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THE LINEAR QUADRATIC REGULATOR PROBLEM FOR A CLASS OF CONTROLLED SYSTEMS MODELED BY SINGULARLY PERTURBED ITÔ DIFFERENTIAL EQUATIONS*

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Abstract. This paper discusses an infinite-horizon linear quadratic (LQ) optimal control problem involving state- and control-dependent noise in singularly perturbed stochastic systems. First, an asymptotic structure along with a stabilizing solution for the stochastic algebraic Riccati equation (ARE) are newly established. It is shown that the dominant part of this solution can be obtained by solving a parameter-independent system of coupled Riccati-type equations. Moreover, sufficient conditions for the existence of the stabilizing solution to the problem are given. A new sequential numerical algorithm for solving the reduced-order AREs is also described. Based on the asymptotic behavior of the ARE, a class of $O(\sqrt{\varepsilon})$ approximate controller that stabilizes the system is obtained. Unlike the existing results in singularly perturbed deterministic systems, it is noteworthy that the resulting controller achieves an $O(\varepsilon)$ approximation to the optimal cost of the original LQ optimal control problem. As a result, the proposed control methodology can be applied to practical applications even if the value of the small parameter ε is not precisely known.

Key words. singularly perturbed control systems, asymptotic behavior, stabilizing solution

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1. Introduction.

The dynamics of many control systems are described by high-order differential equations. However, the behavior is governed by a few dominant parameters, with a relatively minor role being played by the remaining parameters, such as small time constants, masses, moments of inertia, inductances, and capacitances. The presence of these "parasitic" parameters is often the source of the increased order and the "stiffness" of the systems. **Singularly perturbed systems** (SPSs) are those whose order is reduced when the parasitic parameter is neglected.

Note that in the vast majority of optimal control and optimization problems, the data/parameters specifying the problem are not precisely known. This imprecision is often captured by incorporating a "disturbance" called a **perturbation** into the problem. In most applications, such a perturbation would be "small" but unknown. A fundamental issue that needs to be understood is the behavior of the solutions as the perturbation tends to zero. This issue is important because for many of the most interesting applications, there is, loosely speaking, a "discontinuity" at the time, which complicates the analysis. These are so-called **singularly perturbed** problems. SPSs and, more generally, multi-time-scale systems, often occur naturally as a result of the presence of small "parasitic" parameters, typically small time constants, masses, etc., multiplying time derivatives or, in a more disguised form, as a result of the pres-

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ence of large feedback gains and weak coupling. The main purpose of the singular perturbation approach to analysis and design is alleviation of the high dimensionality and ill conditioning that results from the interaction of slow and fast dynamic modes. This time-scale approach is asymptotic, that is, it is exact in the limit as the ratio ε of the speeds of the slow versus the fast dynamics tends to zero. When ε is small, approximations are obtained from reduced-order models in separate time scales [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]. A **singularly perturbed control system** (**SPCS**) evolving in a discrete time scale arises in many applications as well as in the construction of the different approximations of SPCS evolving in continuous time. An important issue in the theory of SPCS is the justification of the so-called **reduction technique approach** (**RTA**). According to this approach, the fast variables are replaced by their steady states obtained with "frozen" slow variables and controls, and the slow dynamics is approximated by the corresponding reduced order system. Although the RTA may fail to provide a proper approximation for the SPCS in a general case [20, 21], the application of RTA has been very successful in many important cases (see [33, 34, 35] and the references therein).

Note that SPSs have been widely used in electric power modeling; in the control of solar thermal central receivers; in battle management command, control, and communication systems; in armature-controlled DC motors; and in electronic RC circuit design.

Recently, there has been an increasing interest in the study of various control and filtering problems for linear stochastic systems modeled by singularly perturbed Itô differential equations with additive and/or multiplicative white noise perturbations. See [7, 9, 10, 12, 13, 15, 28, 29, 31, 37, 38, 39, 40, 41] and the references therein.

In this paper, we will study the problem of a **linear quadratic regulator** (**LQR**) for a class of controlled systems modeled by singularly perturbed Itô differential equations [36]. The problem investigated in this paper is for a new class of singularly perturbed controlled stochastic systems, namely, singularly perturbed linear stochastic systems with both state- and control-dependent multiplicative white noise. As is known from the existing literature (see, e.g., [36] for the stochastic version of Tikhonov's theorem, or [3, 19] for the analysis of the exponential stability in mean square), a scaling of the diffusion part of the fast differential equations is introduced in the case of the singularly perturbed stochastic systems with multiplicative and/or additive white noise. The scaling parameters, which are usually functions of the singular perturbation $\varepsilon > 0$, are introduced in order to guarantee the correct placement of the time-scale separation. The scaling technique of the magnitude of the diffusion part of the fast equations of a singularly perturbed system of Itô differential equations was successfully used in the asymptotic analysis of some linear quadratic Gaussian (LQG) problems for singularly perturbed linear time invariant systems affected by additive white noise [12, 37, 38, 39].

In this paper, we consider an optimization problem described by a controlled system modeled by singularly perturbed Itô differential equations with the diffusion part of the fast equation of the order of magnitude $\sqrt{\varepsilon}$ and a quadratic functional without sign of the weights. It is known that for each fixed value of the singular perturbation $\varepsilon > 0$, the LQ optimization problem has an optimal control if a certain Riccati-type matrix equation has a stabilizing solution [6, 17]. In many applications, the value of the small parameter $\varepsilon > 0$ is not precisely known; alternatively, even if it is known, its presence in the coefficients of Riccati equation produces ill conditioning of the numerical computation of the desired solution. Therefore, it follows that it is useful to know the dependence with respect to the small parameter ε of the stabilizing solution of the Riccati equation involved in the construction of the optimal control of the problem under consideration. As such, the main goal of this paper is to deduce the asymptotic structure when the small parameter $\varepsilon \to 0$ of the stabilizing solution of the Riccati equation arising in connection with the considered LQR problem.

It is known from [2, 15, 16] that in the deterministic case, the dominant part of the stabilizing solution of the original Riccati equation is constructed based on the stabilizing solutions of two uncoupled AREs of lower dimensions that are independent of the small parameter ε. The two Riccati equations of lower dimensions are associated with two LQ optimization problems obtained from the original problem by simply neglecting the small parameter ε. In the present paper, we show that in the case of the LQR problem associated with a singularly perturbed system of stochastic equations with state- and control-dependent multiplicative white noise, the dominant part of the stabilizing solution of the associated ARE one constructs based on a suitable solution of a system of strongly interconnected algebraic Riccati-type equations called the reduced system of AREs (see system (3.9)). By rewriting this system of matrix nonlinear equations in the form of a Riccati-type equation on an ordered Banach space, we are able to introduced the concept of stabilizing the solution of the reduced system of AREs and to provide a set of necessary and sufficient conditions for the existence of such a solution. These conditions are expressed in terms of the solvability of a system of some suitable linear matrix inequalities (LMIs). The solvability of the same system of LMIs provides a set of sufficient conditions for the existence of the stabilizing solution of the ARE associated with the original problem. The dominant part of the stabilizing gain matrix is used to construct a near-optimal stabilizing control, which does not depend upon the small parameter ε.

The outline of this paper is as follows. Section 2 presents the problem formulation. In the first part of section 3, we show how we can associate the system of reduced AREs corresponding to the singularly perturbed Riccati equation described in the previous section. Further, we show how we can rewrite the system of reduced AREs as a Riccati-type equation on an ordered Banach space. Following the ideas from [5, 11], we introduce the concept of stabilizing the solution of the system of reduced AREs and provide a set of necessary and sufficient conditions for the existence of this stabilizing solution of this system of strongly interconnected Riccati equations. In the final part of section 3, we provide an iterative procedure that allows us to compute the stabilizing solution of the reduced AREs. The asymptotic structure with respect to the small parameter $\varepsilon > 0$ of the stabilizing solution of the ARE of stochastic control as well as of the asymptotic structure of the corresponding stabilizing feedback gain is studied in section 4 (Theorem 4.1). In addition, we show that the control constructed based on the dominant part of the stabilizing feedback gain still stabilizes the full controlled system. Finally, we analyze the level of suboptimality achieved by this control.

2. Problem formulation. Let us consider a controlled system modeled by singularly perturbed Itô differential equations of the following form:

$$\begin{aligned} dx_1(t) &= [A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t)]dt \\ &\quad + [C_{11}x_1(t) + C_{12}x_2(t) + D_1u(t)]dw(t),\ x_1(0) = x_{10}, \end{aligned} \tag{2.1a}$$

$$\begin{aligned} \varepsilon dx_2(t) &= [A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t)]dt \\ &\quad + \sqrt{\varepsilon}[C_{21}x_1(t) + C_{22}x_2(t) + D_2u(t)]dw(t),\ x_2(0) = x_{20}, \end{aligned} \tag{2.1b}$$

where $x(t) = \begin{bmatrix} x_1^T(t) & x_2^T(t) \end{bmatrix}^T \in \Re^{n_1} \oplus \Re^{n_2}$ is a state vector, $u(t) \in \Re^m$ is the vector of control parameters, A_{ij}, C_{ij}, B_i, D_i are the given real matrices of appropriate dimensions, and $\varepsilon > 0$ is a small parameter. In (2.1) $\{w(t)\}_{t\geq 0}$ is a standard scalar process on a given probability space $(\Omega,\ \mathcal{F},\ \mathcal{P})$. Without loss of generality, it is assumed that the small parameter $\sqrt{\varepsilon}$ appears at the diffusion coefficient of the equation for the fast subsystems [36].

It should be noted that although the LQR problems for a class of SPS and multiparameter SPS have been investigated, the state-dependent noise has only been theoretically considered [31, 32]. Moreover, for the coefficient matrices of the diffusion term, a conservative condition has been imposed.

The problem of LQR for system (2.1) requires minimization of the cost functional

$$J(u) = E\int_0^\infty [x^T(t)Qx(t) + u^T(t)Ru(t)]dt \tag{2.2}$$

along the trajectories of system (2.1) determined by the admissible controls. The class of admissible controls consists of the set of measurable stochastic processes, $u = \{u(t)\}_{t\geq 0}$, which are adapted to the filtration generated by the Wiener process $\{w(t)\}_{t\geq 0}$ and which satisfy the following additional properties:

$$E\int_0^\infty |u(t)|^2 dt < +\infty$$

and

$$\lim_{t\to\infty} E|x_u(t,\ x_0)|^2 = 0,$$

$x_u(\cdot,\ x_0)$ being the trajectory of (2.1) determined by the input $u(t)$ starting from x_0 at time $t_0 = 0$.

E stands for the mathematical expectation. In (2.2), $R = R^T$ and $Q = Q^T$. In [17], it was shown that if the problem of LQR has an optimal control, then it is in a state feedback form

$$\tilde{u}(t) = \tilde{F}x(t) \tag{2.3}$$

with the gain matrix $\tilde{F}$ given by

$$\tilde{F} = -[R + D^T(\varepsilon)\tilde{X}D(\varepsilon)]^{-1}[B^T(\varepsilon)X + D^T(\varepsilon)\tilde{X}C(\varepsilon)]. \tag{2.4}$$

In (2.4), $\tilde{X}$ is the stabilizing solution of the ARE

$$\begin{aligned} &A^T(\varepsilon)X + XA(\varepsilon) + C^T(\varepsilon)XC(\varepsilon) + Q \\ &\quad - [XB(\varepsilon) + C^T(\varepsilon)XD(\varepsilon)]\Delta(\varepsilon)^{-1}[B^T(\varepsilon)X + D^T(\varepsilon)XC(\varepsilon)] = 0, \end{aligned} \tag{2.5}$$

which satisfies the following sign condition:

$$\Delta(\varepsilon) = R + D^T(\varepsilon)XD(\varepsilon) > 0. \tag{2.6}$$

In (2.4)–(2.6), we have

$$\begin{aligned} &A(\varepsilon) = \begin{bmatrix} A_{11} & A_{12} \\ \varepsilon^{-1}A_{21} & \varepsilon^{-1}A_{22} \end{bmatrix},\ C(\varepsilon) = \begin{bmatrix} C_{11} & C_{12} \\ \sqrt{\varepsilon}^{-1}C_{21} & \sqrt{\varepsilon}^{-1}C_{22} \end{bmatrix}, \\ &B(\varepsilon) = \begin{bmatrix} B_1 \\ \varepsilon^{-1}B_2 \end{bmatrix},\ D(\varepsilon) = \begin{bmatrix} D_1 \\ \sqrt{\varepsilon}^{-1}D_2 \end{bmatrix}. \end{aligned} \tag{2.7}$$

To compute the stabilizing solution of (2.5) and (2.6), a procedure based on solving a suitable **semidefinite programming** is proposed in [17], whereas in [4], an iterative procedure based on solving Lyapunov equations is provided. In the special case when $D_j = 0$, $j = 1,\ 2$, an iterative procedure to solve (2.5) is provided in [14].

It is known that the presence of the small parameter ε in the matrix coefficients of the system provides an ill conditioning of the numerical computations of the stabilizing solution of the ARE. Therefore, it is desired to obtain the asymptotic structure with respect to the small parameter ε of the stabilizing solution of (2.5) and (2.6) when the coefficient matrices have the structure given in (2.7). Together with the asymptotic structure of the stabilizing solution, we shall provide a set of conditions independent of the small parameter ε that guarantee the existence of the stabilizing solution of (2.5) and (2.6). Finally, we shall use the dominant part of the stabilizing solution of (2.5) to construct a suboptimal control whose feedback gain does not depend upon the small parameter ε.

We shall see that, unlike in the deterministic framework, in the case of the optimization problem described by (2.1) and (2.2), we cannot associate, in a visible way, two optimization problems of lower dimension. However, one may associate the so-called reduced system of ARE (2.5) that extends to this framework both the reduced (slow) ARE and the boundary layer (fast) ARE equation from the deterministic framework.

3. The reduced system of AREs.

3.1. The derivation of the reduced system of ARE.

Set

$$F = -[R + D^T(\varepsilon)XD(\varepsilon)]^{-1}[B^T(\varepsilon)X + D^T(\varepsilon)XC(\varepsilon)],$$

and note that if X is a solution of (2.5), then (X, F) is a solution of the following system:

$$\begin{aligned} &A^T(\varepsilon)X + XA(\varepsilon) + C^T(\varepsilon)XC(\varepsilon) \\ &\quad + Q - F^T[R + D^T(\varepsilon)XD(\varepsilon)]F = 0, \end{aligned} \tag{3.1a}$$

$$B^T(\varepsilon)X + D^T(\varepsilon)XC(\varepsilon) = -[R + D^T(\varepsilon)XD(\varepsilon)]F. \tag{3.1b}$$

Conversely, if (X, F) is a solution of the system (3.1) such that $R + D^T(\varepsilon)XD(\varepsilon)$ is an invertible matrix, then X is a solution of (2.5). Choose

$$X = \begin{bmatrix} X_{11} & \varepsilon X_{12} \\ \varepsilon X_{12}^T & \varepsilon X_{22} \end{bmatrix},\ F = \begin{bmatrix} F_1 & F_2 \end{bmatrix}$$

with $X_{ii} = X_{ii}^T \in \Re^{n_i \times n_i}$, $i = 1,\ 2$, $X_{12} \in \Re^{n_1 \times n_2}$, and $F_i \in \Re^{m \times n_i}$, $i = 1,\ 2$. With this notation, one obtains the following partition of (3.1):

$$\begin{aligned} &A_{11}^T X_{11} + X_{11}A_{11} + A_{21}^T X_{12}^T + X_{12}A_{21} + C_{11}^T X_{11} C_{11} \\ &\quad + \sqrt{\varepsilon}(C_{21}^T X_{12}^T C_{11} + C_{11}^T X_{12} C_{21}) + C_{21}^T X_{22} C_{21} + Q_{11} - F_1^T \Delta(\varepsilon) F_1 = 0, \end{aligned} \tag{3.2a}$$

$$\begin{aligned} &\varepsilon A_{11}^T X_{12} + X_{11}A_{12} + A_{21}^T X_{22} + X_{12}A_{22} + C_{11}^T X_{11} C_{12} \\ &\quad + \sqrt{\varepsilon}(C_{21}^T X_{12}^T C_{12} + C_{11}^T X_{12} C_{22}) + C_{21}^T X_{22} C_{22} + Q_{12} - F_1^T \Delta(\varepsilon) F_2 = 0, \end{aligned} \tag{3.2b}$$

$$\begin{aligned} &A_{22}^T X_{22} + X_{22}A_{22} + \varepsilon(A_{12}^T X_{12} + X_{12}^T A_{12}) + C_{12}^T X_{11} C_{12} \\ &\quad + \sqrt{\varepsilon}(C_{22}^T X_{12}^T C_{12} + C_{12}^T X_{12} C_{22}) + C_{22}^T X_{22} C_{22} + Q_{22} - F_2^T \Delta(\varepsilon) F_2 = 0, \end{aligned} \tag{3.2c}$$

$$\begin{aligned}&B_1^T X_{11} + B_2^T X_{12}^T + D_1^T X_{11} C_{11} + \sqrt{\varepsilon}(D_2^T X_{12}^T C_{11} + D_1^T X_{12} C_{21})\\ &\quad + D_2^T X_{22} C_{21} + \Delta(\varepsilon) F_1 = 0, \qquad \text{(3.2d)}\\ &\varepsilon B_1^T X_{12} + B_2^T X_{22}^T + D_1^T X_{11} C_{12} + \sqrt{\varepsilon}(D_2^T X_{12}^T C_{12} + D_1^T X_{12} C_{22})\\ &\quad + D_2^T X_{22} C_{22} + \Delta(\varepsilon) F_2 = 0, \qquad \text{(3.2e)}\end{aligned}$$

where $\Delta(\varepsilon) = R + D_1^T X_{11} D_1 + \sqrt{\varepsilon}(D_2^T X_{12}^T D_1 + D_1^T X_{12} D_2) + D_2^T X_{22} D_2$.

Here, $\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}$ is the partition of Q that is compatible with the coefficient structure (2.7).

Setting $\varepsilon = 0$ in (3.2), one obtains

$$\begin{aligned}&A_{11}^T \bar{X}_{11} + \bar{X}_{11} A_{11} + A_{21}^T \bar{X}_{12}^T + \bar{X}_{12} A_{21} + C_{11}^T \bar{X}_{11} C_{11}\\ &\quad + C_{21}^T \bar{X}_{22} C_{21} + Q_{11} - \bar{F}_1^T \bar{\Delta} \bar{F}_1 = 0, \qquad \text{(3.3a)}\\ &\bar{X}_{11} A_{12} + A_{21}^T \bar{X}_{22} + \bar{X}_{12} A_{22} + C_{11}^T \bar{X}_{11} C_{12}\\ &\quad + C_{21}^T \bar{X}_{22} C_{22} + Q_{12} - \bar{F}_1^T \bar{\Delta} \bar{F}_2 = 0, \qquad \text{(3.3b)}\end{aligned}$$

$$A_{22}^T \bar{X}_{22} + \bar{X}_{22} A_{22} + C_{12}^T \bar{X}_{11} C_{12} + C_{22}^T \bar{X}_{22} C_{22} + Q_{22} - \bar{F}_2^T \bar{\Delta} \bar{F}_2 = 0, \qquad \text{(3.3c)}$$

$$B_1^T \bar{X}_{11} + B_2^T \bar{X}_{12}^T + D_1^T \bar{X}_{11} C_{11} + D_2^T \bar{X}_{22} C_{21} = -\bar{\Delta} \bar{F}_1, \qquad \text{(3.3d)}$$

$$B_2^T \bar{X}_{22}^T + D_1^T \bar{X}_{11} C_{12} + D_2^T \bar{X}_{22} C_{22} = -\bar{\Delta} \bar{F}_2, \qquad \text{(3.3e)}$$

where $\bar{\Delta} = R + D_1^T \bar{X}_{11} D_1 + D_2^T \bar{X}_{22} D_2$.

It may be noted that $\bar{X}_{11}$, $\bar{X}_{21}$, $\bar{X}_{22}$, $\bar{F}_1$, and $\bar{F}_2$ are called 0th order solutions.

Let us recall some useful equalities known from the deterministic case (see [2, 15, 16]). They also play an important role in the stochastic case.

LEMMA 3.1. *If A_{22} and $A_{22} + B_2 \bar{F}_2$ are invertible matrices, then we have the following:*

(i) *$I_m + \bar{F}_2 A_{22}^{-1} B_2$ is an invertible matrix and $(I_m + \bar{F}_2 A_{22}^{-1} B_2)^{-1} = I_m - \bar{F}_2 (A_{22} + B_2 \bar{F}_2)^{-1} B_2$.*

(ii) *$(A_{22} + B_2 \bar{F}_2)^{-1} = A_{22}^{-1} - A_{22}^{-1} B_2 (I_m + \bar{F}_2 A_{22}^{-1} B_2)^{-1} \bar{F}_2 A_{22}^{-1}$.*

Assuming that A_{22} is invertible, we introduce the notation:

$$\begin{aligned}A_s &= A_{11} - A_{12} A_{22}^{-1} A_{21}, \; B_s = B_1 - A_{12} A_{22}^{-1} B_2,\\ C_{1s} &= C_{11} - C_{12} A_{22}^{-1} A_{21}, \; C_{2s} = C_{21} - C_{22} A_{22}^{-1} A_{21},\\ D_{1s} &= D_1 - C_{12} A_{22}^{-1} B_2, \; D_{2s} = D_2 - C_{22} A_{22}^{-1} B_2,\\ Q_s &= Q_{11} - Q_{12} A_{22}^{-1} A_{21} - A_{21}^T A_{22}^{-T} Q_{12}^T + A_{21}^T A_{22}^{-T} Q_{22} A_{22}^{-1} A_{21},\\ L_s &= (A_{21}^T A_{22}^{-T} Q_{22} - Q_{12}) A_{22}^{-1} B_2, \; R_s = R + B_2^T A_{22}^{-T} Q_{22} A_{22}^{-1} B_2. \qquad \text{(3.4)}\end{aligned}$$

Regarding the solutions of the system (3.3), we have the following.

PROPOSITION 3.2. *If A_{22} is an invertible matrix, then the following are true:*

(i) *If $(\bar{X}_{11}, \bar{X}_{12}, \bar{X}_{22}, \bar{F}_1, \bar{F}_2)$ is a solution of the system (3.3) such that $A_{22} + B_2 \bar{F}_2$ is an invertible matrix, then $(\bar{X}_{11}, \bar{X}_{22}, \bar{F}_1, \bar{F}_2) \in \mathcal{S}_{n_1} \oplus \mathcal{S}_{n_2} \oplus \Re^{m \times n_1} \oplus \Re^{m \times n_2}$ is a solution of the system*

$$\begin{aligned}&A_s^T \bar{X}_{11} + \bar{X}_{11} A_s + C_{1s}^T \bar{X}_{11} C_{1s} + C_{2s}^T \bar{X}_{22} C_{2s}\\ &\quad - F_s^T (R_s + D_{1s}^T \bar{X}_{11} D_{1s} + D_{2s}^T \bar{X}_{22} D_{2s}) F_s + Q_s = 0, \qquad \text{(3.5a)}\end{aligned}$$

$$A_{22}^T \bar{X}_{22} + \bar{X}_{22} A_{22} + C_{12}^T \bar{X}_{11} C_{12} + C_{22}^T \bar{X}_{22} C_{22} - F_2^T \bar{\Delta} F_2 + Q_{22} = 0, \qquad \text{(3.5b)}$$

$$\begin{aligned}&B_s^T \bar{X}_{11} + D_{1s}^T \bar{X}_{11} C_{1s} + D_{2s}^T \bar{X}_{22} C_{2s} + L_s^T\\ &\quad + (R_s + D_{1s}^T \bar{X}_{11} D_{1s} + D_{2s}^T \bar{X}_{22} D_{2s}) F_s = 0, \qquad \text{(3.5c)}\end{aligned}$$

$$B_2^T \bar{X}_{22} + D_1^T \bar{X}_{11} C_{12} + D_2^T \bar{X}_{22} C_{22} + \bar{\Delta} F_2 = 0, \qquad \text{(3.5d)}$$

where

$$\bar{F}_s = (I_m + \bar{F}_2 A_{22}^{-1} B_2)^{-1} (\bar{F}_1 - \bar{F}_2 A_{22}^{-1} A_{21}). \tag{3.6}$$

(ii) *If* $(\bar{X}_{11},\ \bar{X}_{22},\ \bar{F}_1,\ \bar{F}_2) \in \mathcal{S}_{n_1} \oplus \mathcal{S}_{n_2} \oplus \Re^{m\times n_1} \oplus \Re^{m\times n_2}$ *is a solution of the system* (3.5) *such that* $A_{22}+B_2\bar{F}_2$ *is an invertible matrix, then* $(\bar{X}_{11},\ \bar{X}_{12},\ \bar{X}_{22},\ \bar{F}_1,\ \bar{F}_2)$ *verifies the system* (3.3)*, where*

$$\bar{F}_1 = (I_m + \bar{F}_2 A_{22}^{-1} B_2)\bar{F}_s + \bar{F}_2 A_{22}^{-1} A_{21}, \tag{3.7}$$

$$\begin{aligned} \bar{X}_{12} = -[&\bar{X}_{11} A_{12} + A_{21}^T \bar{X}_{22} + C_{11}^T \bar{X}_{11} C_{12} + C_{21}^T \bar{X}_{22} C_{22} \\ &+ Q_{12} - \bar{F}_1^T \bar{\Delta} \bar{F}_2] A_{22}^{-1}. \end{aligned} \tag{3.8}$$

The proof can be obtained from direct but tedious algebraic calculations, the details of which are omitted.

In the statement of Proposition 3.2 as well as in the remainder of the paper, $\mathcal{S}_{n_i} \subset \Re^{n_i \times n_i}$, $i = 1,\ 2$, are subspaces of symmetric matrices.

In $\mathcal{S}_{n_1} \oplus \mathcal{S}_{n_2}$, we consider the following subset:

$$\begin{aligned} \mathbf{Dom}(\mathcal{R}_0) = \{&(X_1,\ X_2) \in \mathcal{S}_{n_1} \oplus \mathcal{S}_{n_2} \mid R_s + D_{1s}^T X_1 D_{1s} + D_{2s}^T X_2 D_{2s} \\ &\text{and } R + D_1^T X_1 D_1 + D_2^T X_2 D_2 \text{ are invertible matrices}\}. \end{aligned}$$

One sees that if $(X_1,\ X_2) \in \mathbf{Dom}(\mathcal{R}_0)$, then (3.5) is equivalent to the following subset of coupled AREs:

$$\begin{aligned} &A_s^T X_1 + X_1 A_s + C_{1s}^T X_1 C_{1s} + C_{2s}^T X_2 C_{2s} \\ &\quad - M_s^T (R_s + D_{1s}^T X_1 D_{1s} + D_{2s}^T X_2 D_{2s})^{-1} M_s + Q_s = 0, \end{aligned} \tag{3.9a}$$

$$A_{22}^T X_2 + X_2 A_{22} + C_{12}^T X_1 C_{12} + C_{22}^T X_2 C_{22} - M_2^T \bar{\Delta}^{-1} M_2 + Q_{22} = 0, \tag{3.9b}$$

where $M_s = B_s^T X_1 + D_{1s}^T X_1 C_{1s} + D_{2s}^T X_2 C_{2s} + L_s^T$ and $M_2 = B_2^T X_2 + D_1^T X_1 C_{12} + D_2^T X_2 C_{22}$.

Let us remark that if $C_{ij},\ D_j,\ i,\ j = 1,\ 2$, vanish, then (3.9) reduces to

$$A_s^T X_1 + X_1 A_s - (X_1 B_s + L_s) R_s^{-1} (B_s^T X_1 + L_s^T) + Q_s = 0 \tag{3.10}$$

and

$$A_{22}^T X_2 + X_2 A_{22} - X_2 B_2 R^{-1} B_2^T X_2 + Q_{22} = 0, \tag{3.11}$$

which appears in the deterministic case, in connection with the investigation of the asymptotic behavior of the stabilizing solution in the corresponding problem of the LQR. The stabilizing solutions of the two AREs of lower dimension are involved in the construction of a suboptimal control in the LQR problem for the deterministic context. It is expected that in the stochastic case considered in this paper, the stabilizing solution of (3.9) plays an important role in the investigation of the asymptotic behavior of the stabilizing solution of (2.5)–(2.6) and in the construction of a suboptimal control for the optimal control problem described by (2.1) and (2.2).

In the following, the system of coupled AREs (3.9) is called a reduced system of AREs.

In the next subsection, we provide a set of necessary and sufficient conditions for the existence of the stabilizing solution of the reduced system of AREs (3.9). In addition, we present a procedure that allows us to compute the stabilizing solution of this system.

3.2. Stabilizing solution of the reduced system of AREs. For a better understanding of the statements in this subsection, we rewrite (3.9) in a compact form as a Riccati-type equation on an ordered Hilbert space. Toward this end, we establish several conventions of notation:

(α) If $\boldsymbol{B} = (B_1,\ B_2) \in \Re^{n_1 \times m_1} \oplus \Re^{n_2 \times m_2}$ and $\boldsymbol{C} = (C_1,\ C_2) \in \Re^{m_1 \times q_1} \oplus \Re^{m_2 \times q_2}$, then $\boldsymbol{D} = \boldsymbol{BC} \in \Re^{n_1 \times q_1} \oplus \Re^{n_2 \times q_2}$ is defined by $\boldsymbol{D} = (D_1,\ D_2)$ with $D_i = B_i C_i \in \Re^{n_i \times q_i}$, $i = 1,\ 2$.

(β) If $\boldsymbol{Z} = (Z_1,\ Z_2) \in \Re^{n_1 \times m_1} \oplus \Re^{n_2 \times m_2}$, then $\boldsymbol{Z}^T$ is defined by $\boldsymbol{Z}^T = (Z_1^T,\ Z_2^T) \in \Re^{m_1 \times n_1} \oplus \Re^{m_2 \times n_2}$.

If $\boldsymbol{R} = (R_1,\ R_2) \in \Re^{m_1 \times m_1} \oplus \Re^{m_2 \times m_2}$ is such that $\mathbf{det} R_i \neq 0$, $i = 1,\ 2$, then $\boldsymbol{R}^{-1}$ is defined by $\boldsymbol{R}^{-1} = (R_1^{-1},\ R_2^{-1})$.

With this convention, the system (3.9) may be written in the form

$$\begin{aligned} &\boldsymbol{A}^T \boldsymbol{X} + \boldsymbol{X}\boldsymbol{A} + \boldsymbol{\Pi_1}(\boldsymbol{X}) - [\boldsymbol{X}\boldsymbol{B} + \boldsymbol{\Pi_2}(\boldsymbol{X}) + \boldsymbol{L}] \\ &\quad \times [\boldsymbol{R} + \boldsymbol{\Pi_3}(\boldsymbol{X})]^{-1}[\boldsymbol{B}^T \boldsymbol{X} + \boldsymbol{\Pi_2}^T(\boldsymbol{X}) + \boldsymbol{L}^T] + \boldsymbol{Q} = 0 \end{aligned} \tag{3.12}$$

with the unknown $\boldsymbol{X} = (X_1,\ X_2) \in \mathcal{S}_{n_1} \oplus \mathcal{S}_{n_2}$ and the coefficients

$$\begin{aligned} &\boldsymbol{A} = (A_s,\ A_{22}) \in \Re^{n_1 \times n_1} \oplus \Re^{n_2 \times n_2}, \\ &\boldsymbol{B} = (B_s,\ B_2) \in \Re^{n_1 \times m} \oplus \Re^{n_2 \times m}, \\ &\boldsymbol{Q} = (Q_s,\ Q_{22}) \in \mathcal{S}_{n_1} \oplus \mathcal{S}_{n_2},\ \boldsymbol{L} = (L_s,\ 0) \in \Re^{n_1 \times m} \oplus \Re^{n_2 \times m}, \\ &\boldsymbol{\Pi_1} : \mathcal{S}_{n_1} \oplus \mathcal{S}_{n_2} \to \mathcal{S}_{n_1} \oplus \mathcal{S}_{n_2},\ \boldsymbol{\Pi_2} : \mathcal{S}_{n_1} \oplus \mathcal{S}_{n_2} \to \Re^{n_1 \times m} \oplus \Re^{n_2 \times m}, \\ &\boldsymbol{\Pi_3} : \mathcal{S}_{n_1} \oplus \mathcal{S}_{n_2} \to \mathcal{S}_m \oplus \mathcal{S}_m \end{aligned} \tag{3.13}$$

are linear operators designed by
$\boldsymbol{\Pi_k}(\boldsymbol{X}) = (\Pi_{ks}(\boldsymbol{X}),\ \Pi_{kf}(\boldsymbol{X}))$, $k = 1,\ 2,\ 3$, where

$$\Pi_{1s}(\boldsymbol{X}) = C_{1s}^T X_1 C_{1s} + C_{2s}^T X_2 C_{2s},\ \Pi_{1f}(\boldsymbol{X}) = C_{12}^T X_1 C_{12} + C_{22}^T X_2 C_{22}, \tag{3.14}$$

$$\Pi_{2s}(\boldsymbol{X}) = C_{1s}^T X_1 D_{1s} + C_{2s}^T X_2 D_{2s},\ \Pi_{2f}(\boldsymbol{X}) = C_{12}^T X_1 D_1 + C_{22}^T X_2 D_2, \tag{3.15}$$

$$\Pi_{3s}(\boldsymbol{X}) = D_{1s}^T X_1 D_{1s} + D_{2s}^T X_2 D_{2s},\ \Pi_{3f}(\boldsymbol{X}) = D_1^T X_1 D_1 + D_2^T X_2 D_2 \tag{3.16}$$

for all $\boldsymbol{X} = (X_1,\ X_2) \in \mathcal{S}_{n_1} \oplus \mathcal{S}_{n_2}$.

Let us denote the linear space $\mathcal{X} = \mathcal{S}_{n_1} \oplus \mathcal{S}_{n_2}$. In $\mathcal{X}$, we consider the inner product

$$\langle \boldsymbol{X},\ \boldsymbol{Y} \rangle = \mathbf{Tr}[X_1 Y_1] + \mathbf{Tr}[X_2 Y_2] \tag{3.17}$$

for all $\boldsymbol{X},\ \boldsymbol{Y} \in \mathcal{X}$.

We introduce the order relation introduced by the closed, solid, convex cone $\mathcal{X}^+ = \mathcal{S}_{n_1}^+ \oplus \mathcal{S}_{n_2}^+$, where for each $i = 1,\ 2$

$$\mathcal{S}_{n_i}^+ = \{\ Y \in \mathcal{S}_{n_i}\ |\ Y \geq 0\ \}.$$

Here, $Y \geq 0$ means that Y is positive semidefinite. One verifies that $|\cdot|$ induced by the inner product (3.17) is monotonic with respect to the cone $\mathcal{X}^+$. That is, $|\boldsymbol{X}| \leq |\boldsymbol{Y}|$ if $0 \leq \boldsymbol{X} \leq \boldsymbol{Y}$.

Based on the operators $\boldsymbol{\Pi_k}(\cdot)$ introduced in (3.14)–(3.16), we define the operator $\boldsymbol{\Pi} : \boldsymbol{X} \to \mathcal{S}_{n_1+m} \oplus \mathcal{S}_{n_2+m}$ by

$$\boldsymbol{\Pi}(\boldsymbol{X}) = (\Pi_s(\boldsymbol{X}),\ \Pi_f(\boldsymbol{X})),$$

where

$$\Pi_s(\boldsymbol{X}) = \begin{bmatrix} \Pi_{1s}(\boldsymbol{X}) & \Pi_{2s}(\boldsymbol{X}) \\ \Pi_{2s}^T(\boldsymbol{X}) & \Pi_{3s}(\boldsymbol{X}) \end{bmatrix} = \begin{bmatrix} C_{1s} & D_{1s} \end{bmatrix}^T X_1 \begin{bmatrix} C_{1s} & D_{1s} \end{bmatrix} + \begin{bmatrix} C_{2s} & D_{2s} \end{bmatrix}^T X_2 \begin{bmatrix} C_{2s} & D_{2s} \end{bmatrix}, \tag{3.18}$$

$$\Pi_f(\boldsymbol{X}) = \begin{bmatrix} \Pi_{1f}(\boldsymbol{X}) & \Pi_{2f}(\boldsymbol{X}) \\ \Pi_{2f}^T(\boldsymbol{X}) & \Pi_{3f}(\boldsymbol{X}) \end{bmatrix} = \begin{bmatrix} C_{12} & D_1 \end{bmatrix}^T X_1 \begin{bmatrix} C_{12} & D_1 \end{bmatrix} + \begin{bmatrix} C_{22} & D_2 \end{bmatrix}^T X_2 \begin{bmatrix} C_{22} & D_2 \end{bmatrix} \quad \forall \boldsymbol{X} \in \mathcal{X}. \tag{3.19}$$

From (3.18) and (3.19), one sees that $\boldsymbol{\Pi}(\boldsymbol{X}) \geq 0$ if $\boldsymbol{X} \geq 0$.

If $F = (F_s,\ F_f) \in \Re^{m\times n_1} \oplus \Re^{m\times n_2}$, we construct the operator $\boldsymbol{\Pi}_F : \mathcal{X} \to \mathcal{X}$ as follows:

$$\boldsymbol{\Pi}_F(\boldsymbol{X}) = (\Pi_{Fs}(\boldsymbol{X}),\ \Pi_{Ff}(\boldsymbol{X}))$$

with

$$\Pi_{Fs}(\boldsymbol{X}) = \begin{bmatrix} I_{n_1} & F_s^T \end{bmatrix} \Pi_s(\boldsymbol{X}) \begin{bmatrix} I_{n_1} & F_s^T \end{bmatrix}^T = (C_{1s} + D_{1s}F_s)^T X_1 (C_{1s} + D_{1s}F_s) + (C_{2s} + D_{2s}F_s)^T X_2 (C_{2s} + D_{2s}F_s), \tag{3.20}$$

$$\Pi_{Ff}(\boldsymbol{X}) = \begin{bmatrix} I_{n_2} & F_f^T \end{bmatrix} \Pi_f(\boldsymbol{X}) \begin{bmatrix} I_{n_2} & F_f^T \end{bmatrix}^T = (C_{12} + D_1F_f)^T X_1 (C_{12} + D_1F_f) + (C_{22} + D_2F_f)^T X_2 (C_{22} + D_2F_f), \quad \forall \boldsymbol{X} \in \mathcal{X}. \tag{3.21}$$

Hence, $\boldsymbol{\Pi}_F(\boldsymbol{X}) \geq 0$ if $\boldsymbol{X} \geq 0$. By direct calculation, one obtains that the corresponding adjoint operator with respect to the inner product (3.17) is given by:

$$\boldsymbol{\Pi}_F^*(\boldsymbol{X}) = (\Pi_{Fs}^*(\boldsymbol{X}),\ \Pi_{Ff}^*(\boldsymbol{X})),$$

where

$$\Pi_{Fs}^*(\boldsymbol{X}) = (C_{1s} + D_{1s}F_s)X_1(C_{1s} + D_{1s}F_s)^T + (C_{12} + D_1F_f)X_2(C_{12} + D_1F_f)^T, \tag{3.22}$$

$$\Pi_{Ff}^*(\boldsymbol{X}) = (C_{2s} + D_{2s}F_s)X_1(C_{2s} + D_{2s}F_s)^T + (C_{22} + D_2F_f)X_2(C_{22} + D_2F_f)^T, \quad \forall \boldsymbol{X} \in \mathcal{X}. \tag{3.23}$$

According to [5] and [11], we introduce the following definition.

Definition 3.3. *We say that the triple* $(\boldsymbol{A}, \boldsymbol{B},\ \boldsymbol{\Pi})$ *is stabilizable if there exists* $F = (F_s,\ F_f) \in \Re^{m\times n_1} \oplus \Re^{m\times n_2}$ *such that the eigenvalues of the operator* $\mathcal{L}_F$ *are in the half plane* $\mathbf{C}^- = \{z \in \mathbf{C} \mid \mathrm{Re}(z) < 0\}$, *where*

$$\mathcal{L}_F(\boldsymbol{X}) = (\boldsymbol{A} + \boldsymbol{B}F)^T \boldsymbol{X} + \boldsymbol{X}(\boldsymbol{A} + \boldsymbol{B}F) + \boldsymbol{\Pi}_F(\boldsymbol{X}) \ \forall \boldsymbol{X} \in \mathcal{X}. \tag{3.24}$$

Definition 3.4. *A solution* $\tilde{\boldsymbol{X}} = (\tilde{X}_1,\ \tilde{X}_2)$ *of* (3.12) *is a stabilizing solution if the eigenvalues of the linear operator* $\mathcal{L}_{\tilde{F}}$ *are in the half plane* $\mathbf{C}^-$, *where* $\mathcal{L}_{\tilde{F}}$ *is* *constructed via* (3.24) *for* F *replaced by* $\tilde{F} = (\tilde{F}_s,\ \tilde{F}_f)$ *defined by*

$$\tilde{F}_s = -(R_s + D_{1s}^T \tilde{X}_1 D_{1s} + D_{2s} \tilde{X} D_{2s})^{-1} \times (B_s^T \tilde{X}_1 + D_{1s}^T \tilde{X}_1 C_{1s} + D_{2s}^T \tilde{X}_2 C_{2s} + L_s^T), \tag{3.25}$$

$$\tilde{F}_f = -(R + D_1^T \tilde{X}_1 D_1 + D_2^T \tilde{X} D_2)^{-1} \times (B_2^T \tilde{X}_1 + D_1^T \tilde{X}_1 C_{12} + D_2^T \tilde{X}_2 C_{22}). \tag{3.26}$$

Let us note that the stabilizing feedback gain introduced by (3.25) and (3.26) may be written in a compact form

$$\tilde{F} = -[\boldsymbol{R} + \boldsymbol{\Pi}_3(\tilde{\boldsymbol{X}})]^{-1}[\tilde{\boldsymbol{X}}\boldsymbol{B} + \boldsymbol{\Pi}_2(\tilde{\boldsymbol{X}}) + \boldsymbol{L}]^T. \tag{3.27}$$

To state in an elegant way a set of necessary and sufficient conditions for the existence of the stabilizing solution of ARE (3.12), or equivalently the stabilizing solution of the reduced system of AREs (3.9), we introduce the dissipation operator defined by the coefficients of (3.9),

$$\mathcal{D} : \mathcal{S}_{n_1} \oplus \mathcal{S}_{n_2} \to \mathcal{S}_{n_1+m} \oplus \mathcal{S}_{n_2+m},$$

as follows:

$$\mathcal{D}(\boldsymbol{X}) = (\mathcal{D}_s(\boldsymbol{X}),\ \mathcal{D}_f(\boldsymbol{X})),$$

where

$$\mathcal{D}_s(\boldsymbol{X}) = \begin{bmatrix} \Lambda_{s1} & \Lambda_{s2} \\ \Lambda_{s2}^T & \Lambda_{s3} \end{bmatrix},\ \mathcal{D}_f(\boldsymbol{X}) = \begin{bmatrix} \Lambda_{f1} & \Lambda_{f2} \\ \Lambda_{f2}^T & \Lambda_{f3} \end{bmatrix}$$

with

$$\Lambda_{s1} = A_s^T X_1 + X_1 A_s + \Pi_{1s}(\boldsymbol{X}) + Q_s,\ \Lambda_{s2} = X_1 B_s + \Pi_{2s}(\boldsymbol{X}) + L_s,$$
$$\Lambda_{s3} = \Pi_{3s}(\boldsymbol{X}) + R_s, \tag{3.28}$$
$$\Lambda_{f1} = A_{22}^T X_2 + X_2 A_{22} + \Pi_{1f}(\boldsymbol{X}) + Q_{22},\ \Lambda_{f2} = X_1 B_2 + \Pi_{2f}(\boldsymbol{X}),$$
$$\Lambda_{f3} = \Pi_{3f}(\boldsymbol{X}) + R, \tag{3.29}$$
$$\forall \boldsymbol{X} = (X_1,\ X_2) \in \mathcal{S}_{n_1} \oplus \mathcal{S}_{n_2}.$$

The next result provides a set of conditions equivalent to the existence of the stabilizing solution of system (3.9).

THEOREM 3.5. *The following are equivalent.*

(i) *The reduced system of ARE* (3.9) *has a stabilizing solution* $\tilde{\boldsymbol{X}} = (\tilde{X}_1,\ \tilde{X}_2)$ *that satisfies the following sign conditions:*

$$R_s + D_{1s}^T \tilde{X}_1 D_{1s} + D_{2s}^T \tilde{X}_2 D_{2s} > 0, \tag{3.30}$$
$$R + D_1^T \tilde{X}_1 D_1 + D_2^T \tilde{X}_2 D_2 > 0. \tag{3.31}$$

(ii) (a) *The triple* $(\boldsymbol{A},\ \boldsymbol{B},\ \boldsymbol{\Pi})$ *is stabilizable.*

(b) *There exists* $\boldsymbol{Y} = (Y_1,\ Y_2) \in \mathcal{X}$ *that verifies the LMIs:*

$$\mathcal{D}_s(\boldsymbol{Y}) > 0,\ \mathcal{D}_f(\boldsymbol{Y}) > 0.$$

The proof may be realized by following step by step the proof of Theorems 4.7 and 5.8 in [4]. For the special case $C_{12} = 0$ and $C_{22} = 0$, see [9].

Remark 3.1. In [9], the stabilizing solution $(\tilde{X}_1,\ \tilde{X}_2)$ is obtained as the limit of a sequence of approximations $\{X_{1k},\ X_{2k}\}_{k\geq 1}$. These approximations are obtained by specializing to the case of the system of AREs (3.9) for the Newton–Kantorovich algorithm. Hence, this method would be used for numerical computation of the stabilizing solution of (3.9). Unfortunately, this procedure requires the solving of some systems of coupled Lyapunov type equations for each step k. In what follows, we provide an iterative procedure for numerical computation of the stabilizing solution of (3.9) based on the solution of a decoupled standard Lyapunov equations.

This procedure is obtained by specializing to the case of (3.12) via the similar procedure given in [4]. The main steps of this procedure are as follows.

STEP 0. Choose a stabilizing feedback gain $\boldsymbol{W} = (W_1,\ W_2) \in \Re^{m\times n_1} \oplus \Re^{m\times n_2}$. One can take, for example, $W_j = V_j Z_j^{-1}, j = 1,\ 2$, where $(Z_j, V_j), j = 1, 2$, is a solution of the system of LMIs (3.40)–(3.41).

STEP 1. Compute $\boldsymbol{X}^1 = (X_1^1,\ X_2^1) \in \mathcal{S}_{n_1} \oplus \mathcal{S}_{n_2}$ as a solution of the following system of LMIs:

$$\begin{aligned}&(A_s + B_sW_1)^TX_1^1 + X_1^1(A_s + B_sW_1)\\&\quad + (C_{1s} + D_{1s}W_1)^TX_1^1(C_{1s} + D_{1s}W_1)\\&\quad + (C_{2s} + D_{2s}W_1)^TX_2^1(C_{2s} + D_{2s}W_1)\\&\quad + Q_s + L_sW_1 + W_1^TL_s^T + W_1^TR_sW_1 + \delta I_{n_1} \leq 0,\end{aligned}\tag{3.32a}$$

$$\begin{aligned}&(A_{22} + B_2W_2)^TX_2^1 + X_2^1(A_{22} + B_2W_2)\\&\quad + (C_{12} + D_1W_2)^TX_1^1(C_{12} + D_1W_2)\\&\quad + (C_{22} + D_2W_2)^TX_2^1(C_{22} + D_2W_2)\\&\quad + Q_{22} + W_2^TRW_2 + \delta I_{n_2} \leq 0,\end{aligned}\tag{3.32b}$$

where $\delta > 0$ is a fixed parameter.

Compute the feedback gains $F^1 = (F_1^1,\ F_2^1)$ by

$$F_1^1 = -[R_s + \Pi_{3s}(\tilde{\boldsymbol{X}}^1)]^{-1}[X_1^1B_s + \Pi_{2s}(\tilde{\boldsymbol{X}}^1) + \boldsymbol{L}_s]^T,\tag{3.33a}$$

$$F_2^1 = -[R + \Pi_{3f}(\tilde{\boldsymbol{X}}^1)]^{-1}[X_2^1B_2 + \Pi_{2f}(\tilde{\boldsymbol{X}}^1)]^T.\tag{3.33b}$$

STEP $\boldsymbol{k}$, $\boldsymbol{k \geq 2}$. Compute $\boldsymbol{X}^k = (X_1^k,\ X_2^k) \in \mathcal{S}_{n_1} \oplus \mathcal{S}_{n_2}$ satisfying the following decoupled standard Lyapunov equations:

$$(A_s + B_sF_1^{k-1})^TX_1^k + X_1^k(A_s + B_sF_1^{k-1}) + Q_1^k = 0,\tag{3.34}$$

$$(A_{22} + B_2F_2^{k-1})^TX_2^k + X_2^k(A_{22} + B_2F_2^{k-1}) + Q_2^k = 0,\tag{3.35}$$

where

$$\begin{aligned}Q_1^k = {}&\frac{\delta}{k}I_{n_1} + Q_s + L_sF_1^{k-1} + (F_1^{(k-1)})^TL_s^T + (F_1^{(k-1)})^TR_sF_1^{k-1}\\&+ (C_{1s} + D_{1s}F_1^{k-1})^TX_1^{k-1}(C_{1s} + D_{1s}F_1^{k-1})\\&+ (C_{2s} + D_{2s}F_1^{k-1})^TX_2^{k-1}(C_{2s} + D_{2s}F_1^{k-1}),\end{aligned}\tag{3.36}$$

$$\begin{aligned}Q_2^k = {}&\frac{\delta}{k}I_{n_2} + Q_{22} + (F_2^{(k-1)})^TRF_2^{k-1}\\&+ (C_{12} + D_1F_2^{k-1})^TX_1^{k-1}(C_{12} + D_1F_2^{k-1})\\&+ (C_{22} + D_2F_2^{k-1})^TX_2^{k-1}(C_{22} + D_2F_2^{k-1}),\end{aligned}\tag{3.37}$$

$$F_1^k = -[R_s + \Pi_{3s}(\tilde{\boldsymbol{X}}^{k-1})]^{-1}[X_1^kB_s + \Pi_{2s}(\tilde{\boldsymbol{X}}^{k-1}) + L_s]^T,\tag{3.38}$$

$$F_2^k = -[R + \Pi_{3f}(\tilde{\boldsymbol{X}}^{k-1})]^{-1}[X_2^kB_2 + \Pi_{2f}(\tilde{\boldsymbol{X}}^{k-1})]^T.\tag{3.39}$$

Remark 3.2. If $(\boldsymbol{A},\ \boldsymbol{B},\ \boldsymbol{\Pi})$ is stabilizable and there exists $\boldsymbol{Y} = (Y_1,\ Y_2)$ verifying $\mathcal{D}_s(\boldsymbol{Y}) > 0$, $\mathcal{D}_f(\boldsymbol{Y}) > 0$, then the sequences $\{\boldsymbol{X}^k\}_{k\geq 1}$ and $\{F^k\}_{k\geq 1}$ are well defined by (3.32)–(3.39). Moreover, we have $X_j^1 \geq \cdots \geq X_j^k \geq X_j^{k+1} \geq \cdots \geq Y_j$, $j = 1,\ 2$.

Setting

$$\tilde{X}_j = \lim_{k\to\infty} X_j^k,\ j = 1,\ 2,$$

$\tilde{\boldsymbol{X}} = (\tilde{X}_1,\ \tilde{X}_2)$ is simply the stabilizing solution of (3.9). The existence of a stabilizing feedback gain $\boldsymbol{W}$ from STEP 0 is closely related to the stabilizability property of the triple $(\boldsymbol{A},\ \boldsymbol{B},\ \boldsymbol{\Pi})$.

The next result provides necessary and sufficient conditions for the stabilizability of this triple.

PROPOSITION 3.6. *For* $(\boldsymbol{A},\ \boldsymbol{B},\ \boldsymbol{\Pi})$ *introduced via* (3.13)–(3.16), *the following are equivalent:*

(i) *The triple* $(\boldsymbol{A},\ \boldsymbol{B},\ \Pi)$ *is stabilizable.*

(ii) *There exist* $\boldsymbol{Z} = (Z_1,\ Z_2) \in \mathcal{S}_{n_1} \oplus \mathcal{S}_{n_2}$ *and* $\boldsymbol{V} = (V_1,\ V_2) \in \Re^{m\times n_1} \oplus \Re^{m\times n_2}$ *that solve the system of LMIs*

$$\begin{bmatrix} \Psi_1(\boldsymbol{Z},\ \boldsymbol{V}) & C_{1s}Z_1 + D_{1s}V_1 & C_{12}Z_2 + D_2V_2 \\ (C_{1s}Z_1 + D_{1s}V_1)^T & -Z_1 & 0 \\ (C_{12}Z_2 + D_2V_2)^T & 0 & -Z_2 \end{bmatrix} < 0, \tag{3.40}$$

$$\begin{bmatrix} \Psi_2(\boldsymbol{Z},\ \boldsymbol{V}) & C_{2s}Z_1 + D_{2s}V_1 & C_{22}Z_2 + D_2V_2 \\ (C_{2s}Z_1 + D_{2s}V_1)^T & -Z_1 & 0 \\ (C_{22}Z_2 + D_2V_2)^T & 0 & -Z_2 \end{bmatrix} < 0 \tag{3.41}$$

with $\Psi_1(\boldsymbol{Z},\ \boldsymbol{V}) = A_sZ_1 + Z_1A_s^T + B_sV_1 + V_1^TB_s^T$ *and* $\Psi_2(\boldsymbol{Z},\ \boldsymbol{V}) = A_{22}Z_2 + Z_2A_{22}^T + B_2V_2 + V_2^TB_2^T$.

If $(\boldsymbol{Z},\ \boldsymbol{V})$ *is a solution of the system* (3.40)–(3.41), *then* $\boldsymbol{W} = (W_1,\ W_2)$ *defined by* $W_j = V_jZ_j^{-1}$, $j = 1,\ 2$, *is a stabilizing feedback gain for the triple* $(\boldsymbol{A},\ \boldsymbol{B},\ \boldsymbol{\Pi})$.

Proof. From Definition 3.3, we know that the stabilizability of the triple $(\boldsymbol{A},\ \boldsymbol{B},\ \boldsymbol{\Pi})$ is equivalent to the existence of a feedback gain $F = (F_1,\ F_2)$ such that the eigenvalues of the linear operator $\mathcal{L}_F$ defined by (3.24) are located in $\mathbf{C}^-$ or, equivalently, the eigenvalues of the adjoint operator $\mathcal{L}_F^*$ are located in the half plane $\mathbf{C}^-$. Applying Theorem 2.11 from [1], we deduce that the fact that the eigenvalues of the operator $\mathcal{L}_F^*$ are in the half plane $\mathbf{C}^-$ is equivalent to the existence of $\mathbf{Z} = (Z_1, Z_2) \in \mathcal{S}_{n_1} \oplus \mathcal{S}_{n_2}, Z_i > 0, i = 1, 2$, such that $\mathcal{L}_F^*(\mathbf{Z}) < 0$. Finally, if we take into account the fact that $\mathcal{L}_F^*(\mathbf{Z}) = (\mathbf{A} + \mathbf{B}F)\mathbf{Z} + \mathbf{Z}(\mathbf{A} + \mathbf{B}F)^T + \Pi_F^*(\mathbf{Z})$, then we obtain via (3.22)–(3.23) together with the Schur complement technique that $\mathcal{L}_F^*(\mathbf{Z}) < 0$ is equivalent to the LMIs (3.40) and (3.41), and thus the proof is complete. ∎

4. The main results.

4.1. The asymptotic structure of the stabilizing solution of ARE (2.5)–(2.6). In this subsection, we provide a set of sufficient conditions that guarantee the existence of the stabilizing solution of ARE (2.5), which verifies the sign condition (2.6) for any $\varepsilon > 0$ that is sufficiently small.

THEOREM 4.1. *Assume the following:*

(a) A_{22} *is invertible.*

(b) *The triple* $(\boldsymbol{A},\ \boldsymbol{B},\ \boldsymbol{\Pi})$ *is stabilizable.*

(c) *There exists* $\boldsymbol{Y} = (Y_1,\ Y_2) \in \mathcal{S}_{n_1} \oplus \mathcal{S}_{n_2}$, *which satisfies the following system of LMIs:*

$$\mathcal{D}_s(\boldsymbol{Y}) > 0,\ \mathcal{D}_f(\boldsymbol{Y}) > 0. \tag{4.1}$$

Under these conditions, there exists $\varepsilon^* > 0$ *with the property that for any* $\varepsilon \in (0,\ \varepsilon^*]$, *ARE* (2.5) *has a stabilizing solution* $\tilde{X}(\varepsilon)$ *that satisfies the sign condition* (2.6). *Moreover, the stabilizing solution* $\tilde{X}(\varepsilon)$ *and the corresponding stabilizing feedback gain* $\tilde{F}(\varepsilon)$ *have an asymptotic structure,*

$$\tilde{X}(\varepsilon) = \begin{bmatrix} \tilde{X}_1 + \sqrt{\varepsilon}\check{X}_{11}(\varepsilon) & \varepsilon(\tilde{X}_{12} + \sqrt{\varepsilon}\check{X}_{12}(\varepsilon)) \\ \varepsilon(\tilde{X}_{12} + \sqrt{\varepsilon}\check{X}_{12}(\varepsilon))^T & \varepsilon(\tilde{X}_2 + \sqrt{\varepsilon}\check{X}_{22}(\varepsilon)) \end{bmatrix}, \tag{4.2}$$

$$\tilde{F}(\varepsilon) = \begin{bmatrix} \tilde{F}_1 + \sqrt{\varepsilon}\check{F}_1(\varepsilon) & \tilde{F}_2 + \sqrt{\varepsilon}\check{F}_2(\varepsilon) \end{bmatrix}, \tag{4.3}$$

where $(\tilde{X}_1,\ \tilde{X}_2) \in \mathcal{S}_{n_1} \oplus \mathcal{S}_{n_2}$ *is the stabilizing solution of the reduced system of AREs* (3.9), *satisfying the sign conditions* (3.30)–(3.31),

$$\tilde{F}_2 = \tilde{F}_f,\ \tilde{F}_1 = (I_m + \tilde{F}_2 A_{22}^{-1} B_2)(\tilde{F}_s + \tilde{F}_2 A_{22}^{-1} A_{21}) \tag{4.4}$$

($\tilde{F}_s$ *is introduced in* (3.25) *and* $\tilde{F}_f$*, in* (3.26)*).*

$$\begin{aligned} \tilde{X}_{12} = &-[A_{22}^T \tilde{X}_2 + \tilde{X}_1 A_{12} - \tilde{F}_1^T (R + D_1^T \tilde{X}_1 D_1 + D_2^T \tilde{X}_2 D_2)\tilde{F}_2 \\ &+ Q_{12} + C_{11}^T \tilde{X}_1 C_{12} + C_{21}^T \tilde{X}_2 C_{22}] A_{22}^{-1} \end{aligned} \tag{4.5}$$

$\varepsilon \to \check{X}_{ij}(\varepsilon)$, $\varepsilon \to \check{F}_j(\varepsilon)$, $i,\ j \in \{1,\ 2\}$, *are bounded functions on* $(0,\ \varepsilon^*]$.

Proof. Setting $\eta = \sqrt{\varepsilon}$, we may rewrite the system (3.2) in the compact form

$$\mathcal{F}(\boldsymbol{Y},\ \eta) = 0, \tag{4.6}$$

where $\boldsymbol{Y} = (X_{11},\ X_{12},\ X_{22},\ F_1,\ F_2) \in \mathcal{Y} = \mathcal{S}_{n_1} \oplus \Re^{n_1 \times n_2} \oplus \mathcal{S}_{n_2} \oplus \Re^{m \times n_1} \oplus \Re^{m \times n_2}$, while $\mathcal{F} : \mathcal{Y} \times \Re \to \mathcal{Y}$ is described by the left-hand side of (3.2), where $\sqrt{\varepsilon}$ is replaced by η and ε by η^2. We apply the implicit function theorem to obtain the existence of the solution with the desired properties of system (3.2).

First, let us observe that $\mathcal{F}(\cdot,\ \cdot)$ is an analytic function. On the other hand, assumptions (a)–(c) guarantee the existence of the stabilizing solution $(\tilde{X}_1,\ \tilde{X}_2)$ of the reduced system of AREs (3.9), which verifies the sign conditions (3.30)–(3.31).

Let $(\tilde{F}_s,\ \tilde{F}_f)$ be the stabilizing feedback gain constructed via (3.25)–(3.26). Take $\tilde{\boldsymbol{Y}} = (\tilde{X}_1,\ \tilde{X}_{12},\ \tilde{X}_2,\ \tilde{F}_1,\ \tilde{F}_2) \in \mathcal{Y}$, where $\tilde{F}_1$ is constructed via (4.4) and $\tilde{X}_{12}$ is constructed via (4.5), respectively.

Using the result of Proposition 3.2(ii), one obtains that

$$\mathcal{F}(\tilde{\boldsymbol{Y}},\ 0) = 0. \tag{4.7}$$

As $\mathcal{Y}$ is a finite dimensional Banach space, it follows that to show that $\boldsymbol{Z} \to \partial\mathcal{F}/\partial\boldsymbol{Y}(\tilde{\boldsymbol{Y}},\ 0)\boldsymbol{Z}$ is an isomorphism, it is sufficient to check that it is an injective map. Based on the fact that

$$\frac{\partial\mathcal{F}}{\partial\boldsymbol{Y}}(\tilde{\boldsymbol{Y}},\ 0)\boldsymbol{Z} = \lim_{h\to 0} \frac{1}{h}[\mathcal{F}(\tilde{\boldsymbol{Y}} + h\boldsymbol{Z},\ 0) - \mathcal{F}(\tilde{\boldsymbol{Y}},\ 0)],$$

one obtains that the equation

$$\frac{\partial\mathcal{F}}{\partial\boldsymbol{Y}}(\tilde{\boldsymbol{Y}},\ 0)\boldsymbol{Z} = 0 \tag{4.8}$$

with unknown $\boldsymbol{Z} = (X_{11},\ X_{12},\ X_{22},\ F_1,\ F_2) \in \mathcal{Y}$ is equivalent to the linear system

$$\begin{aligned}
&A_{11}^T X_{11} + X_{11}A_{11} + A_{21}^T X_{12}^T + X_{12}A_{21} + C_{11}^T X_{11}C_{11} + C_{21}^T X_{22}C_{21}\\
&\quad - \tilde{F}_1^T(R + D_1^T\tilde{X}_1D_1 + D_2^T\tilde{X}_2D_2)F_1 - F_1^T(R + D_1^T\tilde{X}_1D_1 + D_2^T\tilde{X}_2D_2)\tilde{F}_1\\
&\quad - \tilde{F}_1^T(D_1^TX_{11}D_1 + D_2^TX_{22}D_2)\tilde{F}_1 = 0, \qquad (4.9a)\\
&A_{21}^T X_{22} + X_{11}A_{12} + X_{12}A_{22} + C_{11}^T X_{11}C_{12} + C_{21}^T X_{22}C_{22}\\
&\quad - F_1^T(R + D_1^T\tilde{X}_1D_1 + D_2^T\tilde{X}_2D_2)\tilde{F}_2 - \tilde{F}_1^T(R + D_1^T\tilde{X}_1D_1 + D_2^T\tilde{X}_2D_2)F_2\\
&\quad - \tilde{F}_1^T(D_1^TX_{11}D_1 + D_2^TX_{22}D_2)\tilde{F}_2 = 0, \qquad (4.9b)\\
&A_{22}^T X_{22} + X_{22}A_{22} + C_{12}^T X_{11}C_{12} + C_{22}^T X_{22}C_{22}\\
&\quad - \tilde{F}_2^T(R + D_1^T\tilde{X}_1D_1 + D_2^T\tilde{X}_2D_2)F_2 - F_2^T(R + D_1^T\tilde{X}_1D_1 + D_2^T\tilde{X}_2D_2)\tilde{F}_2\\
&\quad - \tilde{F}_2^T(D_1^TX_{11}D_1 + D_2^TX_{22}D_2)\tilde{F}_2 = 0, \qquad (4.9c)\\
&B_1^T X_{11} + B_2^T X_{12}^T + D_1^T X_{11}C_{11} + D_2^T X_{22}C_{21}\\
&\quad + (D_1^TX_{11}D_1 + D_2^TX_{22}D_2)\tilde{F}_1 + (R + D_1^T\tilde{X}_1D_1 + D_2^T\tilde{X}_2D_2)F_1 = 0, \qquad (4.9d)\\
&B_2^T X_{22} + D_1^T X_{11}C_{12} + D_2^T X_{22}C_{22}\\
&\quad + (D_1^TX_{11}D_1 + D_2^TX_{22}D_2)\tilde{F}_2 + (R + D_1^T\tilde{X}_1D_1 + D_2^T\tilde{X}_2D_2)F_2 = 0. \qquad (4.9e)
\end{aligned}$$

Because A_{22} is invertible, we may use the second equation of system (4.9) to eliminate X_{12} from the other equation of that system. By some algebraic laborious calculation, based on the identities from Lemma 3.1, one obtains that if $(X_{11},\ X_{12},\ X_{22},\ F_1,\ F_2)$ is a solution of the system (4.9), then $(X_{11},\ X_{22})$ is a solution of the system

$$\begin{aligned}
&(A_s + B_s\tilde{F}_s)^T X_{11} + X_{11}(A_s + B_s\tilde{F}_s) + (C_{1s} + D_{1s}\tilde{F}_s)^T X_{11}(C_{1s} + D_{1s}\tilde{F}_s)\\
&\quad + (C_{2s} + D_{2s}\tilde{F}_s)^T X_{22}(C_{2s} + D_{2s}\tilde{F}_s) = 0, \qquad (4.10a)\\
&(A_{22} + B_2\tilde{F}_f)^T X_{22} + X_{22}(A_{22} + B_2\tilde{F}_f) + (C_{12} + D_1\tilde{F}_f)^T X_{11}(C_{12} + D_1\tilde{F}_f)\\
&\quad + (C_{22} + D_2\tilde{F}_f)^T X_{22}(C_{22} + D_2\tilde{F}_f) = 0. \qquad (4.10b)
\end{aligned}$$

System (4.10) may be rewritten in a compact form as follows: $\mathcal{L}_{\tilde{F}}^*(\mathbf{X}) = 0$. If we take into account Definition 3.4 of the stabilizing solution of (3.12), we deduce that the eigenvalues of the operators $\mathcal{L}_{\tilde{F}}$ and $\mathcal{L}_{\tilde{F}}^*$ are in the half plane $\mathbf{C}^-$. Applying Theorem 4.5 in [8], we deduce that (4.10) has a unique solution. As $\hat{X}_1 = 0, \hat{X}_2 = 0$ verifies (4.10), we may conclude that $X_{11} = 0,\ X_{22} = 0$. Furthermore, from the last two equations in (4.9), one obtains $F_1 = 0$ and $F_2 = 0$. Finally, from the second equation of (4.9), one obtains $X_{12} = 0$. We have shown that (4.8) has only the solution $\boldsymbol{Z} = 0$, which confirms the injectivity of the map $\boldsymbol{Z} \to \partial\mathcal{F}/\partial\boldsymbol{Y}(\tilde{\boldsymbol{Y}},\ 0)\boldsymbol{Z}$.

Using the simple form of $\mathcal{F}(\cdot,\ \cdot)$, one easily proves the continuity of $(\boldsymbol{Y},\ \varepsilon) \to \partial\mathcal{F}/\partial\boldsymbol{Y}(\boldsymbol{Y}, \varepsilon)$. As such, we have shown that the assumptions of the implicit function theorem [18] are verified for (4.6).

Thus, we deduce that there exist $\eta_1 > 0$ and an analytic function $\eta \to \boldsymbol{Y}(\eta) = (X_{11}(\eta),\ X_{12}(\eta),\ X_{22}(\eta),\ F_1(\eta),\ F_2(\eta)) : (-\eta_1,\ \eta_1) \to \mathcal{Y}$ that verify $\mathcal{F}(\boldsymbol{Y}(\eta),\ \eta) = 0$ for all $|\eta| \le \eta_1$.

We also have the asymptotic structure

$$\begin{aligned}
&X_{ii}(\eta) = \tilde{X}_i + \eta\check{X}_{ii}(\eta),\ F_i(\eta) = \tilde{F}_i + \eta\check{F}_i(\eta),\ i = 1,\ 2,\\
&X_{12}(\eta) = \tilde{X}_{12} + \eta\check{X}_{12}(\eta)\ \forall\eta \in (-\eta_1,\ \eta_1). \qquad (4.11)
\end{aligned}$$

We set

$$\tilde{X}(\varepsilon) = \begin{bmatrix} X_{11}(\sqrt{\varepsilon}) & \varepsilon X_{12}(\sqrt{\varepsilon}) \\ \varepsilon X_{12}^T(\sqrt{\varepsilon}) & \varepsilon X_{22}(\sqrt{\varepsilon}) \end{bmatrix}, \qquad (4.12)$$

$$\tilde{F}(\varepsilon) = \left[\begin{array}{cc} F_1(\sqrt{\varepsilon}) & F_2(\sqrt{\varepsilon}) \end{array} \right], \quad 0 \le \varepsilon \le \varepsilon_1 = \eta_1^2. \tag{4.13}$$

Using (3.1) and (3.2), we deduce that $\tilde{X}(\varepsilon)$ is a solution of ARE (2.5) and $\tilde{F}(\varepsilon)$ is a corresponding feedback gain associated via (2.4). Combining (3.31) and (4.11), we deduce that there exists $0 < \varepsilon_2 \le \varepsilon_1$ such that

$$\begin{aligned} &R + D_1^T X_{11}(\sqrt{\varepsilon}) D_1 + \sqrt{\varepsilon}(D_2^T X_{12}^T(\sqrt{\varepsilon}) D_1 + D_1^T X_{12}(\sqrt{\varepsilon}) D_2) \\ &\quad + D_2^T X_{22}(\sqrt{\varepsilon}) D_2 > 0 \; \forall \varepsilon \in (0,\ \varepsilon_2]. \end{aligned} \tag{4.14}$$

Thus, from (4.11) and (4.14), it follows that $\tilde{X}$ is a solution of ARE (2.5), which satisfies the sign condition (2.6) for $0 < \varepsilon < \varepsilon_2$ and has the asymptotic structure (4.2) and (4.3). It remains to be shown that there exists $\varepsilon^* \in (0,\ \varepsilon_2]$ such that $\tilde{X}(\varepsilon)$ is the stabilizing solution of (2.5) for any $\varepsilon \in (0,\ \varepsilon^*]$. Toward this end, we show that the linear equation on $\mathcal{S}_n$, $n = n_1 + n_2$,

$$\begin{aligned} &[A(\varepsilon) + B(\varepsilon)\tilde{F}(\varepsilon)]^T U + U[A(\varepsilon) + B(\varepsilon)\tilde{F}(\varepsilon)] \\ &\quad + [C(\varepsilon) + D(\varepsilon)\tilde{F}(\varepsilon)]^T U [C(\varepsilon) + D(\varepsilon)\tilde{F}(\varepsilon)] + I_n = 0 \end{aligned} \tag{4.15}$$

has a solution $U(\varepsilon) > 0$. Taking

$$U(\varepsilon) = \begin{bmatrix} U_{11} & \varepsilon U_{12} \\ \varepsilon U_{12}^T & \varepsilon U_{22} \end{bmatrix}$$

and using the structure (2.7) of the coefficients of (4.15), we obtain the following partition of this equation:

$$\begin{aligned} &[A_{11} + B_1 F_1(\sqrt{\varepsilon})]^T U_{11} + U_{11}[A_{11} + B_1 F_1(\sqrt{\varepsilon})] \\ &\quad + [A_{21} + B_2 F_1(\sqrt{\varepsilon})]^T U_{12}^T + U_{12}[A_{21} + B_2 F_1(\sqrt{\varepsilon})] \\ &\quad + [C_{11} + D_1 F_1(\sqrt{\varepsilon})]^T U_{11} [C_{11} + D_1 F_1(\sqrt{\varepsilon})] \\ &\quad + \sqrt{\varepsilon}\big([C_{21} + D_2 F_1(\sqrt{\varepsilon})]^T U_{12}^T [C_{11} + D_1 F_1(\sqrt{\varepsilon})] \\ &\qquad\quad + [C_{11} + D_1 F_1(\sqrt{\varepsilon})]^T U_{12} [C_{21} + D_2 F_1(\sqrt{\varepsilon})]\big) \\ &\quad + [C_{21} + D_2 F_1(\sqrt{\varepsilon})]^T U_{22} [C_{21} + D_2 F_1(\sqrt{\varepsilon})] + I_{n_1} = 0, \end{aligned} \tag{4.16a}$$

$$\begin{aligned} &\varepsilon[A_{11} + B_1 F_1(\sqrt{\varepsilon})]^T U_{12} + [A_{21} + B_2 F_1(\sqrt{\varepsilon})]^T U_{22} \\ &\quad + U_{11}[A_{12} + B_1 F_2(\sqrt{\varepsilon})] + U_{12}[A_{22} + B_2 F_2(\sqrt{\varepsilon})] \\ &\quad + [C_{11} + D_1 F_1(\sqrt{\varepsilon})]^T U_{11} [C_{12} + D_1 F_2(\sqrt{\varepsilon})] \\ &\quad + \sqrt{\varepsilon}\big([C_{21} + D_2 F_1(\sqrt{\varepsilon})]^T U_{12}^T [C_{12} + D_1 F_2(\sqrt{\varepsilon})] \\ &\qquad\quad + [C_{11} + D_1 F_1(\sqrt{\varepsilon})]^T U_{12} [C_{22} + D_2 F_2(\sqrt{\varepsilon})]\big) \\ &\quad + [C_{21} + D_2 F_1(\sqrt{\varepsilon})]^T U_{22} [C_{22} + D_2 F_2(\sqrt{\varepsilon})] = 0, \end{aligned} \tag{4.16b}$$

$$\begin{aligned} &[A_{22} + B_2 F_2(\sqrt{\varepsilon})]^T U_{22} + U_{22}[A_{22} + B_2 F_2(\sqrt{\varepsilon})] \\ &\quad + \varepsilon\big([A_{12} + B_1 F_2(\sqrt{\varepsilon})]^T U_{12} + U_{12}^T [A_{12} + B_1 F_2(\sqrt{\varepsilon})]\big) \\ &\quad + [C_{12} + D_1 F_2(\sqrt{\varepsilon})]^T U_{11} [C_{12} + D_1 F_2(\sqrt{\varepsilon})] \\ &\quad + \sqrt{\varepsilon}\big([C_{22} + D_2 F_2(\sqrt{\varepsilon})]^T U_{12}^T [C_{12} + D_1 F_2(\sqrt{\varepsilon})] \\ &\qquad\quad + [C_{12} + D_1 F_2(\sqrt{\varepsilon})]^T U_{12} [C_{22} + D_2 F_2(\sqrt{\varepsilon})]\big) \\ &\quad + [C_{22} + D_2 F_2(\sqrt{\varepsilon})]^T U_{22} [C_{22} + D_2 F_2(\sqrt{\varepsilon})] + I_{n_2} = 0. \end{aligned} \tag{4.16c}$$

Taking $\varepsilon = 0$ in (4.16), one obtains via (4.11) the following system:

$$
\begin{aligned}
&(A_{11} + B_1\tilde{F}_1)^T\bar{U}_{11} + \bar{U}_{11}(A_{11} + B_1\tilde{F}_1) \\
&\quad + (A_{21} + B_2\tilde{F}_1)^T\bar{U}_{12}^T + \bar{U}_{12}(A_{21} + B_2\tilde{F}_1) \\
&\quad + (C_{11} + D_1\tilde{F}_1)^T\bar{U}_{11}(C_{11} + D_1\tilde{F}_1) \\
&\quad + (C_{21} + D_2\tilde{F}_1)^T\bar{U}_{22}(C_{21} + D_2\tilde{F}_1) + I_{n_1} = 0, \qquad (4.17a)\\
&(A_{21} + B_2\tilde{F}_1)^T\bar{U}_{22} + \bar{U}_{11}(A_{12} + B_1\tilde{F}_2) + \bar{U}_{12}(A_{22} + B_2\tilde{F}_2) \\
&\quad + (C_{11} + D_1\tilde{F}_1)^T\bar{U}_{11}(C_{12} + D_1\tilde{F}_2) \\
&\quad + (C_{21} + D_2\tilde{F}_1)^T\bar{U}_{22}(C_{22} + D_2\tilde{F}_2) = 0, \qquad (4.17b)\\
&(A_{22} + B_2\tilde{F}_2)^T\bar{U}_{22} + \bar{U}_{22}(A_{22} + B_2\tilde{F}_2) \\
&\quad + (C_{12} + D_1\tilde{F}_2)^T\bar{U}_{11}(C_{12} + D_1\tilde{F}_2) \\
&\quad + (C_{22} + D_2\tilde{F}_2)^T\bar{U}_{22}(C_{22} + D_2\tilde{F}_2) + I_{n_2} = 0. \qquad (4.17c)
\end{aligned}
$$

By direct calculations based on the identities given in Lemma 3.1 as well as $A_{22}+B_2\tilde{F}_2$ as a stable matrix, one obtains that if $(\bar{U}_{11},\ \bar{U}_{12},\ \bar{U}_{22})$ is a solution of system (4.17), then $(\bar{U}_{11},\ \bar{U}_{22})$ is a solution of the system

$$
\begin{aligned}
&(A_s + B_s\tilde{F}_s)^T\bar{U}_{11} + \bar{U}_{11}(A_s + B_s\tilde{F}_s) + (C_{1s} + D_{1s}\tilde{F}_s)^T\bar{U}_{11}(C_{1s} + D_{1s}\tilde{F}_s) \\
&\quad + (C_{2s} + D_{2s}\tilde{F}_s)^T\bar{U}_{22}(C_{2s} + D_{2s}\tilde{F}_s) + H_s = 0, \qquad (4.18a)\\
&(A_{22} + B_2\tilde{F}_2)^T\bar{U}_{22} + \bar{U}_{22}(A_{22} + B_2\tilde{F}_2) + (C_{12} + D_1\tilde{F}_2)^T\bar{U}_{11}(C_{12} + D_1\tilde{F}_2) \\
&\quad + (C_{22} + D_2\tilde{F}_2)^T\bar{U}_{22}(C_{22} + D_2\tilde{F}_2) + I_{n_2} = 0, \qquad (4.18b)
\end{aligned}
$$

where

$$(4.19)\quad H_s = I_{n_1} + (A_{21} + B_2\tilde{F}_1)^T(A_{22} + B_2\tilde{F}_2)^{-T}(A_{22} + B_2\tilde{F}_2)^{-1}(A_{21} + B_2\tilde{F}_1)^T$$

and $\tilde{F} = (\tilde{F}_s,\ \tilde{F}_f)$ is the stabilizing feedback gain correspond to the solution $(\tilde{X}_1,\ \tilde{X}_2)$.

System (4.18) can be regarded as a system on the linear space $\mathcal{X} = \mathcal{S}_{n_1} \oplus \mathcal{S}_{n_2}$ of the form

$$\mathcal{L}_{\tilde{F}}(\boldsymbol{U}) + \boldsymbol{H} = 0, \qquad (4.20)$$

where $\boldsymbol{U} = (\bar{U}_{11},\ \bar{U}_{22}) \in \mathcal{S}_{n_1} \oplus \mathcal{S}_{n_2}$ and $\boldsymbol{H} = (H_s,\ I_{n_2})$ and $\mathcal{L}_{\tilde{F}}$ are linear operators (3.24) with F replaced by $\tilde{F}$. Based on Definition 3.2 of the stabilizing solution of the reduced system of ARE (3.12), we deduce that the eigenvalues of the linear operator $\mathcal{L}_{\tilde{F}}$ are in the half plane $\mathbf{C}^-$. On the other hand, from (4.19), we have that $\boldsymbol{H} > 0$. Thus, applying Theorem 4.5(iii) in [5], we deduce that (4.20) has a unique solution $\tilde{\boldsymbol{U}} = (\tilde{U}_1,\ \tilde{U}_2)$, and this solution satisfies $\tilde{U}_i > 0$, $i = 1,\ 2$. Set $\tilde{U}_{12} = -[(A_{21} + B_2\tilde{F}_1)^T\tilde{U}_2 + \tilde{U}_1(A_{12} + B_2\tilde{F}_2) + (C_{11} + D_1\tilde{F}_1)^T\tilde{U}_1(C_{12} + D_1\tilde{F}_2) + (C_{21} + D_2\tilde{F}_1)^T\tilde{U}_2(C_{22} + D_2\tilde{F}_2)](A_{22} + B_2\tilde{F}_2)^{-1}$. One obtains by direct calculations that the triple $(\tilde{U}_1,\ \tilde{U}_{12},\ \tilde{U}_2)$ is a solution of system (4.17).

Moreover, the fact that $(\tilde{U}_1,\ \tilde{U}_2)$ is the unique solution of (4.18) guarantees that $(\tilde{U}_1,\ \tilde{U}_{12},\ \tilde{U}_2)$ is a unique solution of (4.17). Therefore, we deduce that there exists $\varepsilon_3 \in (0,\ \varepsilon_2]$ such that for $0 \le \varepsilon \le \varepsilon_3$, system (4.16) has a unique solution $(U_{11}(\varepsilon),\ U_{12}(\varepsilon),\ U_{22}(\varepsilon))$ with the property that

$$\lim_{\varepsilon \to 0+} U_{ii}(\varepsilon) = \tilde{U}_i,\ i = 1,\ 2, \qquad (4.21)$$

$$\lim_{\varepsilon \to 0+} U_{12}(\varepsilon) = \tilde{U}_{12}. \qquad (4.22)$$

This means that there exists $\varepsilon^* \le \varepsilon_3$ such that

$$U_{22}(\varepsilon) > 0,\ U_{11}(\varepsilon) - \varepsilon U_{12}(\varepsilon)U_{22}^{-1}(\varepsilon)U_{12}^T(\varepsilon) > 0\ \forall\varepsilon \in (0,\ \varepsilon^*]. \tag{4.23}$$

Setting

$$U(\varepsilon) = \begin{bmatrix} U_{11}(\varepsilon) & \varepsilon U_{12}(\varepsilon) \\ \varepsilon U_{12}^T(\varepsilon) & \varepsilon U_{22}(\varepsilon) \end{bmatrix},$$

one obtains via (4.23) that $U(\varepsilon)$ is the positive definite solution of (4.15). Thus, the proof is complete. □

COROLLARY 4.2. *Assume that the assumptions of Theorem* 4.1 *are fulfilled. Let* $(\tilde{X}_1,\ \tilde{X}_2)$ *be the stabilizing solution of the reduced system of AREs* (3.9) *and let* $(\tilde{F}_s,\ \tilde{F}_f)$ *be the corresponding feedback gains constructed via* (3.25)–(3.26). *Let* $\tilde{F}_1$ *and* $\tilde{F}_2$ *be constructed via* (4.4). *Under these conditions, there exists* $0 < \varepsilon^{**} \le \varepsilon^*$ *with the property that the control*

$$u_{\mathrm{app}}(t) = \tilde{F}_1 x_1(t) + \tilde{F}_2 x_2(t) \tag{4.24}$$

stabilizes system (2.1) *for arbitrary* $0 < \varepsilon \le \varepsilon^{**}$.

The proof can be obtained by repeating the reasoning from the last part of the proof of Theorem 4.1 by replacing (4.15) with

$$\begin{aligned} &[A(\varepsilon) + B(\varepsilon)\tilde{\tilde{F}}]^T U + U[A(\varepsilon) + B(\varepsilon)\tilde{\tilde{F}}] \\ &\quad + [C(\varepsilon) + D(\varepsilon)\tilde{\tilde{F}}]^T U[C(\varepsilon) + D(\varepsilon)\tilde{\tilde{F}}] + I_n = 0, \end{aligned} \tag{4.25}$$

where $\tilde{\tilde{F}} = [\tilde{F}_1\ \tilde{F}_2]$.

Remark 4.1.

(a) Control (4.24) does not depend upon ε, but it stabilizes system (2.1) for a sufficiently small $\varepsilon > 0$. The level of suboptimality with respect to the optimal value of the cost functional (2.2) achieved by control (4.24) will be analyzed in the next subsection.

(b) The result proved in Theorem 4.1 shows that in the stochastic framework of systems of type (2.1), the dominant parts of the stabilizing solution and of the stabilizing feedback constructed via the stabilizing solution of the reduced system of AREs achieve an approximation of order $O(\sqrt{\varepsilon})$ of the exact stabilizing solution and of the exact stabilizing feedback gain, respectively. It is worth mentioning that in the deterministic framework, such deviations are of the order $O(\varepsilon)$.

Taking into account the analyticity of the solution $\boldsymbol{Y}(\eta)$ of the implicit function problem (4.6), we may write

$$\boldsymbol{Y}(\eta) = \tilde{\boldsymbol{Y}} + \sum_{k=1}^{\infty} \eta^k \boldsymbol{Y}^k,$$

where $\tilde{\boldsymbol{Y}} = (\tilde{X}_1,\ \tilde{X}_{12},\ \tilde{X}_2,\ \tilde{F}_1,\ \tilde{F}_2)$ and $\boldsymbol{Y}^k$ are obtained, for example, by successively differentiating (4.6) verified by $\boldsymbol{Y}(\eta)$.

Therefore, for $k = 1$, $\boldsymbol{Y}^1$ is obtained as solution of the linear equation

$$\frac{\partial \mathcal{F}}{\partial \boldsymbol{Y}}(\tilde{\boldsymbol{Y}},\ 0)\boldsymbol{Y}^1 + \frac{\partial \mathcal{F}}{\partial \eta}(\tilde{\boldsymbol{Y}},\ 0) = 0. \tag{4.26}$$

Setting $\boldsymbol{Y}^1 = (X_1^1,\ X_{12}^1,\ X_2^1,\ F_1^1,\ F_2^1)$, one obtains the following equation (4.26):

$$\begin{aligned}
&A_{11}^T X_1^1 + X_1^1 A_{11} + A_{21}^T X_{12}^{1T} + X_{12}^1 A_{21} + C_{11}^T X_1^1 C_{11} + C_{21}^T X_2^1 C_{21} \\
&\quad - \tilde{F}_1^T (R + D_1^T \tilde{X}_1 D_1 + D_2^T \tilde{X}_2 D_2) F_1^1 - F_1^{1T} (R + D_1^T \tilde{X}_1 D_1 + D_2^T \tilde{X}_2 D_2) \tilde{F}_1 \\
&\quad - \tilde{F}_1^T (D_1^T X_1^1 D_1 + D_2^T X_2^1 D_2) \tilde{F}_1 + M_1 = 0, \qquad (4.27a)\\
&A_{21}^T X_2^1 + X_1^1 A_{12} + X_{12}^1 A_{22} + C_{11}^T X_1^1 C_{12} + C_{21}^T X_2^1 C_{22} \\
&\quad - F_1^{1T} (R + D_1^T \tilde{X}_1 D_1 + D_2^T \tilde{X}_2 D_2) \tilde{F}_2 - \tilde{F}_1^T (R + D_1^T \tilde{X}_1 D_1 + D_2^T \tilde{X}_2 D_2) F_2^1 \\
&\quad - \tilde{F}_1^T (D_1^T X_1^1 D_1 + D_2^T X_2^1 D_2) \tilde{F}_2 + M_{12} = 0, \qquad (4.27b)\\
&A_{22}^T X_2^1 + X_2^1 A_{22} + C_{12}^T X_1^1 C_{12} + C_{22}^T X_2^1 C_{22} \\
&\quad - \tilde{F}_2^T (R + D_1^T \tilde{X}_1 D_1 + D_2^T \tilde{X}_2 D_2) F_2^1 - F_2^{1T} (R + D_1^T \tilde{X}_1 D_1 + D_2^T \tilde{X}_2 D_2) \tilde{F}_2 \\
&\quad - \tilde{F}_2^T (D_1^T X_1^1 D_1 + D_2^T X_2^1 D_2) \tilde{F}_2 + M_2 = 0, \qquad (4.27c)\\
&B_1^T X_1^1 + B_2^T X_{12}^{1T} + D_1^T X_1^1 C_{11} + D_2^T X_2^1 C_{21} + (D_1^T X_1^1 D_1 + D_2^T X_2^1 D_2) \tilde{F}_1 \\
&\quad + (R + D_1^T \tilde{X}_1 D_1 + D_2^T \tilde{X}_2 D_2) F_1^1 + M_3 = 0, \qquad (4.27d)\\
&B_2^T X_2^1 + D_1^T X_1^1 C_{12} + D_2^T X_2^1 C_{22} + (D_1^T X_1^1 D_1 + D_2^T X_2^1 D_2) \tilde{F}_2 \\
&\quad + (R + D_1^T \tilde{X}_1 D_1 + D_2^T \tilde{X}_2 D_2) F_2^1 + M_4 = 0, \qquad (4.27e)
\end{aligned}$$

where

$$\begin{aligned}
M_1 &= C_{21}^T \tilde{X}_{12}^T C_{11} + C_{11}^T \tilde{X}_{12} C_{21} - \tilde{F}_1^T (D_2^T \tilde{X}_{12}^T D_1 + D_1^T \tilde{X}_{12} D_2) \tilde{F}_1, \qquad (4.28a)\\
M_{12} &= C_{21}^T \tilde{X}_{12}^T C_{12} + C_{11}^T \tilde{X}_{12} C_{22} - \tilde{F}_1^T (D_2^T \tilde{X}_{12}^T D_1 + D_1^T \tilde{X}_{12} D_2) \tilde{F}_2, \qquad (4.28b)\\
M_2 &= C_{22}^T \tilde{X}_{12}^T C_{12} + C_{12}^T \tilde{X}_{12} C_{22} - \tilde{F}_2^T (D_2^T \tilde{X}_{12}^T D_1 + D_1^T \tilde{X}_{12} D_2) \tilde{F}_2, \qquad (4.28c)\\
M_3 &= D_2^T \tilde{X}_{12}^T C_{11} + D_1^T \tilde{X}_{12} C_{21} - (D_2^T \tilde{X}_{12}^T D_1 + D_1^T \tilde{X}_{12} D_2) \tilde{F}_1, \qquad (4.28d)\\
M_4 &= D_2^T \tilde{X}_{12}^T C_{12} + D_1^T \tilde{X}_{12} C_{22} - (D_2^T \tilde{X}_{12}^T D_1 + D_1^T \tilde{X}_{12} D_2) \tilde{F}_2. \qquad (4.28e)
\end{aligned}$$

Using the second equation of (4.27), we eliminate X_{12}^1 from the other equation of this system. Furthermore, taking into account (3.30)–(3.31), we conclude that if $(X_1^1,\ X_{12}^1,\ X_2^1,\ F_1^1,\ F_2^1)$ is a solution of (4.27), then $(X_1^1,\ X_2^1)$ verifies system

$$\begin{aligned}
&(A_s + B_s \tilde{F}_s)^T X_1^1 + X_1^1 (A_s + B_s \tilde{F}_s) + (C_{1s} + D_{1s} \tilde{F}_s)^T X_1^1 (C_{1s} + D_{1s} \tilde{F}_s) \\
&\quad + (C_{2s} + D_{2s} \tilde{F}_s)^T X_2^1 (C_{2s} + D_{2s} \tilde{F}_s) + \tilde{M}_1 = 0, \qquad (4.29a)\\
&(A_{22} + B_2 \tilde{F}_f)^T X_2^1 + X_2^1 (A_{22} + B_2 \tilde{F}_f) + (C_{12} + D_1 \tilde{F}_f)^T X_1^1 (C_{12} + D_1 \tilde{F}_f) \\
&\quad + (C_{22} + D_2 \tilde{F}_f)^T X_2^1 (C_{22} + D_2 \tilde{F}_f) + \tilde{M}_2 = 0, \qquad (4.29b)
\end{aligned}$$

where $\tilde{M}_1$ and $\tilde{M}_2$ are computed based on M_1, M_{12}, M_2, M_3, and M_4 in (4.28). System (4.29) has a unique solution $(X_1^1,\ X_2^1) \in \mathcal{S}_{n_1} \oplus \mathcal{S}_{n_2}$ because $(\tilde{F}_s,\ \tilde{F}_f)$ is the stabilizing feedback gain. Furthermore, the last two equations of (4.27) allow us to uniquely compute the matrices F_1^1 and F_2^1. Finally, from the second equation of (4.27), one uniquely determines X_{12}^1. Therefore, we have proved the following.

COROLLARY 4.3. *Under the assumptions of Theorem* 4.1, *there exists* $\varepsilon^* > 0$ *with the property that for any* $\varepsilon \in (0,\ \varepsilon^*]$, *the stabilizing solution* $\tilde{X}(\varepsilon)$ *of ARE* (2.5), *which verifies the sign condition* (2.6), *exists and has an asymptotic structure,*

$$\tilde{X}(\varepsilon) = \begin{bmatrix} \tilde{X}_1 + \sqrt{\varepsilon} X_1^1 + O(\varepsilon) & \varepsilon[\tilde{X}_{12} + \sqrt{\varepsilon} X_{12}^1 + O(\varepsilon)] \\ \varepsilon[\tilde{X}_{12} + \sqrt{\varepsilon} X_{12}^1 + O(\varepsilon)]^T & \varepsilon[\tilde{X}_2 + \sqrt{\varepsilon} X_2^1 + O(\varepsilon)] \end{bmatrix}.$$

The corresponding stabilizing feedback gain has an asymptotic structure,

$$\tilde{F}(\varepsilon) = \begin{bmatrix} \tilde{F}_1 + \sqrt{\varepsilon} F_1^1 + O(\varepsilon) & \tilde{F}_2 + \sqrt{\varepsilon} F_2^1 + O(\varepsilon) \end{bmatrix},$$

where $\tilde{X}_1$, $\tilde{X}_{12}$, $\tilde{X}_2$, $\tilde{F}_1$ *and* $\tilde{F}_2$ *are those from Theorem* 4.1, *where* $(X_1^1,\ X_{12}^1,\ X_2^1,\ F_1^1,\ F_2^1)$ *is a solution of* (4.27)–(4.28).

At the end of this subsection, let us note that as a result of the sign conditions (3.30)–(3.31), we may eliminate the variables F_1^1 and F_2^1 from systems (4.27)–(4.28). Thus, we obtain

$$\begin{aligned}
&(A_{11} + B_1\tilde{F}_1)^T X_1^1 + X_1^1(A_{11} + B_1\tilde{F}_1)\\
&\quad + (A_{21} + B_2\tilde{F}_1)^T X_{12}^{1T} + X_{12}^1(A_{21} + B_2\tilde{F}_1)\\
&\quad + (C_{11} + D_1\tilde{F}_1)^T X_1^1(C_{11} + D_1\tilde{F}_1) + (C_{21} + D_2\tilde{F}_1)^T X_2^1(C_{21} + D_2\tilde{F}_1)\\
&\quad +(C_{21} + D_2\tilde{F}_1)^T \tilde{X}_{12}^T(C_{11} + D_1\tilde{F}_1)\\
&\quad + (C_{11} + D_1\tilde{F}_1)^T \tilde{X}_{12}(C_{21} + D_2\tilde{F}_1) = 0, \qquad (4.30a)\\
&(A_{21} + B_2\tilde{F}_1)^T X_2^1 + X_1^1(A_{12} + B_1\tilde{F}_2) + X_{12}^1(A_{22} + B_2\tilde{F}_2)\\
&\quad + (C_{11} + D_1\tilde{F}_1)^T X_1^1(C_{12} + D_1\tilde{F}_2) + (C_{21} + D_2\tilde{F}_1)^T X_2^1(C_{22} + D_2\tilde{F}_2)\\
&\quad + (C_{21} + D_2\tilde{F}_1)^T \tilde{X}_{12}^T(C_{12} + D_1\tilde{F}_2)\\
&\quad + (C_{11} + D_1\tilde{F}_1)^T \tilde{X}_{12}(C_{22} + D_2\tilde{F}_2) = 0, \qquad (4.30b)\\
&(A_{22} + B_2\tilde{F}_2)^T X_2^1 + X_2^1(A_{22} + B_2\tilde{F}_2) + (C_{12} + D_1\tilde{F}_2)^T X_1^1(C_{12} + D_1\tilde{F}_2)\\
&\quad + (C_{22} + D_2\tilde{F}_2)^T X_2^1(C_{22} + D_2\tilde{F}_2) + (C_{22} + D_2\tilde{F}_2)^T \tilde{X}_{12}^T(C_{12} + D_1\tilde{F}_2)\\
&\quad + (C_{12} + D_1\tilde{F}_2)^T \tilde{X}_{12}(C_{22} + D_2\tilde{F}_2) = 0. \qquad (4.30c)
\end{aligned}$$

4.2. The estimation of the performance lostness. In this subsection, we analyze the deviation from the optimal value of the cost (2.2) when control (4.24) is used instead of optimal controls (2.3)–(2.4).

THEOREM 4.4. *Under the assumption from Theorem* 4.1, *we have*

$$0 \le J_{\text{app}} - J_{\text{opt}} \le \rho\varepsilon|x_0|^2 \ \forall \varepsilon \in (0,\ \varepsilon^{**}), \qquad (4.31)$$

where $J_{\text{app}} = J(u_{\text{app}})$, $J_{\text{opt}} = J(\tilde{u})$ *is the optimal value of the cost functional,* $x(0) = x_0 = [\ x_1^T(0) \quad x_2^T(0)\ ]^T$, $\rho > 0$ *is a constant independent of* x_0, ε, *and* ε^{**} *is given in Corollary* 4.2.

Proof. Applying Corollary 4.2, we deduce that the control $u_{\text{app}}(t)$ stabilizes system (2.1) for arbitrary $0 < \varepsilon \le \varepsilon^{**}$. Hence, the eigenvalues of the Lyapunov-type operator $Y \to [A(\varepsilon)+B(\varepsilon)\hat{F}]^T Y + Y[A(\varepsilon)+B(\varepsilon)\hat{F}] + [C(\varepsilon)+D(\varepsilon)\hat{F}]^T Y[C(\varepsilon)+D(\varepsilon)\hat{F}]$ are in the half plane $\mathbf{C}^-$. Thus, we obtain via Theorem 4.5(i) and (iii) in [8] that the linear equation on $\mathcal{S}_n$,

$$\begin{aligned}
&[A(\varepsilon) + B(\varepsilon)\hat{F}]^T V(\varepsilon) + V(\varepsilon)[A(\varepsilon) + B(\varepsilon)\hat{F}]\\
&\quad + [C(\varepsilon) + D(\varepsilon)\hat{F}]^T V(\varepsilon)[C(\varepsilon) + D(\varepsilon)\hat{F}] + Q + \hat{F}^T R\hat{F} = 0, \qquad (4.32)
\end{aligned}$$

has a unique solution $V(\varepsilon)$, where $\hat{F} = (\tilde{F}_1 \quad \tilde{F}_2)$ is the gain matrix of (4.24). We have $J(u_{\text{app}}(t)) = x_0^T V(\varepsilon)x_0$. In order to prove (4.31), we have to estimate $\|V(\varepsilon) - \tilde{X}(\varepsilon)\|$. The unique solution of (4.32) is given by

$$V(\varepsilon) = E\int_0^\infty \Phi^T(s,\ 0,\ \varepsilon)[Q + \hat{F}^T R\hat{F}]\Phi(s,\ 0,\ \varepsilon)ds, \qquad (4.33)$$

where $\Phi(s,\ 0,\ \varepsilon)$ is the matrix solution of the initial value problem

$$d\Phi(s,\ 0,\ \varepsilon) = [A(\varepsilon) + B(\varepsilon)\hat{F}]\Phi(s,\ 0,\ \varepsilon)ds + [C(\varepsilon) + D(\varepsilon)\hat{F}]\Phi(s,\ 0,\ \varepsilon)dw(s),\ s > 0,$$

where $\Phi(0,\ 0,\ \varepsilon) = I_n$.

Let

$$\Phi(t,\ 0,\ \varepsilon) = \begin{bmatrix} \Phi_{11}(t,\ 0,\ \varepsilon) & \Phi_{12}(t,\ 0,\ \varepsilon) \\ \Phi_{21}(t,\ 0,\ \varepsilon) & \Phi_{22}(t,\ 0,\ \varepsilon) \end{bmatrix}$$

be the partition of the matrix solution $\Phi(t, 0, \varepsilon)$ compatible with the partition of the coefficients of system (2.1). Applying Theorem 4.4 in [3], we obtain the estimates

$$\begin{aligned} &E[|\Phi_{k1}(t,\ 0,\ \varepsilon)|^2] \le \beta_{k1} e^{-\alpha_1 t}, k \in \{1,\ 2\}, \\ &E[|\Phi_{12}(t,\ 0,\ \varepsilon)|^2] \le \beta_{12}\varepsilon e^{-\alpha_1 t}, \\ &E[|\Phi_{22}(t,\ 0,\ \varepsilon)|^2] \le \beta_{22}(e^{-\alpha_2 \frac{t}{\varepsilon}} + \varepsilon e^{-\alpha_1 t}) \end{aligned}$$

for all $t \ge 0$, where β_{lj}, α_j, $l,\ j \in \{1, 2\}$, are positive constants independent of t and ε. In (4.33), using the above estimates for the block components of $\Phi(s,\ 0,\ \varepsilon)$, one obtains that $V(\varepsilon)$ has structure

$$V(\varepsilon) = \begin{bmatrix} V_{11}(\varepsilon) & \varepsilon V_{12}(\varepsilon) \\ \varepsilon V_{12}^T(\varepsilon) & \varepsilon V_{22}(\varepsilon) \end{bmatrix}. \tag{4.34}$$

Substituting (4.34) into (4.32), one obtains a linear system with unknowns V_{11}, V_{12}, and V_{22}.

The coefficients of this system are analytic functions with respect to the parameter $\eta = \sqrt{\varepsilon}$. This means that the unique solution of this system is an analytic function with respect to η. Hence,

$$V_{ij}(\varepsilon) = V_{ij}^0 + \sqrt{\varepsilon} V_{ij}^1 + \sqrt{\varepsilon} \sum_{k=0}^{\infty} \sqrt{\varepsilon}^k V_{ij}^{k+2},\ i,\ j = 1,\ 2.$$

By standard calculations, which are omitted for brevity, one obtains that V_{ij}^0, i, $j = 1,\ 2$, verifies system (3.9) completed with (3.25), (3.26), and (4.5), whereas $(V_{11}^1,\ V_{12}^1,\ V_{22}^1)$ verifies system (4.30). The uniqueness of the stabilizing solution of system (3.9) together with the uniqueness of the solution of system (4.30) leads to

$$V_{11}^0 = \tilde{X}_1,\ V_{22}^0 = \tilde{X}_2,\ V_{12}^0 = \tilde{X}_{12}, \tag{4.35}$$

and

$$V_{11}^1 = X_1^1,\ V_{22}^1 = X_2^1,\ V_{12}^1 = X_{12}^1. \tag{4.36}$$

Combining the result from Corollary 4.2 with (4.34), (4.35), and (4.36), we deduce

$$\|V(\varepsilon) - \tilde{X}(\varepsilon)\| \le \rho\varepsilon,\ (\forall)\ \ 0 < \varepsilon \le \varepsilon^{**}.$$

Thus, the proof is complete. ◻

5. Conclusions. In this paper, several aspects of the problem of an LQ optimal regulator for a class of stochastic controlled linear systems modeled by systems of singularly perturbed Itô differential equations were considered. The asymptotic structure of the stabilizing solution of the ARE associated with this problem was derived. The dominant part of this solution involved solving a system of coupled Riccati-type equations that is not dependent upon the small parameter ε that was used to construct a suboptimal control. As in the deterministic case, the presence of the small parameter ε in the structure of the coefficients of the ARE leads to an ill conditioning of numerical computations, affecting the accuracy of the existing algorithms for the computation of the stabilizing solution of the ARE associated with the problem of LQR. It is worth mentioning that in many applications, the value of the small parameter ε is not precisely known. This is another argument in favor of the derivation of the asymptotic structure of the stabilizing solution of the ARE in order to be able to construct a near-optimal control law. It is known from [2, 15, 16] that in the deterministic case, two control problems of lower dimension are associated with the original problem, namely, the reduced LQ problem and the boundary layer LQ problem. This is done by simply neglecting the small parameter ε in the controlled system followed by some simple algebraic computations. The stabilizing solutions of the Riccati equations associated with the reduced problem and the boundary layer problem play an important role in the construction of the dominant part of the stabilizing solution of the ARE of the original problem. Unlike the deterministic case, in the stochastic context considered in this paper, we cannot associate a reduced LQ problem and a boundary layer LQ problem by simply neglecting the small parameter ε arising in system (2.1). However, we can associate a system of coupled algebraic Riccati-type equations that are not dependent upon ε, i.e., the so-called reduced system of AREs (3.9). In the case when the matrix coefficients of the diffusion part of the controlled system vanish, system (3.9) reduces to the two AREs of lower dimension known from the deterministic case. As for the system of coupled AREs (3.9), we introduced the concept of the stabilizing solution and provided a set of necessary and sufficient conditions that guarantee the existence of the stabilizing solution of (3.9). An iterative procedure for numerical computation of the stabilizing solution of (3.9) was described. It should be noted that the dominant part of the stabilizing solution of the original ARE constructed based on the stabilizing solution of the reduced system of AREs (3.9) provides an approximation of order $O(\sqrt{\varepsilon})$ of the stabilizing solution associated with the original problem. This guarantees a level of suboptimality of order $O(\varepsilon)$ achieved by the control u_{app} constructed based on the stabilizing solution of the reduced system of AREs (3.9). Finally, we note that the assumptions in Theorem 4.1 also provide a set of sufficient conditions (independent of the small parameter ε) that guarantee that the existence of the stabilizing solution of ARE is associated with the considered problem of LQR. The techniques developed in this paper can be used to derive similar results for other types of Riccati equations associated with controlled linear systems modeled by singularly perturbed Itô differential equations with one or more small parameters. Moreover, the developed methodology can also be further used to solve the H_2 filtering problem for systems governed by singularly perturbed Itô differential equations.

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