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THE LIMIT POINTS OF A NORMALIZED RANDOM WALK[1]

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1. Introduction and statement of results. This paper deals with the set of accumulation points of $n^{-\alpha}S_n$ for a one-dimensional random walk S_n, $n \geqq 1$. S_n is called a random walk if $S_n = \sum_{i=1}^{n} X_i$ for a sequence $\{X_i\}_{i \geqq 1}$ of independent, identically distributed random variables. The (random) set of accumulation points of $n^{-\alpha}S_n$ will be denoted by

(1.1) $A(S_n, \alpha) = \text{set of accumulation points of } n^{-\alpha}S_n, n \geqq 1 = \bigcap_m \overline{\{n^{-\alpha}S_n : n \geqq m\}}.$

The bar in the last member of (1.1) denotes closure in the extended real line $\overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty, +\infty\}$ with its usual topology. This meaning for a bar over a subset of $\overline{\mathbf{R}}$ will be maintained throughout; F will always denote the common distribution function of the X_i.

The motivation for this study lies in two recent results. Firstly, a condition of K. G. Binmore and M. Katz (private communication) for a point b to be an accumulation point of S_n/n. Secondly, a necessary and sufficient condition of Stone [16] for $+\infty$ (or $-\infty$) to belong to $A(S_n, \frac{1}{2})$.

In Section 2 we first prove that $A(S_n, \alpha)$ is w.p.1[2] equal to a fixed (non-random) closed set $B(\alpha)$. Of course $B(\alpha)$ depends on F, and in fact can be viewed as a characteristic of F. In particular $B(1)$ is a sort of generalized mean; it consists only of the number $\int x\,dF(x)$ whenever this integral is meaningful. Next derive two forms of a necessary and sufficient condition for a point b to belong to $B(\alpha)$ (Theorems 2 and 3). The first form of the conditions and its proof (Theorem 2 with Corollaries 1 and 2) are essentially due to K. G. Binmore and M. Katz. In Section 3 we use these conditions to derive the possible forms of $B(\alpha)$ for $0 < \alpha < \frac{1}{2}$, and in part for $\alpha = \frac{1}{2}$. Specifically we prove

THEOREM 4. *Assume* $F(0) - F(0-) < 1$. *Let* $0 < \alpha < \frac{1}{2}$. *If* $n^{-\alpha}S_n$ *has* w.p.1 *a finite limit point, then* w.p.1 *all real numbers are limit points of* $n^{-\alpha}S_n$ (*i.e., if* $B(\alpha) \cap \mathbf{R} \neq \varnothing$ *then* w.p.1 $A(S_n, \alpha) = \overline{\mathbf{R}}$).

If $\alpha = \frac{1}{2}$ *and* $n^{-\frac{1}{2}}S_n$ *has* w.p.1 *a finite limit point, then* w.p.1 $n^{-\frac{1}{2}}S_n$ *has at least a half line* $[b, \infty]$ *or* $[-\infty, b]$ *as limit points.*

We conjecture that even for $\alpha = \frac{1}{2}$ $B(\frac{1}{2}) = \overline{\mathbf{R}}$ as soon as $B(\frac{1}{2}) \cap \mathbf{R} \neq \varnothing$. If correct this result would be an extension of part of Stone's result in [16]. Indeed, the result of [16] implies that if $B(\frac{1}{2}) \cap \mathbf{R} \neq \varnothing$ then $B(\frac{1}{2})$ contains $+\infty$ and $-\infty$. In Section 4 we sharpen Stone's result in another direction. We prove that if $EX_1^+ = +\infty$ and if $n^{-1}S_n \geqq a$ i.o.[2] w.p.1 for some fixed $a \in \mathbf{R}$, then $\limsup_{n\to\infty} n^{-1}S_n = +\infty$ w.p.1

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[2] w.p.1 = with probability 1; we shall occasionally leave out the expression w.p.1 when there is no risk of confusion. i.o. = infinitely often.

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(i.e., either $n^{-1}S_n \to -\infty$ or $+\infty \in A(S_n, 1)$). Combined with a theorem of Spitzer's [14], this result shows that, in case $EX_1^+ = \infty$, $\limsup_{n\to\infty} S_n/n = \infty$ w.p.1 if and only if

$$\sum_{n=1}^{\infty} n^{-1}P\{S_n > 0\} = \infty.$$

Lastly, in Section 5 we prove the somewhat surprising

THEOREM 7. *For any closed set $C \subset \bar{\mathbf{R}}$ which contains $+\infty$ and $-\infty$ there exists a random walk S_n such that*

$$A(S_n, 1) = C \qquad \text{w.p.1.}$$

If $0 \in C$ one can even take S_n recurrent.

To close this introduction we list several problems suggested by this work.

(1). Is $B(\frac{1}{2}) = \bar{\mathbf{R}}$ as soon as $B(\frac{1}{2}) \cap \mathbf{R} \neq \varnothing$? (See Remark 3 for partial results.)

(2). What are the possible structures of $B(\alpha)$ for $\alpha > \frac{1}{2}$, $\alpha \neq 1$, or for any $\alpha > 0$ in the case of higher dimensional random walks?

(3). Find necessary and sufficient conditions on the distribution of X_1 for $B(1) = \bar{\mathbf{R}}$.

(4). Can one find the set of functions $f: [0, 1] \to \bar{\mathbf{R}}$ which are in a suitable topology accumulation points of (the linear interpolations of) the functions f_n, $n \geqq 1$, defined by $f_n(k/n) = k^{-1}S_k$ or $n^{-1}S_k$, $0 \leqq k \leqq n$? In this question we think of analogues of Strassen's determination of the accumulation points for $\{S_k(n \log\log n)^{-\frac{1}{2}}\}_{1 \leqq k \leqq n}$ when $EX_1 = 0$, $EX_1^2 < \infty$ (see [17]). In particular it would be interesting to obtain a clear idea how $n^{-1}S_n$ can have two points a and b as accumulation points, while avoiding all the points between a and b.

2. General conditions for accumulation points of $n^{-\alpha}S_n$. We begin by proving the non-random character of $A(S_n, \alpha)$. More precisely we prove

THEOREM 1. *If $\gamma(n) \to \infty$ $(n \to \infty)$ and $X_1, X_2, \cdots$ are independent random variables each with the same distribution F, then there exists a non-random closed set*

$$B = B(F, \{\gamma(n)\}) \subset \bar{\mathbf{R}} \tag{2.1}$$

such that w.p.1

$$\text{set of accumulation points of } \gamma(n)^{-1}S_n \equiv \bigcap_m \overline{\{\gamma(n)^{-1}S_n : n \geqq m\}} = B. \tag{2.2}$$

The set B is given by

$$B = \{b : P\{\gamma(n)^{-1}S_n \text{ has } b \text{ as accumulation point}\} = 1\}. \tag{2.3}$$

PROOF. By the Hewitt–Savage zero-one law we have for any point b and any interval I

$$P\{b \text{ is an accumulation point of } \gamma(n)^{-1}S_n\} = 0 \text{ or } 1 \tag{2.4}$$

and

$$P\{\gamma(n)^{-1}S_n \in I \text{ i.o.}\} = 0 \text{ or } 1. \tag{2.5}$$

Let

$$B_0 = \{b : P\{b \text{ is an accumulation point of } \gamma(n)^{-1}S_n\} = 1\}$$

and $\mathscr{C}$ = the collection of all open rational intervals I for which

$$P\{\gamma(n)^{-1}S_n \in I \text{ i.o.}\} = 1. \tag{2.6}$$

Since there are only countably many rational intervals, (2.5) implies

(2.7) $P\{\gamma(n)^{-1}S_n \in I$ i.o. for all $I \in \mathscr{C}$, but for no other open rational interval$\} = 1$.

We shall now show that if I is an open rational interval then

$$I \cap \overline{B_0} \neq \varnothing \quad \text{implies} \quad I \in \mathscr{C} \tag{2.8a}$$

and

$$\bar{I} \cap B_0 = \varnothing \quad \text{implies} \quad I \notin \mathscr{C}. \tag{2.8b}$$

(2.7) and (2.8) together will prove (2.2) with $B = \overline{B_0}$, since w.p.1 $\gamma(n)^{-1}S_n$ will enter i.o. any open neighborhood of any point $b \in \overline{B_0}$, and for any $b \notin \overline{B_0}$ there will be an open neighborhood U such that $\gamma(n)^{-1}S_n \in U$ only finitely often. At the same time this argument shows that $b \in B_0$ as soon as $b \in \overline{B_0}$, i.e., $B_0 = \overline{B_0}$, and therefore (2.3) will follow at the same time.

To prove (2.8) observe that b is an accumulation point of $\gamma(n)^{-1}S_n$ if and only if $\gamma(n)^{-1}S_n \in I$ i.o. for every open rational interval I containing b. Thus $b \in B_0$ if and only if

$$P\{\gamma(n)^{-1}S_n \text{ infinitely often in every } I \text{ containing } b\} = 1.$$

In particular, if I is an open interval for which $I \cap \overline{B_0} \neq \varnothing$ then I contains some b from B_0 and (2.6) holds, i.e., $I \in \mathscr{C}$. This proves (2.8a). Conversely, let I be an open rational interval for which $\bar{I} \cap B_0 = \varnothing$. We shall prove (2.8b) by deriving a contradiction from (2.6) for such an interval. Indeed, let I_0 and I_1 be the left and right half of I (put the center point of I in I_0), and in general, let $I_{\varepsilon_1, \cdots, \varepsilon_k, 0}$ and $I_{\varepsilon_1, \cdots, \varepsilon_k, 1}$ be the left and right halves of $I_{\varepsilon_1, \cdots, \varepsilon_k}$ ($\varepsilon_i = 0$ or 1). If (2.6) holds there exists a sequence $\eta_1, \eta_2, \cdots, \eta_i = 0$ or 1, such that for all k

$$P\{\gamma(n)^{-1}S_n \in \bar{I}_{\eta_1, \cdots, \eta_k} \text{ i.o.}\} = 1.$$

Consequently, if $b \in \bigcap_k \bar{I}_{\eta_1, \cdots, \eta_k} \subset \bar{I}$, then

$$P\{b \text{ is an accumulation point of } \gamma(n)^{-1}S_n\}$$
$$\geqq P\{\gamma(n)^{-1}S_n \in \bar{I}_{\eta_1, \cdots, \eta_k} \text{ i.o. for all } k\} = 1.$$

This would mean

$$b \in B_0 \cap \bar{I}$$

contrary to our assumption. Thus (2.8b) also holds and the proof is complete.

As stated in the introduction we find that w.p.1 $A(S_n, \alpha)$ equals a non-random set which we shall denote by $B(\alpha)$ (instead of the more cumbersome $B(F, \{n^\alpha\})$ used in (2.1)). We turn to Binmore and Katz' condition for $b \in B(\alpha)$. Actually we consider somewhat more general normalizing sequences $\gamma(n)$ than n^α.

THEOREM 2. *Let* $\gamma(n) > 0$, $n \geqq 1$, *and* $\gamma(n) \to \infty$ $(n \to \infty)$ *and assume*

$$\lim_{\eta \downarrow 0} \sup_{1 \leqq n_1 \leqq \eta n_2} \gamma(n_1)/\gamma(n_2) = 0 \tag{2.9}$$

and

$$\lim_{\eta \downarrow 0} \sup_{1 \leqq n_1 \leqq n_2 \leqq n_1(1+\eta)} \left| \frac{\gamma(n_2)}{\gamma(n_1)} - 1 \right| = 0. \tag{2.10}$$

If $D > 1$, $\varepsilon > 0$ *and* $-\infty < a < b < \infty$ *are fixed, then* (i) *implies* (ii) *and* (ii) *implies* (iii), *where*

(i) $P\{\gamma(n)^{-1} S_n \in (a, b) \text{ i.o.}\} = 1$,
(ii) $\sum_{r=0}^{\infty} P\{\gamma(n)^{-1} S_n \in (a, b)$ *for some* $n \in [D^r, D^{r+1})\} = \infty$,
(iii) $P\{\gamma(n)^{-1} S_n \in (a-\varepsilon, b+\varepsilon) \text{ i.o.}\} = 1$.

PROOF. (i) implies (ii) by the Borel–Cantelli lemma. Assume then that (ii) holds. Write I for (a, b) and fix an integer s so large that

$$\left| \frac{\gamma(n_2)}{\gamma(n_1)} - 1 \right| < \frac{\varepsilon}{2}(|a|+|b|)^{-1} \quad \text{for} \quad 1 \leqq n_1 \leqq n_2 \leqq n_1(1-D^{1-s})^{-1} \tag{2.11}$$

and

$$\frac{\gamma(n_1)}{\gamma(n_2)} < \frac{\varepsilon}{2}(|a|+|b|)^{-1} \quad \text{for} \quad 1 \leqq n_1 \leqq n_2(D^{s-1}-1)^{-1}. \tag{2.12}$$

By (ii) we can then also find an integer t, $0 \leqq t < s$, such that

$$\textstyle\sum_{r=0}^{\infty} P\{\gamma(n)^{-1} S_n \in I \quad \text{for some} \quad n \in [D^{rs+t}, D^{rs+t+1})\} = \infty. \tag{2.13}$$

Fix such a t and introduce the events

$$E_r = \{\gamma(n)^{-1} S_n \in I \quad \text{for some} \quad n \in [D^{rs+t}, D^{rs+t+1}) \quad \text{but} \quad \gamma(n)^{-1} S_n \notin I \quad \text{for all} \quad n \geqq D^{(r+1)s+t}\}.$$

Then

$$\begin{aligned} P\{E_r\} &= \textstyle\sum_{D^{rs+t} \leqq l < D^{rs+t+1}} \int_{u \in I} P\{\gamma(n)^{-1} S_n \notin I \quad \text{for} \quad D^{rs+t} \leqq n < l, \gamma(l)^{-1} S_l \in du\} \\ &\quad \cdot P\{\gamma(n)^{-1} S_n \notin I \quad \text{for} \quad n \geqq D^{(r+1)s+t} \mid \gamma(l)^{-1} S_l = u\}. \end{aligned} \tag{2.14}$$

Moreover,

$$\begin{aligned} &P\{\gamma(n)^{-1} S_n \notin I \quad \text{for} \quad n \geqq D^{(r+1)s+t} \mid \gamma(l)^{-1} S_l = u\} \\ &\quad = P\{\gamma(n-l)^{-1} S_{n-l} \notin \gamma(n-l)^{-1}(\gamma(n)I - \gamma(l)u) \text{ for } n-l \geqq D^{(r+1)s+t} - l\}. \end{aligned} \tag{2.15}$$

But for $l \leqq D^{rs+t+1}$, $n \geqq D^{(r+1)s+t}$ one has

$$n-l \geqq D^{(r+1)s+t}-l \geqq (D^{s-1}-1)D^{rs+t+1} \geqq (D^{s-1}-1)l \tag{2.16}$$

and

$$n \leqq (1-D^{1-s})^{-1}(n-l). \tag{2.17}$$

If, in addition, $u \in I$, then by (2.11), (2.12), (2.16) and (2.17)

$$|\gamma(n-l)^{-1}(\gamma(n)a-\gamma(l)u)-a| \leqq |a|\left|\frac{\gamma(n)}{\gamma(n-l)}-1\right|+(|a|+|b|)\frac{\gamma(l)}{\gamma(n-l)} < \varepsilon, \tag{2.18}$$

as well as

$$|\gamma(n-l)^{-1}(\gamma(n)b-\gamma(l)u)-b| < \varepsilon. \tag{2.19}$$

(2.18) and (2.19) imply

$$\gamma(n-l)^{-1}(\gamma(n)I-\gamma(l)u) < (a-\varepsilon, b+\varepsilon),$$

and (2.15) therefore gives

$$P\{\gamma(n)^{-1}S_n \notin I \quad \text{for} \quad n > D^{(r+1)s+t} \mid \gamma(l)^{-1}S_l = u\}$$
$$\geqq P\{\gamma(k)^{-1}S_k \notin (a-\varepsilon, b+\varepsilon) \quad \text{for} \quad k \geqq D^{s-1}-1\}.$$

When this is substituted into (2.14) we finally obtain

$$\begin{aligned} P\{E_r\} &\geqq P\{\gamma(k)^{-1}S_k \notin (a-\varepsilon, b+\varepsilon) \quad \text{for} \quad k \geqq D^{s-1}-1\} \\ &\quad \cdot \textstyle\sum_{D^{rs+t} \leqq l < D^{rs+t+1}} P\{\gamma(n)^{-1}S_n \notin I \quad \text{for} \quad D^{rs+t} \leqq n < l \quad \text{but} \quad \gamma(l)^{-1}S_l \in I\} \\ &= P\{\gamma(k)^{-1}S_k \notin (a-\varepsilon, b+\varepsilon) \quad \text{for} \quad k \geqq D^{s-1}-1\} \\ &\quad \cdot P\{\gamma(n)^{-1}S_n \in I \quad \text{for some} \quad n \in [D^{rs+t}, D^{rs+t+1})\}. \end{aligned}$$

Since the events E_r, $r \geqq 0$, are disjoint we have

$$\begin{aligned} 1 \geqq \textstyle\sum_{r \geqq 0} P\{E_r\} &\geqq P\{\gamma(k)^{-1}S_k \notin (a-\varepsilon, b+\varepsilon) \quad \text{for} \quad k \geqq D^{s-1}-1\} \\ &\quad \cdot \textstyle\sum_{r \geqq 0} P\{\gamma(n)^{-1}S_n \in I \quad \text{for some} \quad [D^{rs+t}, D^{rs+t+1})\}. \end{aligned}$$

Together with (2.13) this implies

$$P\{\gamma(k)^{-1}S_k \notin (a-\varepsilon, b+\varepsilon) \quad \text{for} \quad k \geqq D^{s-1}-1\} = 0$$

for all large s which satisfy (2.11), (2.12). Thus (iii) must hold.

COROLLARY 1. *Let $\alpha > 0$, $D > 1$ be fixed. Then $b \in B(\alpha)$ if and only if for all $\varepsilon > 0$*

$$\textstyle\sum_{r=0}^{\infty} P\{|n^{-\alpha}S_n - b| < \varepsilon \quad \textit{for some} \quad n \in [D^r, D^{r+1})\} = \infty. \tag{2.20}$$

PROOF. $\gamma(n) = n^{\alpha}$ satisfies (2.9) and (2.10), and $b \in B(\alpha)$ if and only if

$$P\{|n^{-\alpha}S_n - b| < \varepsilon \text{ i.o.} \quad \text{for all} \quad \varepsilon > 0\} = 1,$$

or, equivalently, if and only if

$$P\{|n^{-\alpha}S_n - b| < \varepsilon \text{ i.o.}\} = 1 \quad \text{for all} \quad \varepsilon > 0. \tag{2.21}$$

By Theorem 2, (2.20) and (2.21) are equivalent.

COROLLARY 2. *If*

$$\sum_{n=1}^{\infty} n^{-1}P\{n^{-\alpha}S_n - b \mid < \varepsilon\} = \infty \tag{2.22}$$

for all $\varepsilon > 0$, *then* $b \in B(\alpha)$.

PROOF. This is immediate from Corollary 1 and the estimate

$$\sum_{D^r \leqq n < D^{r+1}} n^{-1}P\{|n^{-\alpha}S_n - b| < \varepsilon\}$$
$$\leqq \frac{D^{r+1}-D^r+1}{D^r} P\{|n^{-\alpha}S_n - b| < \varepsilon \quad \text{for some} \quad n \in [D^r, D^{r+1})\}.$$

REMARK 1. Condition (2.22) is not necessary for $b \in B(\alpha)$, as can be seen from the example at the end of this section.

For the second version of conditions for $b \in B(\alpha)$ we need the following lemma which is some form of the maximum principle.

LEMMA 1. *For* $x \in R$, $a, b, C \in [0, \infty)$ *and* $n, m \in Z^{+}$[3] *one has*

$$\begin{aligned} E \# \{k : S_k \in (x-aC, x+aC), n \leqq k < n+bm\} &\leqq 2(a+1)(b+1)E \# \{k : S_k \in (-C, +C), 0 \leqq k < m\} \\ &= 2(a+1)(b+1)\sum_{k=0}^{m-1} P\{|S_k| < C\}. \end{aligned} \tag{2.23}$$

PROOF. We can always cover $(x-aC, x+aC)$ by at most $2[a]+2$ intervals of length C, i.e.,

$$(x-aC, x+aC) \subset \bigcup_{i=1}^{2[a]+2} (y_i - \tfrac{1}{2}C, y_i + \tfrac{1}{2}C) \tag{2.24}$$

for suitable choices of y_i. (2.24) implies

$$\begin{aligned} E \# \{k : S_k \in (x-aC, x+aC), n \leqq k < n+m\} \leqq \sum_{i=1}^{2[a]+2} E \# \{k : S_k \in (y_i - \tfrac{1}{2}C, y_i + \tfrac{1}{2}C), n \leqq k < n+m\}. \end{aligned} \tag{2.25}$$

Now for fixed y_i and n define

$$T_i = \inf\{k : k \geqq n, S_k \in (y_i - \tfrac{1}{2}C, y_i + \tfrac{1}{2}C)\}.$$

Then

$$\begin{aligned} E \# \{k : S_k \in (y_i - \tfrac{1}{2}C, y_i + \tfrac{1}{2}C), n \leqq k < n+m\} &= \sum_{l=n}^{n+m-1} \int_{|u-y_i| < \frac{1}{2}C} P\{T_i = l, S_l \in du\} \\ &\quad \cdot E \# \{k : S_k \in (y_i - \tfrac{1}{2}C, y_i + \tfrac{1}{2}C), l \leqq k < n+m \mid S_l = u\} \\ &= \sum_{l=n}^{n+m-1} \int_{|u-y_i| < \frac{1}{2}C} P\{T_i = l, S_l \in du\} \\ &\quad \cdot E \# \{r : S_r \in (y_i - u - \tfrac{1}{2}C, y_i - u + \tfrac{1}{2}C), 0 \leqq r < n+m-l\}. \end{aligned} \tag{2.26}$$

[3] $Z^+ = \{0, 1, 2, \cdots\}$; $\# A$ stands for the number of elements of the set A. Throughout S_0 is taken to be 0.

But for $n \leqq l < n+m$ and $|u-y_i| < \frac{1}{2}C$ one has $n+m-l \leqq m$ and $(y_i-u-\frac{1}{2}C, y_i-u+\frac{1}{2}C) \subset (-C, +C)$, so that

$$(2.27) \qquad E \# \{r: S_r \in (y_i-u-\tfrac{1}{2}C, y_i-u+\tfrac{1}{2}C), 0 \leqq r < n+m-l\} \\ \leqq E \# \{k: S_k \in (-C, +C), 0 \leqq k < m\}.$$

(2.25)-(2.27), together with the estimate $\sum_{l \geqq n} P\{T_i = l\} \leqq 1$, prove

$$(2.28) \qquad E \# \{k: S_k \in (x-aC, x+aC), n \leqq k < n+m\} \\ \leqq 2(a+1)E \# \{k: S_k \in (-C, +C), 0 \leqq k < m\}.$$

If we replace n by $n+jm$ in (2.28) and sum over $j = 0, 1, \cdots [b]$, then we obtain the inequality in (2.23). The equality of the second and third members of (2.23) is obvious.

THEOREM 3. *Let $\gamma(n)$ satisfy the conditions of Theorem 2 and in addition*

$$(2.29) \qquad \sup_{1 \leqq n_1 \leqq n_2} \frac{\gamma(n_1)}{\gamma(n_2)} = K_1 < \infty.$$

Then

$$(2.30) \qquad b \in B(F, \{\gamma(n)\})$$

if and only if

$$(2.31) \qquad \sum_{n=1}^{\infty} \frac{P\{|\gamma(n)^{-1}S_n - b| < \varepsilon\}}{\sum_{l=0}^{n-1} P\{|S_l| < \gamma(n)\}} = \infty \quad \textit{for all} \quad \varepsilon > 0.$$

(Note that the summands in the denominator contain $\gamma(n)$, not $\gamma(l)$.)

PROOF. By Theorem 2, (2.30) implies

$$(2.32) \qquad \textstyle\sum_{r=0}^{\infty} P\{|\gamma(n)^{-1}S_n - b| < \varepsilon \quad \text{for some} \quad n \in [D^r, D^{r+1})\} = \infty$$

for all $\varepsilon > 0$ and $D > 1$. Conversely, if (2.32) holds for all $\varepsilon > 0$ and some $D > 1$, then (2.30) holds (compare proof of Corollary 1). We now estimate the summands in (2.32) for suitable D in terms of certain expected numbers. Put

$$p(\varepsilon, r, D) = P\{|\gamma(n)^{-1}S_n - b| < \varepsilon \quad \text{for some} \quad n \in [D^r, D^{r+1})\}$$

and

$$K_2 = \sup_{1 \leqq n_1 \leqq n_2 \leqq 4n_1} \frac{\gamma(n_2)}{\gamma(n_1)}.$$

By (2.10) $K_2 < \infty$. Also define the stopping times

$$U_r = U_r(\varepsilon, D) = \inf\{n: n \geqq D^r, |\gamma(n)^{-1}S_n - b| < \varepsilon\}.$$

Then

$$(2.33)\quad E\,\#\,\{n: |\gamma(n)^{-1}S_n - b| < (K_1+K_2+|b|)\varepsilon, D^r \leqq n < D^{r+2}\}$$
$$\geqq \textstyle\sum_{D^r \leqq l < D^{r+1}} \int_{|u-b| < \varepsilon} P\{U_r = l, \gamma(l)^{-1}S_l \in du\}$$
$$\cdot E\,\#\,\{n: |\gamma(n)^{-1}S_n - b| < (K_1+K_2+|b|)\varepsilon, l \leqq n < D^{r+2} \mid \gamma(l)^{-1}S_l = u\}.$$

Also

$$(2.34)\quad E\,\#\,\{n: |\gamma(n)^{-1}S_n - b| < (K_1+K_2+|b|)\varepsilon, l \leqq n < D^{r+2} \mid \gamma(l)^{-1}S_l = u\}$$
$$= E\,\#\,\{m: |\gamma(m+l)^{-1}(S_m + \gamma(l)u) - b| < (K_1+K_2+|b|)\varepsilon,$$
$$0 \leqq m < D^{r+2} - l\}$$

and

$$(2.35)\quad |\gamma(m+l)^{-1}(S_m + \gamma(l)u) - b|$$
$$\leqq \gamma(m+l)^{-1}|S_m| + \frac{\gamma(l)}{\gamma(m+l)}|u-b| + \left|\frac{\gamma(m+l)-\gamma(l)}{\gamma(m+l)}\right||b|.$$

For $D^r \leqq l < D^{r+1}$, $0 \leqq m \leqq D^{r+2} - l \leqq D^{r+2}$, $1 < D \leqq 2$ and $|u-b| < \varepsilon$, the right-hand side of (2.35) is bounded by

$$(2.36)\qquad K_2\gamma([D^{r+2}])^{-1}|S_m| + K_1\varepsilon + |b| \sup_{1 \leqq n_1 \leqq n_2 \leqq D^2 n_1} \left|1 - \frac{\gamma(n_1)}{\gamma(n_2)}\right|.$$

By virtue of (2.10) we can take $D = 1+\delta$ with $0 < \delta \leqq \delta_0(\varepsilon)$ so small that the last term of (2.36) is at most $|b|\varepsilon$. With such a choice of D

$$|\gamma(m+l)^{-1}(S_m + \gamma(l)u) - b| < (K_1+K_2+|b|)\varepsilon$$

whenever $\gamma([D^{r+2}])^{-1}|S_m| < \varepsilon$. Consequently, for the indicated values of l and u (see (2.34))

$$(2.37)\qquad E\,\#\,\{n: |\gamma(n)^{-1}S_n - b| < (K_1+K_2+|b|)\varepsilon, l \leqq n < D^{r+2} \mid \gamma(l)^{-1}S_l = u\}$$
$$\geqq E\,\#\,\{m: \gamma([D^{r+2}])^{-1}|S_m| < \varepsilon, 0 \leqq m < D^{r+2} - D^{r+1}\}.$$

Next, an application of Lemma 1 shows that the last member of (2.37) is at least

$$(2.38)\quad \left\{2\left(\frac{1}{\varepsilon}+1\right)\left(\frac{D^{r+2}}{D^{r+2}-D^{r+1}}+1\right)\right\}^{-1} E\,\#\,\{m: |S_m| < \gamma([D^{r+2}]), 0 \leqq m < D^{r+2}\}.$$

We introduce the abbreviation

$$K_3 = K_3(\varepsilon, D) = 2\left(\frac{1}{\varepsilon}+1\right)\left(\frac{D}{D-1}+1\right)$$

and use (2.37) and (2.38) to estimate the right-hand side of (2.33). This yields

$$\begin{aligned} E \# \{n: |\gamma(n)^{-1}S_n - b| < (K_1+K_2+|b|)\varepsilon, D^r \leqq n < D^{r+2}\} \\ \geqq \textstyle\sum_{D^r \leqq l < D^{r+1}} \int_{|u-b|<\varepsilon} P\{U_r = l, \gamma(l)^{-1}S_l \in du\} \\ \cdot K_3^{-1} E \# \{m: |S_m| < \gamma([D^{r+2}]), 0 \leqq m < D^{r+2}\} \\ = K_3^{-1} p(\varepsilon, r, D) E \# \{m: |S_m| < \gamma([D^{r+2}]), 0 \leqq m < D^{r+2}\}. \end{aligned}$$

Upon using the definition of K_1, and Lemma 1 once more, we finally obtain, for D sufficiently close to 1,

$$\begin{aligned} p(\varepsilon, r, D) &\leqq K_3 E \# \{n: |\gamma(n)^{-1}S_n - b| < (K_1+K_2+|b|)\varepsilon, D^r \leqq n < D^{r+2}\} \\ &\quad \cdot (E \# \{m: |S_m| < \gamma([D^{r+2}]), 0 \leqq m < D^{r+2}\})^{-1} \\ &\leqq K_3 \sum_{D^r \leqq n < D^{r+2}} \frac{P\{|\gamma(n)^{-1}S_n - b| < (K_1+K_2+|b|)\varepsilon\}}{E \# \{m: |S_m| < K_1^{-1}\gamma(n), 0 \leqq m < n\}} \\ &\leqq 2K_3(K_1+1) \sum_{D^r \leqq n < D^{r+2}} \frac{P\{|\gamma(n)^{-1}S_n - b| < (K_1+K_2+|b|)\varepsilon\}}{\sum_{l=0}^{n-1} P\{|S_l| < \gamma(n)\}}. \end{aligned}$$

Consequently, if (2.30) holds, and hence (2.32) or

$$\textstyle\sum_{r=0}^{\infty} p(\varepsilon, r, D) = \infty \quad \text{for all} \quad \varepsilon > 0, D > 1,$$

then (2.31) follows.

The estimate in the other direction is similar, but fortunately somewhat simpler. Indeed

$$\begin{aligned} (2.39) \quad & E \# \{n: |\gamma(n)^{-1}S_n - b| < \varepsilon, D^r \leqq n < D^{r+1}\} \\ &\leqq \textstyle\sum_{D^r \leqq l < D^{r+1}} \int_{|u-b|<\varepsilon} P\{U_r = l, \gamma(l)^{-1}S_l \in du\} \\ &\quad \cdot E \# \{n: |\gamma(n)^{-1}S_n - b| < \varepsilon, l \leqq n < D^{r+1} \mid \gamma(l)^{-1}S_l = u\} \\ &\leqq \textstyle\sum_{D^r \leqq l < D^{r+1}} \int_{|u-b|<\varepsilon} P\{U_r = l, \gamma(l)^{-1}S_l \in du\} \\ &\quad \cdot E \# \{m: |\gamma(m+l)^{-1}(S_m + \gamma(l)u)| < |b| + \varepsilon, 0 \leqq m < D^{r+1} - l\} \\ &\leqq p(\varepsilon, r, D) E \# \{m: |S_m| \leqq 2K_1(|b|+\varepsilon)\gamma([D^{r+1}]), 0 \leqq m < D^{r+1}\}. \end{aligned}$$

By definition of K_2 we have for $1 < D \leqq 2$ and $D^r \leqq n < D^{r+1}$, $\gamma([D^{r+1}]) \leqq K_2\gamma(n)$, and thus by Lemma 1 and (2.39)

$$(2.40) \quad p(\varepsilon, r, D) \geqq \{8K_1K_2(|b|+\varepsilon)+2\}^{-1} \sum_{D^r \leqq n < D^{r+1}} \frac{P\{|\gamma(n)^{-1}S_n - b| < \varepsilon\}}{\sum_{l=0}^{n-1} P\{|S_l| < \gamma(n)\}}.$$

(2.40) shows that (2.31) implies (2.32) for $1 < D \leqq 2$, and hence (2.30). The proof is complete.

REMARK 2. Virtually the same proof leads to the following criterion: If $\{\gamma(n)\}$ satisfies the conditions of Theorem 3, then

$$P\{\liminf_{n\to\infty} \gamma(n)^{-1}|S_n| < \infty\} = 1$$

if and only if

$$\sum_{n=1}^{\infty} \frac{P\{|S_n| < K\gamma(n)\}}{\sum_{l=0}^{n-1} P\{|S_l| < \gamma(n)\}} = \infty \quad \text{for some} \quad K > 0.$$

EXAMPLE. To illustrate the above results we show that a transient random walk can go to infinity very slowly. Specifically we show: Let $X_1, X_2, \cdots$ be independent identically distributed integer valued random variables with a symmetric distribution satisfying

$$P\{X_1 = k\} = P\{X_1 = -k\} \sim [C(\log k)^2/k^2]\,(k \to \infty) \tag{2.41}$$

for some $C > 0$. Then S_n is transient (i.e., $|S_n| \to \infty$ w.p.1), but for all $0 < \alpha \leqq 1$

$$A(S_n, \alpha) = \bar{\mathbf{R}} \qquad \text{w.p.1.}$$

In particular, one has w.p.1

$$\liminf_{n\to\infty} \frac{|S_n|}{n^\alpha} = 0 \quad \text{for all} \quad \alpha > 0.$$

PROOF. By a well-known limit theorem (see [8], Theorem 35.2 with proof and footnote, p. 175, or [6], Theorems XVII. 5, 1 and 2) one has

$$\lim_{n\to\infty} P\left\{\frac{S_n}{C^* n(\log n)^2} \leqq x\right\} = \frac{1}{\pi}\int_{-\infty}^{x} \frac{dt}{1+t^2},$$

where

$$C^* = C\int_{-\infty}^{+\infty} \frac{1-\cos x}{x^2}\,dx = \pi C.$$

From the local limit theorem in Section 50 of [8] we then conclude

$$\pi C n(\log n)^2 \left| P\{S_n = k\} - \frac{1}{\pi}\left\{1 + \left(\frac{k}{\pi C n(\log n)^2}\right)^2\right\}^{-1} \right| \to 0$$

as $n \to \infty$, uniformly in k. Thus we have

$$P\{S_n = k_n\} \sim \frac{1}{\pi^2 C n(\log n)^2}$$

when $n \to \infty$, $k_n/[n(\log n)^2] \to 0$. In particular $\sum P\{S_n = 0\} < \infty$. So that S_n is transient ([3], Theorem 2) and for $0 < \alpha \leqq 1$ and fixed $b \in \mathbf{R}$

$$P\{|n^{-\alpha}S_n - b| < \varepsilon\} = \sum_{n^\alpha(b-\varepsilon)<k<n^\alpha(b+\varepsilon)} P\{S_n = k\} \tag{2.42}$$

$$\sim \frac{2\varepsilon n^{\alpha-1}}{\pi^2 C(\log n)^2}.$$

Also, for $0 < \alpha < 1$

$$(2.43)\quad \sum_{l=0}^{n-1} P\{|S_l| < n^\alpha\} = \sum_{0 \leqq l < n^\alpha(\log n)^{-3/2}} + \sum_{n^\alpha(\log n)^{-3/2} \leqq l < n} P\{|S_l| < n^\alpha\}$$

$$= O(n^\alpha(\log n)^{-\frac{3}{2}}) + (1+o(1)) \sum_{n^\alpha(\log n)^{-3/2} \leqq l \leqq n} \frac{2n^\alpha}{\pi^2 C l(\log l)^2}$$

$$= O(n^\alpha(\log n)^{-\frac{3}{2}}) + (1+o(1)) \frac{2(1-\alpha)}{\pi^2 C\alpha} \frac{n^\alpha}{\log n}$$

$$\sim \frac{2(1-\alpha)}{\pi^2 C\alpha} \frac{n^\alpha}{\log n}.$$

Similarly, for $\alpha = 1$

$$(2.44)\quad \sum_{l=0}^{n-1} P\{|S_l| < n\} \sim \frac{4}{\pi^2 C} \frac{n}{(\log n)^2} \log\log n.$$

It is easily seen from (2.42)–(2.44) that

$$\sum_{n=1}^{\infty} \frac{P\{|n^{-\alpha}S_n - b| < \varepsilon\}}{\sum_{l=0}^{n-1} P\{|S_l| < n^\alpha\}} = \infty$$

for $b \in \mathbf{R}$, $0 < \alpha \leqq 1$ and all $\varepsilon > 0$, so that our claims follow from Theorem 3.

3. The limit points of $n^{-\alpha}S_n$ for $\alpha \leqq \frac{1}{2}$. This section is devoted to the proof of Theorem 4, which has already been stated in the introduction. The proof is broken down into several steps, some of which ((b)–(d)) will be used again in later sections.

(a) As usual we denote the distribution function of the X_i by F. If F is concentrated on $[0, \infty)$ but not on $\{0\}$ then $0 < EX_i \leqq \infty$ and by the strong law of large numbers $n^{-1}S_n \to EX_1 > 0$ w.p.1 so that $n^{-\alpha} S_n$ does not have any finite limit points for $\alpha \leqq \frac{1}{2}$; similarly if F is concentrated on $(-\infty, 0]$. Thus we may assume that F has at least two points of increase $-u_1$, $+u_2$ with $u_i > 0$. It is then possible to decompose F as

$$(3.1)\quad F = pG + (1-p)H$$

for some $0 < p < 1$ and nondegenerate distribution functions G, H such that the support of H is bounded and

$$(3.2)\quad \int x\, dH(x) = 0.$$

(E.g., if $A_1 = [-u_1 - 1, -u_1/2]$, $A_2 = [u_2/2, u_2 + 1]$ we can take[4]

$$H(B) = \alpha_1 F(A_1 \cap B) + \alpha_2 F(A_2 \cap B)$$

[4] If H is a distribution function and C a Borel set, then we use $H(C)$ to denote the measure assigned to C by the Borel measure induced by H.

with $\alpha_1, \alpha_2 > 0$ such that

$$\alpha_1 \int_{A_1} x\, dF(x) + \alpha_2 \int_{A_2} x\, dF(x) = 0,$$

$$\alpha_1 F(A_1) + \alpha_2 F(A_2) = 1;$$

then $0 < 1-p < \min(\alpha_1^{-1}, \alpha_2^{-1})$.)

(b) Let $X_1, X_2, \cdots$ be a sequence of independent identically distributed random variables with a distribution function F of the form

$$F = \sum_{k=0}^{\infty} p_k F_k \tag{3.3}$$

where $p_k \geqq 0$, $\sum_{k=0}^{\infty} p_k = 1$ and F_k a distribution function. It is then possible to construct a probability space with an array of random variables $\{X_i^j\}_{i \geqq 1, j \geqq 0}$, $\{\eta_i\}_{i \geqq 1}$, such that all these random variables are independent and such that

$$P\{\eta_i = k\} = p_k, \qquad k \geqq 0, \quad i \geqq 1$$

and such that X_i^j has distribution function F_j. The random variables $\{X_i^{\eta_i}\}_{i \geqq 1}$ are then also independent and have the distribution

$$P\{X_i^{\eta_i} \leqq x\} = \sum_{k=0}^{\infty} P\{\eta_i = k\} P\{X_i^k \leqq x\} = \sum_{k=0}^{\infty} p_k F_k(x) = F(x).$$

Thus the probability structure of the original random walk S_n is the same as that of

$$\tilde{S}_n = \sum_{i=1}^{n} X_i^{\eta_i}, \qquad n \geqq 1. \tag{3.4}$$

In particular, the set of accumulation points of $n^{-\alpha} \tilde{S}_n$ is (w.p.1) the same as that of $n^{-\alpha} S_n$. For later use we shall also introduce[5]

$$U_n^j = \sum_{i=1}^{n} X_i^{\eta_i} I[\eta_i = j]. \tag{3.5}$$

Note that U_n^j, $n \geqq 1$, is for each fixed j a random walk as well. U_n^j is the sum of the n independent random variables $X_i^{\eta_i} I[\eta_i = j]$, $1 \leqq i \leqq n$, each of which has the distribution function

$$(1-p_j)\varepsilon_0 + p_j F_j. \tag{3.6}$$

(ε_0 is the degenerate distribution which assigns mass 1 to $\{0\}$.)

(c) In view of (a) and (b) it suffices to prove Theorem 4 with S_n replaced by $\tilde{S}_n$ as in (3.4) with the following choices of parameters

$$p_0 = p, \qquad p_1 = (1-p), \qquad F_0 = G, \qquad F_1 = H$$

(p, G and H as in (3.1)). We shall make this replacement, but drop the tilde and again write S_n (instead of $\tilde{S}_n$). Then, in the notation of (3.5) $S_n = U_n^0 + U_n^1$ and the increments of U_n^1 have distribution $p\varepsilon_0 + (1-p)H$ which has zero mean and bounded support.

---

[5] $$I[\eta_i \in \Lambda] = 1 \text{ if } \eta_i \in \Lambda$$
$$= 0 \text{ if } \eta_i \notin \Lambda.$$

(d) Let $1 \leqq n_1 < n_2 < \cdots$ be the successive indices n for which $\eta_n = 1$. The random variables $n_i - n_{i-1}$, $i \geqq 1$ ($n_0 = 0$), are independent, all with a geometrical distribution with success probability $p_1 = 1-p$. Even stronger, when

$$\mathscr{F}_n = \sigma\text{-field generated by } \{\eta_i, X_i^j : j = 0 \text{ or } 1,\ i \leqq n\}$$

then

$$P\{\text{smallest } n_i > n \text{ exceeds } n+k \mid \mathscr{F}_n\} = p^k.$$

Thus if $N(n) = \inf\{n_i : n_i > n\}$, then $P\{N(n) < (1+\varepsilon)n \mid \mathscr{F}_n\} \to 1$ and

$$P\{|n^{-\alpha}\textstyle\sum_{i=n+1}^{N(n)} X_i| < \varepsilon \mid \mathscr{F}_n\} \to 1 \qquad (n \to \infty, \quad \varepsilon > 0, \quad \alpha > 0)$$

and it follows from a well-known extension of the Borel–Cantelli lemma, [2], problem 5.6.9,[6] that

$$b \in B(\alpha) \tag{3.7}$$

if and only if

$$P\{b \text{ is an accumulation point of } n_i^{-\alpha} S_{n_i}, i \geqq 1\} = 1. \tag{3.8}$$

Note now that

$$T_k \equiv S_{n_k} = \textstyle\sum_{l=1}^{k} \sum_{i=n_{l-1}+1}^{n_l} X_i^{\eta_i}$$

can be written as the sum $T_k = \sum_{l=1}^{k}(V_l + W_l)$, where

$$V_l = \textstyle\sum_{n_{l-1} < i < n_l} X_i^{\eta_i} = \sum_{n_{l-1} < i < n_l} X_i^0,$$

$$W_l = X_{n_l}^{\eta_{n_l}} = X_{n_l}^1.$$

All the variables $V_1, V_2, \cdots, W_1, W_2, \cdots$ are independent; each V_l has the distribution function

$$\textstyle\sum_{k=0}^{\infty} p^k(1-p)G^{*k} \tag{3.9}$$

and each W_l has the distribution function H. Thus also T_k, $k \geqq 1$, is a random walk (with increments $V_k + W_k$). Lastly, by the strong law of large numbers

$$\lim_{k\to\infty} n_k/k = 1/(1-p) \qquad \text{w.p.1},$$

so that (3.8) holds if and only if

(3.10) $P\{b(1-p)^{-\alpha}$ is an accumulation point of

$$k^{-\alpha}T_k = k^{-\alpha}\textstyle\sum_{l=1}^{k}(V_l + W_l), k \geqq 1\} = 1.$$

By the equivalence of (3.7) and (3.8) it suffices to prove the theorem with S_n replaced by T_n. E.g., for $0 < \alpha < \frac{1}{2}$ we have to show that if $n^{-\alpha}T_n$ has a finite accumulation point w.p.1, then w.p.1 all real numbers are accumulation points of $n^{-\alpha}T_n$.

[6] Take for Breiman's X_i the triple (i, S_i, η_i) and Breiman's $A = \{(n, S, \eta) : |n^{-\alpha}S - b| < \varepsilon\}$, $B = \{(n, S, \eta) : |n^{-\alpha}S - b| < 3\varepsilon, \eta = 1\}$.

(e) Since H has mean zero and bounded support we have by the central limit theorem

$$P\{k^{-\frac{1}{2}}\textstyle\sum_{l=1}^{k} W_l \leqq x\} \to \Phi(x/\sigma) \tag{3.11}$$

where

$$\sigma^2 = \int x^2 H(dx), \qquad \Phi(x) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{x} e^{-t^2/2}\,dt.$$

We shall now show that we may in a certain fashion act as if $P\{k^{-\frac{1}{2}}\sum_{l=1}^{k} W_l \leqq x\}$ equals $\Phi(x/\sigma)$. First we have to "discard" values of $(\sum_{l=1}^{k} W_l)$ which exceed $4\sigma(k \log\log k)^{\frac{1}{2}}$. More precisely, put

$$c_n = c_n(\alpha) = (\textstyle\sum_{l=0}^{n-1} P\{|T_l| < n^{\alpha}\})^{-1}.$$

We claim for any fixed b

$$\textstyle\sum_{n=1}^{\infty} c_n P\{|n^{-\alpha}T_n - b| < \varepsilon, \qquad |\sum_{l=1}^{n} W_l| > 4\sigma(n\log\log n)^{\frac{1}{2}}\} < \infty. \tag{3.12}$$

Indeed, c_n is decreasing and when we introduce

$$R_r = \inf\{n : n \geqq 2^r, |n^{-\alpha}T_n - b| < \varepsilon, |\textstyle\sum_{l=1}^{n} W_l| > 4\sigma(n\log\log n)^{\frac{1}{2}}\},$$

we can write

$$\begin{aligned}
&\textstyle\sum_{2^r \leqq n < 2^{r+1}} c_n P\{|n^{-\alpha}T_n - b| < \varepsilon, |\sum_{l=1}^{n} W_l| > 4\sigma(n\log\log n)^{\frac{1}{2}}\} \\
&\quad\leqq c_{2^r} E \,\#\, \{n : |n^{-\alpha}T_n - b| < \varepsilon, |\textstyle\sum_{l=1}^{n} W_l| > 4\sigma(n\log\log n)^{\frac{1}{2}}, 2^r \leqq n < 2^{r+1}\} \\
&\quad= c_{2^r} \textstyle\sum_{2^r \leqq l < 2^{r+1}} \int_{|u-b|<\varepsilon} P\{R_r = l, l^{-\alpha}T_l \in du\} E \,\#\, \{n : |n^{-\alpha}T_n - b| < \varepsilon, \\
&\qquad\qquad\qquad\qquad l \leqq n < 2^{r+1} \mid l^{-\alpha}T_l = u\}.
\end{aligned}$$

Just as in (2.39) we have for $2^r \leqq l < 2^{r+1}$, $|u-b| < \varepsilon$,

$$\begin{aligned}
E \,\#\, &\{n : |n^{-\alpha}T_n - b| < \varepsilon, l \leqq n < 2^{r+1} \mid l^{-\alpha}T_l = u\} \\
&\leqq E \,\#\, \{m : |T_m| < 2K_1(|b|+\varepsilon)2^{\alpha(r+1)}, 0 \leqq m < 2^{r+1}\} \\
&\leqq 2(2^{1+\alpha}K_1(|b|+\varepsilon)+1)(2+1)E \,\#\, \{m : |T_m| < 2^{\alpha r}, 0 \leqq m < 2^r\} \\
&= 6(2^{1+\alpha}K_1(|b|+\varepsilon)+1)(c_{2^r})^{-1}.
\end{aligned}$$

Consequently

$$\begin{aligned}
&\textstyle\sum_{2^r \leqq n < 2^{r+1}} c_n P\{|n^{-\alpha}T_n - b| < \varepsilon, |\sum_{l=1}^{n} W_l| > 4\sigma(n\log\log n)^{\frac{1}{2}}\} \\
&\qquad\leqq 6(2^{1+\alpha}K_1(|b|+\varepsilon)+1)\textstyle\sum_{2^r \leqq l < 2^{r+1}} P\{R_r = l\}.
\end{aligned} \tag{3.13}$$

But, by the usual reflection principle (see [13], Lemma VII.9.1)

$$\begin{aligned}
\textstyle\sum_{2^r \leqq l < 2^{r+1}} P\{R_r = l\} &\leqq P\{\max_{2^r \leqq n < 2^{r+1}} |\textstyle\sum_{l=1}^{n} W_l| > 2^{\frac{3}{2}}\sigma(2^{r+1}\log\log 2^r)^{\frac{1}{2}}\} \\
&\leqq \tfrac{4}{3}P\{|\textstyle\sum_{l=1}^{2^{r+1}} W_l| > 2\sigma(2^{r+1}\log\log 2^r)^{\frac{1}{2}}\}
\end{aligned} \tag{3.14}$$

for sufficiently large r. Now, by Bernstein's inequality, [13], Theorem VII. 4.1,

$$\sum_{r=0}^{\infty} P\{|\sum_{l=1}^{2^{r+1}} W_l| > 2\sigma(2^{r+1}\log\log 2^r)^{\frac{1}{2}}\} = O(\sum_{r=0}^{\infty}\exp\{-\tfrac{3}{2}\log\log 2^r\}) < \infty \tag{3.15}$$

so that $\sum_{r=0}^{\infty}\sum_{2^r \leqq l < 2^{r+1}} P\{R_r = l\} < \infty$, and (3.12) follows from (3.13).

With the large values of $|\sum_{l=1}^{n} W_l|$ out of the way we can now apply Esseen's central limit theorem. For simplicity we assume that H, the distribution of W_1, is non-lattice. If H is a lattice distribution, Theorem 43.1 in [8] has to take the place of (3.16) below. In the non-lattice case Theorem 42.2 of [8] reads in our notation

$$P\{x_1\sigma\sqrt{n} < \sum_{l=1}^{n} W_l < x_2\sigma\sqrt{n}\} = \Phi(x_2)-\Phi(x_1)+\frac{e^{-\frac{1}{2}(x_2^2)}}{\sqrt{(2\pi)}}\frac{Q_1(x_2)}{\sqrt{n}}-\frac{e^{-\frac{1}{2}(x_1^2)}}{\sqrt{(2\pi)}}\frac{Q_1(x_1)}{\sqrt{n}}+o(n^{-\frac{1}{2}}) \tag{3.16}$$

where $Q_1(x)$ is a quadratic polynomial of x and $n^{\frac{1}{2}}o(n^{-\frac{1}{2}}) \to 0$ as $n \to \infty$, uniformly in x_1, x_2. In particular, for $0 < \alpha < \frac{1}{2}$, $x_2 = \sigma^{-1}n^{-\frac{1}{2}}(n^\alpha(b+\varepsilon)-y)$, $x_1 = \sigma^{-1}n^{-\frac{1}{2}}(n^\alpha(b-\varepsilon)-y)$ and $|y| \leqq 5\sigma(n\log\log n)^{\frac{1}{2}}$ we have

$$P\left\{-y+(b-\varepsilon)n^\alpha < \sum_{l=1}^{n} W_l < -y+(b+\varepsilon)n^\alpha\right\} = (1+o(1))\frac{e^{-(y^2/2n\sigma^2)}}{\sqrt{(2\pi)}}\frac{2\varepsilon n^{\alpha-\frac{1}{2}}}{\sigma}. \tag{3.17}$$

It is now easy to complete the proof of Theorem 4 for $0 \leqq \alpha < \frac{1}{2}$. Indeed, for any $b, b' \in \mathbf{R}$, some $|\theta_1|$, $|\theta_2| < 1$ and large n,

$$\begin{aligned}
P\{|n^{-\alpha}T_n-b| < \varepsilon\} &= \int_{|y|\leqq 5\sigma(n\log\log n)^{1/2}} P\{\textstyle\sum_{l=1}^{n} V_l \in dy\} \\
&\quad \cdot P\{-y+(b-\varepsilon)n^\alpha < \textstyle\sum_{l=1}^{n} W_l < -y+(b+\varepsilon)n^\alpha\} \\
&\quad +\theta_1 P\{|n^{-\alpha}T_n-b| < \varepsilon, |\textstyle\sum_{l=1}^{n} W_l| > 4\sigma(n\log\log n)^{\frac{1}{2}}\} \\
&= (1+o(1))\int_{|y|\leqq 5\sigma(n\log\log n)^{1/2}} P\left\{\sum_{l=1}^{n} V_l \in dy\right\}\frac{2\varepsilon n^{\alpha-\frac{1}{2}}}{\sigma\sqrt{(2\pi)}}\exp\left\{-\frac{y^2}{2n\sigma^2}\right\} \\
&\quad +\theta_1 P\{|n^{-\alpha}T_n-b| < \varepsilon, |\textstyle\sum_{l=1}^{n} W_l| > 4\sigma(n\log\log n)^{\frac{1}{2}}\} \\
&= (1+o(1))P\{|n^{-\alpha}T_n-b'| < \varepsilon\} \\
&\quad +\theta_1 P\{|n^{-\alpha}T_n-b| < \varepsilon, |\textstyle\sum_{l=1}^{n'} W_l| > 4\sigma(n\log\log n)^{\frac{1}{2}}\} \\
&\quad +2\theta_2 P\{|n^{-\alpha}T_n-b'| < \varepsilon, |\textstyle\sum_{l=1}^{n} W_l| > 4\sigma(n\log\log n)^{\frac{1}{2}}\}.
\end{aligned} \tag{3.18}$$

By virtue of (3.12) we may conclude from (3.18) that for $0 < \alpha < \frac{1}{2}$

$$\sum_{n=1}^{\infty} c_n P\{|n^{-\alpha}T_n-b| < \varepsilon\} \tag{3.19}$$

and

$$\sum_{n=1}^{\infty} c_n P\{|n^{-\alpha}T_n-b'| < \varepsilon\} \tag{3.20}$$

converge or diverge together. Since the divergence of (3.19) and (3.20) for all $\varepsilon > 0$ are equivalent to[7] $b \in B(T_n, \alpha)$ respectively $b' \in B(T_n, \alpha)$ (by Theorem 3) we see that if $B(T_n, \alpha)$ contains any $b \in R$, it contains all of $\mathbf{R}$. This proves Theorem 4 for $0 < \alpha < \frac{1}{2}$.

(f) For $\alpha = \frac{1}{2}$ the conclusion (3.17) from (3.16) is not permissible anymore. In this case (3.16) only gives

$$\begin{aligned}(3.21)\quad & P\{-y+(b-\varepsilon)n^{\frac{1}{2}} < \textstyle\sum_{l=1}^{n} W_l < -y+(b+\varepsilon)n^{\frac{1}{2}}\} \\ &= (2\pi)^{-\frac{1}{2}} \int_{|-y\sigma^{-1}n^{-1/2}+b\sigma^{-1}-t| < \varepsilon\sigma^{-1}} e^{-\frac{1}{2}t^2}\,dt \\ &\quad + O\{n^{-\frac{1}{2}}(1+|b|^2+y^2n^{-1}\} \\ &\qquad \cdot \{\exp(-\tfrac{1}{2}\sigma^{-2}(-yn^{-\frac{1}{2}}+b-\varepsilon)^2)+\exp(-\tfrac{1}{2}\sigma^{-2}(-yn^{-\frac{1}{2}}+b+\varepsilon)^2)\} \\ &\quad + o(n^{-\frac{1}{2}}).\end{aligned}$$

Again we restrict y to

$$(3.22)\qquad |y| \leqq 5\sigma(n \log\log n)^{\frac{1}{2}}.$$

From the well-known asymptotic relation, [9] problem 1.4.1,

$$\frac{1}{\sqrt{(2\pi)}}\int_{t \geqq x} e^{-t^2/2}\,dt \sim \frac{1}{x\sqrt{(2\pi)}} e^{-x^2/2} \qquad (x \to \infty)$$

we see that the right-hand side of (3.21) for $|y|n^{-\frac{1}{2}}$ and n large (under the condition (3.22)), and $\varepsilon > 0$ fixed, equals

$$(1+o(1))\frac{\sigma}{|y|}\left(\frac{n}{2\pi}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{2\sigma^2}\right)\min((-yn^{-\frac{1}{2}}+b-\varepsilon)^2, (-yn^{-\frac{1}{2}}+b+\varepsilon)^2).$$

More generally, there exist constants $0 < K_4 < K_5 < \infty$ depending on ε, b and σ, but not on n, such that for all y satisfying (3.22) and n sufficiently large

$$\begin{aligned}(3.23)\quad & \frac{K_4}{\sqrt{(2\pi)}}\left(1+\frac{|y|}{\sigma n^{\frac{1}{2}}}\right)^{-1} \exp\left(-\frac{1}{2\sigma^2}\right)\min((-yn^{-\frac{1}{2}}+b-\varepsilon)^2, (-yn^{-\frac{1}{2}}+b+\varepsilon)^2) \\ &\leqq P\{-y+(b-\varepsilon)n^{\frac{1}{2}} < \textstyle\sum_{l=1}^{n} W_l < -y+(b+\varepsilon)n^{\frac{1}{2}}\} \\ &\leqq \frac{K_5}{\sqrt{(2\pi)}}\left(1+\frac{|y|}{\sigma n^{\frac{1}{2}}}\right)^{-1} \\ &\qquad \cdot \exp\left(-\frac{1}{2\sigma^2}\right)\min((-yn^{-\frac{1}{2}}+b-\varepsilon)^2, (-yn^{-\frac{1}{2}}+b+\varepsilon)^2).\end{aligned}$$

Now assume $b \in B(T_n, \frac{1}{2})$. Then (3.19) with $\alpha = \frac{1}{2}$ diverges for all $\varepsilon > 0$. Assume for the sake of definiteness that

$$(3.24)\qquad \textstyle\sum_{n=1}^{\infty} c_n P\{|n^{-\frac{1}{2}}T_n - b| < \varepsilon, \quad \sum_{l=1}^{n} V_l \geqq 0\} = \infty \quad \text{for all} \quad \varepsilon > 0.$$

---

[7] Even though this is an abuse of notation, it hardly needs saying that $b \in B(T_n, \alpha)$ means $P\{b$ is an accumulation point of $n^{-\alpha}T_n\} = 1$.

(If (3.24) fails then the analogue with $\sum_{l=1}^{n} V_l \leqq 0$ instead of $\sum_{l=1}^{n} V_l \geqq 0$ must hold.) In view of (3.12) and (3.23) we must then also have for all $\varepsilon > 0$ (compare (3.18))

$$\sum_{n=1}^{\infty} c_n \int_{0 \leqq y \leqq 5\sigma(n \log\log n)^{1/2}} P\left\{\sum_{l=1}^{n} V_l \in dy\right\}\left(1+\frac{|y|}{\sigma n^{\frac{1}{2}}}\right)^{-1} \cdot \exp\left(-\frac{1}{2\sigma^2}\right)(-yn^{-\frac{1}{2}}+b+\varepsilon)^2 = \infty.$$

But then also for any $b' \geqq b$ and $\varepsilon > 0$

$$\sum_{n=1}^{\infty} c_n \int_{0 \leqq y \leqq 5\sigma(n \log\log n)^{1/2}} P\left\{\sum_{l=1}^{n} V_l \in dy\right\}\left(1+\frac{|y|}{\sigma n^{\frac{1}{2}}}\right)^{-1} \cdot \exp\left(-\frac{1}{2\sigma^2}\right)(-yn^{\frac{1}{2}}+b'+\varepsilon)^2 = \infty,$$

and a fortiori (by (3.23) again)

$$\sum_{n=1}^{\infty} c_n P\{|n^{-\frac{1}{2}}T_n - b'| < \varepsilon\} = \infty \quad \text{for all} \quad \varepsilon > 0.$$

Thus if $b \in B(T_n, \frac{1}{2})$ and (3.24) holds, then $[b, \infty] \subset B(T_n, \frac{1}{2})$. If the analogue of (3.24) holds with $\sum_{l=1}^{n} V_l \leqq 0$ instead of $\sum_{l=1}^{n} V_l \geqq 0$ then $b \in B(T_n, \frac{1}{2})$ implies $[-\infty, b] \subset B(T_n, \frac{1}{2})$. This completes the proof of Theorem 4 for $\alpha = \frac{1}{2}$.

REMARK 3. Assume one could prove

$$P\{\liminf_{n\to\infty} n^{-\frac{1}{2}} |\textstyle\sum_{l=1}^{n} V_l| < \infty\} = 1. \tag{3.25}$$

Then, by the Hewitt–Savage zero-one law there exists a $c \geqq 0$ $s \cdot t$

$$\liminf_{n\to\infty} n^{-\frac{1}{2}} |\textstyle\sum_{l=1}^{n} V_l| = c \qquad \text{w.p.1}$$

and thus the stopping times

$$N_r = \inf\{n : n \geqq 2^r, n^{-\frac{1}{2}} |\textstyle\sum_{l=1}^{n} V_l| \leqq c+1\}$$

are well defined. Moreover, by the Borel–Cantelli lemma

$$\sum_{r=0}^{\infty} P\{2^r \leqq N_r < 2^{r+1}\} = \infty. \tag{3.26}$$

But since the V_i and W_i are independent we have (using the central limit theorem (3.11))

$$\begin{aligned} &P\{|n^{-\frac{1}{2}}T_n - b'| < \varepsilon \quad \text{for some} \quad 2^r \leqq n < 2^{r+1}\} \\ &\geqq \textstyle\sum_{2^r \leqq l < 2^{r+1}} \int_{|y| \leqq c+1} P\{N_r = l, n^{-\frac{1}{2}} \sum_{l=1}^{n} V_l \in dy\}. \\ &\quad \cdot P\{n^{-\frac{1}{2}} \textstyle\sum_{l=1}^{n} W_l \in (b'-y-\varepsilon, b'-y+\varepsilon)\} \\ &\geqq \textstyle\sum_{2^r \leqq l < 2^{r+1}} \int_{|y| \leqq c+1} P\{N_r = l, n^{-\frac{1}{2}} \sum_{l=1}^{n} V_l \in dy\} K_6 \\ &= K_6 P\{2^r \leqq N_r < 2^{r+1}\} \end{aligned} \tag{3.27}$$

for some $K_6 = K_6(\varepsilon, b'c) > 0$. From (3.26), (3.27) and Theorem 3 we conclude $b' \in B(T_n, \frac{1}{2})$ for any $b' \in \mathbf{R}$. Thus, if one could prove (3.25) from the fact that some $b \in B(T_n, \frac{1}{2})$, then we could conclude $B(\frac{1}{2}) = \overline{\mathbf{R}}$ as soon as $B(\frac{1}{2}) \cap \mathbf{R} \neq \emptyset$. Using Remark 2 this proof can indeed be carried out when F is symmetric, or even when only

$$\left|\operatorname{Im} E\, e^{i\theta X_1}\right| = o(1 - \operatorname{Re} E\, e^{i\theta X_1}) \quad \text{as} \quad \theta \to 0.$$

(3.25) is also easily obtained from [15] when S_n is recurrent. Thus *the answer to problem* 1 *is in the affirmative for a symmetric or a recurrent random walk*. We have, however, been unable to prove (3.25) in general from $B(T_n, \frac{1}{2}) \cap \mathbf{R} \neq \emptyset$ alone.

4. The infinite limit points of $n^{-1}S_n$. In this section we find conditions for $+\infty$ and/or $-\infty$ to be limit points of $n^{-1}S_n$. We do this by investigating the positive and negative contributions to S_n. More precisely, we shall obtain $+\infty \in B(1)$ as a consequence of

$$\limsup_{n\to\infty} \frac{X_n^+}{\sum_{i=1}^n X_i^-} = +\infty \qquad \text{w.p.1}$$

(under suitable conditions). This approach leads to the interesting corollary that if $EX_1^+ = \infty$ and $\limsup S_n > -\infty$ then $\limsup n^{-1} S_n = +\infty$. The heart of the matter is contained in

THEOREM 5. *If* $EX_1^+ = \infty$ *and* $P\{S_n > 0 \text{ i.o.}\} = 1$, *then*

$$\limsup_{n\to\infty} \frac{X_n^+}{\sum_{i=1}^n X_i^-} = +\infty \qquad \text{w.p.1.} \tag{4.1}$$

REMARK 4. It is much simpler to prove a two-sided analogue, namely that $E|X| = \infty$ implies

$$\limsup_{n\to\infty} \frac{|X_n|}{\sum_{i=1}^{n-1} |X_i|} = \infty \qquad \text{w.p.1.}$$

(see [11]). (4.1) and this two-sided analogue express in some way the preponderance of the maximal term in S_n when $E|X| = \infty$. This phenomenon has been investigated more systematically by Darling [4] and Feller [5], [7].

PROOF. If $X_1^- = 0$ w.p.1 then there is nothing to prove, provided we interpret $X_n^+/0$ as $+\infty$ whenever $X_n^+ > 0$. Assume then that $P\{X_1 < 0\} > 0$ and let $1 \leqq n_1 < n_2 < \cdots$ be the successive indices n with $X_n > 0$. The situation is the same as in step (d) of Section 3. The random variables

$$V_l = \sum_{n_{l-1} < i < n_l} X_i = -\sum_{n_{l-1} < i < n_l} X_i^- \quad \text{and} \quad W_l = X_{n_l} = X_{n_l}^+$$

are all independent, all V_l have the same distribution which is concentrated on $(-\infty, 0]$, and all W_l have the same distribution, namely the conditional distribution of X_1, given $X_1 > 0$. Again

$$T_k = \sum_{l=1}^k (V_l + W_l), \qquad k \geqq 1,$$

is a random walk and it is clear that $\{S_n > 0 \text{ i.o.}\}$ implies that $S_n > 0$ for infinitely many n following a positive step. Thus we may assume

(4.2) $$P\{S_{n_k} > 0 \text{i.o.}\} = P\{T_k > 0 \text{i.o.}\} = 1.$$

Thus, we have w.p.1

$$\textstyle\sum_{l=1}^k W_l > -\sum_{l=1}^k V_l \geqq 0 \qquad \text{i.o.}$$

and a fortiori

(4.3) $$\textstyle\sum_{l=1}^k W_l > k \inf_{i \geqq k} -1/i \sum_{l=1}^i V_l \qquad \text{i.o.}$$

Introduce the random variables

(4.4) $$a_k = k \inf_{i \geqq k} -1/i \textstyle\sum_{l=1}^i V_l.$$

By definition

(4.5) $$a_k/k \quad \text{is increasing}$$

and w.p.1

(4.6) $$\lim_{k\to\infty} a_k/k = E(-V_1) = \textstyle\sum_{j=0}^{\infty} P\{X_1 \leqq 0\}^j P\{X_1 > 0\} j E\{X_1^- \mid X_1 \leqq 0\}.$$

If $EX_1^- < \infty$ then

(4.7) $$n^{-1} \textstyle\sum_{i=1}^n X_i^- \to EX_1^- < \infty \qquad \text{w.p.1}$$

whereas it is well known that $EX_1^+ = \infty$ implies (see [2], problem 3.3.10)

(4.8) $$\limsup_{n\to\infty} X_n^+/n = \infty \qquad \text{w.p.1.}$$

Thus (4.1) is trivial if $EX_1^- < \infty$, and we may therefore assume $EX_1^- = \infty$ and a fortiori $E(-V_1) = \infty$. By (4.5) and (4.6) we then have

(4.9) $$a_k/k \uparrow \infty \qquad \text{w.p.1.}$$

Note that the a_k depend on the V_i only and that by (4.3)

(4.10) $$P\{\textstyle\sum_{l=1}^k W_l > a_k \text{i.o.} \mid V_1, V_2, \cdots\} = 1$$

for almost all sequences $V_1, V_2, \cdots$. Consider now a fixed sequence $V_1, V_2, \cdots$ for which (4.9), (4.10) hold. Since the W_l are independent of the V's, $W_1, W_2, \cdots$ are still independent, identically distributed random variables when $V_1, V_2, \cdots$ are fixed. Thus Theorem 2 of Feller [5] applies, and we conclude

$$P\{W_k > a_k \text{i.o.} \mid V_1, V_2, \cdots\} = 1$$

for any sequence $V_1, V_2, \cdots$ which satisfies (4.9) and (4.10). Since this is the case for almost all sequences $V_1, V_2, \cdots$ we must have

(4.11) $$P\{W_k > a_k \text{i.o.}\} = 1.$$

We shall now prove in separate lemmas that (4.11) implies

$$P\{W_k > -\tfrac{1}{4}\textstyle\sum_{l=1}^{k} V_j \text{ i.o.}\} = 1$$

and then, for any $t > 0$

$$P\{W_k > -t\textstyle\sum_{l=1}^{k} V_j \text{ i.o.}\} = 1.$$

For this purpose we introduce the events

$$E(t) = \{W_k > tk \inf_{i \geqq k} - i^{-1}\textstyle\sum_{l=1}^{i} V_l \text{ i.o.}\} \quad \text{and}$$

$$F(t) = \{W_k > -t\textstyle\sum_{l=1}^{k} V_l \text{ i.o.}\}.$$

LEMMA 2. $P\{F(t)\} = 1$ *or* 0 *according as*

$$\textstyle\sum_{k=1}^{\infty} \int P\{W_1 \in dw\} P\{w > -t\textstyle\sum_{l=1}^{k} V_l\} \tag{4.12}$$

is infinite or finite.

PROOF.

$$P\{W_k > -t\textstyle\sum_{l=1}^{k} V_l\} = \int P\{W_1 \in dw\} P\{w > -t\textstyle\sum_{l=1}^{k} V_l\}$$

so that convergence of (4.12) implies $P\{F(t)\} = 0$ by the convergence part of the Borel–Cantelli lemma. When (4.12) diverges we use Kochen and Stone's [12] generalization of the Borel–Cantelli lemma. Note that for $k_1 < k_2$

$$\begin{aligned} &P\{W_{k_1} > -t\textstyle\sum_{l=1}^{k_1} V_l, W_{k_2} > -t\textstyle\sum_{l=1}^{k_2} V_l\} \\ &\quad \leqq P\{W_{k_1} > -t\textstyle\sum_{l=1}^{k_1} V_l, W_{k_2} > -t\textstyle\sum_{l=k_1+1}^{k_2} V_l\} \\ &\quad = P\{W_{k_1} > -t\textstyle\sum_{l=1}^{k_1} V_l\} P\{W_{k_2-k_1} > -t\textstyle\sum_{l=1}^{k_2-k_1} V_l\} \end{aligned}$$

(because $-V_l \geqq 0$). Thus if we write

$$G(k) = G(k,t) = \{W_k > -t\textstyle\sum_{l=1}^{k} V_l\}$$

then, for $k_1 < k_2$,

$$P\{G(k_1) \cap G(k_2)\} \leqq P\{G(k_1)\} P\{G(k_2 - k_1)\}.$$

Moreover, the divergence of (4.12) states

$$\textstyle\sum_{k=1}^{\infty} P\{G(k)\} = \infty.$$

By part (iii) of the theorem in [12] (with X_n taken to be $\#\{k : 1 \leqq k \leqq n$ and $G(k)$ occurs$\}$) we therefore have (if we put $P\{G(0)\} = 1$)

$$\begin{aligned} P\{F(t)\} &= P\{G(k) \text{ i.o.}\} \\ &\geqq \limsup_{n\to\infty} (\textstyle\sum_{k=1}^{n} P\{G(k)\})^2 (2\textstyle\sum_{1 \leqq k_1 \leqq k_2 \leqq n} P\{G(k_1) \cap G(k_2)\})^{-1} \\ &\geqq \tfrac{1}{2} \limsup_{n\to\infty} (\textstyle\sum_{k=1}^{n} P\{G(k)\})^2 (\textstyle\sum_{k_1=1}^{n} P\{G(k_1)\} \textstyle\sum_{k_2=k_1}^{n} P\{G(k_2 - k_1)\})^{-1} \\ &\geqq \tfrac{1}{2}. \end{aligned}$$

The Hewitt–Savage zero-one law then shows that $P\{F(t)\} = 1$ as desired.

LEMMA 3. *If* $P\{F(t)\} = 0$ *then* $P\{E(4t)\} = 0$.

PROOF. For any fixed $w \geqq 0$

$$P\{w > 4tk\inf_{i\geqq k} -i^{-1}\textstyle\sum_{l=1}^{i} V_l\} \tag{4.13}$$
$$\leqq \textstyle\sum_{n=0}^{\infty} P\{w > 4tk\min_{2^n k\leqq i<2^{n+1}k} -i^{-1}\sum_{l=1}^{i} V_l\}$$
$$\leqq \sum_{n=0}^{\infty} P\left\{w > -\frac{4tk}{2^{n+1}k}\sum_{l=1}^{2^n k} V_l\right\}$$
$$= \sum_{n=0}^{\infty} P\left\{w > -\frac{2t}{2^n}\sum_{l=1}^{2^n k} V_l\right\},$$

because $-\sum_{l=1}^{i} V_l$ increases with i. Now $-\sum_{l=1}^{2^n k} V_l$ is the sum of the 2^n positive, independent, identically distributed random variables

$$Z_i = -\textstyle\sum_{l=(i-1)k+1}^{ik} V_l, \qquad 1 \leqq i \leqq 2^n.$$

Thus, if we put

$$p = p(w,k) = P\{Z_1 \geqq w/t\} = P\{w \leqq -t\textstyle\sum_{l=1}^{k} V_l\}$$ then

$$P\left\{w > -\frac{2t}{2^n}\sum_{l=1}^{2^n k} V_l\right\} \leqq P\left\{Z_i \geqq \frac{w}{t} \text{ for at most } 2^{n-1} \text{ values of } i\in[1,2^n]\right\} \tag{4.14}$$
$$= \textstyle\sum_{0\leqq r\leqq 2^{n-1}} \binom{2^n}{r} p^r(1-p)^{2^n-r}.$$

By Chebychev's inequality

$$1-p = P\{w > -t\textstyle\sum_{l=1}^{k} V_l\} \leqq \frac{1}{4} \tag{4.15}$$

implies

$$\sum_{0\leqq r\leqq 2^{n-1}} \binom{2^n}{r} p^r(1-p)^{2^n-r} \leqq \frac{2^n p(1-p)}{(2^{n-2})^2} \leqq 16(1-p)2^{-n}. \tag{4.16}$$

(4.13)–(4.16) combined give

$$P\{w > 4tk\inf_{i\geqq k} -i^{-1}\textstyle\sum_{l=1}^{i} V_l\} \leqq 16(1-p)\sum_{n=0}^{\infty} 2^{-n} = 32P\{w > -t\sum_{l=1}^{k} V_l\},$$

and the inequality between the first and last members remains trivially valid when (4.15) fails. Thus

$$P\{W_k > 4tk\inf_{i\geqq k} -i^{-1}\textstyle\sum_{l=1}^{i} V_l\} \leqq 32\int P\{W_k\in dw\}P\{w > -t\sum_{l=1}^{k} V_l\} \tag{4.17}$$
$$= 32P\{W_k > -t\textstyle\sum_{l=1}^{k} V_l\}.$$

By virtue of Lemma 2 the assumption $P\{F(t)\} = 0$ entails the convergence of (4.12) and hence

$$\textstyle\sum_{k=1}^{\infty} P\{W_k > 4tk\inf_{i\geqq k} -i^{-1}\sum_{l=1}^{i} V_l\} < \infty.$$

This shows $P\{E(4t)\} = 0$ as desired.

LEMMA 4. *If* $P\{F(t)\} > 0$ *for some* $t > 0$, *then* $P\{F(t')\} = 1$ *for all* $t' > 1$.

PROOF. Since $P\{F(t)\}$ is clearly decreasing in t it suffices to prove $P\{F(2t)\} = 1$ whenever $P\{F(t)\} > 0$. But

$$\begin{aligned} P\{W_k > -2t\textstyle\sum_{l=1}^k V_l\} &= P\{W_{2k} > -2t\textstyle\sum_{l=1}^k V_l\} \\ &\geqq P\{W_{2k} > -t\textstyle\sum_{l=1}^{2k} V_l \quad \text{and} \quad -\textstyle\sum_{l=1}^k V_l \leqq -\textstyle\sum_{l=k+1}^{2k} V_l\} \\ &= P\{W_{2k} > -t\textstyle\sum_{l=1}^{2k} V_l\} P\{\textstyle\sum_{l=1}^k V_l \geqq \textstyle\sum_{l=k+1}^{2k} V_l\} \\ &\geqq \tfrac{1}{2}P\{W_{2k} > -t\textstyle\sum_{l=1}^{2k} V_l\}. \end{aligned}$$

Similarly

$$P\{W_k > -2t\textstyle\sum_{l=1}^k V_l\} \geqq \tfrac{1}{2}P\{W_{2k+1} > -t\textstyle\sum_{l=1}^{2k+1} V_l\}.$$

Thus

$$\textstyle\sum_{k=1}^{\infty} P\{W_k > -2t\sum_{l=1}^k V_l\} = \infty$$

as soon as $\sum_{k=1}^{\infty} P\{W_k > -t\sum_{l=1}^k V_l\} = \infty$, and by virtue of Lemma 2 this gives the desired conclusion.

It is now easy to complete the proof of Theorem 5. In the present notation (4.11) states $P\{E(1)\} = 1$. By Lemmas 3 and 4 we therefore have $P\{F(\frac{1}{4})\} > 0$ and $P\{F(t)\} = 1$ for all $t > 0$. In terms of our original variables this says

$$P\{X_{n_k}^+ \geqq t\textstyle\sum_{i=1}^{n_k} X_i^- \text{ i.o.}\} = 1 \quad \text{for all} \quad t > 0,$$

which is equivalent to (4.1).

Theorem 5 has various simple consequences with a more immediate interpretation than the theorem itself.

THEOREM 6. *If* $EX_1^+ = \infty$ *and* $a \in \mathbf{R}$ *then the following statements are equivalent*:

(a) $P\{\limsup_{n\to\infty} S_n > -\infty\} = 1$,
(b) $P\{\limsup_{n\to\infty} S_n/n > -\infty\} = 1$,
(c) $P\{\limsup_{n\to\infty} S_n/n = +\infty\} = 1$,
(d) $P\{\limsup_{n\to\infty} S_n = +\infty\} = 1$,
(e) $\sum_{n=1}^{\infty} 1/n\, P\{S_n - an > 0\} = \infty$.

PROOF. Clearly (a) implies (b). If (b) holds, then there exists a constant K, $0 \leqq K < \infty$, for which

$$P\left\{\limsup_{n\to\infty} \frac{S_n + Kn}{n} > 0\right\} = 1,$$

because $\limsup n^{-1}S_n$ is a constant w.p.1 (by the Hewitt–Savage zero-one law). In particular, it follows that w.p.1

$$S_n + Kn = \textstyle\sum_{i=1}^n (X_i + K) > 0 \qquad \text{i.o.}$$

Since also $E(X_1+K)^+ = \infty$ we can apply (4.1) to the (X_i+K). We now obtain (c) from the strong law of large numbers. Indeed

$$S_n+nK \geqq (X_n+K)^+ - \textstyle\sum_{i=1}^n (X_i+K)^-, \tag{4.18}$$

and thus by (4.1)

$$\limsup \frac{S_n+nK}{\sum_{i=1}^n (X_i+K)^-} = \infty \qquad \text{w.p.1.} \tag{4.19}$$

If $E(X_1+K)^- > 0$ then (c) follows from (4.19) and

$$\lim_{n\to\infty} n^{-1} \textstyle\sum_{i=1}^n (X_i+K)^- = E(X_1+K)^- > 0 \qquad \text{w.p.1.}$$

On the other hand, if $E(X_1+K)^- = 0$ and hence $\sum_{i=1}^n (X_1+K)^- = 0$, (c) follows from (4.18) and (4.8). It is also clear that (c) implies (d), and (d) implies (a). The equivalence of (e) and (a)–(d) now follows from Theorem 4.1 of Spitzer [14], which states the equivalence of (e) and $P\{\limsup_{n\to\infty}(S_n-an) = +\infty\} = 1$.

REMARK 5. Since (e) is equivalent to (a)–(d) for any choice of $a \in \mathbf{R}$ we see that if $EX_1^+ = \infty$ then the series in (e) either converges for *all* $a \in \mathbf{R}$ or diverges for *all* $a \in \mathbf{R}$. In particular this shows (after an interchange of positive and negative) that in the theorem of Binmore and Katz [1], if $EX_1^- = \infty$ one only has to check the divergence of (1) for one (arbitrary) a.

Theorem 6 is also a strengthening of Stone's result [16] as can be seen by rephrasing its main conclusion as

COROLLARY 3. *If EX_1 is well defined* ($+\infty$ or $-\infty$ *permitted as values of* EX_1) *then*

$$\lim_{n\to\infty} S_n/n = EX_1 \qquad \text{w.p.1.} \tag{4.20}$$

If

$$EX_1^+ = EX_1^- = \infty, \tag{4.21}$$

then one of the following three cases must prevail:

(i) $\lim_{n\to\infty} S_n/n = +\infty$ w.p.1,
(ii) $\lim_{n\to\infty} S_n/n = -\infty$ w.p.1,
(iii) $\liminf_{n\to\infty} S_n/n = -\infty$ *and* $\limsup_{n\to\infty} S_n/n = +\infty$ w.p.1.

In particular, if $n^{-1}S_n$ has any finite limit point then both $+\infty$ and $-\infty$ have to be limit points of $n^{-1}S_n$ (under condition (4.21)).

PROOF. If EX_1 is well defined the strong law of large numbers gives (4.20), but if (4.21) applies and (ii) does not hold then the Hewitt–Savage zero-one law and the equivalence of (b) and (c) in Theorem 6 give $\limsup n^{-1}S_n = \infty$ w.p.1. If (i) does not hold either, $\liminf n^{-1}S_n = -\infty$ w.p.1, again by Theorem 6 (with positive and negative interchanged).

5. Construction of a random walk with prescribed $B(1)$. The set of accumulation points of $n^{-1}S_n$ is necessarily closed. If $C = \{c\}$, $c \in \overline{\mathbf{R}}$, is a one-point set in $\overline{\mathbf{R}}$ then we only have to take

$$\int_{-\infty}^{+\infty} x\,dF(x) = c$$

to assure

$$A(S_n, 1) = B(F, \{n\}) = C \qquad \text{w.p.1.} \tag{5.1}$$

If C contains more than one point, then the strong law of large numbers rules out (5.1) as long as the distribution F of X_1 has a well-defined mean. To obtain (5.1) we must at least have

$$EX_1^{\,+} = EX_1^{\,-} = \infty. \tag{5.2}$$

Under (5.2) we must have $+\infty$ and $-\infty$ in $B(F, \{n\})$ as soon as $B(F, \{n\})$ contains a finite point (by Corollary 3). Thus if $B(F, \{n\})$ is not a one-point set then it must contain $+\infty$ and $-\infty$. If $C = \{-\infty, +\infty\}$ there clearly is a choice of F which makes (5.1) valid. In fact, if we take for F the symmetric stable distribution of index β, with characteristic function $\exp -|\theta|^\beta$, then for $0 < \beta < 1$

$$P\{|S_n| < Kn\} = O(Kn^{1-1/\beta})(n \to \infty). \tag{5.3}$$

(5.3) is well known, but also immediate from the fact that S_n has the same distribution as $n^{1/\beta}X_1$. For $\beta < \frac{1}{2}$

$$\sum_{n=1}^{\infty} P\{|S_n| < Kn\} < \infty \quad \text{for all} \quad K > 0,$$

so that

$$|S_n/n| \to \infty \qquad \text{w.p.1.}$$

By the symmetry and the Hewitt–Savage zero-one law we must have

$$P\{+\infty \quad \text{and} \quad -\infty \quad \text{are accumulation points of} \quad S_n/n\} = 1.$$

This leaves us with the construction of an F for which (5.1) holds when C is a closed set of $\overline{\mathbf{R}}$ which contains $+\infty$, $-\infty$ and at least one finite point. This will be the burden of the proof below.

THEOREM 7. *For any closed subset C of $\overline{\mathbf{R}}$ which is either a one-point set or contains $+\infty$ and $-\infty$ there is a random walk for which* (5.1) *is satisfied. If $0 \in C$ one can even take S_n recurrent.*

PROOF. Apart from the recurrence statement, we have handled above all cases where C does not contain $+\infty$, $-\infty$ and a finite point. If $C = \{0\}$ then a random walk with zero mean satisfies (5.1) and is recurrent, [3] Theorem 4. From now on we shall therefore assume that C contains $+\infty$, $-\infty$ and at least one finite point. We pick an arbitrary sequence of points $\{c_k\}_{k \geq 1} \subset C \cap \mathbf{R}$ such that

$$|c_k| \leqq k, \tag{5.4}$$

$$\text{for any} \quad m, \{c_k : k \geqq m\} \quad \text{is dense in} \quad C \cap \mathbf{R}. \tag{5.5}$$

We shall now construct (by a rather horrible method) an F of the form

(5.6) $$F = \sum_{n=0}^{\infty} p_k F_k,$$

where

(5.7) $$p_k \geqq 0, \qquad \sum_{n=0}^{\infty} p_k = 1$$ and

(5.8) $$F_0 = \text{distribution concentrated at } 0,$$

(5.9) $$F_k(x) = G_k\left(\frac{x-b_k}{a_k}\right), \qquad k \geqq 1,$$

for suitable $b_k \in \mathbf{R}$ and $a_k > 0$ and G_k the uniform distribution on the integers in $[-r_k, +r_k]$ for a suitable integer $r_k > 0$, i.e.,

$$G_k(x) = (2r_k+1)^{-1} \sum_{-r_k \leqq l \leqq \min(x, r_k)} 1.$$

The sequences $\{a_k\}$, $\{b_k\}$, $\{p_k\}$, $\{r_k\}$ will be chosen inductively in a manner to be described in a little while. First we note that (5.6) is a decomposition of the form (3.3) and we shall accordingly take as our basic probability space the space described under (b) in Section 3. In particular, $\{X_i^j\}$, $\{\eta_i\}$ $i \geqq 1, j \geqq 0$ will be independent random variables with distributions as described in Section 3 (b). $X_i^{\eta_i}$ will have distribution F and we shall write

$$S_n = \sum_{i=1}^{n} X_i^{\eta_i} \qquad \text{(rather than } \tilde{S}_n\text{)}.$$

U_n^j will be the random walk defined by (3.5), and in addition we shall need the random walks

(5.10) $$V_n^j = \sum_{i=1}^{n} X_i^{\eta_i} I[\eta_i \leqq j-2].$$

If we put

(5.11) $$N_k = \inf\{i : \eta_i = k\}$$

then clearly

(5.12) $$S_n = V_n^k + U_n^{k-1} + U_n^k \quad \text{for} \quad n < \inf\{N_r : r > k\}.$$

The program for the construction is to choose the parameters such that w.p.1

(5.13) $$N_k < N_{k+1} \quad \text{eventually}$$

as well as the appropriate parts of (5.15)–(5.18) below.

(5.13) guarantees the decomposition

(5.14) $$S_n = V_n^k + U_n^{k-1} + U_n^k \quad \text{on} \quad N_k \leqq n < N_{k+1},$$

and the parameters will have to be such that U_n and V_n satisfy for any $\varepsilon > 0$ w.p.1 on $N_k \leqq n < N_{k+1}$ and k sufficiently large

(5.15) $$|n^{-1} V_n^k - c_{k-2}| \leqq \varepsilon,$$

(5.16) $$|n^{-1} U_n^{k-1} - (c_{k-1} - c_{k-2})| \leqq \tfrac{1}{8} a_k p_{k+1} k^{-1},$$

and one of (5.17a), (5.17b) or (5.18).

$$(5.17a) \qquad n^{-1}U_n^k-(c_k-c_{k-1}) \geqq \tfrac{3}{8}a_k p_{k+1} k^{-1} \geqq k^2,$$

$$(5.17b) \qquad n^{-1}U_n^k-(c_k-c_{k-1}) \leqq -\tfrac{3}{8}a_k p_{k+1} k^{-1} \leqq -k^2,$$

$$(5.18) \quad |n^{-1}U_n^k-(c_k-c_{k-1})| \leqq \varepsilon \quad \text{as well as} \quad |n^{-1}U_n^{k-1}-(c_{k-1}-c_{k-2})| \leqq \varepsilon.$$

Clearly, when (5.14)–(5.16) and (5.17a) hold, then

$$(5.19) \quad n^{-1}S_n-c_k = (n^{-1}V_n^k-c_{k-2})+(n^{-1}U_n^{k-1}-(c_{k-1}-c_{k-2})) + (n^{-1}U_n^k-(c_k-c_{k-1})) \geqq (\tfrac{3}{8}-\tfrac{1}{8})a_k p_{k+1}/k-\varepsilon \geqq \tfrac{1}{2}k^2.$$

Similarly, when (5.14)–(5.16) and (5.17b) hold

$$(5.20) \qquad n^{-1}S_n-c_k \leqq -\tfrac{1}{2}k^2.$$

On the other hand, when (5.14), (5.15) and (5.18) hold,

$$(5.21) \qquad |n^{-1}S_n-c_k| \leqq 3\varepsilon.$$

In view of (5.4), (5.19) implies

$$(5.22a) \qquad n^{-1}S_n \geqq \tfrac{1}{2}k^2-k \geqq \tfrac{1}{4}k^2 \quad \text{eventually,}$$

and

$$(5.22b) \qquad n^{-1}S_n \leqq -\tfrac{1}{4}k^2$$

when (5.20) applies. On the other hand, (5.21) implies

$$(5.23) \qquad \inf_{c\in C}|n^{-1}S_n-c| \leqq 3\varepsilon$$

because $c_k \in C$. Since one of these relations has to hold eventually, no matter what $\varepsilon > 0$ is, such a construction will indeed guarantee

$$(5.24) \qquad A(S_n, 1) \subset \bar{C}\cup\{-\infty, +\infty\} = C \qquad \text{w.p.1.}$$

To make sure that there is equality in (5.24) we have to make sure that there exists for each $\varepsilon > 0$ w.p.1 for all sufficiently large k an $n \in [N_k, N_{k+1})$ for which (5.18) holds as well as infinitely many k for which (5.17a) holds for some $n \in [N_k, N_{k+1})$ and similarly for (5.17b). Indeed this will guarantee that for each $\varepsilon > 0$ and sufficiently large k (5.21) (and hence (5.23)) occurs for some $n \in [N_k, N_{k+1})$ and also that (5.22a) and (5.22b) will occur for arbitrarily large k. In view of (5.5) this will make every $c \in C\cap\mathbf{R}$ and $+\infty$, $-\infty$ accumulation points of $n^{-1}S_n$ so that (5.1) will indeed hold.

We are almost ready to specify the parameters. First, we put

$$Y_i^0 = 0, \qquad Y_i^j = a_j^{-1}(X_i^j-b_j), \qquad j \geqq 1.$$

The $\{Y_i^j\}$, $\{\eta_i\}$, $i \geqq 1$, $j \geqq 0$, are still independent and Y_i^j has the distribution function G_j. Thus

$$(5.25) \qquad T_n^j = \textstyle\sum_{i=1}^n Y_i^j, \qquad n \geqq 1,$$

is an integer-valued recurrent random walk for each fixed j (it has zero mean). Together with the sequences $\{a_k\}$, $\{b_k\}$, $\{p_k\}$, $\{r_k\}$ we shall choose two auxiliary sequences of integers $\{\lambda_k\}_{k \geqq 1}$, $\{\nu_k\}_{k \geqq 1}$ such that $0 < \lambda_k \leqq \nu_k$. In addition, the parameters will be taken so as to satisfy (5.7) and (5.26)–(5.32), where for $k \geqq 1$

$$b_k = p_k^{-1}[c_k - \textstyle\sum_{i=1}^{k-1} p_i b_i], \tag{5.26}$$

$$\lambda_k \geqq \max\{20 \log k, 16k^2 p_k p_{k-1}(a_{k-1}^2 r_{k-1}^2 + b_{k-1}^2), 8k^2 p_k^2 b_k^2\}, \tag{5.27}$$

$$P\{T_n^k = 0 \text{ for some } \lambda_k \leqq n \leqq \nu_k\} \geqq 1 - k^{-2}, \tag{5.28}$$

$$2r_k + 1 \geqq 3k^2 \lambda_k^{\frac{1}{2}}, \tag{5.29}$$

$$p_{k+1} \leqq \min\{\tfrac{1}{2} p_k k^{-3}, (3k^5 \textstyle\sum_{j=1}^{k-1} p_j\{a_j^2 r_j^2 + b_j^2\})^{-1}, \tfrac{1}{2} p_k (\nu_k k^2)^{-1}, k^{-2} f(k)^{-1}\}, \tag{5.30}$$

where $f(k)$ is large enough to insure

$$P\{\textstyle\sum_{i=1}^n Y_i^k I[\eta_i = k] = 0 \quad \text{and} \quad |\sum_{i=1}^n \{X_i^{\eta_i} I[\eta_i \leqq k-1] + b_k I[\eta_i = k] - c_k\}| \leqq 1 \quad \text{for some} \quad n \in [kp_k^{-1}, f(k)]\} \geqq 1 - k^{-2}, \tag{5.31}$$

and

$$a_k \geqq \max\{p_{k+1}^{-1}(512k^6 p_k p_{k-1}(a_{k-1}^2 r_{k-1}^2 + b_{k-1}))^{\frac{1}{2}}, \tfrac{8}{3}k^3 p_{k+1}^{-1}, 8k|b_k| p_{k+1}^{-1}\}. \tag{5.32}$$

Our first task is to show that it is indeed possible to find a set of parameters satisfying (5.7) and (5.26)–(5.32). We show that they can be picked inductively. The c_i already having been chosen, we begin with taking $a_0 = b_0 = 0$, $r_0 = \lambda_0 = \nu_0 = 1$[8] and $p_1 = \frac{1}{4}$. Assume that at some stage $b_i, \lambda_i, \nu_i, r_i, p_{i+1}, a_i$ for $0 \leqq i \leqq k-1$ have already been found in such a way that (5.26)–(5.32) hold for $1 \leqq i \leqq k-1$, and such that $p_k > 0$. Then (5.26) determines b_k and one can successively choose λ_k, r_k and ν_k large enough to satisfy (5.27)–(5.29). (5.28) can be satisfied because T_n^k, $n \geqq 1$, is recurrent. For the same reason, one can determine an $f(k)$ satisfying (5.31) because the two-dimensional random variables

$$(Y_i^k I[\eta_i = k], \qquad X_i^{\eta_i} I[\eta_i \leqq k-1] + b_k I[\eta_i = k] - c_k), \qquad i \geqq 1, \tag{5.33}$$

are independent identically distributed bounded random variables, whose first component is integer-valued with zero mean and whose second component has expectation

$$\begin{aligned} &\textstyle\sum_{j=0}^{k-1} P\{\eta_i = j\} EX_i^j + P\{\eta_i = k\} b_k - c_k \\ &= \sum_{j=1}^{k-1} p_j \int x \, dG_j\left(\frac{x - b_j}{a_j}\right) + p_k b_k - c_k \qquad \text{(see (5.8), (5.9))} \\ &= \textstyle\sum_{j=1}^k p_j b_j - c_k = 0 \qquad \text{(see (5.26))}. \end{aligned}$$

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[8] This is just for convenience. $a_0, b_0, r_0, \lambda_0, \nu_0$ do not occur in F, but in (5.27) etc. Notice, though, that the choice $a_0 = b_0 = 0$ makes it unnecessary to know p_0 for any of the choices below.

Thus the two-dimensional random walk

$$(\sum_{i=1}^{n} Y_i^k I[\eta_i = k], \qquad \sum_{i=1}^{n} \{X_i^{\eta_i} I[\eta_i \leqq k-1] + b_k I[\eta_i = k] - c_k\})$$

is recurrent (see [3], Theorem 5) and $f(k) < \infty$ can be found. Note that $X_0 = 0$ w.p.1 so that the distribution of the variable in (5.33) depends only on p_i, b_i, r_i, $1 \leqq i \leqq k$, and $a_1, \cdots, a_{k-1}$, all of which have been determined at this stage. Thus also, $f(k)$ depends only on parameters which are already chosen. Lastly, we can choose $p_{k+1} > 0$ small enough and a_k large enough to satisfy (5.30) and (5.32). After this we repeat the cycle. Since $p_1 = \frac{1}{4}$ and $p_{k+1} \leqq \frac{1}{2} p_k k^{-3}$ one has $\sum_{k=1}^{\infty} p_k \leqq \frac{1}{2}$, and (5.7) will therefore be automatic if one chooses p_0 last of all as

$$p_0 = 1 - \sum_{k=1}^{\infty} p_k .$$

From now on we may and shall assume that we have a set of parameters satisfying (5.7) and (5.26)–(5.32), and we shall complete our task by showing that (5.13)–(5.18) hold for any such set of parameters and that (5.17a), (5.17b) and (5.18) occur sufficiently often. Firstly, from the definition (5.11)

$$P\{N_k \leqq m\} \leqq m p_k \tag{5.34}$$

and

$$P\{N_k > m\} \leqq (1-p_k)^m \tag{5.35}$$

are immediate. Thus from the Borel–Cantelli lemma one has w.p.1 eventually

$$\frac{1}{k^2 p_k} \leqq N_k \leqq \frac{k}{p_k} . \tag{5.36}$$

(5.36) entails (5.13) (and automatically (5.14)) because $k p_k^{-1} < (k+1)^{-2} p_{k+1}^{-1}$ eventually by (5.30). (5.15), (5.16) and the last part of (5.18) will all be proved as an application of Kolmogorov's inequality. To do this observe that if $\{\gamma_i\}_{i \geqq 1}$ is an arbitrary sequence of independent, identically distributed random variables with

$$E\gamma_i = 0, \qquad \sigma^2 = E\gamma_i^2 < \infty,$$

then one has for

$$\Gamma_n = \sum_{i=1}^{n} \gamma_i$$

$$P\{|\Gamma_n| \geqq n\varepsilon \text{ for some } n \geqq A\} \leqq \sum_{l=0}^{\infty} P\{\max_{2^l A \leqq n < 2^{l+1} A} |\Gamma_n| \geqq 2^l A \varepsilon\} \tag{5.37}$$
$$\leqq \sum_{l=0}^{\infty} \frac{2^{l+1} A \sigma^2}{(2^l A \varepsilon)^2} = \frac{4\sigma^2}{A\varepsilon^2} .$$

This is first applied to the random walk

$$V_n^k - n c_{k-2} = \sum_{i=1}^{n} (X_i^{\eta_i} I[\eta_i \leqq k-2] - c_{k-2})$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{k-2} \{a_j Y_i^j I[\eta_i = j] + b_j (I[\eta_i = j] - p_j)\},$$

which has zero mean and a variance for each increment of

$$\begin{aligned} E(V_1{}^k - c_{k-2})^2 &\leqq (k-2)\textstyle\sum_{j=1}^{k-2} E\{a_j Y_i^j I[\eta_i = j] + b_j(I[\eta_i = j] - p_j)\}^2 \\ &\leqq 2k \textstyle\sum_{j=1}^{k-2} \{a_j{}^2 p_j E(Y_i^j)^2 + b_j{}^2 p_j\} \\ &\leqq 2k \textstyle\sum_{j=1}^{k-2} p_j\{a_j{}^2 r_j{}^2 + b_j{}^2\} \leqq (k^4 p_k)^{-1} \qquad \text{(see (5.30))}. \end{aligned}$$

Thus, by (5.34) and (5.37)

$$\begin{aligned} (5.38) \quad & P\{|n^{-1}V_n^k - c_{k-2}| > \varepsilon \text{ for some } n \geqq N_k\} \\ &\leqq P\{N_k \leqq (k^2 p_k)^{-1}\} + P\{|V_n^k - nc_{k-2}| > n\varepsilon \text{ for some } n \geqq (k^2 p_k)^{-1}\} \\ &\leqq k^{-2} + 4k^2 p_k(\varepsilon^2 k^4 p_k)^{-1} \leqq (1+4\varepsilon^{-2})k^{-2}. \end{aligned}$$

(5.38), together with the Borel–Cantelli lemma, shows that (5.15) holds w.p.1 for sufficiently large k and $n \geqq N_k$. In exactly the same way one shows

$$\begin{aligned} (5.39) \quad & P\{|n^{-1}U_n{}^{k-1} - (c_{k-1} - c_{k-2})| > \tfrac{1}{8}a_k p_{k+1}/k \quad \text{for some} \quad n \geqq N_k\} \\ &\leqq k^{-2} + 4k^2 p_k 64k^2(a_k p_{k+1})^{-2} E\{a_{k-1} Y_1{}^{k-1} I[\eta_1 = k-1] \\ &\qquad + b_{k-1}(I[\eta_1 = k-1] - p_{k-1})\}^2 \\ &\leqq k^{-2} + 512k^4 p_k(a_k p_{k+1})^{-2} p_{k-1}(a_{k-1}^2 r_{k-1}^2 + b_{k-1}^2) \leqq 2k^{-2} \end{aligned}$$

(use (5.32)), which guarantees (5.16) for large k and $n \geqq N_k$.

Slightly more is needed for (5.17) and (5.18). Define $t_i^k = i$th index n with $\eta_n = k$, i.e., $t_i^k = m$ if $\eta_m = k$ and $\eta_l = k$ for exactly $i-1$ values of $l \in [1, m-1]$. In particular, $N_k = t_1{}^k$. Define also

$$L_k = t_{\lambda_k}^k \quad \text{and} \quad M_k = t_{\nu_k}^k.$$

First we use Bernstein's inequality to bound the distribution of L_k and M_k. Specifically, for $\lambda < +1$ one has

$$\begin{aligned} (5.40) \qquad E\exp(\lambda p_k t_i^k) &= \{E\exp(\lambda p_k t_1{}^k\}^i \\ &= \{\textstyle\sum_{r=1}^{\infty} (1-p_k)^{r-1} p_k \exp(\lambda p_k r)\}^i \\ &= \{1 - p_k{}^{-1}(1-\exp(-\lambda p_k))\}^{-i}. \end{aligned}$$

Since

$$P\{t_i^k \leqq i/2p_k\} \leqq e^{\frac{1}{4}i} E\exp(-\tfrac{1}{2}p_k t_i^k)$$

we obtain from (5.40) with $\lambda = -\frac{1}{2}$

$$\begin{aligned} P\{t_i^k \leqq i/2p_k\} &\leqq e^{\frac{1}{4}i}\{1 - p_k{}^{-1}(1-\exp(\tfrac{1}{2}p_k))\}^{-i} \\ &\leqq e^{\frac{1}{4}i}(1+\tfrac{1}{2})^{-i} \leqq e^{-\frac{1}{8}i}. \end{aligned}$$

Similarly, (using (5.40) with $\lambda = \frac{1}{4}$), $P\{t_i^k \geqq 2i/p_k\} \leqq e^{-\frac{1}{8}i}$.
In particular,

(5.41) $$P\{\tfrac{1}{2}\lambda_k/p_k \leqq L_k \leqq 2\lambda_k/p_k\} \geqq 1-2e^{-\frac{1}{8}\lambda_k} \geqq 1-k^{-2}$$

and

(5.42) $$P\{\tfrac{1}{2}\nu_k/p_k \leqq M_k \leqq 2\nu_k/p_k\} \geqq 1-k^{-2}.$$

Another observation is needed. Since the Y_i^k are integer-valued one has

(5.43a) $$a_k \sum_{i=1}^n Y_i^k I[\eta_i = k] \geqq a_k$$

whenever

(5.44a) $$\sum_{i=1}^n Y_i^k I[\eta_i = k] > 0.$$

Also

(5.43b) $$a_k \sum_{i=1}^n Y_i^k I[\eta_i = k] \leqq -a_k$$

whenever

(5.44b) $$\sum_{i=1}^n Y_i^k I[\eta_i = k] < 0.$$

Moreover, we can easily obtain information on the distribution of the zeroes of

(5.45) $$\sum_{i=1}^n Y_i^k I[\eta_i = k].$$

Indeed, for

(5.46) $$t_l^k \leqq n < t_{l+1}^k$$

the sum in (5.45) equals

(5.47) $$\sum_{j \in \{t_1^k, \cdots, t_l^k\}} Y_j^k$$

which, by the independence of the Y_i^j and the η_i has the distribution of $T_l^k = \sum_{i=1}^l Y_i^k$.

This is even true for the conditional distribution of (5.45) or (5.47) given $t_1^k, \cdots, t_l^k$ and (5.46). We therefore have $P\{\sum_{i=1}^n Y_i^k I[\eta_i = k] = 0$ for some $n \leqq L_k = t_{\lambda_k}^k\} \leqq P\{T_l^k = 0$ for some $l \leqq \lambda_k\} \leqq \sum_{l=1}^{\lambda_k} P\{T_l^k = 0\}$. Now the Y_i^k are independent, integer-valued random variables with

$$\sup_s P\{Y_i^k = s\} = (2r_k+1)^{-1}.$$

By Corollary 1 of [10] there exists a constant C_0 such that

(5.48) $$P\{T_l^k = 0\} \leqq \frac{C_0}{l^{\frac{1}{2}}} \sup_{i,s} P\{Y_i^k = s\} \leqq \frac{C_0}{(2r_k+1)l^{\frac{1}{2}}}$$

(since the distribution of Y_i^k is so simple one can also check (5.48) directly, without recourse to [10], by means of characteristic functions). Thus

(5.49) $$P\{\sum_{i=1}^n Y_i^k I[\eta_i = k] = 0 \text{ for some } n \leqq L_k = t_{\lambda_k}^k\}$$
$$\leqq \frac{C_0}{(2r_k+1)} \sum_{l=1}^{\lambda_k} l^{-\frac{1}{2}} \leqq \frac{3C_0 \lambda_k^{\frac{1}{2}}}{(2r_k+1)} \leqq C_0 k^{-2} \quad \text{(see (5.29))}.$$

By (5.28) we have also

$$(5.50)\quad P\{\textstyle\sum_{i=1}^{n} Y_i^k I[\eta_i = k] = 0 \quad \text{for some} \quad L_k \leqq n \leqq M_k\} = P\{T_l^k = 0 \quad \text{for some} \quad \lambda_k \leqq l \leqq \nu_k\} \geqq 1-k^{-2}.$$

We are now ready to check (5.17) and (5.18).

$$(5.51)\qquad U_n^{\ k} - n(c_k - c_{k-1}) = \textstyle\sum_{i=1}^{n} \{a_k Y_i^k I[\eta_i = k] + b_k(I[\eta_i = k] - p_k)\},$$

and trivially on $N_k \leqq n < N_{k+1} \leqq (k+1)p_{k+1}^{-1}$

$$(5.52)\qquad |\textstyle\sum_{i=1}^{n} b_k(I[\eta_i = k] - p_k)| \leqq n\,|b_k| \leqq (k+1)\,|b_k|\,p_{k+1}^{-1} \leqq \tfrac{1}{4}a_k$$

(see (5.32)). (5.51), (5.52), (5.36) and (5.32) show that for sufficiently large k and $N_k \leqq n < N_{k+1}$ one has

$$n^{-1}U_n^{\ k} - (c_k - c_{k-1}) \geqq n^{-1}(a_k - \tfrac{1}{4}a_k) \geqq \tfrac{3}{4}N_{k+1}^{-1}\,a_k \geqq \tfrac{3}{8}a_k\,p_{k+1}\,k^{-1} \geqq k^2,$$

i.e., (5.17a), whenever (5.44a) holds. Similarly (5.17b) holds eventually for any $n \in [N_k, N_{k+1})$ for which (5.44b) holds. In particular, (5.17a) or (5.17b) will hold eventually on $N_k \leqq n \leqq L_k$ since by (5.49) and the Borel–Cantelli lemma

$$P\{\textstyle\sum_{i=1}^{n} Y_i^k I[\eta_i = k] \neq 0 \,\text{for}\, N_k \leqq n \leqq L_k \,\text{eventually}\} = 1.$$

In view of the symmetry of the $Y_i^{\ k}$, this also shows for large k

$$(5.53)\qquad P\{(5.17\text{a}) \text{ holds for some } N_k \leqq n \leqq L_k\} \geqq \tfrac{1}{2}P\{\textstyle\sum_{i=1}^{n} Y_i^k I[\eta_i = k] \neq 0 \text{ for some } N_k \leqq n \leqq L_k\} - P\{N_{k+1} > (k+1)p_{k+1}^{-1}\} \geqq \tfrac{1}{4}.$$

As for $n \in [L_k, N_{k+1})$, we again apply (5.37). As in (5.38), (5.39)

$$(5.54)\qquad P\{|n^{-1}U_n^{\ k-1} - (c_{k-1} - c_{k-2})| > \varepsilon \quad \text{for some} \quad n \geqq L_k\} \leqq P\{L_k < \tfrac{1}{2}(\lambda_k/p_k)\} + (\lambda_k\varepsilon^2)^{-1}16p_k\,p_{k-1}(a_{k-1}^2\,r_{k-1}^2 + b_{k-1}^2) \leqq k^{-2}(1+\varepsilon^{-2}) \qquad (\text{by (5.41) and (5.27)}).$$

Thus the second part of (5.18) will hold eventually on $n \geqq L_k$. Similarly,

$$(5.55)\quad P\{n^{-1}\,|\textstyle\sum_{i=1}^{n} b_k(I[\eta_i = k] - p_k)) > \varepsilon \text{ for some } n \geqq L_k\} \leqq P\{L_k < \tfrac{1}{2}(\lambda_k/p_k)\} + (\lambda_k\varepsilon^2)^{-1}8p_k p_k\,b_k^{\ 2} \leqq k^{-2}(1+\varepsilon^{-2}) \qquad (\text{by (5.41) and (5.27)}).$$

Together with (5.51), (5.55) shows that eventually the first part of (5.18) holds when $n \geqq L_k$ and

$$(5.56)\qquad \textstyle\sum_{i=1}^{n} Y_i^k[\eta_i = k] = 0.$$

Summarizing, we have (5.17a) or (5.17b) on $N_k \leqq n < N_{k+1}$ whenever (5.56) fails, in particular on $N_k \leqq n \leqq L_k$. On $L_k < n < N_{k+1}$ (5.17a) or (5.17b) holds whenever

(5.56) fails, and (5.18) holds whenever (5.56) occurs. This completes the proof of (5.13)–(5.18). *Inter alia* we proved that (5.17a), (5.17b) and (5.18) occur "sufficiently often". In fact (5.53) shows

$$P\{(5.17a) \text{ occurs for infinitely many } k \text{ and } N_k \leqq n < N_{k+1}\} \geqq \tfrac{1}{4}, \tag{5.57}$$

and by the Hewitt–Savage law the left-hand side of (5.57) equals 1. The same is true if (5.17a) is replaced by (5.17b). Lastly,

$$\begin{aligned} P\{&(5.18) \text{ occurs for some } N_k \leqq n < N_{k+1}\} \\ &\geqq P\{(5.56) \text{ occurs for some } L_k \leqq n < N_{k+1}\} - 2(1+\varepsilon^{-2})k^{-2} \\ &\geqq P\{\textstyle\sum_{i=1}^n Y_i^k[\eta_i = k] = 0 \text{ for some } L_k \leqq n \leqq M_k\} \\ &\quad - P\{M_k \geqq N_{k+1}\} - 2(1+\varepsilon^{-2})k^{-2} \\ &\geqq P\{T_l^k = 0 \text{ for some } \lambda_k \leqq l \leqq \nu_k\} - P\{M_k > 2\nu_k/p_k\} \\ &\quad - P\{N_{k+1} \leqq 2\nu_k/p_k\} - 2(1+\varepsilon^{-2})k^{-2} \\ &\geqq 1 - k^{-2}(4+2\varepsilon^{-2}) - 2\nu_k p_{k+1} p_k^{-1} \qquad (\text{by } (5.42), (5.50) \text{ and } (5.34)) \\ &\geqq 1 - k^{-2}(5+2\varepsilon^{-2}) \qquad (\text{by } (5.30)). \end{aligned}$$

Again by the Borel–Cantelli lemma we have

$$P\{\text{for all large } k \text{ (5.18) occurs for some } n \in [N_k, N_{k+1})\} = 1. \tag{5.58}$$

As pointed out before, (5.13)–(5.18), (5.58) together with the fact that (5.17a) and (5.17b) occur infinitely often, imply (5.1). This proves the theorem, except for the recurrence of S_n when $0 \in C$. This last gap is easily filled though. Indeed, when $0 \in C$, we can take $c_k = 0$ for infinitely many k. For any k with $c_k = 0$ we have

$$\begin{aligned} P\{&|S_n| \leqq 1 \text{ for some } N_k \leqq n < N_{k+1}\} \\ &= P\{|\textstyle\sum_{i=1}^n \{X_i^{\eta_i} I[\eta_i \leqq k-1] + a_k Y_i^k I[\eta_i = k] + b_k I[\eta_i = k] - c_k\}| \\ &\leqq 1 \text{ for some } N_k \leqq n < N_{k+1}\} \\ &\geqq P\{\textstyle\sum_{i=1}^n Y_i^k I[\eta_i = k] = 0 \text{ and } |\sum_{i=1}^n \{X_i^{\eta_i} I[\eta_i \leqq k-1] \\ &\quad + b_k I[\eta_i = k] - c_k\}| \leqq 1 \text{ for some } k p_k^{-1} \leqq n \leqq f(k)\} \\ &\quad - P\{N_k > k p_k^{-1}\} - P\{N_{k+1} \leqq f(k)\} \\ &\geqq 1 - 2k^{-2} - f(k) p_{k+1} \qquad (\text{by } (5.31), (5.34) \text{ and } (5.35)) \\ &\geqq 1 - 3k^{-2} \qquad (\text{by } (5.30)). \end{aligned}$$

Again the Borel–Cantelli lemma shows

$$P\{|S_n| \leqq 1 \text{ i.o.}\} = 1,$$

which means that S_n is a recurrent random walk (see [3]).

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