THE LIMITING DISTRIBUTION OF THE SERIAL CORRELATION COEFFICIENT IN THE EXPLOSIVE CASE

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1. Introduction and summary. Several authors have studied the discrete stochastic process (x_t) in which the x's are related by the stochastic difference equation

$$x_t = \alpha x_{t-1} + u_t, \qquad t = 1, 2, \cdots, T, \tag{1.1}$$

where the u's are unobservable disturbances, independent and identically distributed with mean zero and variance σ^2, and α is an unknown parameter.

The statistical problem is to find some appropriate function of the x's as an estimator for α and examine its properties.

We may rewrite (1.1) as

$$x_t = u_t + \alpha u_{t-1} + \cdots + \alpha^{t-1} u_1 + \alpha^t x_0 . \tag{1.2}$$

From (1.2) we see that the distribution of the successive x's is not uniquely determined by that of the u's alone. The distribution of x_0 must also be specified. Three distributions which have been proposed for x_0 are the following:

(A) $x_0 =$ a constant (with probability one),

(B) x_0 is normally distributed with mean zero and variance $\sigma^2/(1 - \alpha^2)$,

(C) $x_0 = x_T$.

Distribution (B) is perhaps the most appealing from a physical point of view, since if x_0 has this distribution and if the u's are normally distributed, then the process is stationary (e.g., see Koopmans [4]). However, there are several analytic difficulties which arise in the statistical treatment of this process. Distribution (C), the so-called circular distribution, has been proposed as an approximation to (B) and is much easier to analyze (e.g., see Dixon [2]). Distribution (A) has been studied extensively by Mann and Wald [5]. An interesting feature of distribution (A) is that α may assume any finite value, while for distributions (B) and (C) α must be between -1 and 1. From (1.2) we see that a process satisfying (1.1) and (A) has

$$\text{var}\,(x_t) = \sigma^2(1 + \alpha^2 + \cdots + \alpha^{2(t-1)}). \tag{1.3}$$

If $|\alpha| \geqq 1$, $\lim_{t=\infty} \text{var}\,(x_t) = \infty$ and the process is said to be "explosive."

Mann and Wald [5] considered only the case $|\alpha| < 1$. They showed that the least squares estimator for α is the serial correlation coefficient[1]

$$\hat{\alpha} = \frac{\sum x_t x_{t-1}}{\sum x_{t-1}^2} \tag{1.4}$$

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[1] In this paper, the summation sign $\sum$ will always refer to summation from $t = 1$ to $t = T$.

and that (for $|\alpha| < 1$) this estimator is asymptotically normally distributed with mean α and variance $(1 - \alpha^2)/T$. Rubin [6] showed that the estimator $\hat{\alpha}$ is consistent (i.e., plim $\hat{\alpha} = \alpha$) for all α.

In this paper the asymptotic distribution of $\hat{\alpha}$ will be studied under the assumption that the u's are normally distributed. For $|\alpha| > 1$, it is shown that the asymptotic distribution of α is the Cauchy distribution. For $|\alpha| = 1$, a moment generating function is found, the inversion of which will yield the asymptotic distribution.

2. The distribution of $\hat{\alpha} - \alpha$. From equation (1.1) and condition (A) the joint distribution of

$$x' = (x_1, x_2, \cdots, x_T)$$

is easily found to be

$$f(x') = \frac{\exp [(-1/2\sigma^2) \sum (x_t - \alpha x_{t-1})^2]}{(2\pi\sigma^2)^{T/2}}. \tag{2.1}$$

The maximum likelihood estimator for α is then the least-squares estimator $\hat{\alpha}$. Since we shall be considering only the distribution of

$$\hat{\alpha} = \frac{\sum x_t x_{t-1}}{\sum x_{t-1}^2},$$

we may, without loss of generality, take $\sigma^2 = 1$. For the time being we shall also set $x_0 = 0$.

We may now write (2.1) in matrix form as follows:

$$f(x') = \frac{\exp(-\frac{1}{2}x'Px)}{(2\pi)^{T/2}}, \tag{2.2}$$

where P is the $T \times T$ matrix

$$P = \begin{bmatrix} 1+\alpha^2 & -\alpha & 0 & 0 & & & & \\ -\alpha & 1+\alpha^2 & -\alpha & 0 & & & & \\ 0 & -\alpha & 1+\alpha^2 & -\alpha & & & & \\ & & & & \cdots & & & \\ & & & & & -\alpha & 1+\alpha^2 & -\alpha \\ & & & & & 0 & -\alpha & 1 \end{bmatrix}. \tag{2.3}$$

Since $\hat{\alpha}$ is a consistent estimator for α, we shall consider the distribution of $\hat{\alpha} - \alpha$ rather than that of α alone. We have

$$\begin{aligned} \hat{\alpha} - \alpha &= \frac{\sum x_t x_{t-1}}{\sum x_{t-1}^2} - \alpha \\ &= \frac{\sum x_t x_{t-1} - \alpha \sum x_{t-1}^2}{\sum x_{t-1}^2} \\ &= \frac{x'Ax}{x'Bx}, \end{aligned} \tag{2.4}$$

where A and B are the $T \times T$ matrices

$$A = -\tfrac{1}{2}\begin{bmatrix} 2\alpha & -1 & 0 & & & & \\ -1 & 2\alpha & -1 & & & & \\ 0 & -1 & 2\alpha & & & & \\ & & & \cdots & & & \\ & & & & -1 & 2\alpha & -1 \\ & & & & 0 & -1 & 0 \end{bmatrix},$$

(2.5)

$$B = \begin{bmatrix} 1 & 0 & 0 & & & & \\ 0 & 1 & 0 & & & & \\ 0 & 0 & 1 & & & & \\ & & & \cdots & & & \\ & & & & 0 & 1 & 0 \\ & & & & 0 & 0 & 0 \end{bmatrix}.$$

Let $m(u, v)$ be the joint moment generating function of $x'Ax$ and $x'Bx$. We have

(2.6)
$$\begin{aligned} m(u, v) &= E\,(\exp\{x'Axu + x'Bxv\}) \\ &= (2\pi)^{-T/2} \int \exp\,(x'Axu + x'Bxv - x'Px/2)\,dx \\ &= (2\pi)^{-T/2} \int \exp\,(-x'Dx/2)\,dx, \end{aligned}$$

where D is the $T \times T$ matrix

(2.7)
$$D = P - 2Au - 2Bv = \begin{bmatrix} p & q & 0 & & & & \\ q & p & q & & & & \\ 0 & q & p & & & & \\ & & & \cdots & & & \\ & & & & q & p & q \\ & & & & 0 & q & 1 \end{bmatrix},$$

$$p = 1 + \alpha^2 - 2v + 2\alpha u, \qquad q = -(\alpha + u).$$

By a well-known integration formula (Cramer [1], Eq. (11.12.2.), p. 120) we have

(2.8)
$$m(u, v) = (2\pi)^{-T/2} \int \exp\left(-\frac{x'Dx}{2}\right) dx = (\det D)^{-\frac{1}{2}}.$$

If we now write $\det D = D(T)$, we note that expanding (2.7) by the elements of the first column gives the difference equation

(2.9)
$$D(T) = pD(T-1) - q^2 D(T-2).$$

From the initial values $D(1) = 1$ and $D(2) = p - q^2$, we obtain

(2.10)
$$D(T) = \frac{1-s}{r-s}\,r^T + \frac{1-r}{s-r}\,s^T,$$

where r and s are roots of the equation $x^2 - px + q^2 = 0$, that is

$$r, s = (p \pm \sqrt{p^2 - 4q^2})/2. \tag{2.11}$$

The inversion of $m(u, v) = D(T)^{-\frac{1}{2}}$ seems out of the question for finite T. The inversion of a certain limiting form of $m(u, v)$ will be discussed in Section 4.

3. The standardizing function $g(T)$. Since $\hat{\alpha}$ is consistent the limiting distribution of $\hat{\alpha} - \alpha$ is the unitary distribution. The first problem then is to find some function of T, say $g(T)$, such that the limiting distribution of $g(T)$ $(\hat{\alpha} - \alpha)$ is non-degenerate. We note that the results of Mann and Wald (Eq. (1.4) above) give $g(T) = (T/\{1 - \alpha^2\})^{\frac{1}{2}}$ for $|\alpha| < 1$, since $(T/\{1 - \alpha^2\})^{\frac{1}{2}}$ $(\hat{\alpha} - \alpha)$ has a limiting normal distribution. The function $g^2(T)$ corresponds roughly to the reciprocal of the asymptotic variance of $(\hat{\alpha} - \alpha)$, or in Fisher's terminology the "information" on α supplied by the sample.

The "information" on α may be obtained explicitly as follows. Let f be the density function (2.1) with $x_0 = 0$ and $\sigma^2 = 1$. The "information," say $I(\alpha)$, is then defined as

$$\begin{aligned} I(\alpha) &= E\left(-\frac{d^2 \log f}{d\alpha^2}\right) \\ &= E\left(\sum x_{t-1}^2\right) \\ &= \frac{1}{1 - \alpha^2}\left(T - \frac{1 - \alpha^{2T}}{1 - \alpha^2}\right) \quad \text{if} \quad |\alpha| \neq 1 \\ &= \frac{T(T - 1)}{2} \quad \text{if} \quad |\alpha| = 1. \end{aligned} \tag{3.1}$$

If the x's had been independent random variables, then $I(\alpha)$ $(\hat{\alpha} - \alpha)$ would be asymptotically N (0, 1) (Cramer [1], Eq.(33.3.4), p. 503). This, of course, is not the case. This approach does, however, give an heuristic method for finding a function $g(T)$ such that $g(T)$ $(\hat{\alpha} - \alpha)$ has a non-degenerate limiting distribution.

We might now take $g(T) = [I(\alpha)]^{\frac{1}{2}}$; however, it will simplify the computations to use slight modifications which are asymptotically equivalent to $[I(\alpha)]^{\frac{1}{2}}$. We choose

$$\begin{aligned} g(T) &= \sqrt{\frac{T}{1 - \alpha^2}} \qquad \text{for } |\alpha| < 1, \\ &= \frac{T}{\sqrt{2}} \qquad \text{for } |\alpha| = 1, \\ &= \frac{|\alpha|^T}{\alpha^2 - 1} \qquad \text{for } |\alpha| > 1. \end{aligned} \tag{3.2}$$

In the next section it will be shown that $g(T)$ $(\hat{\alpha} - \alpha)$ has a non-degenerate distribution for all values of α.

4. The limiting distribution of $g(T)$ $(\hat{\alpha} - \alpha)$. We shall first consider the joint distribution of $x'Ax/g(T)$ and $x'Bx/g^2(T)$. Let $M(U, V)$ be the joint moment generating function of these two statistics. We then have

$$M(U, V) = E[\exp x'AxU/g(T) + x'BxV/g^2(T)] = m[U/g(T), V/g^2(T)], \tag{4.1}$$

where $m(u, v)$ is the joint moment generating function (2.6).

From (2.10) and (2.11) with $g = g(T)$, $u = U/g$ and $v = V/g^2$, we have

$$M(U, V) = D(T)^{-\frac{1}{2}} = \frac{1 - s}{r - s} r^T + \frac{1 - r}{s - r} s^T, \tag{4.2}$$

$$r, s = \tfrac{1}{2}[1 + \alpha^2 + 2\alpha U/g - 2V/g^2 \pm \{(1 - \alpha^2)^2 - 4\alpha(1 - \alpha^2)U/g - 4(1 - \alpha^2)U^2/g^2 - 4(1 + \alpha^2)V/g^2 - 8\alpha UV/g^3 + 4V^2/g^4\}^{1/2}]. \tag{4.3}$$

For sufficiently large T and $|\alpha| \neq 1$, we may factor $(1 - \alpha^2)$ out of the radical in (4.3) and expand the remaining radical by the binomial theorem. We then have, up to terms of order $O(g^{-3})$

$$r, s = \frac{1}{2}\left[1 + \alpha^2 + 2\alpha U/g - 2V/g^2 \pm \left\{1 - \alpha^2 - 2\alpha U/g - \frac{2(1 + \alpha^2)V}{(1 - \alpha^2)g^2} - \frac{2U^2}{(1 - \alpha^2)g^2} + O(g^{-3})\right\}\right]. \tag{4.4}$$

Taking r with the plus sign and s with the minus sign we have

$$r = 1 - \frac{U^2 + 2V}{(1 - \alpha^2)g^2} + O(g^{-3}),$$
$$s = \alpha^2 + 2\alpha U/g + \frac{U^2 + 2\alpha^2 V}{(1 - \alpha^2)g^2} + O(g^{-3}). \tag{4.5}$$

Substituting the appropriate values of $g(T)$ from (3.2), we have

$$r = 1 - \frac{U^2 + 2V}{T} + O(T^{-\frac{3}{2}}) \qquad \text{for } |\alpha| < 1,$$
$$s = \alpha^2 + 2\alpha\sqrt{\frac{1 - \alpha^2}{T}}\, U + \frac{U^2 + 2\alpha^2 V}{T} + O(T^{-\frac{3}{2}}). \tag{4.6}$$

$$r = 1 + \frac{(U^2 + 2V)(\alpha^2 - 1)}{\alpha^{2T}} + O(|\alpha|^{-3T}) \qquad \text{for } |\alpha| > 1,$$
$$s = \alpha^2 + \frac{2\alpha U(\alpha^2 - 1)}{|\alpha|^T} - \frac{(U^2 + 2\alpha^2 V)(\alpha^2 - 1)}{\alpha^{2T}} + O(|\alpha|^{-3T}). \tag{4.7}$$

If $|\alpha| = 1$, the expansion in (4.4) is not valid; however, from (4.3), we have

$$(4.8)\qquad \begin{aligned} r &= 1 + \frac{\sqrt{2}\alpha U}{T} + \frac{2i\sqrt{V}}{T} + O(T^{-2}) \qquad \text{for } |\alpha| = 1, \\ s &= 1 + \frac{\sqrt{2}\alpha U}{T} - \frac{2i\sqrt{V}}{T} + O(T^{-2}). \end{aligned}$$

Substituting these results in (4.2), we have

$$(4.9)\qquad \begin{aligned} \lim M(U, V) &= \exp(V + U^2/2) \qquad \text{for } |\alpha| < 1, \\ &= (1 - U^2 - 2V)^{-1/2} \qquad \text{for } |\alpha| > 1, \\ &= \exp(\sqrt{2}\alpha U)\left(\cos 2\sqrt{V} - \frac{\sqrt{2}\alpha U}{2\sqrt{V}} \sin 2\sqrt{V}\right)^{-\frac{1}{2}} \\ &\qquad \text{for } |\alpha| = 1. \end{aligned}$$

The next problem is to obtain the limiting distribution of $g(T)(\hat{\alpha} - \alpha)$ from $\lim M(U, V)$. Since $g(T)(\hat{\alpha} - \alpha) = g(T)x'Ax/x'bX$, the problem is one of finding the distribution of the ratio of two random variables. One method of solution has been proposed by Gurland [3]. Let X and Y be two random variables, Prob $(Y > 0) = 1$. We wish to determine the distribution of $Z = X/Y$. Let $W = W_z = X - zY$. Then we have

$$(4.10)\qquad \begin{aligned} \text{Prob}(Z < z) &= \text{Prob}(X/Y < z) \\ &= \text{Prob}(X - zY < 0) \\ &= \text{Prob}(W_z < 0). \end{aligned}$$

If the distribution of W can be found, the distribution of Z will immediately follow. Frequently the distribution of W can be found from that of X and Y by means of moment generating functions. Let

$$(4.11)\qquad m(w) = E(\exp\{Ww\}), \qquad m^*(u, v) = E(\exp\{Xu + Yv\}),$$

then

$$m(w) = E(\exp\{X - zY\}w) = E(\exp\{Xw - Yzw\}) = m^*(w, -zw).$$

To apply this technique to the problem at hand, we set $W = x'Ax/g - zx'Bx/g^2$. From (4.1), (4.2) and (4.9) we have

$$(4.12)\qquad \begin{aligned} m(w) &= M(w, -zw), \\ \lim m(w) &= \exp(-zw + w^2/2) \qquad \text{for } |\alpha| < 1, \\ &= (1 + 2zw - w^2)^{-\frac{1}{2}} \qquad \text{for } |\alpha| > 1, \\ &= \left\{\exp(\sqrt{2}\alpha w)\left(\cos 2\sqrt{-zw} - \frac{\sqrt{2}\alpha w}{2\sqrt{-zw}} \sin 2\sqrt{-zw}\right)\right\}^{-1/2} \\ &\qquad \text{for } |\alpha| = 1. \end{aligned}$$

The inversion of lim $m(w)$ is trivial for $|\alpha| < 1$. The moment generating function $\exp(-zw+w^2/2)$ is immediately recognized as that of a random variable which is normally distributed with mean $-z$ and variance 1. Hence we have

$$\text{(4.13)}\qquad \begin{aligned}\lim \text{Prob } (W < 0) &= (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{0} \exp\left(-\{t + z\}^2/2\right) dt \\ &= (2\pi)^{-1/2} \int_{-\infty}^{z} \exp(-t^2/2)\, dt \\ &= \lim \text{Prob } \{g(T)(\hat{\alpha} - \alpha) < z\},\end{aligned}$$

i.e., $g(T)\ (\hat{\alpha} - \alpha)$ is asymptotically normal with mean 0 and variance 1.

For $|\alpha| > 1$, the inverse of lim $m(w)$ might be obtained directly in terms of Bessel functions; however, it is more appealing from a statistical point of view to proceed as follows. Let X and Y be independent chi-squared variables with one degree of freedom. Then $E(\exp\{Xw\}) = E(\exp\{Yw\}) = (1 - 2w)^{-1/2}$ is their common moment generating function. Now set $R = aX - bY$, the moment generating function of R will be

$$\text{(4.14)}\qquad \begin{aligned}m_R(w) = E(\exp\{Rw\}) &= E(\exp\{aX - bY\}w) \\ &= (\{1 - 2aw\}\{1 + 2bw\})^{-1/2}.\end{aligned}$$

In particular if we set

$$\text{(4.15)}\qquad 2a = \sqrt{1 + z^2} - z, \qquad 2b = \sqrt{1 + z^2} + z,$$

we have

$$\text{(4.16)}\qquad m_R(w) = (1 + 2zw - w^2)^{1/2} = \lim m(w).$$

Hence, the limiting distribution of W, for $|\alpha| > 1$, is the same as the distribution of $R = aX - bY$. We then have

$$\text{(4.17)}\qquad \begin{aligned}\lim \text{Prob } (W < 0) &= \text{Prob } (aX - bY < 0) \\ &= \text{Prob } (X < bY/a) \\ &= \frac{1}{2\pi} \int_0^\infty \int_0^{by/a} \frac{\exp(-x/2 - y/2)}{\sqrt{xy}}\, dx\, dy \\ &= \lim \text{Prob } \{g(T)(\hat{\alpha} - \alpha) < z\} = \text{say } F(z).\end{aligned}$$

The density function corresponding to $F(z)$ is

$$\text{(4.18)}\qquad \begin{aligned}f(z) = \frac{dF(z)}{dz} &= \frac{1}{2\pi} \int_0^\infty \sqrt{a/b}\, \exp(-by/2a - y/2) \left\{\frac{d(b/a)}{dz}\right\} dy \\ &= \frac{1}{2\pi} \sqrt{a/b}\, \frac{2}{1 + (b/a)}\, \frac{d(b/a)}{dz} \\ &= \frac{1}{\pi}\, \frac{1}{1 + z^2} \qquad \text{(by (4.15))}.\end{aligned}$$

Hence the limiting distribution of $g(T)(\hat{\alpha} - \alpha)$, for $|\alpha| > 1$, is the Cauchy distribution.

We have been unable to invert $\lim m(w)$ when $|\alpha| = 1$. In the next section certain results concerning this limit and more general problems of this type will be discussed.

If we now let $x_0 = c$, a non-zero constant, the analysis proceeds much as before. Let A, B, P, and D be the $T \times T$ matrices defined in (2.3), (2.5) and (2.7). We then have, analogous to (2.1) and (2.4),

$$f(x') = (2\pi)^{-T/2} \exp (cx_1\alpha - \alpha^2c^2/2 - x'Px/2),$$
$$\hat{\alpha} - \alpha = \frac{x'Ax + cx_1 - \alpha c^2}{x'Bx + c^2}. \tag{4.19}$$

The joint moment generating function of $x'Ax + cx_1 - \alpha^2c^2$ and $x'Bx + c^2$ is

$$\begin{aligned} m(u, v) &= E (\exp \{(x'Ax + cx_1 - \alpha c^2)u + (x'Bx + c^2)v\}) \\ &= \left\{\exp \left(c^2v - c^2\alpha u - \frac{\alpha^2c^2}{2}\right)\right\} (2\pi)^{-T/2} \\ &\qquad \cdot \int \exp \left([u + \alpha]cx_1 - \frac{x'Dx}{2}\right) dx \\ &= \exp \left(c^2v - c^2\alpha u - \frac{\alpha^2c^2}{2}\right) \exp\left\{(u + \alpha)^2\frac{c^2}{2}\frac{D(T-1)}{D(T)}\right\} D(T)^{-1/2}, \end{aligned} \tag{4.20}$$

$$\begin{aligned} \lim m(U/g, V/g^2) &= \lim M(U, V) \\ &= \lim\left\{D(T)^{-\frac{1}{2}} \exp \left(\frac{-\alpha^2c^2}{2}\left[1 - \frac{D'(T-1)}{D(T)}\right]\right)\right\}, \end{aligned} \tag{4.21}$$

where $D(T)$ is as defined in (4.2) while $D'(T - 1)$ is defined in a similar fashion but with $g = g(T)$.

For $|\alpha| \leqq 1$, it follows from (4.6) and (4.8) that, since $g(T)$ and $g(T - 1)$ are of the same order,

$$\lim D(T) = \lim D'(T - 1)$$

and hence

$$\lim m(U/g, V/g^2) = \lim M(U, V) = \lim D(T)^{-1/2}. \tag{4.22}$$

We see that this limit is the same as that for $x_0 = 0$ as given in (4.9) and hence the limiting distribution of $g(T)(\hat{\alpha} - \alpha)$ does not depend on the initial value x_0 for $|\alpha| \leqq 1$.

For $|\alpha| > 1$ we have, from (4.7),

$$\lim D(T) = 1 - (U + 2V),$$
$$\lim D'(T - 1) = \frac{(U + 2V)}{\alpha^2}; \tag{4.23}$$

and in place of (4.22) we have

$$\begin{aligned}\lim M(U, V) &= \lim D(T)^{-1/2} \exp\left(-\frac{\alpha^2 c^2}{2}\left[1 - \frac{D'(T-1)}{D(T)}\right]\right) \\ &= (1 - U^2 - 2V)^{-1/2} \exp\left\{\frac{(\alpha^2 - 1)c^2}{2}\left(\frac{U^2 + 2V}{1 - U^2 - 2V}\right)\right\}.\end{aligned} \tag{4.24}$$

This moment generating function may be inverted by the methods of Section 4 to give

$$f(x) = \frac{e^{-q}}{\sqrt{\pi}(1 + x^2)} \sum_{k=0}^{\infty} \left(\frac{q}{1 + x^2}\right)^k \frac{1}{\Gamma(k + \frac{1}{2})}, \qquad q = \frac{c^2(\alpha^2 - 1)}{2}, \tag{4.25}$$

as the limiting distribution of $g(T)(\hat{\alpha} - \alpha)$. We note that for $c = 0$, $f(x)$ is the Cauchy distribution as obtained in (4.18).

6. Final remarks. The results of Mann and Wald [5] show that the limiting distribution of $g(T)(\hat{\alpha} - \alpha)$, for $|\alpha| < 1$, is also $N(0, 1)$ if, rather than assuming that the "errors" u_t are normally distributed, we merely assume that all of the moments of the u's are finite. This is another example of an invariance principle which seems to hold quite generally for the limiting distributions of function of random variables. Roughly speaking, there seems to be an unproved (and unstated) theorem that the limiting distribution of a function of a sequence of independent random variables, with suitable restrictions on these random variables, depends only on the form of the function and is the same as the distribution of a related functional on a stochastic process.

A general result of this form is Donsker's Theorem [7] which gives the limiting distribution of any function of sums of independent identically distributed random variables with finite variances as the distribution of a corresponding functional on the Wiener process. It is conjectured that this type of reasoning will show that the results of Mann and Wald will still hold if the u's are merely assumed to have finite variances.

For $\alpha = 1$, application of Donsker's Theorem shows that the limiting distribution of $g(T)(\hat{\alpha} - \alpha)$ is the same as the distribution of the functional

$$G[x(\cdot)] = \frac{\int_0^1 x(t)\, dx(t)}{\int_0^1 x^2(t)\, dt} = \frac{\frac{1}{2}x^2(1) - \frac{1}{2}}{\int_0^1 x^2(t)\, dt}$$

on the Wiener process, independent of the distribution of the u's. This distribution will be considered in a future paper.

REFERENCES

[1] H. Cramer, *Mathematical Methods of Statistics*, Princeton University Press, 1946.

[2] W. J. Dixon, "Further contributions to the problem of serial correlation," *Ann. Math. Stat.*, Vol. 15 (1944), pp. 119–144.

[3] J. GURLAND, "Inversion formulae for the distribution of ratios," *Ann. Math. Stat.*, Vol. 19 (1948), pp. 228–237.
[4] T. KOOPMANS, "Serial correlation and quadratic forms in normal variables," *Ann. Math. Stat.*, Vol. 13 (1942), pp. 1–42.
[5] H. B. MANN AND A. WALD, "On the statistical treatment of linear stochastic difference equations," *Econometrica*, Vol. 11 (1943), pp. 173–200.
[6] H. RUBIN, "Consistency of maximum likelihood estimates in the explosive case," *Statistical Inference in Dynamic Economic Models*, T. Koopmans (ed.), John Wiley & Sons, 1950.
[7] M. D. DONSKER, "An invariance principle for certain probability limit theorems," *Mem. Am. Math. Soc.*, No. 6, 1951, pp. 1–12.