The Lindenbaum Construction and Decidability

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Let L be a set of sentences in a formal language — propositional, predicate, modal, epistemic, etc.[1] It is assumed that the syntax of the language is effective and that it includes the standard sentential connectives. It is also assumed that the language has a deductive system in which the axioms and rules of inference are recursively enumerable and all truth-functional tautologies are theorems.

The set L is said to be *Post-complete* if L is consistent and has no consistent, proper extension. That is, if L is Post-complete then for every sentence Φ, either Φ is in L or the set $L \cup \{\Phi\}$ is not consistent.[2]

Lemma 1   *If a recursively enumerable set of sentences is Post-complete, then it is recursive.*

Proof: Let L be recursively enumerable and Post-complete. To determine if a given sentence Φ is in L, simultaneously enumerate L and enumerate the consequences of $L \cup \{\Phi\}$. If Φ is in L, then eventually Φ will appear in the first enumeration. If Φ is not in L, then, by the Post-completeness of L, a contradiction will eventually appear in the second enumeration.

The Lindenbaum proof of maximal consistency occurs in several contexts in mathematical logic (see, for example, [2], pp. 64–65). In virtually all cases, the technique yields a proof that every consistent set of sentences has a Post-complete extension. It is often noted that the proof is not constructive. This amounts to an observation that the proof does not provide an effective method for enumerating the members of the indicated Post-complete extension, even if an enumeration procedure is available for the given set of sentences.

The Rosser extension of the Gödel incompleteness theorem (see [2], pp. 145–148) indicates that in the case of full predicate logic, there cannot be a general constructive proof of the Lindenbaum theorem. Indeed, the Rosser theorem

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indicates that no consistent, recursively enumerable extension of arithmetic is complete.

The purpose of this note is to consider the circumstances in which the Lindenbaum theorem has no constructive proof. First, it is shown that in any language and deductive system that satisfies a certain general condition, there is a consistent, recursively enumerable set of sentences that has no consistent, recursive extension and, thus, by Lemma 1, has no recursively enumerable, Post-complete extension. Moreover, the condition is met by virtually every standard formal language. Attention is then turned to sets of sentences which are deductively closed and recursive. For such sets, in standard systems that do not have a rule of substitution and satisfy the deduction theorem, the Lindenbaum theorem *is* constructive. That is, in these cases, every consistent, recursive, deductively closed set of sentences has a recursive, Post-complete extension. A rule of substitution, however, does make a difference. The final result is that in propositional modal logic, with a rule of substitution, there is a consistent, recursive, deductively closed set of sentences that has no recursively enumerable, Post-complete extension.

Let A be a set of sentences. Define the *negation* of A, written $\neg A$, to be the set $\{\neg\Phi \mid \Phi \in A\}$. A collection of sentences C is defined to be a *set of atoms* if, for any pair A, B of disjoint subsets of C, the set $A \cup \neg B$ is consistent. Let N be the set of natural numbers.

Examples: In a standard propositional or predicate system *without* a rule of substitution, the collection of propositional variables is a set of atoms. In a standard predicate system, let Px be a monadic predicate and $\{a_i \mid i \in N\}$ be a set of constants. Then $\{Pa_i \mid i \in N\}$ is a set of atoms.

In what follows, it is assumed that the language and deductive system contains an infinite, recursively enumerable set C of atoms. Fix an effective enumeration of C (without repetition) — for each natural number n, let Φn be the nth member of C.

Theorem 1   *There is a consistent, recursively enumerable set of sentences that has no consistent, recursive extension.*

Proof: The following is a well-known result in the theory of computability (for a proof, see [3], pp. 170–171).

Lemma 2   *There is a pair $P1, P2$ of recursively enumerable sets of natural numbers, such that there is no recursive set which contains $P1$ and is disjoint from $P2$.*

Let $A1$ be $\{\Phi n \mid n \in P1\}$ and $A2$ be $\{\Phi m \mid m \in P2\}$ with $P1, P2$ as in Lemma 2. Let $L1$ be $A1 \cup \neg A2$. That is, let $L1$ be $\{\Phi n \mid n \in P1\} \cup \{\neg\Phi m \mid m \in P2\}$. One may also include the deductive consequences of $L1$. Since $P1$ and $P2$ are disjoint, so are $A1$ and $A2$. Since C is a set of atoms, $L1$ is consistent. Since $P1$ and $P2$ are both recursively enumerable, so are $A1$ and $\neg A2$. Thus, $L1$ is recursively enumerable. Suppose, finally, that $L1 \subseteq L$ and L is recursive. It follows that $Q = \{n \mid \Phi n \in L\}$ is a recursive set of natural numbers. Since L extends $A1$, $P1 \subseteq Q$. By Lemma 2, Q cannot be disjoint from $P2$. So let $m \in P2$ and $m \in Q$. Since L extends $\neg A2$, $\neg\Phi m \in L$, but since $m \in Q$, $\Phi m \in L$. Thus, L is not consistent.

Corollary 1   *There is a consistent, recursively enumerable set of sentences that has no recursively enumerable, Post-complete extension.*

We now turn to sets of sentences that are deductively closed and recursive. Define a language and deductive system to have the *deduction property* if, for any set L of sentences and any pair of sentences Φ, ψ, $L \cup \{\Phi\} \vdash \psi$ iff $L \vdash \Phi \rightarrow \psi$. Most systems that do not include a rule of substitution have the deduction property.[3] Examples include standard propositional logic and predicate logic.

Theorem 2 ([4], pp. 15–16)   *In a language and deductive system with the deduction property, any consistent, deductively closed, recursive set of sentences has a recursive, Post-complete extension.*

Proof: The Lindenbaum construction does the trick. Let L be consistent, deductively closed, and recursive. Fix an effective enumeration of all of the sentences of the language — for each natural number n, let ψn be the nth sentence.[4] Consider the following sequence of sets:

$$M_0 = L.$$

M_{n+1}: *If ψn is consistent with M_n, then M_{n+1} is the deductive closure of $M_n \cup \{\psi n\}$. Otherwise, $M_{n+1} = M_n$.*

Let $M = \cup \{M_n \mid n \in N\}$. The standard Lindenbaum proof indicates that M is Post-complete. Here it is shown that M is recursive. It suffices to show that the above procedure is effective, or, in other words, that there is an effective procedure to determine, for each natural number n, whether ψn is consistent with M_n. Notice, first, that if P is any consistent, deductively closed set of sentences and Φ is any sentence, then Φ is consistent with P iff $\neg \Phi$ is not in P. Indeed, if Φ is not consistent with P, then $P \cup \{\Phi\} \vdash \neg \Phi$, so by the deduction property $P \vdash \neg \Phi$, and thus $\neg \Phi \in P$. The converse is trivial. If, in addition, P is recursive, it follows that there is an effective procedure to determine whether a given sentence is consistent with P. Now, M_0 is L which, by hypothesis, is recursive. Suppose that one had a decision procedure for M_n. By the sketch just completed, one could effectively determine whether ψn is consistent with M_n. If it is not, $M_{n+1} = M_n$ and thus the procedure for M_n will decide membership in M_{n+1}. On the other hand, if ψn is consistent with M_n, then M_{n+1} is the deductive closure of $M_n \cup \{\psi n\}$. But in this case, the deduction property entails that a given sentence Φ is in M_{n+1} iff $\psi n \rightarrow \Phi$ is in M_n. Hence, in either case, one can effectively decide membership in M_{n+1}, and, thus, whether a given formula is consistent with it.

Of course, there are some rather simple systems that do not have the deduction property. The final result is that in at least some such cases, the Lindenbaum theorem is not constructive, even if one begins with a recursive, deductively closed set.

Define a *modal language* to be a standard propositional language augmented with the sentential operator $\square$. A *modal logic* is a set of sentences of a modal language that contains all propositional tautologies and is closed under modus ponens and substitution.

The following lemmas are useful:

Lemma 3   *Let P be a set of sentences of the modal language. Then a sentence Φ is in the smallest modal logic containing P iff there is a finite set $\psi_1, \ldots, \psi_n$, of substitution instances of members of P, such that*

$$(\psi_1 \ \& \ldots \& \ \psi_n) \to \Phi$$

is a (propositional) tautology.

Proof: Define a proof (from P) to be *substitution-initial* if it begins with (zero or more) members of P and (zero or more) applications of substitution, following which only tautologies and applications of modus ponens occur. It is straightforward, but tedious, to verify that for any sentence Φ, if Φ is in the smallest modal logic containing P, then there is a substitution-initial proof of Φ from P. Let $\psi_1, \ldots, \psi_n$ be the members and substitution instances of P that occur in such a proof. The lemma follows from the observation that the indicated substitution-initial proof can be seen as a proof of Φ from $\{\psi_1, \ldots, \psi_n\}$ in ordinary propositional calculus without substitution.

Notice that when the modal language is considered from the point of view of the propositional calculus, sentences beginning with a $\Box$ are regarded as "atomic". For example, if Φ is a sentence then neither $\Box\Phi$ & $\Box\neg\Phi$ nor $\Box(\Phi$ & $\neg\Phi)$ are contradictions.

If Φ is a sentence and n a natural number, then let $\Box^n\Phi$ be the result of prefixing Φ with n $\Box$'s. The next lemma follows from the last:

Lemma 4   *Let p be a propositional variable. Then $\{\Box^{n+1}(p \to p) \mid n \in N\}$ is a set of atoms.*

Thus, Theorem 1 applies to the modal language—there is a consistent, recursively enumerable modal logic that has no recursively enumerable, Post-complete extension. Here, however, there is more to be said.

Theorem 3   *There is a consistent, recursive modal logic which does not have a recursively enumerable, Post-complete extension.*

Proof: Let $P1$ and $P2$ be as in Lemma 2. Let $B1^- = \{\Box^{n+1}(p \to p) \mid n \in P1\}$ and $B2^- = \{\Box^{n+1}(p \to p) \mid n \in P2\}$. Let $L2^-$ be the deductive closure of $B1^- \cup \neg B2^-$. From Lemma 4, $L2^-$ is consistent.

If Φ is a sentence of the modal language, let $\#(\Phi)$ be the number of symbols in Φ. Fix procedures for enumerating $P1$ and $P2$. For each $n \in P1$, let $p_1(n)$ be the number of steps required to enumerate $P1$ until n appears, and for each $m \in P2$, let $p_2(m)$ be the number of steps required to enumerate $P2$ until m appears.

Let $B1 = \{\Box^{n+1}(\Phi \to \Phi) \mid n \in P1 \text{ and } p_1(n) \le \#(\Phi)\}$, and let $B2 = \{\Box^{m+1}(\Phi \to \Phi) \mid m \in P2 \text{ and } p_2(m) \le \#(\Phi)\}$.

Let $L2$ be the deductive closure of $B1 \cup \neg B2$.

$L2$ is consistent: $L2$ is a subset of the consistent set $L2^-$.

$L2$ is recursive: Notice, first, that $B1$ is recursive. Indeed, if ψ is a sentence in the modal language, to determine whether ψ is in $B1$, one need only determine whether ψ is of the form $\Box^{n+1}(\Phi \to \Phi)$ and, if it is, enumerate $P1$ for $\#(\Phi)$ steps. Similarly, $B2$ is recursive. Now, let ψ be an arbitrary sentence. Let

$\alpha 1, \ldots, \alpha s$ be all the members of $B1$ that occur in ψ, and let $\beta 1, \ldots, \beta t$ be all the members of $B2$ that occur in ψ. It is clear that if

$$(\alpha 1 \And \ldots \And \alpha s \And \neg \beta 1 \And \ldots \And \neg \beta t) \to \psi$$

is a tautology, then ψ is in $L2$. The converse holds as well. Indeed, if ψ is in $L2$, then by Lemma 3 (noting that $B1$ and $\neg B2$ are both closed under substitution), there is a list $\gamma 1, \ldots, \gamma u$ of members of $b1$ and a list $\delta 1, \ldots, \delta v$ of members of $B2$ such that

$$(\gamma 1 \And \ldots \And \gamma u \And \neg \delta 1 \And \ldots \And \neg \delta v) \to \psi$$

is a tautology. Since the members of $B1$ and $B2$ all begin with a $\square$, they may be regarded as "propositional atoms". Thus, the latter formula remains a tautology if the conjuncts in the antecedent that correspond to members of $B1$ and $B2$ that do not occur in ψ are dropped and if other "atoms" are added. Thus,

$$(\alpha 1 \And \ldots \And \alpha s \And \neg \beta 1 \And \ldots \And \neg \beta t) \to \psi$$

is a tautology. The recursiveness of $L2$ follows.

$L2$ has no recursively enumerable, Post-complete extension: Suppose that L is a recursively enumerable, Post-complete extension of $L2$. By Lemma 1, L is recursive. Thus the set $Q = \{n \mid \square^{n+1}(p \to p) \in L\}$ is a recursive set of natural numbers. Hence, by Lemma 2, either $P1$ is not a subset of Q, or else Q is not disjoint from $P2$. I show that in either case, L is not consistent. (1) Suppose, first, that $P1$ is not a subset of Q. Let n be in $P1$ but not in Q. Then $\square^{n+1}(p \to p)$ is not in L. Hence, by the Post-completeness of L, $\square^{n+1}(p \to p)$ cannot be consistently added to L. Thus, $L \cup \{\square^{n+1}(p \to p)\}$ entails a contradiction, say $(p \And \neg p)$. Applying Lemma 3 to $L \cup \{\square^{n+1}(p \to p)\}$, there is a finite set of substitution instances $\square^{n+1}(\alpha 1 \to \alpha 1), \ldots, \square^{n+1}(\alpha s \to \alpha s)$, of $\square^{n+1}(p \to p)$ and (noting that L is closed under substitution) a finite set $\Phi 1, \ldots, \Phi t$ of members of L such that

$$[\Phi 1 \And \ldots \And \Phi t \And \square^{n+1}(\alpha 1 \to \alpha 1) \And \ldots \And \square^{n+1}(\alpha s \to \alpha s)] \to (p \And \neg p)$$

is a tautology. It follows that

$$[\square^{n+1}(\alpha 1 \to \alpha 1) \And \ldots \And \square^{n+1}(\alpha s \to \alpha s)] \to (p \And \neg p)$$

is in L. Let $\beta 1, \ldots, \beta s$, π be uniform substitution instances of $\alpha 1, \ldots, \alpha s$, $(p \And \neg p)$, respectively, such that $p_1(n) \le \#(\beta i)$ for $1 \le i \le s$. Notice that for each i, $\square^{n+1}(\beta i \to \beta i)$ is in $B1$ and, hence, in $L2$ and L. Thus, by modus ponens, π is in L, but π is (a substitution instance of) a contradiction. (2) Suppose now that Q and $P2$ are not disjoint. Then let $m \in Q$ and $m \in P2$. Thus, $\square^{m+1}(p \to p)$ is in L and, for some Φ, $\neg \square^{m+1}(\Phi \to \Phi)$ is in $L2$, hence in L. But, by substitution on the former, $\square^{m+1}(\Phi \to \Phi)$ is also in L. Thus, L is not consistent.

It might be noted in closing that there is nothing essentially "modal" in this example. The central elements in the proof of Theorem 3 are Lemmas 3 and 4. In particular, the theorem applies to any language and deductive system that includes a rule of substitution, satisfies Lemma 3, and has a set S of atoms such that every substitution instance of every member of S is "propositionally atomic".

Moreover, as a modal logic, $L2$ is rather bizarre. It does not satisfy the rule of necessitation, $\Phi \vdash \Box\Phi$, nor does it contain the K-axiom, $\Box(p \to q) \to (\Box p \to \Box q)$. In short, it is a modal logic in name only. However, no more natural modal logic would suffice. It follows from the result of [1] that any modal logic (whether recursively enumerable or not) that contains the K-axiom and is closed under the rule of necessitation is a subset of one of three recursive, Post-complete modal logics. In fact, Makinson's result applies to any modal logic closed under the rule $\Phi \to \psi \vdash \Box\Phi \to \Box\psi$.

NOTES

1. Here, a *sentence* is a well-formed formula which, if it is in a predicate language, has no free variables.

2. The phrase "maximally consistent" is perhaps more common than "Post-complete," but the former is sometimes defined to entail that if L is maximally consistent and Φ is any sentence, then either Φ is in L or $\neg\Phi$ is in L. In some of the systems considered here, there are Post-complete sets of sentences that do not have this property.

3. It might be added that modal systems that include the rule of necessitation, $\Phi \vdash \Box\Phi$, generally do not have the deduction property.

4. This should not be confused with the previous enumeration of the members of a given set of atoms.

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