THE LIMIT AS $p \to \infty$ FOR THE EIGENVALUE PROBLEM OF THE 1-HOMOGENEOUS p-LAPLACIAN

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ABSTRACT. In this paper we study asymptotics as $p \to \infty$ of the Dirichlet eigenvalue problem for the 1-homogeneous p-Laplacian, that is,

$$\begin{cases} -\frac{1}{p}|Du|^{2-p}\mathrm{div}\,(|Du|^{p-2}Du) = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Here Ω is a bounded starshaped domain in $\mathbb{R}^n$ and $p > n$. There exists a principal eigenvalue $\lambda_{1,p}(\Omega)$, which is positive, and has associated a non-negative nontrivial eigenfunction. Moreover, we show that $\lim_{p\to\infty} \lambda_{1,p}(\Omega) = \lambda_{1,\infty}(\Omega)$, where $\lambda_{1,\infty}(\Omega)$ is the first eigenvalue corresponding to the 1-homogeneous infinity Laplacian, that is, $-\left(D^2u\frac{Du}{|Du|}\right)\cdot\frac{Du}{|Du|} = \lambda u$.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

In this paper we analyze the eigenvalue problem corresponding to the 1-homogeneous p-Laplacian,

$$\begin{cases} -\Delta_p^N u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

Key words and phrases. Infinity Laplacian, 1-homogeneous p-Laplacian.
2000 *Mathematics Subject Classification.* 35A02, 35B51, 35J60.
P.J.M.-A. supported by MICINN Ministerio de Ciencia e Innovación (Spain) MTM2009-10878 and Junta de Andalucía FQM-116. M.P.-L. and J.D.R. supported by project MTM2010-18128 (Spain).

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $p > n$ and the 1-homogeneous p-Laplacian is given by

$$\begin{aligned}\Delta_p^N u := & \frac{1}{p} |Du|^{2-p} \cdot \operatorname{div}\big(|Du|^{p-2} Du\big) \\ = & \frac{1}{p} \operatorname{trace}\left[\left(I + (p-2)\frac{Du \otimes Du}{|Du|^2}\right) D^2 u\right] = \frac{1}{p}\Delta u + \frac{p-2}{p}\Delta_\infty u.\end{aligned}$$

This operator appears naturally when one considers Tug-of-War games with noise, see [15, 16, 18, 19, 20], where the Poisson problem is studied. Moreover, the sublinear problem associated to the 1-homogeneous p-Laplacian, namely, the problem with right-hand side λu^q for $0 < q < 1$, has been studied in [17].

Our main goal here is to analyze the limit as $p \to \infty$ for the eigenvalue problem corresponding to the 1-homogeneous p-Laplacian (1.1).

Eigenvalue problems appear in several contexts in the theory of second order elliptic partial differential equations. It is well-known that, for linear elliptic operators in divergence form, $Lu = \operatorname{div}(A(x)Du)$, the following Dirichlet problem

$$\begin{cases} Lu + \lambda u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

has a first eigenvalue, that is, a nontrivial solution (u, λ) with u non-negative and nontrivial. More precisely, the first eigenvalue is given by

$$\lambda_1 = \inf_{u \in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\int_\Omega \big(A(x)Du(x)\big)\cdot Du(x)\,dx}{\int_\Omega u^2\,dx},$$

and the corresponding eigenfunctions are the minimizers of this functional. However, this result is based on variational methods that are not adequate for operators in non-divergence form.

In [3] the authors deal with this difficulty and prove that the number

$$\lambda_1 = \sup\big\{\lambda \in \mathbb{R} : \exists v(x) > 0\ \forall x \in \Omega \text{ such that } (L+\lambda)v \le 0\big\},$$

turns out to be the smallest eigenvalue of L. In this direction, in [11] the following eigenvalue problem is studied

$$\begin{cases} -\Delta_\infty u(x) = \lambda u(x), & \text{in } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where the operator

$$\Delta_\infty u = \left(D^2 u \frac{Du}{|Du|}\right) \cdot \frac{Du}{|Du|} \tag{1.3}$$

is known as the 1-homogeneous infinity Laplacian, see the survey [1]. In [11] it is proved that the principal eigenvalue is given by

$$\begin{aligned}\lambda_{1,\infty}(\Omega) = \sup\big\{\lambda \in \mathbb{R} :\ & \exists v \in \mathcal{C}(\overline{\Omega}) \text{ such that } v(x) > 0\ \forall x \in \overline{\Omega} \\ & \text{and } -\Delta_\infty v \ge \lambda v \text{ in the viscosity sense}\big\}.\end{aligned} \tag{1.4}$$

Moreover, it is proved that $\lambda_{1,\infty}(\Omega) > 0$ and it is computed explicitly in the case of a ball.

For the problem under consideration here, in [4] and [5] the authors show existence of a first eigenvalue with a positive eigenfunction (in fact, the results contained there cover more general fully nonlinear operators). Let

$$\lambda_{1,p}(\Omega) = \sup\big\{\lambda \in \mathbb{R} : \ \exists v \in \mathcal{C}(\overline{\Omega}) \text{ such that } v(x) > 0 \ \forall x \in \overline{\Omega} \text{ and } -\Delta_p^N v \geq \lambda v \text{ in the viscosity sense}\big\}. \tag{1.5}$$

In [4] and [5] it is proved that this number is the principal eigenvalue of problem (1.1).

Theorem 1.1. *([4], [5]) There exists a first eigenvalue for (1.1), which is given by (1.5). Moreover, this eigenvalue has associated a non-negative and nontrivial eigenfunction.*

For completeness of this paper and since some care has to be taken when defining viscosity solutions to this operator, we provide a short proof of this result. In the case of star-shaped domains we prove here (see Proposition 3.7) that the existence of a nonnegative nontrivial eigenfunction characterizes the first eigenvalue.

Next, we analyze the limit as $p \to \infty$ and we obtain our main result:

Theorem 1.2. *Assume that Ω is star-shaped, then it holds that*

$$\lim_{p\to\infty} \lambda_{1,p}(\Omega) = \lambda_{1,\infty}(\Omega) \tag{1.6}$$

where $\lambda_{1,\infty}(\Omega)$, given by (1.4), is the first eigenvalue for the infinity Laplacian.

Remark 1.3. *We prove that the limit as p goes to infinity of the eigenvalue problem for the 1-homogeneous p-Laplacian (1.1) is (1.2) in contrast to the case of the variational p-Laplacian (see [13]),*

$$-\text{div}\,(|Du|^{p-2}Du) = \lambda|u|^{p-2}u,$$

where the limit problem is given by

$$\min\{|Du| - \lambda u, \ -\Delta_\infty u\} = 0.$$

Concerning methods and ideas used in the proofs we just mention that the equation under consideration is nonlinear and it is not in divergence form. In addition, it is undefined when Du vanishes. Therefore we use the concept of viscosity solutions, see [8], and we have to take care in the proof of a comparison principle that allows us to compare super and subsolutions. We also need a Harnack inequality to pass to the limit in certain approximating problems. This Harnack inequality was proved in [6]. To deal with the limit as $p \to \infty$ we use the uniform estimates proved in [7].

The paper is organized as follows, in Section 2 we give some necessary definitions and we show a Comparison Principle; in Section 3 we deal with the problem for $p < \infty$ fixed, and we prove that (1.5) is the first eigenvalue for (1.1); finally, in Section 4 we show the convergence of the eigenvalues as $p \to \infty$ given in (1.6).

2. Preliminaries

We devote this section to state precisely the notion of solution of problem (1.1). Note that, if $Du = 0$ the operator $\frac{1}{p}|Du|^{2-p}\mathrm{div}\,(|Du|^{p-2}Du)$ is undefined even if u is regular, and we need to deal with this fact. As mentioned in the introduction, if u is smooth and $Du \neq 0$ we have

$$\frac{1}{p}|Du|^{2-p}\mathrm{div}\,(|Du|^{p-2}Du) = \frac{(p-2)}{p}\Delta_\infty u(x) + \frac{1}{p}\Delta u(x), \tag{2.1}$$

where $\Delta_\infty u$ is the $1-$homogeneous infinity Laplacian given by (1.3).

We observe that, in order to define the $1-$homogeneous infinity Laplacian, we have to give a meaning when $\xi = 0$ to the following function,

$$F(\xi, X) = \left(X \frac{\xi}{|\xi|} \right) \cdot \frac{\xi}{|\xi|}, \qquad \xi \in \mathbb{R}^n,\ X \in \mathbb{S}_N.$$

By $\mathbb{S}_N$ we denote the set of symmetric matrices in $\mathbb{R}^{n\times n}$ and by $M(A)$ and $m(A)$ the largest and smallest eigenvalues of $A \in \mathbb{S}_N$, respectively, i.e.

$$M(A) = \max_{|\eta|=1}(A\eta)\cdot\eta \qquad \text{and} \qquad m(A) = \min_{|\eta|=1}(A\eta)\cdot\eta.$$

Taking into account (2.1), problem (1.1) can be rewritten as follows

$$\begin{cases} -\frac{(p-2)}{p}\Delta_\infty u(x) - \frac{1}{p}\Delta u(x) = \lambda u(x), & \text{in } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

and we consider the following standard definitions of viscosity sub and supersolutions of (2.2) that use the upper and lower semicontinuous envelopes (relaxations) of the operator (see [8, Section 9]).

Definition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $p \geq 2$ and $\lambda \in \mathbb{R}$. We say that an upper semicontinuous function $u : \Omega \to \mathbb{R}$ is a viscosity subsolution of* (2.2) *if, $u|_{\partial\Omega} \leq 0$ and, whenever $x_0 \in \Omega$ and $\psi \in C^2(\Omega)$ are such that $u(x_0) = \psi(x_0)$ and $u(x) < \psi(x)$, if $x \neq x_0$, then*

$$\begin{cases} -\frac{(p-2)}{p}\Delta_\infty \psi(x_0) - \frac{1}{p}\Delta\psi(x_0) \leq \lambda\psi(x_0), & \textit{if } D\psi(x_0) \neq 0, \\ -\frac{(p-2)}{p}M(D^2\psi(x_0)) - \frac{1}{p}\Delta\psi(x_0) \leq \lambda\psi(x_0), & \textit{if } D\psi(x_0) = 0. \end{cases}$$

We say that a lower semicontinuous function $u : \Omega \to \mathbb{R}$ is a viscosity supersolution of (2.2) *if, $u|_{\partial\Omega} \geq 0$ and, whenever $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that $u(x_0) = \varphi(x_0)$ and $u(x) > \varphi(x)$, if $x \neq x_0$, then*

$$\begin{cases} -\frac{(p-2)}{p}\Delta_\infty \varphi(x_0) - \frac{1}{p}\Delta\varphi(x_0) \geq \lambda\varphi(x_0), & \textit{if } D\varphi(x_0) \neq 0, \\ -\frac{(p-2)}{p}m(D^2\varphi(x_0)) - \frac{1}{p}\Delta\varphi(x_0) \geq \lambda\varphi(x_0), & \textit{if } D\varphi(x_0) = 0. \end{cases}$$

Finally, a continuous function $u : \Omega \to \mathbb{R}$ is a viscosity solution of (2.2) *if it is both, a viscosity supersolution and a viscosity subsolution.*

For the sake of clarity, we will keep in the sequel the notation used in the above definitions. That is, we will denote by φ the test functions touching the graph of u from below, and by ψ the test functions touching the graph of u from above.

Note that the contact condition in the above definition is local in the sense that it is required to hold only in a neighborhood of x_0. Hence, it is possible to relax the strict inequality, we refer to [8] for more details about general theory of viscosity solutions, and [10, 12] for viscosity solutions related to the Infinity Laplacian and the $p-$Laplacian operators.

At this point we are ready to prove a comparison principle for problem (2.2) inspired by the ideas in [11].

Proposition 2.2. *Let $\mu < \lambda$ and $v \in \mathcal{C}(\overline{\Omega})$ such that $v > 0$ in $\overline{\Omega}$ and $-\Delta_p^N v \geq \lambda v$. If $u \in \mathcal{C}(\overline{\Omega})$ verifies $-\Delta_p^N u \leq \mu u$ and $u \leq 0$ on $\partial\Omega$, then $u \leq 0$ in Ω.*

Proof. Arguing by contradiction, suppose that there exists an interior point at which u is strictly positive. Since $u \leq 0$ on $\partial\Omega$ and $v > 0$ in Ω, this implies that there exists $x_0 \in \Omega$ where $\frac{u(x)}{v(x)}$ attains a positive maximum. Denote by Ω_0 an open set containing x_0, where u is positive. We consider $w(x) = \log(u(x))$ in Ω_0 and $g(x) = \log(v(x))$ in Ω. It is not difficult to see that

$$
\begin{aligned}
&-\tfrac{(p-1)}{p}|Dw|^2 - \tfrac{(p-2)}{p}\Delta_\infty w - \tfrac{1}{p}\Delta w \leq \mu, && \text{in } \Omega_0,\\
&-\tfrac{(p-1)}{p}|Dg|^2 - \tfrac{(p-2)}{p}\Delta_\infty g - \tfrac{1}{p}\Delta g \geq \lambda, && \text{in } \Omega.
\end{aligned}
\tag{2.3}
$$

Note that x_0 is also a positive maximum of $w(x) - g(x) = \log\left(\frac{u(x)}{v(x)}\right)$. Then, using arguments in [11], we consider

$$
\psi_j(x,y) = w(x) - g(y) - \theta_j(x,y), \quad j \in \mathbb{N}, \quad \theta_j(x,y) = \frac{j}{4}|x-y|^4, \tag{2.4}
$$

and let $(x_j, y_j) \in \Omega_0 \times \Omega_0$ such that

$$
\psi_j(x_j, y_j) = \sup_{(x,y)\in\Omega_0\times\Omega_0} \psi_j(x,y).
$$

Then we have that

$$
x_j \to x_0, \quad y_j \to x_0, \quad j|x_j - y_j|^4 \to 0, \quad \text{as } j \to \infty.
$$

Thus we can assume that $\psi_j(x,y)$ attains a positive maximum at (x_j, y_j) for j large. Applying the maximum principle for semicontinuous functions we obtain that there exist symmetric matrices $X_j, Y_j \in \mathbb{S}_N$, such that

$$
(\eta_j, X_j) \in \overline{J}^{2,+} w(x_j), \quad (\eta_j, Y_j) \in \overline{J}^{2,-} g(y_j), \tag{2.5}
$$

being $\eta_j = j|x_j - y_j|^2(x_j - y_j)$, and

$$
\begin{pmatrix} X_j & 0 \\ 0 & -Y_j \end{pmatrix} \leq D^2\theta_j(x_j, y_j) + \frac{1}{j}\Big(D^2\theta_j(x_j, y_j)\Big)^2.
$$

After some computations and denoting $z_j = x_j - y_j$, the previous inequality reads as follows

$$
\begin{aligned}
\begin{pmatrix} X_j & 0 \\ 0 & -Y_j \end{pmatrix} \leq\ & j(|z_j|^2 + 2|z_j|^4)\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \\
&+16j|z_j|^2 \begin{pmatrix} z_j \otimes z_j & -z_j \otimes z_j \\ -z_j \otimes z_j & z_j \otimes z_j \end{pmatrix}.
\end{aligned}
\tag{2.6}
$$

Evaluating these quadratic forms at $(\xi,\xi)\in\mathbb{R}^{2N}$ it leads to

$$(X_j\xi)\cdot\xi \le (Y_j\xi)\cdot\xi, \qquad \text{for all } \xi\in\mathbb{R}^n, \tag{2.7}$$

that is, $Y_j - X_j$ is positive semidefinite. From (2.3) and (2.5) together with (2.7), if $x_j \neq y_j$ we can conclude that

$$\begin{aligned}\lambda &\le -\frac{(p-1)}{p}|\eta_j|^2 - \frac{(p-2)}{p}\left(Y_j\frac{\eta_j}{|\eta_j|}\right)\cdot\frac{\eta_j}{|\eta_j|} - \frac{1}{p}\mathrm{trace}(Y_j)\\ &\le -\frac{(p-1)}{p}|\eta_j|^2 - \frac{(p-2)}{p}\left(X_j\frac{\eta_j}{|\eta_j|}\right)\cdot\frac{\eta_j}{|\eta_j|} - \frac{1}{p}\mathrm{trace}(X_j) \le \mu,\end{aligned}$$

which is a contradiction with the assumption $\lambda > \mu$.

If $x_j = y_j$ then $\eta_j = z_j = 0$ and by (2.6) we get that $X_j \le 0 \le Y_j$. Taking into account (2.3) and (2.5), it implies that

$$\lambda \le -\frac{(p-2)}{p}m(Y_j) - \frac{1}{p}\mathrm{trace}(Y_j) \le -\frac{(p-2)}{p}M(X_j) - \frac{1}{p}\mathrm{trace}(X_j) \le \mu,$$

getting again a contradiction. □

In particular, for our purposes we have the following corollary. Recall that $\lambda_{1,p}(\Omega)$ is given by (1.5).

Corollary 2.3. *Let $\mu < \lambda_{1,p}(\Omega)$ and $u \in \mathcal{C}(\overline{\Omega})$ satisfying $-\Delta_p^N u \le \mu u$ and $u \le 0$ on $\partial\Omega$. Then $u \le 0$ in Ω.*

Moreover, we have also obtained that any real number $\lambda < \lambda_{1,p}(\Omega)$ cannot be an eigenvalue.

Corollary 2.4. *Let $u \in \mathcal{C}(\overline{\Omega})$ a non-negative viscosity solution of*

$$\begin{cases} -\Delta_p^N u(x) = \lambda u(x), & \text{in } \Omega,\\ u(x) = 0, & \text{on } \partial\Omega,\end{cases}$$

with $\lambda < \lambda_{1,p}(\Omega)$. Then $u \equiv 0$. In particular, λ is not an eigenvalue.

If we show that $\lambda_{1,p}(\Omega)$ is indeed an eigenvalue, then Corollary 2.4 will lead to our characterization for the first eigenvalue.

We will use the following result, see [6], Theorem 4.3 (see also Theorem 4.8).

Lemma 2.5. *(Harnack inequality) Let u be a solution of the $1-$homogeneous $p-$Laplacian, if $x_0 \in \Omega$ and $0 < r < R \le dist(x_0, \partial\Omega)$, then there exists a constant C such that*

$$\sup_{y\in B(x_0,r)} u(y) \le C \inf_{y\in B(x_0,r)} u(y).$$

Now we need to establish a comparison result for a related problem. We use the ideas in [11].

Theorem 2.6. *Let $\lambda < \lambda_{1,p}(\Omega)$, and let u and v be a viscosity subsolution and a supersolution, respectively, of the equation*

$$-\Delta_p^N \phi(x) = \lambda\phi(x) + f(x),$$

where $f \in \mathcal{C}(\Omega)$. *Suppose that either*

$$f(x) > 0, \quad \forall x \in \Omega$$

or

$$f(x) \geq 0, \quad \forall x \in \Omega \text{ and } \lambda > 0.$$

Then, if both $v \geq u$ *and* $v > 0$ *on* $\partial\Omega$, *we have* $v \geq u$ *in* Ω.

Proof. Arguing by contradiction, we assume that $\{x \in \Omega : u(x) > v(x)\} \neq \emptyset$. Applying Proposition 2.2 to $-v$ we deduce that v is non-negative. In fact $v > 0$ in $\overline{\Omega}$, if $\lambda \geq 0$, thanks to the Harnack inequality (see Lemma 2.5). In the case $\lambda < 0$, just notice that the trivial function is a test function (from below) at the points where v vanishes and then use the assumption $f(x) > 0$ for all $x \in \Omega$. We consider $\tilde{x} \in \Omega$ as a point verifying

$$1 < \frac{u(\tilde{x})}{v(\tilde{x})} = \sup_{x \in \overline{\Omega}} \frac{u(x)}{v(x)}. \tag{2.8}$$

Let us assume that $u > v > 1$ in some neighborhood $\tilde{\Omega}$ of $\tilde{x}$ (otherwise we can rescale f). As in Proposition 2.2, a simple computation shows that the functions $w(x) = \log(u(x))$ and $g(x) = \log(v(x))$, are a subsolution and a supersolution, respectively, of

$$-\frac{(p-1)}{p}|D\phi(x)|^2 - \frac{(p-2)}{p}\Delta_\infty\phi(x) - \frac{1}{p}\Delta\phi(x) - \lambda - f(x)e^{-\phi(x)} = 0 \tag{2.9}$$

in the subdomain $\tilde{\Omega}$. Now we apply the maximum principle for semicontinuous functions as in (2.4). If $x_j \neq y_j$, it follows from $X_j \leq Y_j$ and the fact that w and g are a subsolution and a supersolution of (2.9) that

$$\begin{aligned}
\lambda + f(y_j)e^{-g(y_j)} &\leq -\frac{(p-1)}{p}|\eta_j|^2 - \frac{(p-2)}{p}\left(Y_j\frac{\eta_j}{|\eta_j|}\right)\cdot\frac{\eta_j}{|\eta_j|} - \frac{1}{p}\mathrm{trace}(Y_j) \\
&\leq -\frac{(p-1)}{p}|\eta_j|^2 - \frac{(p-2)}{p}\left(X_j\frac{\eta_j}{|\eta_j|}\right)\cdot\frac{\eta_j}{|\eta_j|} - \frac{1}{p}\mathrm{trace}(X_j) \\
&\leq \lambda + f(x_j)e^{-w(x_j)}.
\end{aligned}$$

On the other hand, if $x_j = y_j$, then $\eta_j = 0$ and we obtain

$$\begin{aligned}
\lambda + f(y_j)e^{-g(y_j)} &\leq -\frac{(p-2)}{p}m(Y_j) - \frac{1}{p}\mathrm{trace}(Y_j) \\
&\leq 0 \leq -\frac{(p-2)}{p}M(X_j) - \frac{1}{p}\mathrm{trace}(X_j) \leq \lambda + f(x_j)e^{-w(x_j)}.
\end{aligned}$$

Hence, in both cases we conclude that $\lambda + f(y_j)e^{-g(y_j)} \leq \lambda + f(x_j)e^{-w(x_j)}$ for each j. Then, if $f(\tilde{x}) > 0$, we let $j \to \infty$ to obtain a contradiction with (2.8).

If $f \geq 0$, in order to obtain a strict inequality, we perturb g, so that it becomes a strict supersolution. More precisely, we consider $G(x) := h(g(x))$ for

$$h(t) = \frac{1}{\alpha}\log(1 + A(e^{\alpha t} - 1)), \quad \alpha > 1,\ A > 1.$$

In [13], the following properties for h can be found. We have that $h'(t) > 1$ and $h'(t) - h'(t)^2 - h''(t) > 0$ for all $t \geq 0$. Moreover, $0 < h(t) - t < \frac{A-1}{\alpha}$

for $t \geq 0$ and thus $h(t) \to t$ uniformly if $A \to 1^+$. Taking into account these properties, we can deduce that G verifies

$$\begin{aligned} &-\frac{(p-1)}{p}|DG|^2 - \frac{(p-2)}{p}\Delta_\infty G - \frac{1}{p}\Delta G \\ &\geq h'(g)\left[\lambda + fe^{-g}\right] + \frac{(p-1)}{p}|Dg|^2\left[h'(g) - h'(g)^2 - h''(g)\right] > \lambda + fe^{-G}. \end{aligned}$$

In the last inequality we have used that $h'(t) > 1$ and $h(t) > t$ for all $t \geq 0$. Since h is smooth and increasing, we obtain that G is a strict supersolution as we wanted, namely

$$-\frac{(p-1)}{p}|DG|^2 - \frac{(p-2)}{p}\Delta_\infty G - \frac{1}{p}\Delta G > \lambda + f(x)e^{-G(x)},$$

in the viscosity sense. By choosing $A > 1$ close enough to one, we have that also $w - G$ achieves its positive maximum at certain $\tilde{x} \in \tilde{\Omega}$. We argue as before, but applying the maximum principle for semicontinuous functions to

$$\Psi(x,y) = w(x) - G(y) - \theta_j(x,y), \quad j \in \mathbb{N}.$$

Again we obtain in both cases ($x_j = y_j$ and $x_j \neq y_j$) that $\lambda + f(y_j)e^{-g(y_j)} < \lambda + f(x_j)e^{-w(x_j)}$ for each j, which leads to a contradiction upon letting $j \to \infty$, so the result follows. □

As a consequence we get the following result.

Corollary 2.7. *Let $\lambda < \lambda_{1,p}(\Omega)$ and assume that $f : \overline{\Omega} \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ are continuous functions such that g is positive and either f is positive in Ω or f is non-negative in Ω and $\lambda > 0$. Then the Dirichlet problem*

$$\begin{cases} -\Delta_p^N \phi(x) = \lambda\phi(x) + f(x), & \text{in } \Omega, \\ \phi(x) = g(x), & \text{on } \partial\Omega, \end{cases}$$

has at most one solution.

3. The principal eigenvalue for fixed p

In this section, our purpose is to show that the principal eigenvalue of (1.1) is given by (1.5). Note that constant functions verify $-\Delta_p^N v = 0$, thus $\lambda_{1,p}(\Omega)$ is well-defined and non-negative. Moreover, if $\Omega_1 \subset \Omega_2$ then

$$\lambda_{1,p}(\Omega_2) \leq \lambda_{1,p}(\Omega_1).$$

This fact will allow us to estimate $\lambda_{1,p}(\Omega)$ for a general domain Ω by the principal eigenvalue in a ball.

3.1. **The principal eigenvalue in a ball.** Consider $\Omega = B_R = B_R(0)$ and let us look for radial solutions of problem (2.2). A simple calculation shows that for radial functions $g = g(r)$ problem (2.2) reads as follows

$$\begin{cases} -\left(\dfrac{p-1}{p}\right) g''(r) - \left(\dfrac{n-1}{p}\right)\dfrac{g'(r)}{r} = \lambda_{1,p}(B_R)\, g(r), & \text{in } [0,R), \\ g(R) = 0, \qquad g'(0) = 0. \end{cases} \tag{3.1}$$

Lemma 3.1. *If* $\Omega = B_R$ *the principal eigenvalue defined in* (1.5) *is given by*

$$\lambda_{1,p}(B_R) = \frac{(p-1)}{p}\left(\frac{t_0}{R}\right)^2, \tag{3.2}$$

where t_0 *is the first zero of the Bessel function solving*

$$t^2\frac{d^2u}{dt^2} + t\frac{du}{dt} + (t^2-\nu^2)u = 0, \quad \nu^2 = \left(\frac{p-n}{p-1}\right)^2. \tag{3.3}$$

Proof. Equation (3.1) is equivalent to

$$-g''(r) - \left(\frac{n-1}{p-1}\right)\frac{g'(r)}{r} = \left(\frac{p}{p-1}\right)\lambda_{1,p}(B_R)\, g(r). \tag{3.4}$$

Hence, we obtain (3.3) by means of the following change of variables

$$\begin{cases} g &= \left(\dfrac{t}{\sqrt{c}}\right)^{-\alpha} u, \\ x &= \dfrac{t}{\sqrt{c}}, \end{cases} \qquad \text{for } \alpha = \frac{n-p}{2(N-1)} < 0 \text{ and } c = \frac{p}{p-1}\lambda_{1,p}(\Omega).$$

Let us denote

$$\mu(R) = \frac{(p-1)}{p}\left(\frac{t_0}{R}\right)^2.$$

Then, by (1.5) it holds that $\lambda_{1,p}(B_R) \geq \mu(R)$. In order to prove the equality, let us assume for the sake of contradiction that $\lambda_{1,p}(B_R) > \mu(R)$. Since μ is non-increasing in R, we can take $0 < \rho < R$ such that $\lambda_{1,p}(B_R) > \mu(\rho) > \mu(R)$ and let

$$w(x) = \begin{cases} g_\rho(|x|), & \text{if } |x| \leq \rho, \\ 0, & \text{if } |x| > \rho, \end{cases}$$

with g_ρ verifying

$$-g''(r) - \left(\frac{n-1}{p-1}\right)\frac{g'(r)}{r} = \mu(\rho)\, g(r) \quad \text{in } B_\rho.$$

Note that $-\Delta_p^N w \leq \mu(\rho)\, w$ in B_R and $w \leq 0$ on ∂B_R. Then, the comparison principle (Corollary 2.3) implies that $w \leq 0$ in B_R, a contradiction. □

Remark 3.2. *Notice that taking limits as* $p \to \infty$ *in* (3.4) *we can see that*

$$\lim_{p\to\infty} \lambda_{1,p}(B_R) = \lambda_{1,\infty}(B_R).$$

This convergence also holds for the case of a general star-shaped domain, see Section 4 for the details.

3.2. **The principal eigenvalue in a general domain.** Our purpose now is to prove that $\lambda_{1,p}(\Omega)$ defined in (1.5) is an eigenvalue. This result is contained in [4] and [5] but we include here a short proof for completeness. Thanks to the results for the radial case and the fact that $\Omega_1 \subset \Omega_2$ implies $\lambda_{1,p}(\Omega_1) \geq \lambda_{1,p}(\Omega_2)$, we will be able to find bounds (uniformly in p) for the principal eigenvalue $\lambda_{1,p}(\Omega)$. Define

$$R_* = \inf\{r > 0 : \Omega \subset B_r(x) \text{ for some } x\}$$

and

$$R^* = \sup\{r > 0 : B_r(x) \subset \Omega \text{ for some } x\}.$$

Then,

$$\lambda_{1,p}(B_{R_*}) \le \lambda_{1,p}(\Omega) \le \lambda_{1,p}(B_{R^*}), \tag{3.5}$$

where $\lambda_{1,p}(B_{R_*})$ and $\lambda_{1,p}(B_{R^*})$ are the radial principal eigenvalues given by (3.2). We emphasize that these values do not depend on p for p large enough.

Remark 3.3. *From* (3.5) *we obtain that* $\lambda_{1,p}(\Omega) > 0$. *In fact, from the ABP-type estimate* [7, Theorem 4.1], *we easily obtain that*

$$\lambda_{1,p}(\Omega) \ge \left(\frac{p-1}{p}\right) \operatorname{diam}(\Omega)^{-2}.$$

We state the main result of this section.

Theorem 3.4. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain. Then, there exists* $w \in \mathcal{C}(\overline{\Omega})$ *such that*

$$\begin{cases} -\Delta_p^N w = \lambda_{1,p}(\Omega) w, & \text{in } \Omega, \\ w > 0, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega. \end{cases}$$

We will need in the sequel the following $\mathcal{C}^\alpha$-estimates that can be found in [7, Section 5].

Lemma 3.5. *Let* Ω *be a bounded domain,* $2 \le n < p < \infty$, *and* u *a viscosity solution of*

$$-\Delta_p^N u = f \quad \text{in } \Omega$$

with $f \in \mathcal{C}(\Omega)$. *Then, for any* $x \in \Omega$ *we have that*

$$\frac{|u(y) - u(x)|}{|y-x|^{1-\alpha_p}} \le \frac{2\|u\|_{L^\infty(\Omega)}}{\operatorname{dist}(x, \partial\Omega)^{1-\alpha_p}} + \frac{C_p}{1-\alpha_p} \operatorname{diam}(\Omega)^{1+\alpha_p} \|f\|_{L^\infty(\Omega)}$$

for every $y \in \overline{\Omega}$, *where* $\alpha_p = \frac{n-1}{p-1}$ *and* $C_p = \frac{p}{p+n-2}$.

Lemma 3.6. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain and* $0 < \lambda < \lambda_{1,p}(\Omega)$. *Then, there exists* $w \in \mathcal{C}(\overline{\Omega})$ *viscosity solution of*

$$\begin{cases} -\Delta_p^N w = 1 + \lambda w, & \text{in } \Omega, \\ w > 0, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega. \end{cases}$$

Proof. The proof follows by the Perron method (see [9, Section 2.4]). Since Theorem 2.6 provides a comparison principle for this problem, we need to construct a viscosity sub and supersolution, $\underline{v}$ and $\overline{v}$. We begin by constructing the supersolution. Note that thanks to (3.5) we can take $0 < \lambda < \lambda_{1,p}(\Omega)$ such that there exists a positive continuous function u, verifying $-\Delta_p^N u \ge \lambda u$ in the viscosity sense in Ω. Let $\eta_0 = \min_{x \in \partial\Omega} u(x) > 0$. For $0 < \eta < \eta_0$ the function $u_\eta = u - \eta$, which is positive by the Maximum Principle, see [2], verifies

$$-\Delta_p^N u_\eta \ge \lambda u_\eta + \lambda \eta.$$

Defining $\hat{u} = \frac{u_\eta}{\lambda\eta}$, then $\hat{u}$ is a supersolution of $-\Delta_p^N v = \lambda v + 1$.

However, $\hat{u}$ does not necessarily vanish on $\partial\Omega$. In order to get a supersolution of this equation with the right boundary data, we define for any given $z \in \partial\Omega$ the function $u_z(x) = |x-z|^{2\alpha}$, which verifies

$$-\Delta_p^N u_z(x) = -\frac{2\alpha[(2\alpha-1)(p-1)+n-1]}{p}|x-z|^{2(\alpha-1)}.$$

Now the choices

$$p > n \quad \text{and} \quad \alpha < \min\left\{1,\ \frac{p-n}{2(p-1)}\right\}$$

ensure that $-2\alpha[(2\alpha-1)(p-1)+n-1] > 0$. Therefore, it follows that

$$-\Delta_p^N u_z(x) = \left(-\frac{2\alpha[(2\alpha-1)(p-1)+n-1]}{p} - \beta\right)\frac{u_z(x)}{|x-z|^2} + \beta|x-z|^{2(\alpha-1)}.$$

Then, choosing β appropriately, since $\alpha < 1$ there exists $\rho = \rho(\lambda) > 0$ such that $-\Delta_p^N u_z \geq \lambda u_z + 1$ in $B_\rho(z) \cap \Omega$. Taking some constant verifying $C(\rho/2)^{1/2} \geq \sup_\Omega \hat{u}$ we get that $\hat{u}(x) \leq C u_z(x)$ outside $B_{\rho/2}(z) \cap \Omega$, thus

$$U(x) = \inf_{z \in \partial\Omega}\Big(\min\{C u_z(x), \hat{u}(x)\}\Big)$$

is the desired positive supersolution of $-\Delta_p^N v = \lambda v + 1$, vanishing on $\partial\Omega$.

We can take the function $u = 0$ as a subsolution of $-\Delta_p^N v = \lambda v + 1$. At this point, the existence of a solution follows from the Perron method. Notice that $v > 0$. Otherwise, there would exist a point $x_0 \in \Omega$ such that $v(x_0) = 0$ and, since $v \geq 0$, we could use 0 as a test function in the definition of viscosity solution, a contradiction. □

Now we are ready to prove Theorem 3.4. The proof of is based in the previous two results.

Proof of Theorem 3.4. Consider a sequence of numbers $\lambda_k \nearrow \lambda_{1,p}(\Omega)$. By Lemma 3.6 we can assure the existence of w_k a positive solution of $-\Delta_p^N w_k = \lambda_k w_k + 1$ vanishing on $\partial\Omega$. First we show that $\sup_\Omega w_k$ is not bounded. If we assume that it is bounded, by Lemma 3.5 we have that the sequence $\{w_k\}$ is locally equicontinuous and hence convergent (up to a subsequence) to a positive viscosity solution w of

$$\begin{cases} -\Delta_p^N w = \lambda_{1,p}(\Omega)\, w + 1, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega. \end{cases}$$

The homogeneous boundary condition is obtained using uniform barriers of the form $x \mapsto C|x-z|$ with $z \in \partial\Omega$. Then, we define $w_\varepsilon = w + \varepsilon$, a positive function in $\overline{\Omega}$ satisfying

$$-\Delta_p^N w_\varepsilon = (1 - \varepsilon\lambda_{1,p}(\Omega)) + \lambda_{1,p}(\Omega) w_\varepsilon \geq \mu w_\varepsilon$$

for any $0 \leq \mu \leq \lambda_{1,p}(\Omega) + \frac{1-\varepsilon\lambda_{1,p}(\Omega)}{\sup_\Omega w_\varepsilon}$, which contradicts the definition of $\lambda_{1,p}(\Omega)$ choosing ε such that $\varepsilon\lambda_{1,p}(\Omega) < 1$.

Now, if we consider $v_k = \frac{w_k}{\sup_\Omega w_k}$, it verifies $-\Delta_p^N v_k = \lambda_k v_k + \frac{1}{\sup_\Omega w_k}$ in the viscosity sense. Since $\sup_\Omega v_k = 1$ then, again by Lemma 3.5 it holds that v_k converges (up to a subsequence) locally uniformly to some positive nontrivial v (the positivity follows from the Harnack inequality in Lemma 2.5). Since $\sup_\Omega w_k \to \infty$ as $k \to \infty$ (up to a subsequence), the limit v solves

$$\begin{cases} -\Delta_p^N v = \lambda_{1,p}(\Omega) v, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega, \end{cases}$$

by the same barrier argument as before. This function v is the desired eigenfunction and the proof is complete. □

In the final result of this section we prove, for star-shaped domains, that any positive eigenfunction necessarily corresponds to the first eigenvalue.

Proposition 3.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded star-shaped domain with respect to a point (that we may assume to be the origin), $\lambda > 0$ and assume that there exists $\phi \in \mathcal{C}(\overline{\Omega})$ such that*

$$\begin{cases} -\Delta_p^N \phi = \lambda \phi, & \text{in } \Omega, \\ \phi > 0, & \text{in } \Omega, \\ \phi = 0, & \text{on } \partial\Omega. \end{cases}$$

Then, necessarily, $\lambda = \lambda_{1,p}(\Omega)$.

Proof. Since Corollary 2.4 implies $\lambda \geq \lambda_{1,p}(\Omega)$, we can assume for the sake of contradiction that $\lambda > \lambda_{1,p}(\Omega)$.

Now, let $\Omega_\mu = \mu\Omega$. Using that Ω is star-shaped with respect to the origin we get $\Omega \subset \Omega_\mu$ for $\mu > 1$. For $x \in \Omega_\mu$ consider the function

$$\psi(x) = \phi(x/\mu)$$

which is a solution to

$$\begin{cases} -\Delta_p^N \psi = \mu^{-2} \lambda \psi, & \text{in } \Omega, \\ \psi > 0, & \text{in } \Omega, \\ \psi > 0, & \text{on } \partial\Omega \end{cases}$$

(note that the operator is homogeneous under dilations and also that we are restricting ψ to Ω).

To obtain the desired contradiction, we only have to take $\mu > 1$ but close to one in such a way that $\mu^{-2}\lambda > \lambda_{1,p}(\Omega)$. □

Remark 3.8. *Note that a similar argument shows that $\lambda_{1,p}(\Omega)$ is continuous with respect to Ω for a star-shaped domain. In fact, if $\nu < 1 < \mu$ and $\tilde{\Omega}$ a domain verifying*

$$\nu\Omega \subset \tilde{\Omega} \subset \mu\Omega,$$

it holds that

$$\mu^{-2}\lambda_{1,p}(\Omega) \leq \lambda_{1,p}(\tilde{\Omega}) \leq \nu^{-2}\lambda_{1,p}(\Omega),$$

and obviously

$$\mu^{-2}\lambda_{1,p}(\Omega) \leq \lambda_{1,p}(\Omega) \leq \nu^{-2}\lambda_{1,p}(\Omega).$$

But then

$$|\lambda_{1,p}(\Omega) - \lambda_{1,p}(\tilde{\Omega})| \leq (\nu^{-2} - \mu^{-2})\lambda_{1,p}(\Omega),$$

which shows that $\lambda_{1,p}(\Omega)$ is continuous with respect to the domain, whenever Ω is star-shaped.

4. Convergence of the eigenvalues as $p \to \infty$

We devote this section to analyze the behaviour of the first eigenvalue and its corresponding eigenfunction as the exponent p goes to infinity.

Proposition 4.1. *Let Ω be a bounded star-shaped domain and u_p a viscosity solution of*

$$\begin{cases} -\Delta_p^N u_p(x) = \lambda_{1,p}(\Omega)\, u_p(x) & \text{in } \Omega, \\ u_p(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

with $\lambda_{1,p}(\Omega)$ defined in (1.5)*. Then,*

$$\lim_{p\to\infty} \lambda_{1,p}(\Omega) = \lambda_{1,\infty}(\Omega),$$

for $\lambda_{1,\infty}(\Omega)$ the first eigenvalue of the 1-homogeneous infinity Laplacian, given in (1.4)*. Moreover, there exists a subsequence $u_{p'}$, that converges uniformly to some $u > 0$, a viscosity solution of*

$$\begin{cases} -\Delta_\infty^N u(x) = \lambda_{1,\infty}(\Omega)\, u(x), & \text{in } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

In addition, $u \in \mathcal{C}^{0,1}(\Omega)$.

Proof. Estimate (3.5) and the fact that $\mu(R^*) \le C$ in the radial case imply that there exists a subsequence still denoted by p such that

$$\lim_{p\to\infty} \lambda_{1,p}(\Omega) = \Lambda(\Omega).$$

for some $\Lambda(\Omega)$. Our aim is to prove that $\Lambda(\Omega) = \lambda_{1,\infty}(\Omega)$.

Let us consider the sequence u_p of first eigenfunctions corresponding to the subsequence above and assume the normalization $\|u_p\|_\infty = 1$. Fix p_0 such that $n < p_0$. Then, for $p \ge p_0$, [7, Corollary 5.5] yields the following estimate

$$\begin{aligned} \|u_p\|_{\mathcal{C}^{0,\frac{p_0-n}{p_0-1}}(\Omega)} &\le \operatorname{diam}(\Omega)^{\left(\frac{p-n}{p-1} - \frac{p_0-n}{p_0-1}\right)} \cdot \|u_p\|_{\mathcal{C}^{0,\frac{p-n}{p-1}}(\Omega)} \\ &\le \operatorname{diam}(\Omega)^{2-\frac{p_0-n}{p_0-1}} \cdot \left(\frac{2\,p}{p-1} \operatorname{diam}(\Omega)^{\frac{p-n}{p-1}} + \frac{p}{p+n-2} \right) \mu(R^*). \end{aligned} \tag{4.3}$$

Note that the right-hand side can be bounded independently of p, thus Arzelà-Ascoli Theorem yields the existence of a further subsequence converging uniformly to some limit $u \in \mathcal{C}(\Omega)$. We will still denote by u_p the subsequence for which we have that $u_p \to u$ uniformly and $\lim_{p\to\infty} \lambda_{1,p}(\Omega) = \Lambda(\Omega)$.

In addition, we may assume that the limit u has a uniform modulus of continuity, this fact follows from (4.3) taking $p \to \infty$.

Now we observe that (also from (4.3)) we get that the maximums of u_p are located at some points x_p that have a unform distance from the boundary. Extracting a subsequence if necessary we may assume that $x_p \to x_0 \in \Omega$ and from the uniform convergence we get $u(x_0) = 1$. Now the uniform modulus of continuity of u implies that $u \ge 1/2$ in a small ball around x_0 and hence u is nontrivial.

Now, we focus on checking that the limit u is a viscosity solution of (4.2). Let $x_0 \in \Omega$ and a function $\varphi \in \mathcal{C}^2(\Omega)$ such that $u-\varphi$ attains a local minimum at x_0. Up to replacing $\varphi(x)$ with $\varphi(x) - |x - x_0|^4$, we can assume without loss of generality the minimum to be strict.

Since u is the uniform limit of the subsequence u_p and x_0 is a strict minimum point, there exists a sequence of points $x_p \to x_0$ as $p \to \infty$, such that $(u_p - \varphi)(x_p)$ is a local minimum for each p in the sequence.

Let us suppose first that $|D\varphi(x_0)| > 0$. Then, $|D\varphi(x_p)| > 0$ for p large enough and, since u_p is a viscosity supersolution of (4.1), we have that,

$$\begin{aligned} -\frac{1}{p}\,\mathrm{trace}\left[\left(I + (p-2)\frac{D\varphi(x_p)\otimes D\varphi(x_p)}{|D\varphi(x_p)|^2}\right) D^2\varphi(x_p)\right] &= -\Delta_p^N \varphi(x_p) \\ &\geq \lambda_{1,p}(\Omega)\, u_p(x_p). \end{aligned}$$

Letting $p \to \infty$ we get

$$-\left\langle D^2\varphi(x_0)\frac{D\varphi(x_0)}{|D\varphi(x_0)|}, \frac{D\varphi(x_0)}{|D\varphi(x_0)|}\right\rangle = -\Delta_\infty \varphi(x_0) \geq \Lambda(\Omega)\, u(x_0).$$

If, on the contrary, we assume that $D\varphi(x_0) = 0$, we have to consider two cases. Suppose first that there exists a subsequence, still indexed by p, such that $|D\varphi(x_p)| > 0$ for all p in the subsequence. Then, by Definition 2.1, we can let $p \to \infty$ to get

$$\begin{aligned} -\liminf_{p\to\infty}\left\langle D^2\varphi(x_p)\frac{D\varphi(x_p)}{|D\varphi(x_p)|}, \frac{D\varphi(x_p)}{|D\varphi(x_p)|}\right\rangle &= -m\big(D^2\varphi(x_0)\big) \\ &= -\Delta_\infty\varphi(x_0) \geq \Lambda(\Omega)\, u(x_0). \end{aligned}$$

If such a subsequence does not exist, according to Definition 2.1, we deduce that

$$-\frac{1}{p}\,\Delta\varphi(x_p) - \frac{(p-2)}{p}\, m\big(D^2\varphi(x_p)\big) \geq -\Delta_p^N\varphi(x_p) \geq \lambda_{1,p}(\Omega)\, u_p(x_p).$$

for every p large enough. Taking $p \to \infty$, we get

$$-m\big(D^2\varphi(x_0)\big) = -\Delta_\infty\varphi(x_0) \geq \Lambda(\Omega)\, u(x_0).$$

Hence u is a viscosity supersolution of (4.2). The proof of the fact that u is a subsolution runs similarly. Moreover, u is the uniform limit of positive functions, thus $u \geq 0$.

Since u is not trivial, by the Harnack inequality for the infinity laplacian, Lemma 5.1 in [11], we deduce that, indeed $u > 0$ in Ω. Finally, we conclude that $u \in \mathcal{C}^{0,1}(\Omega)$ either letting $p_0 \to \infty$ or using that u is a solution of (4.2), and hence the estimates in [7, Corollary 5.5] apply.

Therefore, it just remains to see that $\Lambda(\Omega) = \lambda_{1,\infty}(\Omega)$. We notice that $\Lambda(\Omega) \geq \lambda_{1,\infty}(\Omega)$. Otherwise Corollary 3.4 in [11] implies that the limit u is non-positive, which is not true. Now, the same argument as in Proposition 3.7 but with $p = \infty$ implies that $\Lambda = \lambda_{1,\infty}(\Omega)$ and the proof is finished. □

Acknowledgements. The authors wish to thank to J. García-Azorero for his useful comments and suggestions.

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