# THE LIMIT SET OF A FUCHSIAN GROUP 

## BY

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## § 1. Introduction

In this paper we shall consider the limit set of a finitely generated Fuchsian group of the second kind. In particular, we shall attempt to calculate the Hausdorff dimension of the limit set. If the group, $G$, has no parabolic elements this is actually achieved in section 4, whereas, if $G$ has parabolic elements we can obtain partial results which are discussed in section 5 . The proof of these results involves the construction of a measure supported on the limit set, from which, at least in principle, we can obtain a lower bound for the Hausdorff dimension. This bound is the same as the upper bound found by Beardon [4].

The actual measure which we construct proves to be intimately related to the theory of the Laplace operator on $\boldsymbol{G} \backslash \mathbf{H}$, and consequently we obtain new insights into both of these. This allows us, for example, to give a new proof of a theorem of Beardon [4] (see the Corollary to Theorem 7.1).

Section 2 recalls some well-known, but difficult to locate, results on the description of the geometry of Fuchsian groups of the second kind. These are used in making various estimates.

By way of notation, we shall take all Fuchsian groups to be finitely generated, although several of the results will be valid without this restriction (in particular those of section 3). These groups will be assumed to act on the unit disc $\Delta$ unless stated otherwise. For a domain $D$ we shall write $\operatorname{Con}(D)$ for the group of conformal homeomorphisms of $D$ onto $D$. We can, as usual, represent an element $g \in \operatorname{Con}(\Delta)$ as a matrix

$$
\left(\begin{array}{ll}
\alpha & \beta \\
\bar{\beta} & \bar{\alpha}
\end{array}\right) \quad\left(\alpha, \beta \in \mathbf{C},|\alpha|^{2}-|\beta|^{2}=1\right),
$$

and if this is so we shall write

$$
\mu(g)=2\left(|\alpha|^{2}+|\beta|^{2}\right)
$$

Likewise, if $g \in \operatorname{Con}(\mathbf{H})$, where $\mathbf{H}$ is the upper half-plane, $g$ can be represented as

$$
\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right) \quad(a, b, c, d \in \mathbf{R}, a d-b c=1) .
$$

We shall write here

$$
\mu(g)=a^{2}+b^{2}+c^{2}+d^{2}
$$

The unit circle, $\partial \Delta$, will be denoted by $S$.
Some of the results of this paper were contained in a Ph.D. thesis submitted to the University of Cambridge. I would like to express my thanks to my research supervisor, Dr A. F. Beardon; some extent of my debt to him will be clear from this paper.

## § 2. Geometry of the fundamental domain and the limit set

Let $G$ be a Fuchsian group acting on $\Delta$. Then there is an (open) fundamental domain $D$ in $\Delta$ so that
(a) $\partial D$ is a finite collection $\left\{u_{j}\right\}$ of geodesic arcs,
(b) $\left\{u_{j}\right\}$ splits uniquely into pairs $u_{k}, u_{k}^{\prime}$ in such a way that there is $\gamma_{k} \in G$ so that $u_{k}=$ $\gamma_{k}\left(u_{k}^{\prime}\right)$, the $\gamma_{k}$ are distinct and generate $G$,
(c) $0 \in D$,
(d) if $u_{j}, u_{k}$ meet at $p \in S$ then $p$ is a parabolic vertex; $u_{j}=u_{k}^{\prime}$ and $\gamma_{k}$ generates $G_{p}$. No other $u_{l}$ meets $G\{p\}$.
A justification of this can be found in [8]. Let $L_{G}$ be the limit set of $G$.
$L_{G}$ is a closed subset of $S$. Thus $\Omega=L_{G}^{c}\left(=S \backslash L_{G}\right)$ is open and so is a countable union of disjoint intervals, say $\Omega=\mathrm{U}_{j} \Omega_{j}$. We shall say that $\Omega_{j}$ and $\Omega_{k}$ are equivalent if there is $g \in G$ so that $g \Omega_{j}=\Omega_{k}$. It is clear, as $L_{G}$ is invariant under $G$, that either $g \Omega_{j}=\Omega_{k}$ or $g \Omega_{j} \cap \Omega_{k}=\varnothing$. Now we have

Theorem 2.1. There are only a finite number of equivalence classes of $\Omega_{j}$. There is a hyperbolic subgroup (but no larger subgroup), $G_{j}$ say, preserving $\Omega_{j}$.

This is proved in [8] but it is not stated formally.
Let $\eta_{j}, \eta_{j}^{\prime}$ be the end-points of $\Omega_{j}$ and let $\lambda_{j}$ be that arc of a circle lying in $\Delta$, joining $\eta_{j}, \eta_{j}^{\prime}$, making an internal angle $\alpha>0$ with $\Omega_{j}$. The collection of all $\lambda_{j}$ is a figure invariant under $G$ as $G$ preserves $\left\{\Omega_{j}\right\}$, angles and orientation. Let $\Lambda_{j}=\Lambda_{j}(\alpha)$ be the open region between $\lambda_{j}$ and $\Omega_{j}$; it is lens-shaped and we call it an $\alpha$-lens. As $G$ is non-elementary the $\Omega_{j}$ have distinct end-points (consider the action of $G_{j}$ on $S \backslash \Omega_{j}$ ) and the $\lambda_{j}$ cannot inter-
sect if $\alpha \leqslant \pi / 2$. In particular, if $\alpha \leqslant \pi / 2$ the $\Lambda_{j}(\alpha)$ are disjoint and are permuted by $G$. Let $K_{G}(\alpha)=\Delta \backslash \bigcup_{j} \Lambda_{j}(\alpha)$. If $\alpha \leqslant \pi / 2, K_{G}(\alpha)$ is hyperbolically convex.

The next point to note is that $D \cap K_{G}(\alpha)$ has finite (hyperbolic) area and if $G$ has no parabolic elements $D \cap K_{G}(\alpha)$ is relatively compact. This follows as the only infinite parts of $D$ are those adjacent to free sides (i.e. $\left\{\Omega_{j} \cap \bar{D}\right\}$ ) and cusps. Any free side is in some $\bar{\Lambda}_{j}$. All this is a direct consequence of the description of $D$ given above.

Suppose now that $G$ has parabolic elements and let $p_{1}, \ldots, p_{r}$ be the parabolic vertices lying on $\partial D$. We call an open disc contained in $\Delta$ and tangent to $S$ at $p$ a horocycle at $p$. Construct horocycles $C_{j}^{\prime}$ at $p_{j}$. Now refer this to $H$ with $p_{1}=\infty$. We know that the diameter of a horocycle $g\left(C_{j}^{\prime}\right)\left(g \in G, g\left(p_{j}\right)=\infty\right)$ is bounded ([9]) and so we can find $C_{1} \subseteq C_{1}^{\prime}$, a horocycle at $\infty$ so that
(a) $C_{1}$ meets no image of $\left\{C_{j}^{\prime}\right\}$ under $G$ other than $C_{1}^{\prime}$ (and hence no image of $\left\{C_{1}, C_{2}^{\prime}, \ldots\right.$, $\left.C_{r}^{\prime}\right\}$, other than $C_{1}$, under $G$ ),
(b) if $\alpha \leqslant(\pi / 2) C_{1}$ meets no $\Lambda_{k}(\alpha)$.
(b) follows as the length of $\Omega_{k}$ is clearly bounded by the translation length of $G_{\infty}$; hence the height of $\Lambda_{k}$ is bounded.

We can repeat this argument to each $j$. Thus

Proposition 2.1. There is a horocycle $C_{p}$ at each parabolic vertex $p$ and a $\pi / 2$-lens $\Lambda_{j}$ on each $\Omega_{j}$ so that
(i) $\left\{C_{p}, \Lambda_{j}\right\}$ are disjoint,
(ii) $C_{g(p)}=g\left(C_{p}\right)$,
(iii) $D \backslash\left(\cup_{p} C_{p} \cup \cup_{j} \Lambda_{j}\right)$ is relatively compact in $\Delta$,
(iv) $D$ meets only a finite number of $C_{p}$ and $\Lambda_{j}$.

If $p$ is any parabolic vertex then $p=g\left(p_{j}\right)$ for some $j$; then define $C_{p}=g\left(C_{p_{j}}\right)$. Then the construction above is sufficient to imply the proposition.

Now let $p$ be a parabolic vertex. If we conjugate $G$ to act on $H$ with $p$ at $\infty$ and $G_{p}$ generated by $z \mapsto z+1$ then $C_{p}$ becomes $\{x: \operatorname{Im}(z)>d\}(d>0)$. ( $G_{p}$ is the subgroup of $G$ fixing $p$ and we have used the same notation for $G$ and its conjugate). A fundamental domain for the action of $G$ on this is

$$
\{z: \operatorname{Im}(z)>d,|\operatorname{Re}(z)|<1 / 2\} .
$$

Such a region, or its image in $\Delta$, we call standard.

Furthermore, if we conjugate $\Omega_{j}$ to be $[0, \infty]$, ( $G$ acting on H) $\Lambda_{j}(\alpha)$ becomes $\{z: 0<\arg (z)<\alpha\}$ and $G_{j}$ is generated by an element of the form

$$
z \mapsto \exp \left(x_{j}\right) z . \quad\left(x_{j}>0\right) .
$$

Then the set

$$
\left\{z: 1<|z|<\exp \left(\varkappa_{j}\right), 0<\arg (z)<\alpha\right\}
$$

is a fundamental domain for the action of $G_{j}$ on $\Lambda_{j}(\alpha)$. This, or its image in $\Delta$, will be called standard.

Now let $A$ be a finite set so that the $\Omega_{j}(j \in A)$ are inequivalent under $G$ and $A$ is maximal subject to this restriction. Let $P$ be a maximal set of inequivalent parabolic vertices; this is also finite. It now follows from Proposition 2.1 that the following assertion holds:

Proposition 2.2. Let the notations as be above. Then there is a (possibly disconnected) fundamental domain $D$ for $G$ of the form

$$
D=K_{\alpha} \cup \bigcup_{j \in A} D_{j}(\alpha) \cup \bigcup_{p \in P} D_{p}^{*}
$$

where
(i) $K_{\alpha}$ is relatively compact in $\Delta$,
(ii) $D_{j}(\alpha)$ is a standard fundamental domain for $G_{j}$ on $\Lambda_{j}(\alpha)$, and,
(iii) $D_{p}^{*}$ is a standard fundamental domain for $G_{p}$ on $C_{p}$.

## § 3. The construction of of a measure

In this section $G$ will denote an arbitrary Fuchsian group acting on the unit disc. If $z, w \in \Delta$ we set,

$$
h(z, w)=\frac{|1-\bar{z} w|^{2}}{\left(1-|z|^{2}\right)\left(1-|w|^{2}\right)} .
$$

From this we form the Dirichlet series for $z, w \in \Delta$,

$$
\begin{equation*}
f(z, w ; s)=\sum_{g \in G} h(z, g(w))^{-s} . \tag{3.1}
\end{equation*}
$$

Let us recall some known facts concerning this series. It converges if $\operatorname{Re}(s)>1$ and diverges for $\operatorname{Re}(s) \leqslant 0$. It has an 'exponent of convergence', $\delta(G)$; it diverges if $\operatorname{Re}(s)<\delta(G)$ and converges if $\operatorname{Re}(s)>\delta(G)$. Trivially $\delta(G) \in[0,1]$. Beardon ([3], [4]) has shown that $\delta(G) \in] 0, \mathrm{I}[$ if $G$ is of the second kind and it is a classical fact that $\delta(G)=1$ if $G$ is of the
first kind. We shall in our work require Beardon's estimate $\delta(G)>0$, which is the easiest part. Beardon has also shown in [3] that if $G$ has parabolic elements $\delta(G)>\frac{1}{2}$.

The construction which we are about to perform relies on the following elementary lemma.

Lemma 3.1. Let $\sum_{n=1}^{\infty} a_{n}^{-s}$ be a Dirichlet series with exponent of convergence $e>0$. There is a positive increasing function $k$ on $[0, \infty]$ so that

$$
\sum_{n=1}^{\infty} \frac{k\left(a_{n}\right)}{a_{n}^{s}}
$$

has exponent of convergence $e$ and diverges at $s=e$. Also if $\varepsilon>0$ is given there is $y_{0}$ so that for $y>y_{0}, x>1$

$$
k(x y) \leqslant x^{\varepsilon} k(y) .
$$

Proof. By reordering the series we may suppose that $\left(a_{n}\right)$ is an increasing sequence and as $e>0, a_{n} \rightarrow \infty$. Let $\left(\varepsilon_{n}\right)$ be a decreasing sequence such that $\varepsilon_{n}>0, \varepsilon_{n} \rightarrow 0$ as $n \rightarrow \infty$. We are going to define a sequence $\left(X_{n}\right)$, with $X_{n} \rightarrow \infty$, and $k$ on the interval [ $X_{n}, X_{n+1}$ ] inductively. Take $X_{1}=1$ and $k(x)=1$ on $[0,1]$.

Suppose $k$ is defined on $\left[0, X_{n}\right]$. Then choose $X_{n+1}$ so that

$$
\begin{equation*}
\frac{k\left(X_{n}\right)}{X_{n}^{\varepsilon_{n}}} p_{p: X_{n}<a_{p} \leqslant X_{n+1}} a_{p}^{-\left(e-\varepsilon_{n}\right)} \geqslant 1 . \tag{3.2}
\end{equation*}
$$

This can always be done as $\Sigma_{p} a_{p}^{-\left(e-\varepsilon_{n}\right)}$ diverges. Now define, for $x \in\left[X_{n}, X_{n+1}\right]$,

$$
\begin{equation*}
k(x)=k\left(X_{n}\right)\left(\frac{x}{X_{n}}\right)^{\varepsilon_{n}} . \tag{3.3}
\end{equation*}
$$

With this choice $\Sigma k\left(a_{p}\right) \cdot a_{p}^{-e}$ diverges as

$$
\sum k\left(a_{p}\right) a_{p}^{-e}=\sum_{n=1}^{\infty} \sum_{p: a_{p} \in\left[X_{n}, X_{n+1]}\right]} k\left(X_{n}\right) \cdot\left(a_{p} / X_{n}\right)^{\varepsilon_{n}} \cdot a_{p}^{-e} \geqslant \sum_{n-1}^{\infty} 1
$$

by (3.2). By (3.3) $k$ is positive and increasing.
Suppose we are given $\varepsilon>0$. Choose $n$ so that $\varepsilon>\varepsilon_{n}$. Then if $x \geqslant X_{n} \log k(x)$ is, by (3.3), a piecewise linear continuous function of $\log x$ and the slope of each component is $\leqslant \varepsilon_{n}$. Hence if $y>X_{n}, x>1$

$$
\log k(x y)-\log k(y) \leqslant \varepsilon \log x
$$

which is precisely what we want.

Finally, we have to show that if $s>e$ then $\Sigma k\left(a_{p}\right) \cdot a_{p}^{-s}$ converges. Choose $\varepsilon>0$ so that $e+\varepsilon<s$. Then, as $p \rightarrow \infty k\left(a_{p}\right)=O\left(a_{p}^{\varepsilon}\right)$ by what we have just proved. The convergence of $\Sigma k\left(a_{p}\right) \cdot a_{p}^{-s}$ follows at once. This proves the lemma.

We apply the lemma to the series (3.1). Hence there is a function $k$ with the properties described in the lemma so that, for fixed $\alpha, \beta \in \Delta$,

$$
M(s)=\sum_{g \in G} k(h(\alpha, g(\beta))) \cdot h(\alpha, g(\beta))^{-s}
$$

has exponent of convergence $\delta(=\delta(G))$ and which diverges at $s=\delta$. As the terms are positive, as $s \rightarrow \delta$,

$$
M(s) \rightarrow \infty .
$$

Let $\delta_{x}$ be the Dirac measure at $x$. For $s>\delta$ we define the following measure on $\Delta$, for fixed $\alpha, \beta \in \Delta$,

$$
\begin{equation*}
\mu_{s}=M(s)^{-1} \sum_{g \in G} k(h(\alpha, g(\beta))) h(\alpha, g(\beta))^{-s} \delta_{g(\beta)} \tag{3.4}
\end{equation*}
$$

$\mu_{s}$ is a family of probability measures supported on $\bar{\Delta}$. We can apply Helly's theorem; that is, if $\left(s_{j}^{\prime}\right)$ is a sequence $s_{j}^{\prime} \rightarrow \delta$ we can extract a subsequence $\left(s_{j}\right)$ so that ( $\mu_{s_{j}}$ ) converges weakly on $\Delta$. Let $\mu$ be the limit on some such sequence.

As there are only a finite number of images of $\beta$ in any closed set in $\Delta$, and as $M\left(s_{j}\right) \rightarrow \infty$, $\mu$ is concentrated on the unit circle $S$. In fact the same reasoning shows that it is concentrated on the limit set $L_{G}$. As $\bar{\Delta}$ is compact $\mu$ is a probability measure.
$\mu$ is the measure we have been seeking. In order to investigate its properties we introduce an auxillary function.

Let us define, for $z \in \Delta, \zeta \in \Delta$, the Poisson 'kernel',

$$
P(z, \zeta)=\frac{1-|z|^{2}}{|1-\bar{z} \zeta|^{2}}
$$

Then we define

$$
\begin{equation*}
F(z)=\int P(z, \zeta)^{\delta} d \mu(\zeta) \tag{3.5}
\end{equation*}
$$

As $\mu_{s_{i}} \rightarrow \mu$ weakly we have

$$
F(z)=\lim _{\jmath \rightarrow \infty} \int P(z, \zeta)^{\delta} d \mu_{s \jmath}(\zeta)
$$

But, for $z$ fixed there is a constant $c$ so that, if $\zeta \in \Delta,\left|\delta-s_{j}\right|<10$,

$$
\left|P(z, \zeta)^{\delta}-P(z, \zeta)^{s_{i}}\right| \leqslant c(z) \cdot\left|s_{j}-\delta\right|
$$

As the $\mu_{s_{j}}$ are probability measures we have

$$
F(z)=\lim _{j \rightarrow \infty} \int P(z, \zeta)^{s_{j}} d \mu_{3 /}(\zeta)
$$

If we take $\alpha=0$ and use (3.4) this becomes

$$
\begin{equation*}
F(z)=\lim _{j \rightarrow \infty} \frac{1}{M\left(s_{j}\right)} \sum_{g \in G} \frac{k(h(0, g(\beta)))}{h(z, g(\beta))^{s_{j}}} \tag{3.6}
\end{equation*}
$$

Let $\varepsilon>0$ and $\gamma \in G$. Then as for any $a, b, c \in \Delta$,

$$
h(a, c) \leqslant 2 h(a, b) h(b, c)
$$

(see [7]) and as $k$ is increasing

$$
\begin{aligned}
F(\gamma(z)) & =\lim _{j \rightarrow \infty} \frac{1}{M\left(s_{j}\right)} \sum_{g \in G} \frac{k(h(0, g(\beta))}{h(\gamma(z), g(\beta))^{s_{j}}} \\
& =\lim _{j \rightarrow \infty} \frac{1}{M\left(s_{j}\right)} \sum_{g \in G} \frac{k(h(0, \gamma g(\beta)))}{h(z, g(\beta))^{s_{j}}} \\
& \leqslant \lim _{j \rightarrow \infty} \frac{1}{M\left(s_{j}\right)} \sum_{g \in G} \frac{k(2 h(0, \gamma(0)) \cdot h(\gamma(0) \gamma g(\beta)))}{h(z, g(\beta))^{s_{j}}} .
\end{aligned}
$$

Now there is $y_{0}$ so that if $y>y_{0}, x \geqslant 1$

$$
k(x y) \leqslant x^{\varepsilon} k(y)
$$

For all but a finite number of $g \in G$

$$
h(\gamma(0), \gamma g(\beta))=h(0, g(\beta)) \geqslant y_{0} .
$$

As $M\left(s_{j}\right) \rightarrow \infty$ as $j \rightarrow \infty$ this finite collection makes no contribution to the limit. Hence

$$
F(\gamma(z)) \leqslant 2 h(0, \gamma(0)))^{\epsilon} \lim _{j \rightarrow \infty} \frac{1}{M\left(s_{j}\right)} \sum_{g \in G} \frac{k(h(0, g(\beta)))}{h(z, g(\beta))^{s_{j}}}=(2 h(0, \gamma(0)))^{2} F(z) .
$$

This is true for any $\varepsilon>0$ and hence

$$
F(\gamma(x)) \leqslant F(z)
$$

Apply this with $\gamma(z)$ replacing $z$ and $\gamma^{-1}$ replacing $\gamma$. Thus

$$
F(\gamma(z)) \geqslant F(z)
$$

Hence

$$
F(\gamma(z))=F(z)
$$

for any $\gamma \in G$. This proves

Theorem 3.1. If $\mu$ is a probability measure constructed as above then
(i) $\mu$ is supported on $L_{G}$,
(ii) $F(z)=\int P(z, \zeta)^{\delta} d \mu(\zeta)$ is automorphic under $G$.

The following facts are worth remarking. Firstly this holds for any Fuchsian group, even infinitely generated ones. Next, $F$ is infinitely differentiable and satisfies

$$
D F=-\delta(1-\delta) F
$$

where $D$ is the Laplace-Beltrami operator. This follows as $P(z, \zeta)^{\delta}$ satisfies the same equation; compare [7]. This means that these problems are susceptible to the methods of Selberg which we shall exploit in later sections.

## § 4. The Hausdorff dimension of the limit set

In this section we use the measure of the last section to calculate the Hausdorff dimension of the limit set of a Fuchsian group of the second kind without parabolic elements. There are two reasons for this last restriction. Firstly, the consideration of parabolic elements makes the estimates much more difficult although they still can be made. Secondly the results are not quite as complete in the excluded case.

In order to explain the genesis of this problem we recall some theorems of Beardon ([2], [3], [4], [5]). In this series of papers he proves that, for a Fuchsian group of the second kind,

$$
\begin{equation*}
0<d\left(L_{G}\right) \leqslant \delta(G)<1 \tag{4.1}
\end{equation*}
$$

and the sharper result if $G$ has parabolic elements

$$
\begin{equation*}
\frac{1}{2}<d\left(L_{G}\right) \leqslant \delta(G)<1 \tag{4.2}
\end{equation*}
$$

Here we have denoted the Hausdorff dimension of a set $E$ by $d(E)$. Clearly, if $G$ is of the first kind

$$
d\left(L_{G}\right)=\delta(G)=1
$$

One would hope that in general $d\left(L_{G}\right)=\delta\left(G^{\prime}\right)$.
Such an assertion has been claimed, for Schottky groups, by Akaza, [1]. The methods of both Beardon and Akaza depend on a close, direct analysis of the action of $G$ on $S$. In this chapter we will use the measure $\mu$, constructed in the last chapter, with the methods of harmonic analysis on thin sets to solve this problem.

Theorem 4.1. If $G$ is of the second kind and has no parabolic elements then

$$
d\left(L_{G}\right)=\delta(G) .
$$

The restriction that $G$ be of the second kind is clearly inessential. If we admit parabolic elements we can only prove that $d\left(L_{G}\right)=\delta(G)$ subject to $\delta(G) \geqslant 2 / 3$, which is unsatisfactory. The reason for this is that $\mu$ is not the equilibrium measure and if $\frac{1}{2}<\delta(G)<\frac{2}{3}$ the difference is too great to give the expected dimension. In general we cannot even do as well as Beardon's estimate (4.2). We shall return to this point in the next section.

For the proof, which will occupy us to the end of the section, we require the new auxillary function

$$
\begin{equation*}
F_{s}(z)=\int P(z, \zeta)^{s} d \mu(\zeta) \quad(z \in \Delta) \tag{4.3}
\end{equation*}
$$

Note that $F(z)=F_{\delta}(z)$.
Proposition 4.1. (i) If $z \in \Delta, g \in G$ then

$$
\begin{equation*}
F_{s}(g(z)) \leqslant \mu(g)^{|s-\delta|} F_{s}(z) . \tag{4.4}
\end{equation*}
$$

(ii) There is a fundamental domain $D$ for $G$ so that $d\left(L_{G}, D\right)>0$ ( $d$ is the Euclidean distance). Fix such a $D$. There is $c_{1}=c_{1}(G, D)>0$ so that if $z \in D, s \in[0,1]$

$$
\begin{equation*}
0<F_{s}(z) \leqslant c_{1}\left(l-|z|^{2}\right)^{s} . \tag{4.5}
\end{equation*}
$$

Proof. We prove (ii) first. The existence of $D$ follows from Proposition 2.2. If $\zeta \in L_{G}$, $z \in D$ then $|\zeta-z| \geqslant d\left(L_{G}, D\right)$ and hence there is $c_{1}>1$ so that

$$
P(z, \zeta) \leqslant c_{1}\left(1-|z|^{2}\right) .
$$

On substituting into (4.3) we obtain (4.5).
Now we prove (i). By Theorem 2.1 if $g \in G$

$$
F(g(z))=F(z) .
$$

Thus as

$$
\begin{aligned}
F(g(z)) & =\int P(g(z), \zeta)^{\delta} d \mu(\zeta) \\
& =\int P(g(z), g(\zeta))^{\delta} d \mu(g(\zeta)) \\
& =\int P(z, \zeta)^{\delta}\left|g^{\prime}(\zeta)\right|^{-\delta} d \mu \circ g(\zeta)
\end{aligned}
$$

it follows that

$$
\int P(z, \zeta)^{\delta} d \mu(\zeta)=\int P(z, \zeta)^{\delta}\left|g^{\prime}(\zeta)\right|^{-\delta} d \mu \circ g(\zeta)
$$

If we write $z=r e^{i \theta}$ then $\int P(z, \zeta)^{\delta} d \mu(\zeta)$ can be expanded as a Fourier series

$$
2 \pi \sum_{n=-\infty}^{+\infty} \frac{\Gamma(\delta+|n|)}{\Gamma(\delta)|n|!} r^{|n|}\left(1-r^{2}\right)^{\delta} F\left(\delta, \delta+|n| ;|n|+1 ; r^{2}\right) \hat{\mu}(n) e^{i n \theta}
$$

(cf. $[7, \S 2]$ ). $\hat{\mu}(n)$ is the $n$th Fourier coefficient of $\mu$.
As $\delta>0$ (by (4.1)) $\int P(z, \zeta)^{\delta} d \mu(\zeta)$ determines $\hat{\mu}$ for all $n$, and hence $\mu$ also. So, if, $g \in G$,

$$
\mu \circ g=\left|g^{\prime}\right|^{\delta} \mu
$$

Observe that here $\mu$ satisfies the sort of functional equation one would get for a $\delta$ dimensional Hausdorff measure; this is the first hint that we are on the right path.

$$
\begin{aligned}
F_{s}(g(z)) & =\int P(g(z), \zeta)^{s} d \mu(\zeta) \\
& =\int P(g(z), g(\zeta))^{s} d \mu \circ g(\zeta) \\
& =\int P(z, \zeta)^{s}\left|g^{\prime}(\zeta)\right|^{\delta-s} d \mu(\zeta)
\end{aligned}
$$

(4.4) will be a result of this and the following lemma.

Lemma 4.1. For any $g \in \operatorname{Con}(\Delta), \zeta \in S$

$$
\mu(g)^{-1} \leqslant\left|g^{\prime}(\zeta)\right| \leqslant \mu(g)
$$

Proof.
Let

$$
g=\left(\begin{array}{ll}
\alpha & \beta \\
\bar{\beta} & \bar{\alpha}
\end{array}\right) .
$$

Then, one the one hand

$$
\left|g^{\prime}(\zeta)\right|^{-1}=|\bar{\beta} \zeta+\bar{\alpha}|^{2} \leqslant 2\left(|\alpha|^{2}+|\beta|^{2}\right)=\mu(g) .
$$

On the other hand, as $|\alpha|^{2}-|\beta|^{2}=1$,

$$
\left|g^{\prime}(\zeta)\right|=|\tilde{\beta} \zeta+\bar{\alpha}|^{-2} \leqslant(|\alpha|-|\beta|)^{-2}=(|\alpha|+|\beta|)^{2} \leqslant 2\left(|\alpha|^{2}+|\beta|^{2}\right)=\mu(g)
$$

This proves the lemma and with it the proposition.

The remainder of the proof rests on an identity: In order to express this we need some notation and preliminaries. Let

$$
d \sigma(z)=\frac{|d z \wedge \bar{d} \bar{z}|}{\left(1-|z|^{2}\right)^{2}}
$$

be the hyperbolic area. Let $\xi, \eta, \zeta$ be three distinct points of $S$. Then the integral

$$
\int P(z, \xi)^{s} P(z, \eta)^{t} P(z, \zeta)^{u} d \sigma(z)
$$

converges if

$$
\begin{equation*}
s+t+u>1, \quad s+t>u, \quad t+u>s, \quad u+s>t \tag{4.6}
\end{equation*}
$$

To see this we consider the integral on $H$ with $\xi=0, \eta=1, \zeta=\infty$ and an elementary manipulation shows that the integral is, up to a constant multiple,

$$
\int_{0}^{\infty} \int_{-\infty}^{+\infty} y^{s+t+u-2}\left(x^{2}+y^{2}\right)^{-s}\left((x-1)^{2}+y^{2}\right)^{-t} d x d y
$$

In $A=\left\{(x, y) \left\lvert\,(x-1)^{2}+y^{2} \leqslant \frac{1}{4}\right.\right\}$ the integral is bounded by

$$
c_{1}(s) y^{s+t+u-2} \cdot\left((a-1)^{2}+y^{2}\right)^{-t} .
$$

Thus the integral over $A$, is bounded by (on changing to polar co-ordinates centred at $(1,0)$ )

$$
c_{1}(s) \int_{0}^{1 / 2} \int_{0}^{\pi}(\sin \theta)^{s+t+u-2} r^{s+t+u-1} d r d \theta
$$

The conditions for this to converge are

$$
s+t+u>1, \quad s+t>u
$$

Consider $B=\left\{(x, y):(x-1)^{2}+y^{2}>\frac{1}{4}\right\}$. In this region, if $t \geqslant 0$, the integrand is bounded by

$$
c_{2}(t) y^{s+t+u-2}\left(x^{2}+y^{2}\right)^{-s} \min \left(1,\left(x^{2}+y^{2}\right)^{-t}\right)
$$

Converting to polars shows that this converges if

$$
s+t+u>1, \quad t+u>s, \quad u+s>t .
$$

The case $t<0$ is incompatible with (4.6).
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Thus we obtain (4.6) as the condition for convergence.

Proposition 4.2. If (4.6) holds there is a function $D(s, t, u)$ so that, if $\xi, \eta, \zeta \in S$ are unequal,

$$
\int_{\Delta} P(z, \xi)^{s} P(z, \eta)^{t} P(z, \zeta)^{u} d \sigma(z)=\frac{D(s, t, u)}{|\xi-\eta|^{s+t-u}|\eta-\zeta|^{t+u-s}|\zeta-\xi|^{u+s-t}} .
$$

This is the identity which we require.

Proof: Let

$$
R(z, \xi, \eta, \zeta)=|\xi-\eta|^{s+t-u}|\eta-\zeta|^{t+u-s}|\zeta-\xi|^{u+s-t} P(z, \xi)^{s} P(z, \eta)^{t} P(z, \zeta)^{u}
$$

If $g \in \operatorname{Con}(\Delta)$ then one checks that

$$
R(g(z), g(\xi), g(\eta), g(\zeta))=R(z, \xi, \eta, \zeta)
$$

Integrating this over $\Delta$ and using the invariance of $\sigma$ shows that the function

$$
S(\xi, \eta, \zeta)=\int R(z, \xi, \eta, \zeta) d \sigma(z)
$$

satisfies

$$
S(g(\xi), g(\eta), g(\zeta))=S(\xi, \eta, \zeta)
$$

But if $\xi, \eta, \zeta \in S$ are unequal there is $g \in \operatorname{Con}(\Delta)$ so that $\{g(\xi), g(\eta), g(\zeta)\}=\{1,-1, i\}$. This shows that

$$
S(\xi, \eta, \zeta)=S(1,-1, i) \text { or } S(-1,1, i)
$$

Conjugating shows these are equal and the conclusion follows on setting

$$
D(s, t, u)=S(1,-1, i) .
$$

Now fix $\zeta \in(\bar{D} \cap S)^{\circ}$. Observe in passing that there is a constant $c_{4}>0$ so that if $z \in D$, $g \in G \backslash\{I\}$ then

$$
\begin{equation*}
|z-g(\zeta)| \geqslant c_{4} \tag{4.7}
\end{equation*}
$$

For we can take $c_{4}$ to be the smallest Euclidean distance to any side (in $\Delta$ ) of $D$.

Suppose $s, t, u$ satisfy (4.6). Then $s, t, u>0$ and

$$
\int_{\Delta} P(z, \xi)^{s} P(z, \eta)^{t} P(z, \zeta)^{u} d \sigma(z) \geqslant 2^{-2 u} D(s, t, u)|\xi-\zeta|^{-(s+t-u)}
$$

as $|\xi-\zeta|,|\zeta-\eta| \leqslant 2$. Integrating this with respect to $d(\mu \otimes \mu)(\xi, \eta)$ and using Tonelli's theorem (the integrand being positive) to interchange the order of integration, we obtain

$$
\begin{equation*}
\int_{\Delta} F_{s}(z) \cdot F_{t}(z) P(z, \zeta)^{u} d \sigma(z) \geqslant 2^{-2 u} D(s, t, u) \int \frac{d \mu(\xi) d \mu(\eta)}{|\xi-\eta|^{s+t-u}}=2^{-2 u} D(s, t, u) I_{s+t-u}(\mu) \tag{4.8}
\end{equation*}
$$

where $I_{k}(\mu)$ is the $k$-energy of $\mu$ (cf. [6, §III]).
Now suppose $s \geqslant \delta, t \geqslant \delta$. Then

$$
\int_{\Delta} F_{s}(z) F_{t}(z) P(z, \zeta)^{u} d \sigma\left(z=\sum_{g \in G} \int_{D} F_{s}(g(z)) \cdot F_{t}(g(z)) P(g(z), \zeta)^{u} d \sigma(z) .\right.
$$

If $g \neq \mathbf{I}$, by (4.7)

$$
\begin{equation*}
P(g(z), \zeta) \leqslant c_{4}^{-2} \cdot\left(1-|g(z)|^{2}\right)=c_{4}^{-2} \cdot\left|g^{\prime}(z)\right|\left(1-|z|^{2}\right) \tag{4.9}
\end{equation*}
$$

We require a lemma.

Lemma 4.2. There is $c_{5}>0$ so that, if $z \in D$,

$$
\left|g^{\prime}(z)\right| \leqslant c_{5} \cdot \mu(g)^{-1}
$$

Proof. $\left\{g^{-1}(\infty) \mid g \in G\right\}$ lies in $\{|z|>1\}$ and accumulates on $L_{G}$. Thus, as $d\left(L_{G}, D\right)>0$, there is $c_{6}>0$ so that, if $z \in D, g \in G$

$$
\left|g^{-1}(\infty)-z\right| \geqslant c_{6} .
$$

But trivially $\left|g^{-1}(\infty)-2\right| \geqslant\left|g^{-1}(\infty)\right|-1$.
Combining these, there is $c_{\eta}>0$ so that

$$
\left|\frac{z}{g^{-1}(\infty)}-1\right| \geqslant c_{\eta} .
$$

It is easy to check that

$$
\left|g^{\prime}(z)\right|=\frac{4}{2+\mu(g)}\left|\frac{z}{g^{-1}(\infty)}-1\right|^{-2}
$$

and the lemma follows immediately.
Applying the lemma to (4.9) we obtain that, if $z \in D, g \in G \backslash\{I\}$

$$
P(g(z), \zeta) \leqslant c_{8} \mu(g)^{-1}\left(1-|z|^{2}\right) .
$$

Recall that by Proposition 4.1 (i), if $s \geqslant \delta, t \geqslant \delta$,

$$
F_{s}(g(z)) \cdot F_{t}(g(z)) \leqslant \mu(g)^{s+t-2 \delta} F_{s}(z) F_{t}(z) .
$$

Combining everything we obtain

$$
\begin{aligned}
& \int_{\Delta} F_{s}(z) F_{t}(z) P(z, \zeta)^{u} d \sigma(z) \leqslant \int_{D} F_{s}(z) F_{t}(z) P(z, \zeta)^{u} d \sigma(z) \\
& \quad+c_{8}^{u} \sum_{g \in G} \mu(g)^{s+t-u-2 s} \int_{D} F_{s}(z) F_{t}(z)\left(1-|z|^{2}\right)^{u} d \sigma(z)
\end{aligned}
$$

Let $\xi, \eta$ be two arbitrary distinct points of $L_{G^{r}}$. Then on $D F_{s}(z)$ (resp. $\left.F_{t}(z)\right)$ is. majorized by $P(z, \xi)^{s}$ (resp. $\left.P(z, \eta)^{t}\right)$ ) (Proposition 4.1 (ii)). Thus, by Proposition 4.2 both integrals are finite if (4.6) holds. The sum $\Sigma_{g \in G} \mu(g)^{s+t-u-2 \delta}$ converges if $s+t-u<\delta$. From (4.8) we have ${ }_{i}$ shown that if

$$
s+t+u>1, \quad s+t>u, \quad s+u>t, \quad t+u>s, \quad s \geqslant \delta, \quad t \geqslant \delta, \quad s+t-u<\delta
$$

then

$$
I_{s+t-u}(\mu)<\infty .
$$

We will now deduce from this that if $k$ is such that $0<k<\delta$ then

$$
\begin{equation*}
I_{l c}(\mu)<\infty . \tag{4.10}
\end{equation*}
$$

Suppose first that $\delta \geqslant \frac{1}{3}$. Let $s=\delta, t=\delta, u=2 \delta-k(>\delta)$. Then $s+t+u=4 \delta-k>3 \delta \geqslant 1$, $s+t-u=k>0$. The other inequalities are trivial and (4.10) follows.

Now suppose that $\delta \leqslant \frac{1}{3}$. Set $s=t=(1+\delta) / 4, u=(1+\delta-2 k) / 2$. Then $s+t+u=$ $(2+2 \delta-2 k) / 2>1 . s+t-u=k$ so that $0<s+t-u<k$. Also $0<(1+\delta-2 \delta) / 2<u$. Finally $s \geqslant \delta$ as $\delta \leqslant \frac{1}{3}$. This verifies the inequalities and (4.10) follows again.

We have shown that if $k<\delta$ then $I_{k}(\mu)<\infty$. As $\mu$ is supported on $L_{G}$ this shows that $d\left(L_{G}\right) \geqslant \delta(G)$, (see [6: § III, Theorem 5; § IV, Theorem 4.1]). By (4.1) $d\left(L_{G}\right) \leqslant \delta(G)$. Hence $d\left(L_{G}\right)=\delta(G)$. This completes the proof of the theorem.

## § 5. Estimates if $\boldsymbol{G}$ has parabolic elements

Now we shall indicate the modifications necessary if $G$ has parabolic elements but is still finitely generated. We shall make use of the fundamental domain described in Proposition 2.2. An immediate consequence of this proposition is the existence of a constant $c>0$ with the property that if $\zeta \in L_{G}, z \in K_{\alpha} \cup \mathrm{U}_{f \in A} D_{j}(\alpha)$ then

$$
|z-\zeta|>c .
$$

The method of proof of Proposition 4.1 now gives

Proposition 5.1. (i) If $z \in \Delta, g \in G$ then

$$
F_{s}(g(z)) \leqslant \mu(g)^{|s-\delta|} F_{s}(z) .
$$

(ii) There is $c_{1}>0$ so that if $z \in K_{\alpha}$ then

$$
0<F_{s}(z) \leqslant c_{1} .
$$

(iii) There is $c_{2}>0$ so that if $z \in D_{j}(\alpha)$ then

$$
0<F_{s}(z)<c_{2}\left(1-|z|^{2}\right)^{s} .
$$

Our main object now is to supplement this in the cusps.
Proposition 5.2. There is $c_{3}>0$ so that if $z \in D_{k}^{*}, s<2$,

$$
0<F_{s}(z)<c_{3}\left(1-|z|^{2}\right)^{28-s-1} .
$$

Proof. We shall work on $\mathbf{H}$ with $p_{k}=\infty$ to prove this proposition. On $\mathbf{H}$ the function corresponding to $h$ is

$$
h_{0}(z, w)=|z-\bar{w}|^{2} /(4 \cdot \operatorname{Im}(z) \cdot \operatorname{Im}(w)) \quad(z, w \in \mathbf{H})
$$

We shall prove the proposition first when $s=\delta$. Equation (3.6) becomes

$$
F_{\delta}(z)=\lim _{j \rightarrow \infty} M\left(s_{j}\right)^{-1} \sum_{\sigma \in G} k\left(h_{0}(i, g(\beta))\right) \cdot h_{0}(z, g(\beta))^{-s_{j}}
$$

where

$$
M(s)=\sum_{g \in G} k\left(h_{0}(i, g(\beta))\right) \cdot h_{0}(i, g(\beta))^{-s}
$$

Suppose the group fixing $\infty$ is generated by $z \mapsto z+\lambda(\lambda>0)$. We shall consider, for $s>\delta$, the series

$$
\begin{aligned}
U(z, s) & =\sum_{g \in G} k\left(h_{0}(i, g(\beta))\right) \cdot h_{0}(z, g(\beta))^{-s} \\
& =\sum_{g \in G \infty \backslash G} \sum_{m=-\infty}^{\infty} k\left(h_{0}(i+m \lambda, g(\beta))\right) \cdot h_{0}(z+m \lambda, g(\beta))^{-s} .
\end{aligned}
$$

If we fix $A>0$ there is a finite number of cosets of $G_{\infty} \backslash G$ so that, if $g$ does not belong to one of these then

$$
\operatorname{Im}(g(\beta))<A^{-1} .
$$

We will choose that representative $g$ of a coset $G_{\infty} g$ so that

$$
-\lambda / 2<\operatorname{Re}(g(\beta)) \leqslant \lambda / 2 .
$$

Also we shall suppose that

$$
-\lambda / 2<\operatorname{Re}(z) \leqslant \lambda / 2,
$$

and that, for some fixed $d>0$,

$$
\operatorname{Im}(z)>d .
$$

From the definition of $h_{0}$ it follows that, if $m \neq 0$,

$$
h_{0}(z+m \lambda, g(\beta))^{-s} \leqslant 4^{s} \operatorname{Im}(g(\beta))^{s} \operatorname{Im}(z)^{s}\left(\operatorname{Im}(z)^{2}+(|m|-1)^{2} \lambda^{2}\right)^{-s},
$$

and that,

$$
h_{0}(z, g(\beta))^{-s} \leqslant 4^{s} \operatorname{Im}(g(\beta))^{s} \operatorname{Im}(z)^{-s}
$$

Further

$$
h_{0}(i+m \lambda, g(\beta)) \leqslant \operatorname{Im}(g(\beta))^{-1}\left(\left(1+\operatorname{Im}(g(\beta))^{2}+(|m|+1)^{2} \lambda^{2}\right)\right.
$$

By $[9 ;$ p. 88$]\{\operatorname{Im}(g(\beta))\}$ is, for fixed $\beta$, bounded above. It follows that there is $c_{4}>0$, depending only on $\beta$ and so that

$$
h_{0}(i+m \lambda, g(\beta)) \leqslant c_{4} h_{0}(i, g(\beta))\left(m^{2}+1\right) .
$$

Choose $\varepsilon>0, \varepsilon \leqslant 1$. Recalling the properties of $k$ as described in section 3 it follows that we can find $A$ so that, if $g$ is not in the finite exceptional set of cosets $G_{\infty} \backslash G$

$$
k\left(h_{0}(i+m \lambda, g(\beta)) \leqslant\left(c_{4}\left(m^{2}+1\right)\right)^{\ell} k\left(h_{0}(i, g(\beta))\right) .\right.
$$

Thus, if $g$ avoids one of the exceptional set of cosets,

$$
\begin{aligned}
\sum_{m=-\infty}^{+\infty} & k\left(h_{0}(i+m \lambda, g(\beta))\right) h_{0}(z+m \lambda, g(\beta))^{-s} \\
& \leqslant c_{4}^{\varepsilon} 4^{s} h\left(h_{0}(i, g(\beta)) \operatorname{Im}(g(\beta))^{s}\left(\operatorname{Im}(z)^{-s}+\operatorname{Im}(z)^{s} R(\operatorname{Im}(z))\right),\right.
\end{aligned}
$$

where

$$
R(y)=\sum_{m \neq 0}\left(m^{2}+1\right)^{\varepsilon}\left(y^{2}+(|m|-1)^{2} \lambda^{2}\right)^{-s}
$$

It is easy to see that $R(y)$ is bounded by

$$
2 \int_{0}^{\infty}\left(x^{2}+1\right)^{\varepsilon}\left(y^{2}+(x-2)^{2}\right)^{-s} d x
$$

Thus if $s \geqslant \delta>1 / 2$ (which we may assume by [3]), $s \leqslant 5,0<\varepsilon \leqslant 1$ we see that there is $c_{5}$ depending only on $\delta, d$ so that if $y>d$

$$
R(y) \leqslant c_{5} y^{1+2 \varepsilon-2 s}
$$

Gathering together our results we obtain, for some $c_{6}$

$$
U(z, s) \leqslant Q(z, s)+c_{6} \operatorname{Im}(z)^{1+2 \varepsilon-\delta} \sum_{g \in G_{\infty} \backslash G} k\left(h_{0}(i, g(\beta))\right) \operatorname{Im}(g(\beta))^{s},
$$

where $Q(z, s)$ is the sum over the set of excluded cosets. $c_{6}$ does not depend on $\varepsilon$. Using the estimate

$$
\operatorname{Im}(g(\beta)) \leqslant c_{7} h_{0}(i, g(\beta))^{-1}
$$

where

$$
c_{7}=\sup \left(|i+g(\beta)|^{2}\right)
$$

and expanding the final sum to be over all $g \in G$ we obtain

$$
U(z, s) \leqslant Q(z, s)+c_{6} c_{7}^{s} \operatorname{Im}(z)^{1+2 \varepsilon-\delta} M(s)
$$

As $Q(z, s)$ is the sum over a finite number of cosets it converges in $s>1 / 2$. Then as

$$
F_{\delta}(z)=\lim U\left(z, s_{j}\right) / M\left(s_{j}\right),
$$

we obtain, if $\operatorname{Im}(z)>d$,

$$
F_{\delta}(z) \leqslant c_{6} c_{7}^{\delta} \operatorname{Im}(z)^{1+2 \varepsilon-\delta}
$$

The constants are independent of $\varepsilon$ and this shows that there is $c_{8}>0$ so that

$$
F_{\delta}(z) \leqslant c_{8} \operatorname{Im}(z)^{1-\delta}
$$

This proves the proposition if $s=\delta$.
Recall that on $\Delta$ we defined

$$
F_{s}(z)=\int P(z, \eta)^{s} d \mu(\eta)
$$

Assume that $p_{k}=1$. Using the map

$$
A: \mathbf{H} \rightarrow \Delta ; w \mapsto(w-i) /(w+i)
$$

we can transfer this definition to $\mathbf{H}$. Let, for $w \in \mathbf{H}, x \in \mathbf{R}$

$$
\begin{gathered}
P_{0}(w, x)=\operatorname{Im}(w) /|w-x|^{2} \\
P_{0}(w, \infty)=\operatorname{Im}(w)
\end{gathered}
$$

be the Poisson kernel on $\mathbf{H}$. Then we find

$$
F_{s}(A(w))=\int P_{0}(w, x)^{s}\left(x^{2}+1\right)^{s} d \mu(A(x))
$$

Let

$$
d \mu_{0}(x)=\left(1+x^{2}\right)^{\delta} d \mu(A(x)) .
$$

As $F_{\delta}$ is automorphic $\mu_{0}$ is periodic with period $\lambda$. As $\mu$ is a probability measure $\mu_{0}$ has finite mass on each period. By abuse of language we shall write $F_{s}(w)$ for $F_{s}(A(w))$. Let $\mu_{0}(\infty)$ be the mass at $\infty$. Then

$$
F_{s}(w)=\mu_{0}(\infty) \operatorname{Im}(w)^{s}+\sum_{m=-\infty}^{+\infty} \int_{0}^{\lambda} P_{0}(w, x+m \lambda)^{s}\left((x+m \lambda)^{2}+1\right)^{s-\delta} d \mu_{0}(x)
$$

By comparison with the corresponding integral we deduce that if $\operatorname{Im}(w)>d$ then, if $2 \geqslant s>\delta>1 / 2$ there is $c_{9}>0$ so that

$$
\sum_{m-=\infty}^{+\infty} P_{0}(w, x+m \lambda)^{s}\left((x+m \lambda)^{2}+1\right)^{s-\delta} \leqslant c_{9} \operatorname{Im}(w)^{1+s-2 \delta}
$$

But, also

$$
F_{s}(w) \geqslant \mu_{0}(\infty) \operatorname{Im}(w)^{s} .
$$

As we have already shown that the proposition is true if $s=\delta$ we deduce that

$$
\mu_{0}(\infty)=0 .
$$

Thus

$$
F_{s}(w) \leqslant \lambda c_{9} \operatorname{Im}(w)^{1+s-2 \delta}
$$

This gives the proposition as stated when we reinterpret it on $\Delta$.
Now we can proceed as in section 4. We have, if

$$
s+t+u>1, \quad s+t>u, \quad t+u>s, \quad u+s>t,
$$

that

$$
2^{-2 u} D(s, t, u) I_{s+t-u}(\mu) \leqslant \sum_{g \in G} \int_{D} F_{s}(g(z)) F_{t}(g(z)) P(g(z), \eta)^{u} d \sigma(z)
$$

where as before $\eta$ is a fixed point of $(\bar{D} \cap S)^{\circ}$. Now let

$$
W(z, \eta)=\sum_{g \in G} \mu(g)^{s-\delta+t-\delta} P(g(z), \eta)^{u}
$$

Then by Proposition 5.1 (i), if $s, t>\delta$,

$$
\begin{equation*}
2^{-2 u} D(s, t, u) I_{s+t-u}(\mu) \leqslant \int_{D} F_{s}(z) F_{t}(z) W(z ; \eta) d \sigma(z) \tag{5.1}
\end{equation*}
$$

It is convenient to specialize now. We choose

$$
s=t=\delta, \quad u>\delta>1 / 2
$$

Now we require bounds on $W(z, \eta)$.
Proposition 5.3. (i) There is a constant $c_{10}>0$ so that if

$$
z \in K_{\alpha} \cup \bigcup_{j=1}^{M} D_{j}(\alpha)
$$

then

$$
W(z, \eta) \leqslant c_{10}\left(P(z, \eta)^{u}+\Sigma_{g \in G} \mu(g)^{-u}\left(1-|z|^{2}\right)^{u}\right)
$$

(ii) There is a constant $c_{11}>0$ so that, if

$$
z \in \bigcup_{k=1}^{N} D_{k}^{*}
$$

then

$$
W(z, \eta) \leqslant c_{11}\left(\Sigma_{g \in G} \mu(g)^{-u}\left(1-|z|^{2}\right)^{u-1}\right) .
$$

Proof. (i) is essentially proved in section 4. (ii) is long and we postpone it to the end of this section.

From Proposition 5.3 the right hand side of (5.1) is finite if

$$
\begin{aligned}
& 2 \delta+u>1 \\
& 2 \delta>u>\delta \\
& u>2-2 \delta
\end{aligned}
$$

Just as at the end of section 4 we deduce from these that

$$
d\left(L_{G}\right) \geqslant \min (\delta, 4 \delta-2)
$$

As

$$
d\left(L_{G}\right) \leqslant \delta
$$

we have now proved
Theorem 5.1. If $G$ has parabolic elements and $\delta \geqslant 2 / 3$ then

$$
d\left(L_{G}\right)=\delta(G)
$$

We shall now complete the proof of Proposition 5.3.
Proof of Proposition 5.3 (ii). This proof is rather similar to that of Proposition 5.2. We prove the corresponding statement for $\mathbf{H}$.

The series to be considered becomes, by abuse of notation

$$
W(z, \eta)=\sum_{g \in G} P_{0}(g(z), \eta)^{u}
$$

where now $z \in \mathbf{H}, \eta \in \mathbf{R}$ and $\infty$ is a parabolic fixed point of $G$. Suppose that $G_{\infty}$ is generated by $z \mapsto z+\lambda(\lambda>0)$. Let $z=x+i y$. Then clearly $W$ is a periodic function of $x$ with period $\lambda$. Thus $W$ can be expanded as a Fourier series

$$
W(z, \eta)=\sum_{n-\infty}^{+\infty} w_{n}(y, \eta, u) \cdot \exp (2 \pi i n x / \lambda)
$$

An easy calculation shows that

$$
\lambda \cdot w_{n}(y, \eta, u)=\sum_{g \in G \backslash G_{\infty}} \exp \left(2 \pi i g^{-1}(\eta) / \lambda\right)\left|\left(g^{-1}\right)^{\prime}(\eta)\right|^{u} \cdot v_{n}(y, u)
$$

where

$$
v_{n}(y, u)=\pi^{u} \sqrt{y}(n / \lambda)^{u-1 / 2} K_{u-1 / 2}(2 \pi n y / \lambda)
$$

We assume that $\operatorname{Im}(z)>d$ for some fixed $d>0$. By the asymptotic expansion of the Bessel function it follows that there are constants $c_{12}, c_{13}>0$ so that if $n \neq 0$ then

$$
\left|v_{n}(y, u)\right|<c_{12} \exp \left(-c_{13} n y\right)
$$

(see [12; p. 374]). On the other hand

$$
v_{0}(y, u)=B(1 / 2, u-1 / 2) y^{1-u} .
$$

Now let

$$
T(\eta, u)=\sum_{g \in G_{\infty} \backslash G}\left|g^{\prime}(\eta)\right|^{u}
$$

Combining all our results to date shows that, for a suitable constant $c_{14}>0$

$$
\begin{equation*}
W(z, \eta) \leqslant c_{14}\left(1+\operatorname{Im}(z)^{1-u}\right) T(\eta, u) . \tag{5.2}
\end{equation*}
$$

We must now obtain bounds for $T(\eta, u)$. To do this let us consider a single coset $G_{\infty} g$. Suppose that $g$ can be represented as a matrix

$$
\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right), \quad(c>0, a d-b c=1)
$$

As $\eta$ is an ordinary point there is a constant $c_{15}>0$ so that

$$
\left|\eta-g^{-1}(\infty)\right|>c_{15}
$$

From this it follows that there is a constant $c_{16}>0$ so that

$$
\begin{equation*}
\left|g^{\prime}(\eta)\right|<c_{16}\left(c^{2}+d^{2}\right)^{-1} \tag{5.3}
\end{equation*}
$$

It is a classical fact (see [9: p. 88]) that there is a constant $c_{17}>0$ so that, for all $g \in G$

$$
\begin{equation*}
c^{2}+d^{2}>c_{17} \tag{5.4}
\end{equation*}
$$

Let

$$
\pi=\left(\begin{array}{ll}
1 & \lambda \\
0 & 1
\end{array}\right)
$$

Then an easy calculation shows

$$
\begin{equation*}
\mu\left(\pi^{n} g\right)=\left(c^{2}+d^{2}\right)(n \lambda+\tau)^{2}+\left(c^{2}+d^{2}\right)+\left(c^{2}+d^{2}\right)^{-1} \tag{5.5}
\end{equation*}
$$

where

$$
\tau=(a c+b d) /\left(c^{2}+d^{2}\right)
$$

From (5.4) and (5.5) it follows that there is a constant $c_{18}>0$ so that

$$
\sum_{n} \mu\left(\pi^{n} g\right)^{-u}>c_{18}\left(c^{2}+d^{2}\right)^{-u}
$$

Combining this with (5.3) shows that

$$
T(\eta, u)<\left(c_{16}^{u} / c_{18}\right) \sum_{g \in G} \mu(g)^{-u}
$$

This and (5.2) together constitute proposition 5.3 (ii) which is thereby proved.

## § 6. Some estimates

In section 3 we introduced the function

$$
\begin{equation*}
f(z, w ; s)=\sum_{g \in G} h(z, g(w))^{-s} \tag{6.1}
\end{equation*}
$$

This function will play an important role in the rest of this paper. In order to use it we need various estimates. Let us assume henceforth that $s \leqslant 2$; this is for convenience only, We shall continue to use the fundamental domain introduced in section 2; we shall also use the notations introduced there. Fix $\alpha \leqslant \pi / 2$. Let

$$
D_{H}=K_{\alpha} \cup \bigcup_{j=1}^{M} D_{j}(\alpha)
$$

and

$$
D_{\mathrm{C}}=\bigcup_{k=1}^{N} D_{k}^{*} .
$$

The object of this section is to prove the following proposition. The methods of proof are much the same as those of Propositions 5.2 and 5.3.

Proposition 6.1. Suppose s is such that (6.1) converges. Then there are constants $c_{1}, c_{2}, \ldots$ so that the following statements hold.
(i) $f(z, w ; s) \geqslant c_{v}\left(1-|z|^{2}\right)^{s}\left(1-|w|^{2}\right)^{s} \sum_{g \in G} \mu(g)^{-s}$.
(ii) If $z, w \in D_{H}$ then

$$
f(z, w ; s) \leqslant c_{2}\left(1-|z|^{2}\right)^{s}\left(1-|w|^{2}\right)^{s}\left(\sum_{g \in G} \mu(g)^{-s}+(1-|z||w|)^{-2 s}\right) .
$$

In particular if $w \in K_{\alpha}$

$$
f(z, w ; s) \leqslant c_{3}\left(1-|z|^{2}\right)^{s}\left(1-|w|^{2}\right)^{s} \sum_{g \in G} \mu(g)^{-s}
$$

(iii) If $z \in D_{H}, w \in D_{C}$ then

$$
f(z, w ; s) \leqslant c_{4}\left(1-|z|^{2}\right)^{s}\left(1-|w|^{2}\right)^{s-1} \sum_{g \in G} \mu(g)^{-s} .
$$

(iv) If $z, w \in D_{k}^{*}$ then

$$
f(z, w ; s) \leqslant c_{5}\left(1-|z|^{2}\right)^{s-1}\left(1-|w|^{2}\right)^{s-1}\left(\sum_{g \in G} \mu(g)^{-s}+\max \left(1-|z|^{2}, 1-|w|^{2}\right)^{1-2 s}\right)
$$

(v) If $z \in D_{j}^{*}, w \in D_{k}^{*}, j \neq k$, then

$$
f(z, w ; s) \leqslant c_{6}\left(1-|z|^{2}\right)^{-s-1}\left(1-|w|^{2}\right)^{s-1} \sum_{g \in G} \mu(g)^{-s} .
$$

Note that the inequality in $(v)$ is not symmetric. This is because it is not the best possible result. The other inequalities are best possible of their kind although this will not be proved now.

Proof. From the definition of the function $h$ and the series (6.1) for $f$ we obtain the following expansion

$$
\begin{equation*}
f(z, w ; s)=\left(1-|z|^{2}\right)^{s}\left(1-|w|^{2}\right)^{s} \sum_{g \in G_{j}}\left|g^{\prime}(w)\right|^{s}| | 1-\left.\bar{z} g(w)\right|^{2 s} . \tag{6.2}
\end{equation*}
$$

Consider a particular group element $g \in G$. Suppose that $g$ is represented by the matrix $\left(\begin{array}{ll}a & b \\ \bar{b} & \bar{a}\end{array}\right)\left(|a|^{2}-|b|^{2}=1\right)$. Then

$$
\begin{equation*}
\left|g^{\prime}(w)\right|=|\bar{b} w+\bar{a}|^{-2} \geqslant 1 /\left(2\left(|a|^{2}+|b|^{2}\right)\right)=\mu(g)^{-1} \tag{6.3}
\end{equation*}
$$

Also

$$
\begin{equation*}
|1-\bar{z} \cdot g(w)|^{2} \leqslant 4 \tag{6.4}
\end{equation*}
$$

(6.2), (6.3) and (6.4) together imply (i) which is now proved.

Suppose now that $w \in D_{H}$. As

$$
\begin{equation*}
\left|g^{\prime}(w)\right|=|a|^{-2}\left|1-z g^{-1}(0)\right|^{-2} \tag{6.5}
\end{equation*}
$$

and as $D_{H}$ is a positive distance from the limit set it follows that there is a constant $c_{7}>0$ so that

$$
\begin{equation*}
\left|g^{\prime}(w)\right| \leqslant c_{7} / \mu(g) . \tag{6.6}
\end{equation*}
$$

Suppose now that $z, w \in D_{H}$. It follows from the description of $D_{H}$ given in section 2 that there is a finite subset $R$ of $G$ and a constant $c_{8}>0$ so that for every element of $G$ except perhaps one element of $R$ we have

$$
\begin{equation*}
|g(w)-z|>c_{8} \tag{6.7}
\end{equation*}
$$

There is then a constant $c_{9}>0$ so that for all $g$ such that (6.7) is true

$$
\begin{equation*}
|1-g(w) \bar{z}|>c_{9} . \tag{6.8}
\end{equation*}
$$

On the other hand there is $c_{10}>0$ so that for the one possible exceptional case

$$
\begin{equation*}
|1-g(w) \bar{z}|>c_{10}(1-|z||w|) . \tag{6.9}
\end{equation*}
$$

Substituting (6.6), (6.8) and (6.9) into (6.2) gives (ii). It is obvious that the second assertion of (ii) is a direct consequence of the first.

It is easy to check that, if $\gamma \in \operatorname{Con}(\Delta), z, w \in \Delta$ then

$$
h(\gamma(z), \gamma(w))=h(z, w)
$$

that is, $h$ is a point-pair invariant. From this it follows that

$$
\begin{equation*}
f(z, w ; s)=f(w, z ; s) \tag{6.10}
\end{equation*}
$$

Now we shall prove (iii). Suppose that $z \in D_{H}$ and that $w \in D_{k}^{*}$ for some $k$. Then by the results of section 2 there is $c_{11}>0$ so that

$$
\begin{equation*}
|1-g(w) \bar{z}|>c_{11} . \tag{6.11}
\end{equation*}
$$

Consequently, if $\eta$ is an ordinary point in $S$ there is $c_{12}>0$ so that

$$
\begin{equation*}
\left|g^{\prime}(w)\right|\left(1-|w|^{2}\right) /|1-g(w) \bar{z}|^{2} \leqslant c_{12} P(w, \eta) \tag{6.12}
\end{equation*}
$$

The conclusion follows from Proposition 5.3, equations (6.2) and (6.12).

Now let us prove (iv). We look at the corresponding series on $\mathbf{H}$ which is (see the proof of Proposition 5.2)

$$
\begin{equation*}
\sum_{g \in G} \operatorname{Im}(z)^{s} \operatorname{Im}(g(w))^{s}| | \bar{z}-\left.g(w)\right|^{2 s} \tag{6.13}
\end{equation*}
$$

We are abusing notations here but it should cause no confusion. Let us suppose that $\infty$ is the parabolic fixed point in question and that the parabolic subgroup of $G$ fixing $\infty$ is generated by $z \mapsto z+1$. We may suppose that

$$
\begin{equation*}
D_{k}^{*}=\{z:|\operatorname{Re}(z)|<1 / 2, \quad \operatorname{Im}(z)>d\} \tag{6.14}
\end{equation*}
$$

Recall that $s \geqslant \delta(G)>1 / 2$.
The function represented by the series (6.13), which we shall also call $f(z, w ; s)$, is clearly periodic in both $z$ and $w$. Thus there is an expansion of the form

$$
\begin{equation*}
f(z, w ; s)=\sum_{m, n} c_{m, n}(\operatorname{Im}(z), \operatorname{Im}(w), s) e^{2 \pi i(m \operatorname{Re}(z)+n \operatorname{Re}(w))} \tag{6.15}
\end{equation*}
$$

The Fourier coefficients, $c_{m, n}$, can be easily calculated.

$$
\begin{align*}
c_{m, n}(x, y, s)= & \delta_{m,-n}(\operatorname{Im}(z) \operatorname{Im}(w))^{s} \operatorname{Im}(z+w)^{1-2 s} D(m \operatorname{Im}(z+w), s) \\
+ & \sum_{G_{\infty} \backslash G_{G} / G_{\infty}} \frac{e^{2 \pi t\left(m g^{-1}(\infty)+n g(\infty)\right)}}{|c(g)|^{2 s}} \\
& \left.\times(\operatorname{Im}(z) \operatorname{Im}(w))^{1-s} D\left(m \operatorname{Im}(z), n \operatorname{Im}(w), 1 / c(g)^{2} \operatorname{Im}(z) \operatorname{Im}(w)\right), s\right) . \tag{6.16}
\end{align*}
$$

In this formula $\delta$ is the Kronecker symbol. $c(g)$ is defined as follows. If $g \in G$ take a representative matrix

$$
\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right) \quad(c>0, a d-b c=1)
$$

$c(g)$ is the " $c$ " in this matrix. In the second term the summation is taken over all double cosets except the one to which the identity belongs. The two functions $D$ are defined as follows.

$$
\begin{gather*}
D(y, s)=\int_{-\infty}^{+\infty} \exp (-2 \pi i x y)\left(1+x^{2}\right)^{-s} d x  \tag{6.17}\\
D(x, y, u, s)=\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \overline{\left((p q+1+u)^{2}+(p-q)^{2}\right)^{s}} d p d q \tag{6.18}
\end{gather*}
$$

By partial integration one can show that, for any integers $M, N \geqslant 0$, as $s \geqslant \delta(G)$ there are constants $c_{13}, c_{14}>0$ ( $c_{14}$ independent of $u$ ) so that

$$
\begin{gather*}
D(y, s) \leqslant c_{13}|y|^{-N}|s|^{N}  \tag{6.19}\\
D(x, y, u, s) \leqslant c_{14}|x|^{-M}|y|^{-N}|s|^{M+N} . \tag{6.20}
\end{gather*}
$$

Combining (6.16) to (6.20) we obtain, if we also assume that $s \leqslant 2$

$$
f(z, w ; s) \leqslant c_{15}\left(\frac{\operatorname{Im}(z)^{s} \operatorname{Im}(w)^{s}}{\operatorname{Im}(z+w)^{2 s-1}}+(\operatorname{Im}(z) \operatorname{Im}(w))^{1-s} \sum|c(g)|^{-2 s}\right)
$$

The technique used in the proof Proposition 5.3 shows that for a suitable constant $c_{16}>0$

$$
\begin{equation*}
\sum_{g \in G_{\infty} \backslash \backslash / G_{\infty}} c(g)^{-2 s} \leqslant c_{16} \sum_{g \in G} \mu(g)^{-s} . \tag{6.21}
\end{equation*}
$$

Combining the last two inequalities gives the assertion of the proposition.
This only leaves (v) to be proved. We continue to work on H. We do not use the full power of our assumptions but merely assume that $z \in D_{k}^{*}$ (as defined by (6.14)) and $\operatorname{Im}(w)<d$. We require an asymptotic estimate for $\operatorname{Im}(w)$ small. Applying (6.15) to (6.20) we obtain

$$
f(z, w ; s) \leqslant c_{17}\left(\frac{\operatorname{Im}(z)^{s} \operatorname{Im}(w)^{s}}{\operatorname{Im}(z+w)^{2 s-1}}+(\operatorname{Im}(z) \operatorname{Im}(w))^{1-s} \operatorname{Im}(w)^{-2} \sum|c(g)|^{-2 s}\right)
$$

Reinterpretating this on $\mathbf{H}$ gives the assertion. This completes the proof of the proposition.

## § 7. An analysis of $F$

In this section we shall only consider groups of the second kind for which $\delta(G)>1 / 2$. In this case, by the results of sections $3,4,5, F=F_{\delta}$ is in $L^{2}(G \backslash \Delta)$ and is an eigenfunction of the Laplace-Beltrami operator. In [10] A. Selberg showed how eigenfunctions of the Laplace-Beltrami operator could also be characterised as eigenfunctions of certain integral operators. The integral operators in question are those with a kernel of the form

$$
\sum_{g \in G} k(x, g(y))
$$

where $k$ is a point-pair invariant. He gives a method of calculating the eigenvalue.
We may apply the considerations to the kernel $f(z, w ; s)$ and the eigenfunction $F(w)$. Using Propositions 4.1, 5.1, 5.2 and 6.1 the following integral converges absolutely.

$$
\int_{D} f(z, w ; s) F(w) d \sigma(w) .
$$

Carrying out the prescription of [10] we obtain

$$
\begin{equation*}
\int_{D} f(z, w ; s) F(w) d \sigma(w)=C(s) \Gamma(s-\delta) \Gamma(s-(1-\delta)) F^{\prime}(z) \tag{7.1}
\end{equation*}
$$

where

$$
C(s)=4^{s-1} \Gamma(1 / 2) \Gamma(s-1 / 2) / \Gamma(2 s-1) \Gamma(s) .
$$

All we need to know about $C(s)$ is that it is an entire function.
Equation (7.1) contains a great deal of information. For instance Proposition 6.1 gives an upper bound for $f$. Substituting this into (7.1) with $z$ fixed shows that there is a constant $c_{1}>0$ so that, for $2 \geqslant s>\delta(G)$

$$
\begin{equation*}
\sum_{g \in G} \mu(g)^{-s} \geqslant c_{1} /(s-\delta(G)) \tag{7.2}
\end{equation*}
$$

If we use Proposition 6.1 (i) we see that there is a positive function $Y$ on $\Delta \times \Delta$ so that

$$
\begin{equation*}
f(z, w ; s) \geqslant Y(z, w) /(s-\delta(G)) \tag{7.3}
\end{equation*}
$$

On the other hand Proposition 6.I (i) gives a lower bound for $f$. Substituting this into (7.1) shows that there is a constant $c_{2}>0$ so that

$$
\begin{equation*}
\sum_{g \in G} \mu(g)^{-s} \leqslant c_{2} /(s-\delta(G)) \tag{7.4}
\end{equation*}
$$

Using the rest of Proposition 6.1 shows that there is a positive function $Z$ on $\Delta \times \Delta$ so that

$$
\begin{equation*}
f(z, w ; s) \leqslant Z(z, w) /(s-\delta(G)) \tag{7,5}
\end{equation*}
$$

The inequalities (7.2) to (7.5) are rather striking. They show that, if we regard $f$ as an analytic function of $s$ it behaves almost as if it has a pole at $s=\delta(G)$. In particular in the considerations of section 3 it appears that in this case (i.e. $\delta(G)>1 / 2$ ) that the introduction of the function $k$ was unnecessary. Thus the results of section 3 can be simplified. In particular, from equation (3.6) we now see that $F$ can be defined as follows. There is a sequence $s_{j} \rightarrow \delta(G)$ so that (up to a constant factor), for some fixed $w$,

$$
\begin{equation*}
F(z)=\lim _{j}\left(s_{j}-\delta(G)\right) f\left(z, w ; s_{j}\right) \tag{7.6}
\end{equation*}
$$

The arguments that we use at each point show even that given a sequence $s_{j}^{\prime \prime} \rightarrow \delta(G)$ there is a subsequence $s_{j}^{\prime}$ on which the limit (7.6) exists.

Suppose $s_{j}^{(1)}$ and $s_{j}^{(2)}$ are two sequences decreasing to $\delta(G)$ on which the limit on the right hand side of (7.6) exists. Let $F_{1}^{\prime}$ and $F_{2}$ be the respective limit functions.

Taking the limit of (7.1) along the sequences $s_{i}^{(1)}$ and $s_{j}^{(2)}$ (this is justified by our estimates and the Lebesgue dominated convergence theorem) we obtain

$$
\begin{equation*}
\int_{D} F_{i}(z) F_{j}(z) d \sigma(z)=C(\delta) \Gamma(2 \delta-1) F_{j}(w) \tag{7.7}
\end{equation*}
$$

Recall that $w$ is a fixed point. From (7.7) it follows that

$$
\int_{D}\left(F_{1}(z)-F_{2}(z)\right)^{2} d \sigma(z)=0,
$$

and so

$$
F_{1}(z)=F_{2}(z) \quad \text { (a.e.). }
$$

By equation (3.5) $F_{1}$ and $F_{2}$ are real-analytic functions of $z$. Hence

$$
F_{1}(z)=F_{2}(z) .
$$

We have now shown that given any sequence $s_{j}$ decreasing to $\delta(G)$ there is a subsequence $s_{j}^{\prime}$ so that

$$
\lim _{j}\left(s_{i}^{\prime}-\delta(G)\right) f\left(z, w ; s_{j}^{\prime}\right)=F_{1}(z) .
$$

From this it follows that

$$
\lim _{s \rightarrow \delta(G)}(s-\delta(G)) f(z, w ; s),
$$

exists and is equal to $F_{1}(z)$. This is true for each fixed value of $w$. Thus there is a function $G(z, w)$ on $\Delta \times \Delta$ so that

$$
\begin{equation*}
\lim (s-\delta(G)) f(z, w ; s)=G(z, w) \tag{7.8}
\end{equation*}
$$

Clearly $G$ is an eigenfunction of the Laplace-Beltrami operator in both variables. Indeed, from (7.8) we have

$$
\begin{equation*}
G(z, w)=G(w, z) . \tag{7.9}
\end{equation*}
$$

Also $G$ is automorphic under $G$ in both variables. Let us state these results formally.
Theorem 7.1. There is a strictly positive function $G(z, w)$ on $\Delta \times \Delta$ so that

$$
\lim _{s \rightarrow \delta(G)}(s-\delta(G)) f(z, w ; s)=G(z, w) .
$$

Corollary. If $G$ is of the second kind then

$$
\delta(G)<1
$$

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Proof. If $\delta(G)=1$ then it would follow that $f(z, w ; s)$ diverged at 1 . But it is a classical fact (see [9: p. 178]) that $f$ converges at $s=1$. Hence $\delta(G)<1$.

This corollary is due to A. F. Beardon (see [4]). His method, which involves a close analysis of the action of $G$ on $\Delta$, is completely different from our own. The assertion of the theorem, that the series diverges at its exponent of convergence is a much finer result than any classical method is capable of producing.

We shall resume the investigation into the nature of $F$. On passing to the limit in (7.1) we now obtain

$$
\begin{equation*}
\int_{D} G(z, w) F(w) d \sigma(w)=C(\delta) \Gamma(2 \delta-1) F(z) \tag{7.10}
\end{equation*}
$$

From Proposition 6.1 we see that there are constants $c_{3}, c_{4}$ so that

$$
\begin{gather*}
G(z, z) \leqslant c_{3}\left(1-|z|^{2}\right)^{2 \delta(G)}, \quad\left(z \in D_{H}\right)  \tag{7.11}\\
G(z, z) \leqslant c_{4}\left(1-|z|^{2}\right)^{2 \delta(G)-2}, \quad\left(z \in D_{C}\right) \tag{7.12}
\end{gather*}
$$

From these it follows

$$
\begin{equation*}
\int_{D} G(z, z) d \sigma(z)<\infty . \tag{7.13}
\end{equation*}
$$

Now let $E_{\delta}$ be the Hilbert subspace of $L^{2}(G \backslash \Delta)$ of eigenfunctions of the Laplace-Beltrami operators with eigenvalue $-\delta(G)(1-\delta(G))$. As $G(z, w)$ as a function of $z$ is in $E_{\delta}$ it follows that the map

$$
\begin{equation*}
f \mapsto \int_{D} G(\cdot, w) f(w) d \sigma(w) ; L^{2}(G \backslash \Delta) \rightarrow L^{2}(G \backslash \Delta) \tag{7.14}
\end{equation*}
$$

maps $L^{2}(G \backslash \Delta)$ into $E_{\delta}$. By (7.10) this map on $E_{\delta}$ is a constant multiple of the identity. By (7.13) it has finite trace. Thus $E_{\delta}$ is a finite dimensional space.

Let us now go back to the methods of section 3. Recall that there we introduced the functions $F$ only after having found a certain measure. Now we may redefine this measure and investigate it somewhat more closely.

Let $w, v$ be fixed points in $\Delta$. Then define

$$
\begin{equation*}
\mu_{s, w, v}=f(w, v ; s)^{-1} \sum_{g \in G} h(w, g(v))^{-s} \delta_{g(v)} . \tag{7.15}
\end{equation*}
$$

We find that if $s_{j}$ is a sequence decreasing to $\delta(G)$ so that $\mu_{s, 0, v}$ converges weakly to $\mu_{v}^{*}$ say then

$$
\begin{equation*}
\int P(z, \eta)^{\delta} d \mu_{v}^{*}(\eta)=\lim _{j} f\left(z, v ; s_{j}\right) / f\left(0, v ; s_{j}\right)=G(z, v) / G(0, v) \tag{7.16}
\end{equation*}
$$

The function on the right hand side here does not depend on the sequence $s_{j}$. As we have already remarked during the proof of Proposition $4.1 \mu_{v}^{*}$ is completely determined by the left hand side, and hence the right hand side of (7.16). Thus all such weak limits of $\mu_{s, 0, v}$ are the same. On the other hand for any sequence $s_{j}^{\prime}$ decreasing to $\delta(G)$ there is, by Helly's theorem, a subsequence $s_{j}$ along which $\mu_{s, 0, v}$ converges weakly. Consequently $\mu_{s, 0, v}$ converges weakly to $\mu_{v}^{*}$ as $s$ decreases to $\delta(G)$.

Lemma 7.1. Let $v_{1}, v_{2} \in \Delta$. Then $\mu_{v_{1}}^{*}$ is absolutely continuous with respect to $\mu_{v_{2}}^{*}$.
Proof. Let $\varphi$ be a continuous function on $\bar{\Delta}$. Then $\varphi$ is uniformly continuous. Choose $\varepsilon>0$. There is $r>0$ so that, if $\left|z_{1}-z_{2}\right|<r$ then $\left|\varphi\left(z_{1}\right)-\varphi\left(z_{2}\right)\right|<\varepsilon$.

Now we find a finite subset $T$ of $G$ so that, if $g \notin G \backslash T$,

$$
\left|g\left(v_{1}\right)-g\left(v_{2}\right)\right|<r .
$$

Recall the inequality

$$
h\left(z_{1}, z_{2}\right) h\left(z_{2}, z_{3}\right) \geqslant h\left(z_{1}, z_{3}\right) .
$$

As

$$
\int \varphi(z) d \mu_{s, 0, v}(z)=f(0, v ; s)^{-1} \sum_{g \in G} h(0, g(v))^{-s} \varphi(g(v)),
$$

we see that

$$
\begin{aligned}
\left|\int \varphi(z) d \mu_{s, 0, v_{1}}(z)\right|= & \left|f\left(0, v_{1} ; s\right)^{-1} \sum_{g \in G} h\left(0, g\left(v_{1}\right)\right)^{-s} \varphi\left(g\left(v_{1}\right)\right)\right| \\
\leqslant & f\left(0, v_{1} ; s\right)^{-1} \sum_{g \in G} h\left(0, g\left(v_{1}\right)\right)^{-s}\left|\varphi\left(g\left(v_{1}\right)\right)-\varphi\left(g\left(v_{2}\right)\right)\right| \\
& +f\left(0, v_{1} ; s\right)^{-1} h\left(v_{1}, v_{2}\right)^{s} \sum_{g \in G} h\left(0, g\left(v_{2}\right)\right)^{-s}\left|\varphi\left(g\left(v_{2}\right)\right)\right|
\end{aligned}
$$

Let us examine the first term as $s \rightarrow \delta(G)$. As $f\left(0, v_{1} ; s\right) \rightarrow \infty$ as $s \rightarrow \delta(G)$ it follows that the first term is

$$
\begin{aligned}
& f\left(0, v_{1} ; s\right)^{-1} \sum_{g \in G \backslash T} h\left(0, g\left(v_{1}\right)\right)^{-1}\left|\varphi\left(g\left(v_{1}\right)\right)-\varphi\left(g\left(v_{2}\right)\right)\right|+o(1) \\
& \quad \leqslant \varepsilon f\left(0, v_{1} ; s\right)^{-1} \sum_{g \in G \backslash T} h\left(0, g\left(v_{1}\right)\right)^{-s}+o(1)=\varepsilon+o(1) .
\end{aligned}
$$

Let $\varphi$ be a positive function. Then from the inequality above we obtain as $s \rightarrow \delta(G)$

$$
\int \varphi(z) d \mu_{v_{1}}^{*}(z) \leqslant h\left(v_{1}, v_{2}\right)^{\delta}\left(G\left(0, v_{2}\right) / G\left(0, v_{1}\right)\right) \int \varphi(z) d \mu_{v_{2}}^{*}(z)+\varepsilon .
$$

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$\varepsilon$ was an arbitrary positive number. So we obtain the above inequality with $\varepsilon=0$. This proves the lemma.

From the lemma and the Radon-Nikodym theorem there is $q \in L^{1}\left(\mu_{v_{1}}^{*}\right)=L^{1}\left(\mu_{v_{2}}^{*}\right)$ so that $\mu_{v_{1}}^{*}=q \cdot \mu_{v_{2}}^{*}$. If $g \in G$ then, as we showed during the proof of Proposition 4.1

$$
\mu_{v_{j}}^{*} \circ g=\left|g^{\prime}\right|^{\delta} \mu_{v_{i}}^{*}
$$

Hence

$$
q \circ g=q
$$

Suppose $A_{1}, \ldots, A_{K}$ are disjoint subsets of $L_{G}$ so that

$$
\mu_{v_{2}}^{*}\left(A_{j}\right)>0 \quad(1 \leqslant j \leqslant K)
$$

and so that the $A_{j}$ are invariant under $G$. Let $\mu_{j}$ be the restriction of $\mu_{v_{1}}^{*}$ to $A_{j}$. Then, as $A_{j}$ is invariant under $G$, for any $g \in G$ we have

$$
\mu_{j} \circ g=\left|g^{\prime}\right|^{\delta} \mu_{j}
$$

Thus

$$
\begin{equation*}
f_{j}(z)=\int P(z, \eta)^{\delta} d \mu_{i}(\eta) \tag{7.17}
\end{equation*}
$$

is a member of $E_{\delta}$. On the other hand, the measures $\mu_{j}$ are linearly independent as the sets $A_{j}$ are disjoint. So, as the integral on the right hand side of (7.17) determines $\mu_{j}$ completely it follows that the $f_{\text {}}$ are linearly independent.

We have shown, however, that $E_{\delta}$ is a finite dimensional space. Thus there is a maximal finite set $B_{1}, \ldots, B_{L}$ of disjoint $G$-invariant subsets of positive $\mu_{v_{1}}^{*}$-measure (at least up to sets of zero $\mu_{v_{1}}^{*}$-mass). Clearly every $G$-invariant subset of $L_{G}$ is, up to a set of $\mu_{v_{1}}^{*}$ measure zero, a union of some of the $B_{j}$.

Let $m_{j}(\mathrm{l} \leqslant j \leqslant L)$ be the restriction of $\mu_{v_{1}}^{*}$ to $B_{j}$. As $B_{1}, \ldots, B_{L}$ is a maximal set we have, up to a set of $\mu_{v_{1}}^{*}$ measure zero,

$$
\begin{equation*}
L_{G}=\bigcup_{j=1}^{L} B_{j} . \tag{7.18}
\end{equation*}
$$

From this it follows at once that

$$
\begin{equation*}
\mu_{v_{1}}^{*}=\sum_{j=1}^{L} m_{j} . \tag{7.19}
\end{equation*}
$$

Recall that we had shown that

$$
\mu_{v_{1}}^{*}=q \mu_{v_{z}}^{*} .
$$

Likewise there is a function $q_{0}$ so that

$$
\begin{equation*}
\mu_{v_{2}}^{*}=q_{0} \mu_{v_{1}}^{*} . \tag{7.20}
\end{equation*}
$$

Clearly $\mu_{v_{1}}^{*}$-almost everywhere

$$
q \cdot q_{0}=1
$$

On the other hand both $q$ and $q_{0}$ are $G$-invariant. Let $I_{j}$ be the characteristic function of $B_{j}$. Then there are constants $a(f), a_{0}(j)(1 \leqslant j \leqslant L)$ so that

$$
q=\sum_{j=1}^{L} a(j) I_{j}
$$

and

$$
q_{0}=\sum_{j=1}^{L} a_{0}(j) I_{j}
$$

As $q \cdot q_{0}=1$ we have that

$$
a(j) a_{0}(j)=1
$$

As $\mu_{v_{j}}^{*}$ is a probability measure we have that the $a(j), a_{0}(j)$ are finite. As both $\mu_{v_{1}}^{*}$ and $\mu_{v_{2}}^{*}$ are positive measures $a(j)$ and $a_{0}(j)$ are both positive. We conclude that

$$
\begin{equation*}
0<a_{0}(j)<\infty . \tag{7.21}
\end{equation*}
$$

From (7.18), (7.19) and (7.20) we conclude

$$
\begin{equation*}
\mu_{v_{\mathrm{a}}}^{*}=\sum_{j=1}^{L} a_{0}(j) m_{j} . \tag{7.22}
\end{equation*}
$$

We can summarise what we have just shown in the following way. For $1 \leqslant j \leqslant L$ define

$$
\begin{equation*}
F_{j}(z)=\int P(z, \eta)^{\delta} d m_{j}(\eta) \tag{7.23}
\end{equation*}
$$

This is a strictly positive element of $E_{\delta}$. Let $v \in \Delta$. We have then shown that there are $L$ strictly positive functions $\varphi_{j}(v)$ so that, in view of (7.16),

$$
\begin{equation*}
G(z, v)=\sum_{j=1}^{L} F_{j}(z) \varphi_{j}(v) . \tag{7.24}
\end{equation*}
$$

We shall now deduce that $L=1$. Suppose that $L>1$. By (7.10), applied to $F_{j}(z)$ we see that

$$
\begin{equation*}
\int_{D} G(z, v) F_{j}(v) d \sigma(v)=C(\delta) \Gamma(2 \delta-1) F_{j}(z) \tag{7.25}
\end{equation*}
$$

If we let

$$
\begin{equation*}
b_{k}=\int_{D} F_{f}(v) \varphi_{k}(v) d \sigma(v) \tag{7.26}
\end{equation*}
$$

then from (7.24) and (7.25) we find

$$
\begin{equation*}
\sum_{k=1}^{L} b_{k} F_{k}(z)=C(\delta) \Gamma(2 \delta-1) F_{j}(z) \tag{7.27}
\end{equation*}
$$

As $F_{j}$ and $\varphi_{k}$ are both strictly positive, from (7.26),

$$
\begin{equation*}
b_{k}>0 \tag{7.28}
\end{equation*}
$$

(7.28) implies that (7.27) is a non-trivial linear relation between the $F_{j}$. This is impossible for suppose we had a relation of the form

$$
\sum_{k=1}^{L} c_{k} \cdot F_{\kappa c}(z)=0
$$

Let

$$
M=\sum_{k=1}^{L} c_{k} \cdot m_{k}
$$

which is a signed measure so that

$$
\int P(z, \eta)^{\delta} d M(\eta)=0
$$

During the proof of Proposition 4.1 we remarked that this implied that $M=0$. From the definition of the $m_{k}$ this shows that, for all $k$,

$$
c_{k}=0
$$

Thus $L=1$. This completes our investigations; the conclusions are expressed in the next theorem.

Theorem 7.2. Let $G$ be a finitely generated Fuchsian group of the second kind with $\delta(G)>1 / 2$. Then there is one, and only one, probability measure $\mu$ supported on $L_{G}$ so that, for every $g \in G$,

$$
\mu \circ g=\left|g^{\prime}\right|^{\delta} \mu
$$

The space $E_{\delta}$ of square-integrable eigenfunctions of the Laplace-Beltrami operator with eigenvalue $-\delta(G)(\mathrm{I}-\delta(G))$ is one-dimensional and is generated by

$$
F(z)=\int P(z, \eta)^{\delta(G)} d \mu(\eta)
$$

There is a constant $\gamma$ so that if $G(z, w)$ is the function appearing in Theorem 7.1

$$
G(z, w)=\gamma F(z) F(w)
$$

If $Q$ is a subset of $L_{G}$ which is G-invariant up to a set of $\mu$-measure zero then

$$
\mu(Q)=0 \text { or } 1
$$

We shall conclude this paper by a few general remarks. If we carry out the construction of $\mu$ in the case of a finitely generated group of the first kind we see that it is the ordinary Lebesgue measure on $S$. The eigenfunction is a constant function. The fact that as $s \rightarrow 1(s-1) f(z, w ; s)$ converges to a constant corresponds to Tsuji's "fundamental theorem" ([11]). The classification of G-invariant sets corresponds to the "easy" ergodic theorem ([9: p. 321]). This paper gives a new proof of these classical theorems.

The results of this paper admit of extensions in two different ways. Firstly, it is very desirable that the relation between the measure that we have constructed and the distribution of orbits of points under the group be clarified and sharpened. It appears that even for groups of the second kind it would be profitable to study the spectral decomposition of the Laplace operator. I hope to deal with this topic later but the reader should consult, on this point, the very interesting papers of Elstrodt [7].

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