| \textstyle\sum_{i=1}^{c_k} Y_i \|_2 \gamma_{n_k}^{-1} > \varepsilon) \le I_{1,k}' + I_{2,k}' + I_{3,k}' \tag{5.16}$$

where

$$\begin{aligned} I_{1,k}' &= P(\| U_{c_k}' - EU_{c_k}' \|_2 > \varepsilon\gamma_{n_k}/3) \\ I_{2,k}' &= P(\| V_{c_k}' - EV_{c_k}' \|_2 > \varepsilon\gamma_{n_k}/3) \\ I_{3,k}' &= P(\| W_{c_k}' - EW_{c_k}' \|_2 > \varepsilon\gamma_{n_k}/3). \end{aligned} \tag{5.17}$$

Hence (5.9) will hold by applying (5.13), provided we show for every $\beta > 1$

(fixed)

$$\lim_r \textstyle\sum_{k\ge r} I_{j,k} = 0 \tag{5.18}$$

for $j = 1, 2, 3$.

The proof of (5.18) for $j = 2, 3$ follows as for the analogous results in the proof of Proposition 2. For $j = 1$ we can also proceed as in Proposition 2 by first pointing out that there is a constant $A < \infty$ such that

$$A \ge n_k E\left(\| Y\|_2^2 I\left(\| Y\|_2 \le d\left(\frac{n_k}{L_2 n_k}\right)\right)\right) \Big/ L_2 n_k d^2\left(\frac{n_k}{L_2 n_k}\right). \tag{5.19}$$

To verify (5.19) we recall $d(n) \approx d_{\|Y\|_2}(n)$ and $E(\| Y\|_2^2 I(\| Y\|_2 \le t))$ is slowly varying (see the argument in Remark (V) following Proposition 2), and hence by (3.52) with $q(\cdot) = \|\cdot\|_2$ we have (5.19) holding. Further, since $\gamma_n \sim \sqrt{2}L_2 n d(n/L_2 n)$ the argument used in Lemma 3.9 yields, by setting $\gamma = 3\sqrt{2}(\varepsilon d(n_k/L_2 n_k))^{-1}$ and $\tau = 1$, that for all k sufficiently large

$$\begin{aligned} I'_{1,k} &\le \exp\{-L_2 n_k[2 - 9(\varepsilon\sqrt{L_2 n_k} d(n_k/L_2 n_k))^{-2} \\ &\qquad \cdot \textstyle\sum_{j=1}^{c_k} E(\| Y_j\|_2^2 I(\| Y_j\|_2 \le d(n_k/L_2 n_k)))\exp(6\sqrt{2}/\varepsilon)]\} \\ &\le \exp\{-L_2 n_k[2 - (9c_k/\varepsilon^2 n_k)A \exp(6\sqrt{2}/\varepsilon)]\} \end{aligned}$$

by (5.19). Since $c_k/n_k \sim \beta/2^{q-1}$ and $\beta > 1$ is fixed we choose q such that $(9c_k A \exp(6\sqrt{2}/\varepsilon)/\varepsilon^2 n_k) < 1/2$, and hence for all k sufficiently large

$$I'_{1,k} \le \exp\{-{}^3/_2 L_2 n_k\}$$

so (5.18) holds for $j = 1$ as well.

Thus the lemma is proved.

For any integer m and $f \in C_B[0, 1]$, we let $\Gamma_m f$ denote the piecewise linear approximation to f such that

$$\Gamma_m f(k/m) = f(k/m) \quad (k = 0, 1, \cdots, m)$$

and $\Gamma_m f$ is linear on each of the subintervals $[k/m, (k+1)/m]$ for $k = 0, \cdots, m-1$.

In view of the equicontinuity of Lemma 5.1 we have

LEMMA 5.2. *With probability one, and for any $\varepsilon > 0$, there is an integer $m_0 = m_0(\omega, \varepsilon)$ such that*

$$\| \Gamma_m \xi_n(\cdot, \omega) - \xi_n(\cdot, \omega)\|_\infty \gamma_n^{-1} < \varepsilon$$

for all $m, n \ge m_0$.

Another lemma is the following.

LEMMA 5.3. *With probability one and for any $\varepsilon > 0$ there is an integer $n_0 = n_0(\omega, \varepsilon)$ such that $\| \xi_n/\gamma_n - \xi_{n'}/\gamma_{n'}\|_\infty < \varepsilon$ for all $n, n' \ge n_0$ provided $| 1 - n'/n | < 1/n_0$.*

The proof of Lemma 5.3 is much the same as that of Corollary 2 in Chover [7] except one must provide some minor changes since the normalizations γ_n equal $\sqrt{2nL_2n}T(d(n/L_2n))$ rather than $\sqrt{2nL_2n}$. Of course, $T \circ d$ is slowly varying, so these adjustments are easy and we omit the proof.

In view of these lemmas we will have

$$\lim_n d(\Pi_N(\eta_n/\gamma_n), \Pi_N \mathscr{K}) = 0 \quad \text{w.p.1} \tag{5.20}$$

if for every $\varepsilon > 0$, $\beta > 1$ and all integers $m \geq 1$ we have

$$\Gamma_m(\xi_{n_k}/\gamma_{n_k}) \in (\Pi_N \mathscr{K})^\varepsilon \tag{5.21}$$

for all k sufficiently large on a set of probability one. Here, of course, $(\Pi_N \mathscr{K})^\varepsilon = \{f: \|f - g\|_\infty < \varepsilon \text{ for some } g \in \Pi_N \mathscr{K}\}$ and the distance d in (5.20) is that given via $\|\cdot\|_\infty$.

Now (5.21) holds if

$$\textstyle\sum_k P(\Gamma_m \xi_{n_k}/\gamma_{n_k} \notin (\Pi_N \mathscr{K})^\varepsilon) < \infty \tag{5.22}$$

and to verify (5.22) we proceed as in [20].

That is, let $E = \Pi_N B$ and consider the Hilbert space

$$E_m = \{(k_1, \cdots, k_m): k_i \in E, i = 1, \cdots, m\}$$

with inner product $|||(k_1, \cdots, k_m)||| = (\sum_{i=1}^m \|k_i\|_2^2)^{1/2}$ where we recall $\|\cdot\|_2 = \|\cdot\|_{H_{\mathscr{L}(G)}}$ on E. Let $K_m = \{(k_1, \cdots, k_m) \in E_m: |||(k_1, \cdots, k_m)||| \leq 1\}$. Let $J_1, \cdots, J_m$ denote the m sets of integers defined by $J_j = \{x \in Z^+: (j-1)r_k < x \leq jr_k\}$ where $r_k = [n_k/m]$. Then $mr_k > n_k - m$ and, of course, $\text{card}(J_j) \sim n_k/m$ as $k \to \infty$. Hence we will have (5.22) by arguing as in [20] if we show

$$\textstyle\sum_k P((\sum_{i\in J_1} Y_i, \sum_{i\in J_2} Y_i, \cdots, \sum_{i\in J_m} Y_i) \notin \gamma_{n_k} K_m^{\varepsilon/2}/m^{1/2}) < \infty. \tag{5.23}$$

To prove (5.23) observe that

$$P((\textstyle\sum_{i\in J_1} Y_i, \cdots, \sum_{i\in J_m} Y_i) \notin \gamma_{n_k} K_m^{\varepsilon/2}/m^{1/2}) \leq I''_{1,k} + I''_{2,k} + I''_{3,k} \tag{5.24}$$

where

$$I''_{1,k} = P((\textstyle\sum_{i\in J_1} (u_i' - E(u_i')), \cdots, \sum_{i\in J_m} (u_i' - E(u_i'))) \notin \gamma_{n_k} K_m^{\varepsilon/6}/m^{1/2})$$

$$I''_{2,k} = P(|||(\textstyle\sum_{i\in J_1} (v_i' - E(v_i')), \cdots, \sum_{i\in J_m} (v_i' - E(v_i')))||| > (\varepsilon/6)\gamma_{n_k}/m^{1/2}) \tag{5.25}$$

$$I''_{3,k} = P(|||(\textstyle\sum_{i\in J_1} (w_i' - E(w_i')), \cdots, \sum_{i\in J_m} (w_i' - E(w_i')))||| > (\varepsilon/6)\gamma_{n_k}/m^{1/2}).$$

Now

$$\lim_r \textstyle\sum_{k\geq r} I''_{j,k} = 0 \tag{5.26}$$

for $j = 2, 3$ as before in Proposition 2 since $d(n) \approx d_{\|Y\|_2}(n)$, but for $j = 1$ we need an argument other than the exponential moment used previously.

To verify (5.26) for $j = 1$, we let $g_1, \cdots, g_{n_k}$ be independent mean zero Gaussian random vectors with values in $\Pi_N B$ and with covariance that of the vector

$$YI(\| Y \|_2 \le d(n_k/L_2 n_k))/T(d(n_k/L_2 n_k)).$$

Then the covariance structure converges to that of $\Pi_N G$ as $k \to \infty$. Since all J_j's are "intervals of integers" with the same number of integers, we can define r_k independent mean zero random vectors with values in E_m as follows:

$$z_j = \frac{(u_j' - Eu_j',\, u_{r_k+j}' - Eu_{r_k+j}',\, \cdots,\, u_{mr_k+j}' - Eu_{mr_k+j}')}{T(d(n_k/L_2 n_k))} \quad (1 \le j \le r_k).$$

Similarly, with the g_j's we obtain independent mean zero Gaussian random vectors

$$G_j = (g_j,\, g_{r_k+j},\, \cdots,\, g_{mr_k+j}) \quad (1 \le j \le r_k)$$

with values in E_m and covariance structure that of z_j $(1 \le j \le r_k)$.

Now let

$$h(x) = \phi(|||\, x \,|||) \quad (x \in E_m)$$

where $\phi(t)$ is three times continuously differentiable on $(0, \infty)$ and such that

$$\phi(t) = \begin{cases} 0 & 0 \le t \le 1 + \varepsilon/12 \\ \text{increasing} & 1 + \varepsilon/12 \le t \le 1 + \varepsilon/6. \\ 1 & t \ge 1 + \varepsilon/6 \end{cases}$$

Then, since $\gamma_{n_k} = \sqrt{2n_k L_2 n_k}\, T(d(n_k/L_2 n_k))$, we have

$$I_{1,k}'' \le E(h(\textstyle\sum_{j=1}^{r_k} z_j/\sqrt{2n_k L_2 n_k/m})).$$

Since $|||\cdot|||$ is a smooth norm and $r_k = [n_k/m] \le n_k/m$, we have by arguing as in [17, pages 73–78] (also see [4, pages 117–118] for a similar adaptation) that

$$E\left(h\left(\textstyle\sum_{j=1}^{r_k} z_j \Big/ \sqrt{\dfrac{2n_k L_2 n_k}{m}}\right)\right)$$

$$\le E\left(h\left(\textstyle\sum_{j=1}^{r_k} G_j \Big/ \sqrt{\dfrac{2n_k L_2 n_k}{m}}\right)\right) + C(\varepsilon) r_k E \left|\left|\left| z_1 \Big/ \sqrt{\dfrac{2n_k L_2 n_k}{m}} \right|\right|\right|^3$$

$$\le P\left(\left|\left|\left| \sum_{j=1}^{r_k} \frac{G_j}{\sqrt{2r_k L_2 n_k}} \right|\right|\right| > 1 + \frac{\varepsilon}{12}\right) + \frac{C(\varepsilon) r_k E\, |||\, z_1 \,|||^3}{(2r_k L_2 n_k)^{3/2}}$$

where $C(\varepsilon) < \infty$.

Since $G_1, \cdots, G_{r_k}$ are identically distributed with values in E_m and covariance structure converging to that given by the identity map on E_m we have

$$\textstyle\sum_k P(|||\, \sum_{j=1}^{r_k} G_j/\sqrt{2r_k L_2 n_k} \,||| > 1 + \varepsilon/12) < \infty$$

by standard arguments, and hence (5.26) holds for $j = 1$ if

$$\textstyle\sum_k r_k E\, |||\, z_1 \,|||^3/(r_k L_2 n_k)^{3/2} < \infty. \tag{5.27}$$

Now

$$E\,|||\, z_1 \,|||^3 = E((\textstyle\sum_{j=1}^m \| u_j' - Eu_j' \|_2^2)^{3/2})/T^3(d(n_k/L_2 n_k))$$

$$(5.28)\qquad \le m2^{m/2} E \| u_1' - Eu_1' \|_2^3/T^3(d(n_k/L_2 n_k))$$

$$\le m2^{m/2+3} E \| u_1' \|_2^3/T^3(d(n_k/L_2 n_k))$$

by the usual c_r-inequality. Hence, since all norms on $\Pi_N B$ are equivalent, we have a constant $1 \le C < \infty$ such that

$$(5.29)\qquad \| \Pi_N x \|_2 \le C \| \Pi_N x \| \le C^2 \| x \| \quad (x \in B),$$

and hence the integrability condition (2.8), which is equivalent to

$$(5.30)\qquad E(\| X \|^2/(L_2 \| X \| \, T^2(\| X \|/L_2 \| X \|))) < \infty,$$

implies

$$(5.31)\qquad E\{\| \Pi_N X \|_2^2 (L_2 \| \Pi_N X \|_2 T^2(\| \Pi_N X \|_2/L_2 \| \Pi_n X \|_2))^{-1}\} < \infty.$$

To see that (5.29) and (5.30) imply (5.31), we observe from (3.5), (3.9), (3.11) and the "equivalence lemma" of Hahn and Klass [10] that as $t \to \infty$, $t^2/T^2(t) \sim g(t)$. Hence $t^2\{L_2 t T^2(t/L_2 t)\} \sim L_2 t g(t/L_2 t)$ and, since $T(t)$ is slowly varying and $L_2 t g(t/L_2 t)$ increases, (5.29) and (5.30) easily yield (5.31). Now

$$(5.32)\qquad E \| u_1' \|_2^3 = E(\| Y \|_2^3 I(\| Y \|_2 \le d(n_k/L_2 n_k)))$$

where $Y = \Pi_N X$ and hence (5.28) and (5.31) will imply (5.27) if we show

$$(5.33)\qquad I(\beta) = \sum_k \frac{E(\| Y \|_2^3 I(\| Y \|_2 \le d(n_k/L_2 n_k)))}{r_k^{1/2}(L_2 n_k)^{3/2} T^3(d(n_k/L_2 n_k))} < \infty.$$

To prove (5.33) we let C denote a finite constant, possibly varying from line to line, and observe that

$$I(\beta) < C \sum_k \sum_{j=1}^{(\alpha^{-1} d\alpha)^2(n_k)} \frac{j^{3/2} P(j-1 < \| Y \|_2^2 \le j)}{n_k^{1/2}(L_2 n_k)^{1/2} T^3(d(\alpha(n_k)))}$$

$$\le C \sum_{j=1}^\infty j^{3/2} P(j-1 < \| Y \|_2^2 \le j) \sum_{n_k \ge (\alpha^{-1} d^{-1} \alpha)(\sqrt{j})} \left\{ n_k^{1/2} (L_2 n_k)^{3/2} T^3\left(d\left(\frac{n_k}{L_2 n_k}\right)\right)\right\}^{-1}$$

$$\le C \sum_{j=1}^\infty \frac{j^{3/2} P(j-1 < \| Y \|_2^2 \le j)}{((\alpha^{-1} d^{-1} \alpha)(\sqrt{j}))^{1/2} (L_2(\alpha^{-1} d^{-1}\alpha)(\sqrt{j}))^{3/2} T^3\left[d\left(\frac{(\alpha^{-1} d^{-1}\alpha)(\sqrt{j})}{L_2(\alpha^{-1} d^{-1}\alpha)(\sqrt{j})}\right)\right]}$$

$$\le C \sum_{j=1}^\infty \frac{j^{3/2} P(j-1 < \| Y \|_2^2 \le j)(L_2 j)^{1/2} T(\sqrt{j}/L_2 j)}{j^{1/2}(L_2 j)^{3/2} T^3(d((j/(L_2 j)^2) T^{-2}(\sqrt{j}/L_2 j)))}$$

since $\alpha^{-1} d^{-1} \alpha(\sqrt{j}) \sim j/((L_2 j) T^{-2}(\sqrt{j}/L_2 j))$, and $T \circ d$ is slowly varying

$$\le C \sum_{j=1}^\infty \frac{jP(j-1 < \| Y \|_2^2 \le j)}{(L_2 j) T^2(\sqrt{j}/L_2 j)} \quad \text{since} \quad d\left(\frac{s^2}{T^2(s)}\right) \sim d(d^{-1}(s)) = s$$

$$\le CE(\| Y \|_2^2/(L_2 \| Y \|_2 T^2(\| Y \|_2/L_2 \| Y \|_2))) < \infty$$

by (5.31). Hence (5.27) holds and as a result (5.20) is proved.

Combining (5.20) and (5.4) we have

$$\lim_n \inf_{g \in \mathscr{K}} \| \eta_n - g \|_{\infty,B} = 0 \quad \text{w.p.1}, \tag{5.34}$$

and recalling (5.5) to complete the proof it suffices to prove the following lemma.

LEMMA 5.4. *Under the previous conditions*

$$C(\{\Pi_N(\eta_n/\gamma_n)\}) = \Pi_N \mathscr{K} \quad w.p.1. \tag{5.35}$$

PROOF. Fix an integer $m > 1$ and let $\beta = m^{k_0}$, $n_k = \beta^k = m^{k_0 k}$, and $r_k = n_k/m = m^{k_0 k - 1}$. Let

$$W_j = (Y_j, Y_{r_k+j}, \cdots, Y_{(m-1)r_k+j}) \quad (1 \le j \le r_k).$$

Then, by the argument in [20, page 404–405], we have (5.35) if for all $(k_1, \cdots, k_m) \in E_m$ such that $|||(k_1, \cdots, k_m)||| < 1$ we have

$$\liminf_k ||| m^{1/2} \textstyle\sum_{j=1}^{r_k} W_j/\gamma_{n_k} - (k_1, \cdots, k_m) ||| = 0 \quad \text{w.p.1}. \tag{5.36}$$

If Z_m denotes a mean zero Gaussian random vector with values in E_m and such that the coordinates are independent random vectors with values in $\Pi_N B$ and law $\Pi_N G$, then $W = W_1 \in \mathrm{DA}(Z_m)$, and, of course, the normalizing constants $d(n)$ of Proposition 1 suffice for W. That is, if one builds normalizing constants $\hat{d}(n)$ from W and Z_m as in the proof of Proposition 1, then Lemma 3.1 applied to W and Z_m easily show $\hat{d}(n) \sim d_{\alpha_j}(n) \sim d(n)$; so we can apply the conclusions of Proposition 1 to W with normalizations $d(n)$ (rather than $\hat{d}(n)$). Hence for every $\varepsilon > 0$ and $U = \{x \in E_m: |||x - (k_1, \cdots, k_m)||| < \varepsilon\}$, we have by (3.4) that

$$\liminf_n \frac{nT^2(d(n/L_2n))}{\gamma_n^2} \log P(\textstyle\sum_{j=1}^n W_j \in U\gamma_n) \ge t^{-2} \log P(Z_m \in tU). \tag{5.37}$$

By the Cameron-Martin translation theorem as used, for example, in [4, page 108], we have

$$\liminf_{t\to\infty} t^{-2} \log P(Z_m \in tU) \ge -\tfrac{1}{2} |||(k_1, \cdots, k_m)|||^2 \tag{5.38}$$

and hence for each $\delta > 0$ there exists $n_0(\delta)$ such that $n \ge n_0$ implies

$$P(\textstyle\sum_{j=1}^n W_j \in U\gamma_n) \ge \exp\{-(1+\delta) |||(k_1, \cdots, k_m)|||^2 L_2 n\}. \tag{5.39}$$

Now Theorem 1 applied to W_1 implies

$$\limsup_k ||| \textstyle\sum_{j=1}^{r_{k-1}} W_j/(\gamma_{n_k} m^{-1/2}) ||| \le m^{-k_0/2}, \tag{5.40}$$

so choosing k_0 such that $m^{-k_0/2} \le \varepsilon$ we have

$$\liminf_k ||| \textstyle\sum_{j+1}^{r_k} W_j/(\gamma_{n_k} m^{-1/2}) - (k_1, \cdots, k_m) ||| \le 2\varepsilon \quad \text{w.p.1} \tag{5.41}$$

provided

$$\liminf_k ||| \textstyle\sum_{j=r_{k-1}}^{r_k} W_j/(\gamma_{n_k} m^{-1/2}) - (k_1, \cdots, k_m) ||| \le \varepsilon \quad \text{w.p.1}. \tag{5.42}$$

Now (5.42) follows from the second half of the Borel-Cantelli Lemma if

$$\textstyle\sum_{k=1}^{\infty} P(\sum_{j=r_{k-1}}^{r_k} W_j/(\gamma_{n_k} m^{-1/2}) \in U) = \infty. \tag{5.43}$$

Since $(r_k - r_{k-1})/r_k = 1 - 1/m^{k_0}$ by choosing $\delta > 0$, k_0 such that

$$(m^{k_0}/(m^{k_0} - 1))(1 + \delta)\,|||\,(k_1, \cdots, k_m)\,|||^2 = \lambda < 1,$$

we have (5.43) from (5.39). Thus Lemma 5.4 holds and Theorem 2 is proved.

6. Some Examples. Using Theorem 1 we see that if $X \in \mathrm{DA}(G)$ where G is Gaussian, then (2.3) holds. In the examples of this section, we will show that K is possibly a proper subset of $C(\{(S_n - nE(x))/\gamma_n\})$ when the integrability condition in (2.5) fails.

EXAMPLE 1. X is assumed to be a symmetric real-valued random variable with probability density function

$$p(x) = \begin{cases} 0 & |x| < 1 \\ 1/x^3 & |x| \ge 1. \end{cases} \tag{6.1}$$

Then, letting $G = N(0, 1)$ and $f(x) = x$, we have $\eta = f(X) = X$. Hence

$$E(|\eta| I(|\eta| > u)) = 2 \int_u^\infty x p(x)\, dx = \begin{cases} 2/u & u \ge 1 \\ 2 & 0 \le u \le 1, \end{cases} \tag{6.2}$$

and using the notation of (3.5), (3.6) and (3.11) we have

$$g(t) = \begin{cases} t/2 & 0 \le t \le 1 \\ t^2/2(1 + \log t) & t \ge 1, \end{cases} \tag{6.3}$$

$$T^2(t) = 2 \int_0^t x^2 p(x)\, dx = 2 \log t \quad (t \ge 1), \tag{6.4}$$

and

$$d(t) = g^{-1}(t) \sim \sqrt{t \log t} \quad (t \to \infty). \tag{6.5}$$

Thus

$$\gamma_n = \sqrt{2nL_2n}\,T(d(n/L_2n)) \sim \sqrt{2nLnL_2n} \tag{6.6}$$

as $n \to \infty$.

LEMMA 6.1. *If X has probability density given by* (6.1) *and $X_1, X_2, \cdots$ are independent copies of X, then*

$$C(\{S_n/\gamma_n\}) = (-\infty, \infty) \quad w.p.1. \tag{6.7}$$

PROOF. Since X is in the domain of attraction of $G = N(0, 1)$, we have $S_n/\gamma_n \to_{\text{prob}} 0$ and hence zero is in the cluster set $C(\{S_n/\gamma_n\})$. In view of symmetry and the separability of $\mathbb{R}^1$, (6.7) will hold if we show each $b > 0$ is in $C(\{S_n/\gamma_n\})$ w.p.1.

Fix $b > 0$ and $\varepsilon > 0$. Then $b \in C(\{S_n/\gamma_n\})$ w.p.1 if

$$P(|S_n/\gamma_n - b| < \varepsilon \text{ i.o.}) = 1. \tag{6.8}$$

Let $F_n = \{|S_{n-1}/\gamma_n| < \varepsilon/2,\ |X_n/\gamma_n - b| \le \varepsilon/2\}$. Then (6.8) holds if

$$P(F_n \text{ i.o.}) = 1. \tag{6.9}$$

Now for $n \neq m$

$$\begin{aligned} P(F_n \cap F_m) &\le P(|X_n/\gamma_n - b| \le \varepsilon/2,\ |X_m/\gamma_m - b| \le \varepsilon/2) \\ &= P(|X_n/\gamma_n - b| \le \varepsilon/2)P(|X_m/\gamma_m - b| \le \varepsilon/2) \\ &\le CP(F_n)P(F_m) \end{aligned} \tag{6.10}$$

for some constant $C < \infty$, since

$$\begin{aligned} P(F_n) &= P(|S_{n-1}/\gamma_n| < \varepsilon/2;\ |X_n/\gamma_n - b| \le \varepsilon/2) \\ &= P(|S_{n-1}/\gamma_n| < \varepsilon/2)P(|X_n/\gamma_n - b| \le \varepsilon/2) \\ &> \tfrac{1}{2}P(|X_n/\gamma_n - b| \le \varepsilon/2) \end{aligned} \tag{6.11}$$

for all $n \ge n_0$ where we use the fact that $S_n/\gamma_n \to_{\text{prob}} 0$ and $\gamma_n/\gamma_{n-1} \to 1$. Hence for $n, m \ge n_0$, we have (6.10) with $C = 4$. Hence there is a $C < \infty$ such that (6.10) holds for all $n \neq m$. Applying a well-known extension of the Borel-Cantelli Lemma (see, for example, [26, page 317]), we have

$$P(F_n \text{ i.o.}) \ge 1/C > 0, \tag{6.12}$$

and since $\{F_n \text{ i.o.}\}$ is a tail even we have (6.9) from (6.12) and the Kolmogorov zero-one law. Thus (6.7) holds and the lemma is proved.

Another result which will be useful follows in the next example.

LEMMA 6.2. *If X has probability density as given by* (6.1) *and $X_1, X_2, \cdots$ are independent copies of X, then for each $\varepsilon \ge 0$, $\rho > 0$ there is a $C < \infty$ such that*

$$P(S_n \ge \varepsilon\gamma_n) \le C[(1/Ln)^{\varepsilon^2(1-\rho)} + L_2n/Ln] \tag{6.13}$$

for all $n \ge 1$. The constant C can be taken uniformly in ε for ε uniformly bounded.

PROOF. Let

$$Y = XI(|X| \le d(\tau n/L_2n)) \tag{6.14}$$

where $\tau > 0$ and set

$$T_n = \textstyle\sum_{j=1}^n X_jI(|X_j| < d(\tau n/L_2n)). \tag{6.15}$$

Then, for fixed $\varepsilon \ge 0$, $\rho > 0$, we have

$$P(S_n > \varepsilon\gamma_n) \le P(T_n \ge \varepsilon\gamma_n) + nP(|X| > d(\tau n/L_2n)) \tag{6.16}$$

for all $n \ge 1$.

Applying Lemma 3.8 we have

$$P(T_n \geq \varepsilon\gamma_n) \leq E(e^{\lambda T_n})e^{-\varepsilon\lambda\gamma_n}$$

$$\leq \exp\left\{\frac{\lambda^2}{2}\, nE(Y^2)\exp\left\{2\lambda d\left(\frac{\tau n}{L_2 n}\right)\right\} - \varepsilon\lambda\gamma_n\right\} \tag{6.17}$$

$$= \exp\left\{\frac{\lambda^2}{2}\, nT^2\left(d\left(\frac{\tau n}{L_2 n}\right)\right)\exp\left\{2\lambda d\left(\frac{\tau n}{L_2 n}\right)\right\} - \varepsilon\lambda\gamma_n\right\}$$

since $nE(Y^2) = nT^2(d(\tau n/L_2 n))$. Setting $\lambda = \sqrt{2}\varepsilon(d(n/L_2 n))^{-1}$ and taking into account that $d(\tau x) \leq \tau^{1/2} d(x)$ for $0 < \tau \leq 1$ and all $x \geq 0$ (see (3.6)), we have

$$J(\lambda, n, \tau, \varepsilon) \equiv \frac{\lambda^2}{2}\, nT^2\left(d\left(\frac{\tau n}{L_2 n}\right)\right)\exp\left\{2\lambda d\left(\frac{\tau n}{L_2 n}\right)\right\} - 2\lambda\gamma_n$$

$$\leq \varepsilon^2 nT^2\left(d\left(\frac{\tau n}{L_2 n}\right)\right)\exp\{2\sqrt{2}\varepsilon\tau^{1/2}\}\left(d\left(\frac{n}{L_2 n}\right)\right)^{-2}$$

$$- 2\varepsilon^2(nL_2 n)^{1/2} T\left(d\left(\frac{n}{L_2 n}\right)\right)\left(d\left(\frac{n}{L_2 n}\right)\right)^{-1} \tag{6.18}$$

$$= -\varepsilon^2 L_2 n\left\{\frac{2\sqrt{n}T(d(n/L_2 n))}{(L_2 n)^{1/2} d(n/L_2 n)}\right.$$

$$\left. - \exp(2\sqrt{2}\varepsilon\tau^{1/2}) nT^2\left(d\left(\frac{\tau n}{L_2 n}\right)\right)\left((L_2 n)^{1/2} d\left(\frac{n}{L_2 n}\right)\right)^{-2}\right\}.$$

Since $d(n) \sim \sqrt{n}\; T(d)(n))$ and $d(\tau x) \leq \tau^{1/2} d(x)$ with $T \nearrow \infty$, we have the right-hand side of (6.18) asymptotic to

$$-\varepsilon^2 L_2 n[2 - \exp(2\sqrt{2}\varepsilon\tau^{1/2}]. \tag{6.19}$$

Hence for $\varepsilon \geq 0$, $\rho > 0$ fixed there is a $\tau > 0$ such that $2 - \exp(2\sqrt{2}\varepsilon\tau^{1/2}) > 1 - \rho/2$ and hence

$$J(\lambda, n, \tau, \varepsilon) \leq -\varepsilon^2 L_2 n(1 - \rho) \tag{6.20}$$

for all $n \geq n_0(\varepsilon, \rho, \tau)$. Note that if $\varepsilon \geq 0$ is uniformly bounded by M then $\tau > 0$ can be taken so that (6.20) is uniform in ε for $\varepsilon \geq M$. Hence for $n \geq n_0(\varepsilon, \rho, \tau)$

$$P(T_n \geq \varepsilon\gamma_n) \leq \exp\{-\varepsilon^2(1 - \rho)L_2 n\}. \tag{6.21}$$

Now for $d(\tau n/L_2 n) \geq 1$

$$nP\left(|X| > d\left(\frac{\tau n}{L_2 n}\right)\right) = 2n\int_{d(\tau n/L_2 n)}^{\infty} y^{-3}\, dy \sim \frac{L_2 n}{\tau L n}, \tag{6.22}$$

and hence for all $n \geq n_1(\varepsilon, \rho, \tau)$ by combining (6.16), (6.21) and (6.22) we have

$$P(S_n > \varepsilon\gamma_n) \leq \exp\{-\varepsilon^2(1 - \rho)L_2 n\} + (2/\tau)(L_2 n/Ln). \tag{6.23}$$

Since ρ, ε are fixed, we now fix $0 < \tau \le 1$ such that $2 - \exp(2\sqrt{2}\varepsilon\tau^{1/2}) > 1 - \rho/2$, and hence (6.23) implies there is a $C = C(\varepsilon, \rho, \tau)$ such that (6.13) holds for all $n \ge 1$. Hence the lemma is proved since C can be chosen uniform in $\varepsilon \ge 0$ for ε uniformly bounded.

EXAMPLE 2. Let $X = (\eta_1, \eta_2)$ be a symmetric $\mathbb{R}^2$-valued random variable where η_1 and η_2 are independent random variables with probability density as in (6.1). Let $G = (g_1, g_2)$ be a mean zero $\mathbb{R}^2$-valued Gaussian random variable where g_1 and g_2 are independent $N(0, 1)$. Then $X \in \mathrm{DA}(G)$ and if $f(x, y) = x$, then $f(X) = \eta_1$ yields a d-function as in Proposition 1 which satisfies (6.3), (6.4), (6.5), and (6.6). We simply denote this function by $d(t)$ as before and let $\{\gamma_n\}$ be as in (6.6).

LEMMA 6.3. *If $X = (\eta_1, \eta_2)$ is as above, $X_1, X_2, \cdots$ are independent copies of X, and $\{\gamma_n\}$ is as in* (6.6), *then*

$$C(\{S_n/\gamma_n\}) = K \cup E_1 \cup E_2 \quad w.p.1 \tag{6.24}$$

where

$$K = \{(x, y): x^2 + y^2 \le 1\}$$
$$E_1 = \{(x, y): y = 0\}$$
$$E_2 = \{(x, y): x = 0\}.$$

PROOF. Since $C(\{S_n/\gamma_n\}) \supseteq K$ by Theorem 1, and $K = \{(x, y): x^2 + y^2 \le 1\}$ in this case, (6.24) will hold if

$$C(\{S_n/\gamma_n\} \supseteq E_1 \cup E_2 \quad \text{w.p.1}, \tag{6.25}$$

and for $(x, y) \notin K \cup E_1 \cup E_2$ we have

$$(x, y) \notin C(\{S_n/\gamma_n\}) \quad \text{w.p.1}. \tag{6.26}$$

Since X is symmetric with identical components, (6.25) will hold if we show

$$(b, 0) \in C(\{S_n/\gamma_n\}) \quad \text{w.p.1}. \tag{6.27}$$

Let $\|\cdot\|$ denote the usual norm on $\mathbb{R}^2$. Then (6.27) holds if for all $\varepsilon > 0$

$$P(\| S_n/\gamma_n - (b, 0) \| < \varepsilon \text{ i.o.}) = 1. \tag{6.28}$$

Now

$$\{\| S_n/\gamma_n - (b, 0) \| < \varepsilon \text{ i.o.}\} \supseteq \{J_n \text{ i.o.}\} \tag{6.29}$$

where

$$J_n = \left\{ \left| \frac{S_{n-1}^{(1)}}{\gamma_n} \right| \le \frac{\varepsilon}{3}, \left| \frac{X_n^{(1)}}{\gamma_n} - b \right| < \frac{\varepsilon}{3}, \left| \frac{S_n^{(2)}}{\gamma_n} \right| < \frac{\varepsilon}{3} \right\}$$

and $(x, y)^{(1)} = x$, $(x, y)^{(2)} = y$ for $(x, y) \in \mathbb{R}^2$. Then, as in Lemma 6.1,

$$P(J_n) \geq \tfrac{1}{2} P(|X_n^{(1)}/\gamma_n - b| < \varepsilon/3) \tag{6.30}$$

for all $n \geq n_0$, and arguing as in Lemma 6.1 we have

$$P(J_n \text{ i.o.}) = 1. \tag{6.31}$$

Thus (6.27) holds and since (6.27) yields (6.25) it now suffices to prove (6.26) for $(x, y) \notin K \cup E_1 \cup E_2$.

Take (x, y) such that $x > 0$, $y > 0$, $x^2 + y^2 > 1$. Let $\rho > 0$ be such that

$$[(x - \rho)^2 + (y - \rho)^2](1 - \rho) > 1. \tag{6.32}$$

Then, for all $n \geq 1$

$$\begin{aligned} P\left(\left\| \frac{S_n}{\gamma_n} - (x, y) \right\| < \rho \right) &\leq P\left(\frac{S_n^{(1)}}{\gamma_n} > x - \rho, \frac{S_n^{(2)}}{\gamma_n} > (y - \rho) \right) \\ &= P\left(\frac{S_n^{(1)}}{\gamma_n} > (x - \rho) \right) P\left(\frac{S_n^{(2)}}{\gamma_n} > (y - \rho) \right) \\ &\leq C^2 \left[\left(\frac{1}{Ln} \right)^{(x-\rho)^2(1-\rho)} + \frac{L_2 n}{Ln} \right] \left[\left(\frac{1}{Ln} \right)^{(y-\rho)^2(1-\rho)} + \frac{L_2 n}{Ln} \right] \end{aligned} \tag{6.33}$$

by the independence of the coordinates of X and Lemma 6.2 applied to each coordinate. Since (6.32) holds, (6.33) implies

$$P(\| S_n/\gamma_n - (x, y) \| \leq \rho) \leq C^2 (1/L_n)^{1+\delta} \tag{6.34}$$

for some $\delta > 0$ and all $n \geq n_0(\delta)$. Now (6.34) implies that

$$\textstyle\sum_{n=1}^{\infty} P(\| S_n/\gamma_n - (x, y) \| < \rho)/n < \infty \tag{6.35}$$

for some $\rho > 0$. Since $S_n/\rho_n \to_{\text{prob}} 0$ and $\{\gamma_n\}$ is sufficiently regular, Theorem 3 of [12, page 1179] can be modified (see [19, page 390] for a similar result) to yield

$$(x, y) \in C(\{S_n/\gamma_n\}) \quad \text{w.p.1} \tag{6.36}$$

iff for all $\varepsilon > 0$

$$\textstyle\sum_{n=1}^{\infty} P(\| S_n/\gamma_n - (x, y) \| < \varepsilon)/n = \infty. \tag{6.37}$$

Hence (6.35) shows that, for all (x, y) such that $x > 0$, $y > 0$ and $x^2 + y^2 > 1$, we have $(x, y) \notin C(\{S_n/\gamma_n\})$. In view of the symmetry of X we thus have (6.26) and Lemma 6.3 is proved.

Acknowledgement. Most of the results of this paper were obtained during the 1982–83 academic year while visiting the University of Minnesota. I benefitted from a number of discussions with Bill Pruitt and Naresh Jain, and, in particular, I am thankful to Bill Pruitt for his suggestions regarding the examples of Section 6. The author is also indebted to the referee for his careful reading of the manuscript.

REFERENCES

[1] DE ACOSTA, A. (1983). A new proof of the Hartman-Wintner law of the iterated logarithm. *Ann. Probab.* **11** 270–276.

[2] DE ACOSTA, A., ARAUJO, A. and GINÉ, E. (1978). On Poisson measures, Gaussian measures, and the central limit theorem in Banach spaces. *Adv. in Probab.* **IV** 1–68. Dekker, New York.

[3] DE ACOSTA, A. and GINÉ, E. (1979). Convergence of moments and related functionals in the general central limit theorem in Banach spaces. *Z. Wahrsch. verw. Gebiete* **48** 213–231.

[4] DE ACOSTA, A. and KUELBS, J. (1983a). Some results on the cluster set $C(\{S_n/a_n\})$ and the LIL. *Ann. Probab.* **11** 102–122.

[5] DE ACOSTA, A. and KUELBS, J. (1983b). Limit theorems for moving averages of independent random vectors. *Z. Wahrsch. verw. Gebiete* **64** 67–123.

[6] ARAUJO, A., AND GINÉ, E. (1980). *The Central Limit Theorem for Real and Banach valued random variables.* Wiley, New York.

[7] CHOVER, J. (1967). On Strassen's version of the log log law. *Z. Wahrsch. verw. Gebiete* **8** 83–90.

[8] FELLER, W. (1971). *An Introduction to Probability Theory and its Applications.* Vol. II. 2nd ed. Wiley, New York.

[9] GOODMAN, V., KUELBS, J. and ZINN, J. (1981). Some results on the LIL in Banach space with applications to weighted empirical processes. *Ann. Probab.* **9** 713–752.

[10] HAHN, M. and KLASS, M. (1980). Matrix normalizations of sums of random vectors in the domain of attraction of the multivariate normal. *Ann. Probab.* **8** 262–280.

[11] HEINKEL, B. (1979). Relation entre théorème central-limite et loi du logarithme itéré dans les espaces de Banach. *Z. Wahrsch. verw. Gebiete* **49** 211–220.

[12] KESTEN, H. (1970). The limit points of a normalized random walk. *Ann. Math. Statist.* **41** 1173–1205.

[13] KESTEN, H. (1972). Sums of independent random variables without moment conditions. *Ann. Math. Statist.* **43** 701–732.

[14] KLASS, M. (1976). Toward a universal law of the iterated logarithm I. *Z. Wahrsch. verw. Gebiete* **36** 165–178.

[15] KLASS, M. (1977). Toward a universal law of the iterated logarithm II. *Z. Wahrsch. verw. Gebiete* **39** 151–165.

[16] KUELBS, J. (1973). The invariance principle for Banach space valued random variables. *J. Multivariate Anal.* **3** 161–172.

[17] KUELBS, J. (1974). An inequality for the distribution of the sum of certain Banach space valued random variables. *Studia Math.* **52** 69–87.

[18] KUELBS, J. (1976). The law of the iterated logarithm and related strong convergence theorems for Banach space valued random variables. *Lecture Notes in Mathematics* **539**. Springer-Verlag, Berlin.

[19] KUELBS, J. (1981). When is the cluster set of $\{S_n/a_n\}$ empty? *Ann. Probab.* **9** 377–394.

[20] KUELBS, J. and KURTZ, T. (1974). Berry-Esséen estimates in Hilbert space and an application to the law of the iterated logarithm. *Ann. Probab.* **2** 387–407.

[21] KUELBS, J. and LEPAGE, R. (1973). The law of the iterated logarithm for Brownian motion in a Banach space. *Trans. Amer. Math. Soc.* **185** 253–264.

[22] KUELBS, J. and ZINN, J. (1983). Some results on LIL behavior. *Ann. Probab.* **11** 506–557.

[23] PRUITT, W. (1981). General one sided laws of the iterated logarithm. *Ann. Probab.* **9** 1–48.

[24] SENETA, E. (1976). Regularly varying functions. *Lecture Notes in Mathematics* **508**. Springer-Verlag, Berlin.

[25] STOUT, W. (1974). *Almost Sure Convergence.* Academic Press, New York.

[26] SPITZER, F. (1964). *Principles of Random Walk.* Van Nostrand, Princeton.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WISCONSIN
MADISON, WISCONSIN 53706