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THE LIMITING UNIQUENESS CRITERION BY VORTICITY FOR NAVIER-STOKES EQUATIONS IN BESOV SPACES

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Abstract. We investigate a limiting uniqueness criterion in terms of the vorticity for the Navier-Stokes equations in the Besov space. We prove that Leray-Hopf's weak solution is unique under an auxiliary assumption that the vorticity belongs to a scale characterized by the Besov space in space, and the Orlicz space in time direction. As a corollary, we give also the uniqueness criterion in terms of bounded mean oscillation (BMO).

1. Introduction to Uniqueness Criterion. We investigate the uniqueness problem for the Navier-Stokes equations:

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \Delta u\,, \qquad t > 0\,, \quad x \in \boldsymbol{R}^n\,, \\ \operatorname{div} u = 0\,, \qquad t > 0, \quad x \in \boldsymbol{R}^n\,, \\ u(0, x) = u_0(x)\,, \end{cases} \tag{1.1}$$

where $u = (u^1(t, x), u^2(t, x), \ldots, u^n(t, x))$ and $p = p(t, x)$ denote the velocity vector and the pressure of fluid at the point $(t, x) \in (0, \infty) \times \boldsymbol{R}^n$, respectively, while $u_0 = (u_0^1(x), u_0^2(x), \ldots, u_0^n(x))$ is a given initial velocity vector.

It is well-known that a weak solution of the energy-class (so-called Leray-Hopf's weak solution) $u \in L^\infty(0, T; L^2(\boldsymbol{R}^n)) \cap L^2(0, T; \dot{H}^1(\boldsymbol{R}^n))$ is unique under the auxiliary assumption

$$u \in L^\theta(0, T; L^p)\,, \quad \frac{2}{\theta} + \frac{n}{p} = 1\,, \quad n \le p\,. \tag{1.2}$$

See Ohyama [26], Serrin [32], Prodi [30], Masuda [24] and Kozono-Sohr [19].

An interesting question is to consider the corresponding condition on the vorticity $\omega = \operatorname{rot} u$. By the Sobolev embedding theorem, the condition corresponding to $|\nabla|^r u$ is known as

$$|\nabla|^r u \in L^\theta(0, T; L^p)\,, \quad \frac{2}{\theta} + \frac{n}{p} = 1 + r\,, \quad \frac{n}{r+1} \le p < \frac{n}{r}\,. \tag{1.3}$$

The conditions (1.2) and (1.3) are closely related to the estimate for the tri-linear form induced by the nonlinear term $u \cdot \nabla v$. Recent development of the study for this term facilitates direct progress on the regularity and decay problem for the Navier-Stokes system. For instance, Chanillo considered the tri-linear estimate [11] via a real analytical argument (see also

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Hélein [17]). Coifman-Lions-Meyer-Semmes [12] also showed the $\mathcal{H}^1$ regularity of the nonlinear coupling $u \cdot \nabla u$ for the Leray-Hopf weak solutions by an elegant proof. Those results showed that the nonlinear term have a slightly better regularity due to its special algebraic structure which is a coupling of divergence-free and rotation-free factors. This was applied to the decay problem in the Hardy space corresponding to the L^p, where $p \leq 1$, by Miyakawa [25]. There is another type of development in estimating the nonlinearity in the setting of Besov and BMO spaces by Furioli-Lemarié Rieusset-Terraneo [15], Cannone-Planchon [8] and Koch-Tataru [18].

Our attention here is devoted to the uniqueness criterion in terms of the vorticity. In view of the above conditions, $\nabla u \in L^1(0, T; L^\infty)$ is considered as the limiting case in obtaining the uniqueness. On the other hand, compared to the result of the break down condition to the Euler equations:

$$
\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p\,, & t > 0\,, \quad x \in \boldsymbol{R}^n\,, \\ \operatorname{div} u = 0\,, \quad t > 0\,, \quad x \in \boldsymbol{R}^n\,, \\ u(0, x) = u_0(x)\,, \end{cases} \tag{1.4}
$$

it is desirable to control the situation in terms of the vorticity of fluid, rot $u(t)$. In the celebrated work of Beale-Kato-Majda [3], a solution of the 3D Euler equations (1.4) is shown to be regular under the condition rot $u(t) \in L^1(0, T; L^\infty)$. This result is extended to a slightly larger class of solutions in Kozono-Taniuchi [22]. They also find the related uniqueness condition to the Navier-Stokes equations in terms of velocity u (cf. [21]). The corresponding uniqueness result by vorticity, however, seems to have difficulty, since $\|\nabla u\|_\infty$ can not be controlled simply by the quantity involving only $\|\omega\|_\infty$. This is because, in the uniqueness problem, one can not assume extra regularity of weak solutions in general. More precisely, in the case of regularity problem of the Euler equations, the solution is assumed to be regular until $t < T$ and shown to be regular after $t = T$. Thanks to the logarithmic Sobolev type inequalities; i.e., for $f = (f_1, f_2, f_3) \in (W^{s,p}(\boldsymbol{R}^3))^3$ with div $f = 0$,

$$
\|\nabla f\|_\infty \leq C(1 + \|\nabla f\|_2 + \|\operatorname{rot} f\|_\infty(1 + \log^+ \|f\|_{W^{s,p}}))\,, \quad s > 3/p + 1\,, \tag{1.5}
$$

in Beale-Kato-Majda [3] it is proved that solutions of the Euler equations can be continued to be regular after $t = T$. By extending the above inequality, Kozono-Taniuchi [22] showed an analogous result for rot $u \in L^1(0, T; BMO)$ (see also Ponce [29] for the condition on the deformation tensor). Chemin [10] also considered a logarithmic singularity for the vortex for the Euler equations in the Zygmund and log-Lipschitz classes. His argument, which was based on a fine analysis using the Bony's paraproduct formula for the nonlinear term, included the logarithmic type functional inequality in terms of the log-Lipschitz semi-norm as well. Vishik [36] also developed this direction in the two dimensional case.

The uniqueness problem of weak solutions to the Navier-Stokes equations, however, is in a slightly different situation. Namely, if u is a weak solution, we can not directly deal with the norm of u appearing in the logarithm function of the Sobolev inequality (1.5). One possibility to avoid this lack of regularity is that we may invoke the aid of a viscosity term in the energy inequality. Then in the following section, we show a uniqueness criterion by yielding the condition for the time regularity to the slightly stronger Orlicz scale $L \log L$, yet keeping the space regularity in the limiting scale BMO for the vorticity $\omega = \mathrm{rot}\, u$. Furthermore, we generalize the space regularity by introducing the homogeneous Besov space $\dot{B}^0_{\infty,\infty}$, where the singular integral operator remains bounded and it is a possible substitute for BMO. This generalization is achieved by establishing a generalized critical Sobolev inequality of logarithmic type in the homogeneous Besov space (cf. [20]), which is shown in the third section.

Here we summarize several notations that will be used throughout this paper.

Let $C^\infty_{0,\sigma}$ denote the set of all C^∞ vector functions $\varphi = (\varphi^1, \varphi^2, \dots, \varphi^n)$ with compact support in $\boldsymbol{R}^n$ such that $\mathrm{div}\, \varphi = 0$. L^r_σ is the closure of $C^\infty_{0,\sigma}$ with respect to the L^r-norm $\|\cdot\|_r$; $(\cdot,\cdot)$ denotes the duality pairing between L^r and $L^{r'}$, where $1/r + 1/r' = 1$. L^r stands for the usual (vector-valued) L^r-space over $\boldsymbol{R}^n$, $1 \le r \le \infty$. H^s_σ denotes the closure of $C^\infty_{0,\sigma}$ with respect to the H^s-norm $\|\varphi\|_{H^s} = \|(1-\Delta)^{s/2}\varphi\|_2$, $s \ge 0$.

2. Uniqueness criterion for Navier-Stokes equations. Before stating our result, we recall some function spaces. Let ϕ_j, $j = 0, \pm1, \pm2, \pm3, \dots$, be the Littlewood-Paley dyadic decomposition of unity that satisfies $\hat{\phi} \in C^\infty_0(B_2 \backslash B_{1/2})$, $\hat{\phi}_j(\xi) = \hat{\phi}(2^{-j}\xi)$ and $\sum_{j=-\infty}^{\infty} \hat{\phi}_j(\xi) = 1$ except $\xi = 0$. To fill the origin, we put a smooth cut off $\psi \in \mathcal{S}(\boldsymbol{R}^n)$ with $\hat{\psi}(\xi) \in C^\infty_0(B_1)$ such that $\hat{\psi} + \sum_{j=0}^{\infty} \hat{\phi}_j(\xi) = 1$.

DEFINITION (cf. [35]). The homogeneous Besov space $\dot{B}^s_{p,\rho} = \{f \in \mathcal{S}';\ \|f\|_{\dot{B}^s_{p,\rho}} < \infty\}$ is introduced by the norm

$$\|f\|_{\dot{B}^s_{p,\rho}} = \left(\sum_{j=-\infty}^{\infty} \|2^{js}\phi_j * f\|_p^\rho \right)^{1/\rho}$$

for $s \in \boldsymbol{R}$, $1 \le p, \rho \le \infty$.

We use the non-negative logarithmic function $\log^+ r$ defined by

$$\log^+ r = \begin{cases} \log r, & e < r, \\ 1, & 0 \le r \le e. \end{cases}$$

DEFINITION. Let X denote a normed space. For $\alpha > 0$, a function $u(t)$ belongs to the class $L(\log L)^\alpha(I; X)$ for an interval I if

$$\int_I \|u(t)\|_X (\log^+ \|u(t)\|_X)^\alpha dt < \infty .$$

In particular, $L \log L(I; X)$ is the space of functions $u(t)$ with

$$\int_I \|u(t)\|_X \log^+ \|u(t)\|_X dt < \infty .$$

We recall the definition of the Leray-Hopf weak solution to the equation (1.1) as follows:

DEFINITION. Let $a \in L^2_\sigma$. A measurable function u on $\boldsymbol{R}^n \times (0,T)$ is called a *weak solution* of (1.1) on $(0,T)$ if

(i) $u \in L^\infty(0,T; L^2_\sigma) \cap L^2(0,T; H^1_\sigma)$;

(ii) $u(t)$ is continuous on $[0,T]$ in the weak topology of L^2_σ;

(iii)

$$\int_s^t \{-(u, \partial_\tau \Phi) + (\nabla u, \nabla \Phi) + (u \cdot \nabla u, \Phi)\} d\tau = -(u(t), \Phi(t)) + (u(s), \Phi(s)) \tag{2.1}$$

for every $0 \le s \le t < T$ and every $\Phi \in H^1((s,t); H^1_\sigma \cap L^n)$.

We give the uniqueness criterion of the weak solution by vorticity as follows.

THEOREM 2.1 (Uniqueness). *Let u and $\tilde{u}$ be weak solutions of the Navier-Stokes system* (1.1) *with the same initial data u_0. For* $1 \le \rho \le \infty$, *we suppose that the vorticity ω of one of the solutions, say u, satisfies* rot $u = \omega \in L(\log L)^{1/\rho'}([0,T]; \dot{B}^0_{\infty,\rho})$ *with* $1/\rho + 1/\rho' = 1$ *and the other solution $\tilde{u}$ satisfies the energy inequality*

$$\|\tilde{u}(t)\|_2^2 + 2\int_0^t \|\nabla \tilde{u}(\tau)\|_2^2 d\tau \le \|u_0\|_2^2, \quad a.e. \quad 0 \le t < T. \tag{2.2}$$

Then $u = \tilde{u}$ on $[0,T)$.

We note that $\|f\|_{\dot{B}^0_{\infty,\rho}} \le C\|f\|_{\dot{B}^s_{p,\rho}}$ for $s = n/p$. Moreover, as far as weak solutions are concerned, solutions are restricted to a subspace of the Besov space. Therefore, the following inclusions holds: $\dot{B}^0_{\infty,1}(\boldsymbol{R}^n) \subset L^\infty(\boldsymbol{R}^n) \subset BMO(\boldsymbol{R}^n) \subset \dot{B}^0_{\infty,\infty}(\boldsymbol{R}^n)$. This observation gives the following corollary to Theorem 2.1.

COROLLARY 2.2. *Let u and $\tilde{u}$ be weak solutions of the Navier-Stokes system* (1.1) *with the same initial data u_0. Suppose that the vorticity of one of the solution u satisfies* rot $u = \omega \in L(\log L)^{1/\rho'}([0,T]; \dot{B}^s_{p,\rho})$ $(s = n/p)$, $1 \le p, \rho \le \infty$, *and the other solution $\tilde{u}$ satisfies the energy inequality* (2.2). *Then $u = \tilde{u}$. In particular, if* rot $u \in L \log L([0,T]; BMO)$ *and if $\tilde{u}$ satisfies the energy inequality* (2.2), *then $u = \tilde{u}$ on* $[0,T)$.

As stated in the introduction, Beale-Kato-Majda [3] showed that a solution of the Euler equation is regular if rot $u \in L^1(0,T; L^\infty)$. In this case, the vorticity rot $u = \omega$ can dominate $\|\nabla u\|_\infty$ by the Biot-Savart law and the extra regularity assumption (see also Ponce [29], Kozono-Taniuchi [22] and Vishik [36]). In our case, however, the regularity can be covered by the viscosity of the equation.

PROOF OF THEOREM 2.1. Since div $u = 0$, the Biot-Savart law implies

$$\partial_i u = R_i \vec{R} \times \text{rot}\, u,$$

where $\vec{R} = (R_1, R_2, \dots, R_n)$ and R_k denotes the Riesz transform, i.e., $R_k = \partial_k(-\Delta)^{-1/2}$. Now we recall that

$$\|\phi_j * R_i f\|_\infty = \|(R_i(\phi_{j-1} + \phi_j + \phi_{j+1})) * \phi_j * f\|_\infty \le C\|\phi_j * f\|_\infty,$$

where the constant C is independent of j. These equalities imply that the assumption

$$\omega = \operatorname{rot} u \in L(\log L)^{1/\rho'}([0,T];\dot{B}^0_{\infty,\rho})$$

is equivalent to

$$\nabla u \in L(\log L)^{1/\rho'}([0,T];\dot{B}^0_{\infty,\rho}) \,. \tag{2.3}$$

Hence it is sufficient to prove Theorem 2.1 under (2.3).

Set $w = u - \tilde{u}$. Since w satisfies

$$\begin{cases} \partial_t w - \Delta w - w\cdot\nabla w + w\cdot\nabla u + u\cdot\nabla w + \nabla(p-q) = 0\,, & t>0\,,\ x\in \boldsymbol{R}^n\,, \\ \operatorname{div} w = 0\,, \quad t>0\,, \quad x\in\boldsymbol{R}^n\,, \\ w(0,x) = 0\,, \end{cases} \tag{2.4}$$

in the sense of distribution, we have the energy identity by a formal argument:

$$\frac{d}{dt}\|w(t)\|_2^2 + 2\|\nabla w\|_2^2 = -2(w\cdot\nabla u, w)\,. \tag{2.5}$$

Indeed, this process can be justified by the following argument.

LEMMA 2.3. *Under the assumption* $\omega = \operatorname{rot} u \in L(\log L)^{1/\rho'}([0,T];\dot{B}^0_{\infty,\rho})$, *the weak solution* u *satisfies the following.*

(1) *For all* $v \in L^\infty(0,T;L^2_\sigma)\cap L^2(0,T;H^1_\sigma)$,

$$v\cdot\nabla u \in L^\infty(0,T;L^2) + L^1(0,T;L^2) + L^\infty(0,T;H^{-1})\,. \tag{2.6}$$

(2) *The energy equality*:

$$\|u(t)\|_2^2 + 2\int_s^t \|\nabla u(\tau)\|_2^2 d\tau = \|u(s)\|_2^2\,, \quad 0\le s\le t<T\,. \tag{2.7}$$

We give the proof of Lemma 2.3 after proving the uniqueness, since the argument is somewhat similar to that for uniqueness. We note that the energy equality (2.7) guarantees strong continuity of $u(t)$ on $[0,T]$ in L^2.

By the definition of weak solutions, we have

$$\int_s^t \{-(\tilde{u},\partial_\tau\Phi) + (\nabla\tilde{u},\nabla\Phi) + (\tilde{u}\cdot\nabla\tilde{u},\Phi)\}d\tau = -(\tilde{u}(t),\Phi(t)) + (\tilde{u}(s),\Phi(s)) \tag{2.8}$$

for every $0\le s\le t<T$ and every $\Phi\in H^1((s,t);H^1_\sigma\cap L^n)$. On the other hand, under the conditions that the energy equality (2.7) holds and that $\omega\in L(\log L)^{1/\rho'}([0,T];\dot{B}^0_{\infty,\rho})\subset L^1(0,T;\dot{B}^0_{\infty,\infty})$, one may show the smoothness of the weak solution except $t=0$, i.e., $u\in C^1((0,T];H^m)$ for all $m\ge 1$ (cf. Kozono-Ogawa-Taniuchi [20]). Then we can take u as a test function in (2.8) to obtain

$$\int_s^t \{-(\tilde{u},\partial_\tau u) + (\nabla\tilde{u},\nabla u) + (\tilde{u}\cdot\nabla\tilde{u},u)\}d\tau = -(\tilde{u}(t),u(t)) + (\tilde{u}(s),u(s)) \tag{2.9}$$

for every $0<s\le t<T$.

Since u satisfies (1.1) in the strong sense on (s, T), we see

$$\int_s^t \{(\partial_\tau u, \tilde{u}) + (\nabla u, \nabla \tilde{u}) + (u \cdot \nabla u, \tilde{u})\} d\tau = 0 . \tag{2.10}$$

From the assumption (2.2), Lemma 2.3 (2.7) together with (2.9) and (2.10), we have

$$\begin{aligned} &\|u(t) - \tilde{u}(t)\|_2^2 + 2\int_0^t (\|\nabla u(\tau)\|_2^2 + \|\nabla \tilde{u}(\tau)\|_2^2) d\tau - 4\int_s^t (\nabla u(\tau), \nabla \tilde{u}(\tau)) d\tau \\ &\quad \le 2\|u_0\|_2^2 - 2(\tilde{u}(s), u(s)) + 2\int_s^t \{(u \cdot \nabla u, \tilde{u}) + (\tilde{u} \cdot \nabla \tilde{u}, u)\} d\tau \\ &\quad = 2\|u_0\|_2^2 - 2(\tilde{u}(s), u(s)) - 2\int_s^t (w \cdot \nabla u, w) d\tau , \end{aligned} \tag{2.11}$$

where $w = u - \tilde{u}$. Since $u(t)$ is strongly and $\tilde{u}(t)$ is weakly continuous in L^2, we may take a limit as $s \to 0$ to obtain

$$\|w(t)\|_2^2 + 2\int_0^t \|\nabla w(\tau)\|_2^2 d\tau \le 2\int_0^t |(w \cdot \nabla u, w)(\tau)| d\tau . \tag{2.12}$$

We should note that the above limiting process can be justified, since (2.6) guarantees $(w \cdot \nabla u, w) \in L^1(0, T)$.

Now, we decompose the smoother solution u into three parts in the phase variables so that

$$\begin{aligned} (w \cdot \nabla u, w) &= -(w \cdot \nabla w, u) \\ &= -(w \cdot \nabla w, \psi_{-N} * u) - \left(w \cdot \nabla w, \sum_{|j| \le N} \phi_j * u\right) - \left(w \cdot \nabla w, \sum_{j > N} \phi_j * u\right) \\ &= -(w \cdot \nabla w, u_l) - (w \cdot \nabla w, u_m) - (w \cdot \nabla w, u_h) , \end{aligned} \tag{2.13}$$

where $\psi_{-N}(x) = 2^{-Nn}\psi(2^{-N}x)$. Then by the Hausdorff-Young inequality, the low frequency part is estimated as

$$\begin{aligned} |(w \cdot \nabla w, u_l)| &\le \|\psi_{-N} * \nabla(w \otimes w)\|_2 \|u\|_2 \\ &\le C\|\nabla \psi_{-N}\|_2 \|w\|_2^2 \|u\|_2 \\ &\le C 2^{-(n+2)N/2} \|w\|_2^2 \|u\|_2 . \end{aligned} \tag{2.14}$$

The second term, giving the core part of the solutions, can be bounded by the logarithmic Sobolev inequality (3.1) in Section 3 so that for small $\varepsilon > 0$,

$$
\begin{aligned}
|(w\cdot\nabla w, u_m)| &= |(w\cdot\nabla u_m, w)| \\
&\le \|w\|_2^2 \Big\| \nabla\Big(\sum_{|j|\le N} \phi_j * u\Big)\Big\|_\infty \\
&\le \|w\|_2^2 \|\nabla u_m\|_{\dot B^0_{\infty,1}} \\
&\le C\|w\|_2^2\|\nabla u_m\|_{\dot B^0_{\infty,\rho}}\left\{1+\left(\frac{1}{\varepsilon}\log^+\frac{\|\nabla u_m\|_{\dot B^{\varepsilon}_{\infty,\rho}}+\|\nabla u_m\|_{\dot B^{-\varepsilon}_{\infty,\rho}}}{\varepsilon^{1/\rho'}\|\nabla u_m\|_{\dot B^0_{\infty,\rho}}}\right)^{1/\rho'}\right\} \\
&\le C\|w\|_2^2\|\nabla u\|_{\dot B^0_{\infty,\rho}}\left\{1+\left(\frac{1}{\varepsilon}\log^+\frac{2^{\varepsilon N}\|\nabla u_m\|_{\dot B^0_{\infty,\rho}}+2^{\varepsilon N}\|\nabla u_m\|_{\dot B^0_{\infty,\rho}}}{\varepsilon^{1/\rho'}\|\nabla u_m\|_{\dot B^0_{\infty,\rho}}}\right)^{1/\rho'}\right\} \\
&\le C(\varepsilon,\rho)\, N^{1/\rho'}\|w\|_2^2\|\nabla u\|_{\dot B^0_{\infty,\rho}}\ .
\end{aligned}
\tag{2.15}
$$

On the other hand, the last term is simply estimated by the Hausdorff-Young inequality as

$$
\begin{aligned}
|(w\cdot\nabla u_h, w)| &= |(w\cdot\nabla w, u_h)| \\
&\le \|w\|_2\|\nabla w\|_2 \Big\|\Big(\sum_{j>N}\phi_j * u\Big)\Big\|_\infty \\
&\le \|w\|_2\|\nabla w\|_2 \sum_{j>N}\|\{(-\Delta)^{-1/2}(\phi_{j-1}+\phi_j+\phi_{j+1})\}*\phi_j*(-\Delta)^{1/2}u\|_\infty \\
&\le C\|w\|_2\|\nabla w\|_2\sum_{j>N}2^{-j}\|\phi_j*(-\Delta)^{1/2}u\|_\infty \\
&\le C\|w\|_2\|\nabla w\|_2\Big\{\sum_{j>N}2^{-j\rho'}\Big\}^{1/\rho'}\Big\{\sum_{j>N}\|\phi_j*\nabla u\|_\infty^\rho\Big\}^{1/\rho} \\
&\le C2^{-N}\|w\|_2\|\nabla w\|_2\|\nabla u\|_{\dot B^0_{\infty,\rho}}.
\end{aligned}
\tag{2.16}
$$

Combining the estimates (2.14) through (2.16) with (2.13) and choosing N properly large so that $2^{-N}\|\nabla u\|_{\dot B^0_{\infty,\rho}} \simeq 1$, we see that

$$
\begin{aligned}
&|(w\cdot\nabla u, w)| \\
&\le C\|w\|_2^2(\|u\|_2+\|\nabla u\|_{\dot B^0_{\infty,\rho}}(\log^+\|\nabla u\|_{\dot B^0_{\infty,\rho}})^{1/\rho'})+C\|w\|_2\|\nabla w\|_2 \\
&\le C\|w\|_2^2(1+\|u\|_2+\|\nabla u\|_{\dot B^0_{\infty,\rho}}(\log^+\|\nabla u\|_{\dot B^0_{\infty,\rho}})^{1/\rho'})+\|\nabla w\|_2^2\,.
\end{aligned}
\tag{2.17}
$$

Hence we obtain from (2.12) and (2.17) that

$$
\begin{aligned}
&\|w(t)\|_2^2+2\int_0^t\|\nabla w\|_2^2 d\tau \\
&\le \int_0^t\{C\|w(\tau)\|_2^2(1+\|\nabla u(\tau)\|_{\dot B^0_{\infty,\rho}}(\log^+\|\nabla u(\tau)\|_{\dot B^0_{\infty,\rho}})^{1/\rho'})+\|\nabla w(\tau)\|_2^2\}d\tau
\end{aligned}
\tag{2.18}
$$

and

$$\|w(t)\|_2^2 \le C \int_0^t \{\|w(\tau)\|_2^2 (1 + \|\nabla u(\tau)\|_{\dot{B}^0_{\infty,\rho}} (\log^+ \|\nabla u(\tau)\|_{\dot{B}^0_{\infty,\rho}})^{1/\rho'})\} d\tau \,. \tag{2.19}$$

The Gronwall argument then gives

$$\|w(t)\|_2^2 \le C\|w(0)\|_2^2 \exp\left(\int_0^t \{(\|\nabla u(\tau)\|_{\dot{B}^0_{\infty,\rho}} (\log^+ \|\nabla u(\tau)\|_{\dot{B}^0_{\infty,\rho}})^{1/\rho'})\} d\tau \right). \tag{2.20}$$

The right hand side is 0 under the condition rot $u \in L(\log L)^{1/\rho'}(0, T; \dot{B}^0_{\infty,\rho})$. This implies $u = \tilde{u}$ on $[0, T)$. □

PROOF OF LEMMA 2.3. Here we employ a similar argument. Since $\nabla u(s) \in \dot{B}^0_{\infty,\rho}$ and $u(s) \in H^1_\sigma$ a.e. $s \in (0, T)$, it is clear that $u(s) \in B^0_{\infty,1}$ and we may decompose

$$u = \psi_{-N} * u + \sum_{|j| \le N} \phi_j * u + \sum_{j > N} \phi_j * u \quad \text{in } L^\infty \,.$$

It follows that for v and $\Phi \in C^\infty_{0,\sigma}$,

$$\begin{aligned} (v \cdot \nabla u, \Phi) &= -(v \cdot \nabla \Phi, u) \\ &= -(v \cdot \nabla \Phi, \psi_{-N} * u) - \Big(v \cdot \nabla \Phi, \sum_{|j| \le N} \phi_j * u\Big) - \Big(v \cdot \nabla \Phi, \sum_{j > N} \phi_j * u\Big) \\ &= -(v \cdot \nabla \Phi, u_l) - (v \cdot \nabla \Phi, u_m) - (v \cdot \nabla \Phi, u_h) \,. \end{aligned} \tag{2.21}$$

Similarly to (2.14)–(2.16), each of them can be bounded as

$$\begin{aligned} |(v \cdot \nabla \Phi, u_l)| &= |(v \cdot \nabla u_l, \Phi)| \le C 2^{-(n+2)N/2} \|v\|_2 \|\Phi\|_2 \|u\|_2 \,, \\ |(v \cdot \nabla \Phi, u_m)| &= |(v \cdot \nabla u_m, \Phi)| \le C N^{1/\rho'} \|v\|_2 \|\Phi\|_2 \|\nabla u\|_{\dot{B}^0_{\infty,\rho}} \,, \\ |(v \cdot \nabla \Phi, u_h)| &\le C 2^{-N} \|v\|_2 \|\nabla \Phi\|_2 \|\nabla u\|_{\dot{B}^0_{\infty,\rho}} \,. \end{aligned} \tag{2.22}$$

Hence we obtain

$$\begin{aligned} |(v \cdot \nabla \Phi, u)| \le & C\{2^{-(n+2)N/2} \|v\|_2 \|\Phi\|_2 \|u\|_2 \\ & + N^{1/\rho'} \|v\|_2 \|\Phi\|_2 \|\nabla u\|_{\dot{B}^0_{\infty,\rho}} + 2^{-N} \|v\|_2 \|\nabla \Phi\|_2 \|\nabla u\|_{\dot{B}^0_{\infty,\rho}}\} \,. \end{aligned} \tag{2.23}$$

Obviously, we can extend these estimates (2.22) and (2.23) for all v and $\Phi \in H^1_\sigma$. Since div $v = 0$, we see that for all $v, \Phi, u \in H^1_\sigma$ with $\nabla u, \nabla \Phi \in \dot{B}^0_{\infty,\rho}$ it holds

$$(v \cdot \nabla \Phi, u) = -(v \cdot \nabla u, \Phi) \,,$$

which implies

$$(v \cdot \nabla u, u) = 0 \,. \tag{2.24}$$

Similarly to (2.17), we choose N such that

$$\begin{aligned} |(v \cdot \nabla u_l, \Phi)| &\le C \|v\|_2 \|\Phi\|_2 \|u\|_2 \,, \\ |(v \cdot \nabla u_m, \Phi)| &\le C \|v\|_2 \|\Phi\|_2 \|\nabla u\|_{\dot{B}^0_{\infty,\rho}} (\log^+ \|\nabla u\|_{\dot{B}^0_{\infty,\rho}})^{1/\rho'} \,, \\ |(v \cdot \nabla u_h, \Phi)| &\le C \|v\|_2 \|\nabla \Phi\|_2 \,. \end{aligned} \tag{2.25}$$

Then, if $u, v \in L^\infty(0, T; L^2_\sigma) \cap L^2(0, T; H^1_\sigma)$ and if rot $u \in L(\log L)^{1/\rho'}(0, T; \dot{B}^0_{\infty,\rho})$, we have (2.6):

$$v \cdot \nabla u \in L^\infty(0, T; L^2) + L^1(0, T; L^2) + L^\infty(0, T; H^{-1}),$$

which implies that $\int_0^T (v \cdot \nabla u, u) d\tau$ is well-defined. Obviously, from (2.24) we obtain

$$\int_0^T (v \cdot \nabla u, u) d\tau = 0 .$$

This equality together with (2.6) and the usual mollifier argument yield the energy equality (2.7) (cf. Serrin [32]). This proves Lemma 2.3. □

3. Logarithmic Sobolev inequality. Here we give a generalization of the logarithmic Sobolev inequality originally due to Brezis-Gallouet [6], Brezis-Wainger [7] and Beale-Kato-Majda [3] (see for other generalization, [14], [34], [28], [10], [22] and [20]). Our generalization is analogous to, but slightly different from an inequality found in [20].

THEOREM 3.1 (cf. [20]). *For any* $p, \rho, q, \nu, \sigma_1, \sigma_2 \in [1, \infty]$; $r_1, r_2 \in [1, \infty]$ *satisfying* $\nu \le \rho$, $1/q = 1/p - s/n$, $r_1, r_2, p \le q$ *and* $1/r_1 - s_1/n > 1/q > 1/r_2 - s_2/n$, *there exists a constant C depending only on n such that for* $f \in \dot{B}^{s_1}_{r_1,\sigma_1} \cap \dot{B}^{s_2}_{r_2,\sigma_2}$ *the following inequality holds. If* $\nu \le \min(\sigma_1, \sigma_2)$, *we have*

$$\|f\|_{\dot{B}^0_{q,\nu}} \le C\|f\|_{\dot{B}^s_{p,\rho}}\left(1+\left(\frac{1}{\kappa}\log^+\frac{\kappa^{1/\sigma_1}\|f\|_{\dot{B}^{s_1}_{r_1,\sigma_1}}+\kappa^{1/\sigma_2}\|f\|_{\dot{B}^{s_2}_{r_2,\sigma_2}}}{\kappa^{1/\nu}\|f\|_{\dot{B}^s_{p,\rho}}}\right)^{1/\nu-1/\rho}\right), \tag{3.1}$$

where $\kappa = \min(n(1/r_1 - 1/q) - s_1, s_2 + n(1/q - 1/r_2))$. *If* $\max(\sigma_1, \sigma_2) \le \nu$, *then we have the slightly simplified inequality*

$$\|f\|_{\dot{B}^0_{q,\nu}} \le C\|f\|_{\dot{B}^s_{p,\rho}}\left(1+\left(\frac{1}{\kappa}\log^+\frac{\|f\|_{\dot{B}^{s_1}_{r_1,\sigma_1}}+\|f\|_{\dot{B}^{s_2}_{r_2,\sigma_2}}}{\|f\|_{\dot{B}^s_{p,\rho}}}\right)^{1/\nu-1/\rho}\right). \tag{3.2}$$

The inequality (3.1) is a sort of the interpolation inequality for functions in the Besov space. Indeed, under the same condition on the exponents, the embedding

$$\dot{B}^{s_1}_{r_1,\sigma_1} \cap \dot{B}^{s_2}_{r_2,\sigma_2} \subset \dot{B}^0_{q,\nu}$$

is well-known. The advantage of the above expression is the explicit form of the logarithmic term: Usual interpolation inequalities do not contain terms like those in (3.1) and (3.2). This term appears only when the inequality involves the critical relation of the exponents $1/q = 1/p - s/n$. Moreover, the power of the logarithmic term is explicitly determined by the second exponents of the Besov norms. If $\nu \ge \rho$, then the inequality trivially holds without the extra logarithmic term. For the other case, we need some extra regularity to compensate the summability of the Besov norm determined by the second exponent ν and ρ. The extra regularity $f \in \dot{B}^{s_1}_{r_1,\sigma_1}$ is used for deriving the regularity of f around the low frequency, and $f \in \dot{B}^{s_2}_{r_2,\sigma_2}$ yields the regularity for high frequency. The proof below shows that one may also prove that

(3.3)

$$\|f\|_{\dot B^0_{q,\nu}} \le C\|f\|_{\dot B^s_{p,\rho}}\left(1+\left(\frac{1}{\kappa}\log^+\frac{\kappa^{1/\sigma_1}\|\psi*f\|_{\dot B^{s_1}_{r_1,\sigma_1}}+\kappa^{1/\sigma_2}\|\mathcal{F}^{-1}(1-\hat\psi)*f\|_{\dot B^{s_2}_{r_2,\sigma_2}}}{\kappa^{1/\nu}\|f\|_{\dot B^s_{p,\rho}}}\right)^{1/\nu-1/\rho}\right),$$

which is a generalization of the known critical Sobolev inequalities in [6], [7], [3], [34], [10] and [22], mentioned above (see [20] for detailed discussions). We remark that the inequality (3.1) is scaling invariant in both the summability and differentiability exponents.

PROOF OF THEOREM 3.1. We verify the inequality for the case $\nu \le \min(\sigma_1, \sigma_2)$. The other case is obtained easily by the same proof. We go back to the definition of the Besov space and divide f into the following three parts. Let $\kappa_1 = n(1/r_1 - 1/q) - s_1$, $\kappa_2 = s_2 + n(1/q - 1/r_2)$ and let $\tilde\phi_j = \phi_{j-1} + \phi_j + \phi_{j+1}$. Then by the Hausdorff-Young inequality and $1/q = 1/p - s/n$, we have

$$\begin{aligned}
\|f\|_{\dot B^0_{q,\nu}} &\le \Big(\sum_{j<-N}\|\phi_j*f\|_q^\nu\Big)^{1/\nu} + \Big(\sum_{|j|\le N}\|\phi_j*f\|_q^\nu\Big)^{1/\nu} + \Big(\sum_{j>N}\|\phi_j*f\|_q^\nu\Big)^{1/\nu}\\
&\le \Big(\sum_{j<-N}\|\tilde\phi_j\|^\nu_{(1/q-1/r_1+1)^{-1}}\|\phi_j*f\|^\nu_{r_1}\Big)^{1/\nu} + \Big(\sum_{|j|\le N}\|\phi_j*f\|_q^\nu\Big)^{1/\nu}\\
&\quad + \Big(\sum_{j>N}\|\tilde\phi_j\|^\nu_{(1/q-1/r_2+1)^{-1}}\|\phi_j*f\|^\nu_{r_2}\Big)^{1/\nu}\\
&\le C\Big(\sum_{j<-N}2^{jn\nu(1/r_1-1/q)}\|\phi_j*f\|^\nu_{r_1}\Big)^{1/\nu} + \Big(\sum_{|j|\le N}1\Big)^{1/\nu-1/\rho}\Big(\sum_{|j|\le N}\|\phi_j*f\|_q^\rho\Big)^{1/\rho}\\
&\quad + C\Big(\sum_{j>N}2^{jn\nu(1/r_2-1/q)}\|\phi_j*f\|^\nu_{r_2}\Big)^{1/\nu},
\end{aligned}$$

where the constant C depends only on $\|\phi\|_1 + \|\phi\|_\infty$. Using the Hölder inequality for the first and third terms of the right hand side, we have

(3.4)

$$\begin{aligned}
\|f\|_{\dot B^0_{q,\nu}} &\le C\Big(\sum_{j<-N}2^{jn(1/r_1-1/q-s_1/n)(1/\nu-1/\sigma_1)^{-1}}\Big)^{1/\nu-1/\sigma_1}\Big(\sum_{j<-N}\|2^{js_1}\phi_j*f\|^{\sigma_1}_{r_1}\Big)^{1/\sigma_1}\\
&\quad + CN^{1/\nu-1/\rho}\Big(\sum_{|j|\le N}\|2^{js}\phi_j*f\|_p^\rho\Big)^{1/\rho}
\end{aligned}$$

$$
\begin{aligned}
&+C\Big(\sum_{j>N}2^{jn(1/r_2-1/q-s_2/n)(1/\nu-1/\sigma_2)^{-1}}\Big)^{1/\nu-1/\sigma_2}\Big(\sum_{j>N}\|2^{js_2}\phi_j*f\|_{r_2}^{\sigma_2}\Big)^{1/\sigma_2}\\
&\le C2^{-\kappa_1 N}\Big(\frac{2^{-\kappa_1(1/\nu-1/\sigma_1)^{-1}}}{1-2^{-\kappa_1(1/\nu-1/\sigma_1)^{-1}}}\Big)^{(1/\nu-1/\sigma_1)}\Big(\sum_{j<-N}\|2^{js_1}\phi_j*f\|_{r_1}^{\sigma_1}\Big)^{1/\sigma_1}\\
&+CN^{1/\nu-1/\rho}\Big(\sum_{|j|\le N}\|2^{js}\phi_j*f\|_p^{\rho}\Big)^{1/\rho}\\
&+C2^{-\kappa_2 N}\Big(\frac{2^{-\kappa_2(1/\nu-1/\sigma_2)^{-1}}}{1-2^{-\kappa_2(1/\nu-1/\sigma_2)^{-1}}}\Big)^{(1/\nu-1/\sigma_2)}\Big(\sum_{j>N}\|2^{js_2}\phi_j*f\|_{r_2}^{\sigma_2}\Big)^{1/\sigma_2}\\
&\le C2^{-\kappa N}\kappa_1^{1/\sigma_1-1/\nu}\Big(\sum_{j<-N}\|2^{js_1}\phi_j*f\|_{r_1}^{\sigma_1}\Big)^{1/\sigma_1}+CN^{1/\nu-1/\rho}\Big(\sum_{|j|\le N}\|2^{js}\phi_j*f\|_p^{\rho}\Big)^{1/\rho}\\
&+C2^{-\kappa N}\kappa_2^{1/\sigma_2-1/\nu}\Big(\sum_{j>N}\|2^{js_2}\phi_j*f\|_{r_2}^{\sigma_2}\Big)^{1/\sigma_2},
\end{aligned}
$$

where

$$\kappa=\min(\kappa_1,\kappa_2)>0\,.$$

Since $1/\sigma_1-1/\nu\le 0$, $1/\sigma_2-1/\nu\le 0$, we obtain from (3.4) that

$$\|f\|_{\dot B^0_{q,\nu}}\le C(2^{-\kappa N}(\kappa^{1/\sigma_1-1/\nu}\|f\|_{\dot B^{s_1}_{r_1,\sigma_1}}+\kappa^{1/\sigma_2-1/\nu}\|f\|_{\dot B^{s_2}_{r_2,\sigma_2}})+N^{1/\nu-1/\rho}\|f\|_{\dot B^s_{p,\rho}})\,. \tag{3.5}$$

Optimizing in N, we obtain from (3.5) the inequality

$$\|f\|_{\dot B^0_{q,\nu}}\le C\|f\|_{\dot B^s_{p,\rho}}\left\{1+\left(\frac{1}{\kappa}\log^+\frac{\kappa^{1/\sigma_1}\|f\|_{\dot B^{s_1}_{r_1,\sigma_1}}+\kappa^{1/\sigma_2}\|f\|_{\dot B^{s_2}_{r_2,\sigma_2}}}{\kappa^{1/\nu}\|f\|_{\dot B^s_{p,\rho}}}\right)^{1/\nu-1/\rho}\right\}, \tag{3.6}$$

where

$$
\begin{gathered}
\frac{1}{r_1}-\frac{s_1}{n}>\frac{1}{q}>\frac{1}{r_2}-\frac{s_2}{n},\\
r_1,r_2,\,p\le q,\\
\nu\le\min(\rho,\sigma_1,\sigma_2)\,.
\end{gathered}
\tag{3.7}
$$

The constant C is dependent only on n. □

REMARK. We may also have by the different choice of N that

(3.8)

$$\|f\|_{\dot B^0_{q,\nu}}\le C\left\{1+\|f\|_{\dot B^s_{p,\rho}}\left(\frac{1}{\kappa}\log^+\left(\kappa^{1/\sigma_1-1/\nu}\|f\|_{\dot B^{s_1}_{r_1,\sigma_1}}+\kappa^{1/\sigma_2-1/\nu}\|f\|_{\dot B^{s_2}_{r_2,\sigma_2}}\right)\right)^{1/\nu-1/\rho}\right\}$$

under the conditions of (3.7).

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