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# THE LINEAR UTILITY MODEL FOR OPTIMAL SELECTION

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A linear utility model is introduced for optimal selection when several subpopulations of applicants are to be distinguished. Using this model, procedures are described for obtaining optimal cutting scores in subpopulations in quota-free as well as quota-restricted selection situations. The cutting scores are optimal in the sense that they maximize the overall expected utility of the selection process. The procedures are demonstrated with empirical data.

Key words: culture-fair selection, threshold utility.

Several models for culture-fair selection have been proposed: The regression model [e.g., Cleary, 1968], the constant ratio model [Thorndike, 1971], the conditional probability model [Cole, 1973], the equal probability model [e.g., Linn, 1973], the equal risk model [Einhorn & Bass, 1971], and the culture-modified criterion model [Darlington, 1971]. Petersen and Novick [1976], in an enlightening review of these models, have shown that some of these models are internally contradictory. Following Gross and Su [1975], they argue that the correct procedure in selection is the decision-theoretic maximization of the expected utility of the selection process.

As Novick and Petersen have demonstrated, the only culture-fair selection models of those previously considered in the literature that are acceptable from a decision-theoretic point of view are the regression and equal risk model. As the regression model will turn out to be a special instance of one of the models in the present paper, it is recalled that the model can be described as follow: Suppose that in a selection procedure $g$ subpopulations are to be distinguished, and that the regression of the criterion variable $Y$ on the predictor variable $X$ for the $i$th subpopulation is linear:

$$E_i(Y \mid X) = \alpha_i + \beta_i X \qquad (i = 1, 2, \ldots, g), \tag{1}$$

where $\alpha_i$ and $\beta_i$ are the intercept and slope of the regression line. Denoting the predictor cutting score for subpopulation $i$ by $x_i'$, the regression model says that a culture-fair selection is attained when the values of $x_i'$ are chosen such that the predicted values of the criterion $Y_i$ are equal to the minimum level of satisfactory criterion performance $y^*$ for $i = 1, 2, \ldots, g$.

Gross and Su [1975] were the first to note that "fair" selection is a question of utilities. Whether a selection procedure is believed to be fair to the various subpopulations which can be distinguished depends on the utilities of those involved in the selection process. The only requirement a selection procedure has to meet to be culture-fair is that its utility

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TABLE 1

Threshold Utility Function

| Criterion Score ($Y$) | Predictor Score ($X$) $X < x_i'$ | $X \geq x_i'$ |
| --- | --- | --- |
| $Y < y^*$ | $u_{i00}$ | $u_{i01}$ |
| $Y \geq y^*$ | $u_{i10}$ | $u_{i11}$ |

structure reflects these utilities, and that, given this structure, the selection decisions maximize the overall expected utility. Gross and Su use a threshold utility function to show how their expected utility approach proceeds. Petersen [1976] provides a Bayesian version of this approach with solutions for the quota-free as well as the quota-restricted case. The threshold utility function is exemplified in Table 1. Note that the table gives the utilities for subpopulation $i$ only and that when using threshold utility, a complete set of these utilities must be specified for each subpopulation. It is precisely this feature that sets the decision-theoretic approach to the culture-fair selection problem apart from other decision-theoretic problems. Cronbach and Gleser [1965], in their classical monograph *Psychological Tests and Personnel Decisions*, make a distinction between an institutional and an invididual selection problem. The culture-fair selection problem is neither of the two but something in between: several *sub*populations are distinguished and for each subpopulation a separate utility function is specified representing the utility of the various decision outcomes for the subpopulation.

It is the goal of the present paper to propose a linear utility function for use in the decision-theoretic approach to the culture-fair testing problem. A linear utility structure fits the problem of optimal selection from several subpopulations rather well and leads to comparably simple solutions for both the case of quota-free and quota-restricted selection. For a general introduction to additive utilities and utilities that are linear in the true state, the reader is referred to Raiffa and Schlaifer (1961, pp. 97–207).

An advantage of the linear utility model is the weakness of its assumptions. In all selection models mentioned earlier, it is assumed that the predictor and criterion variables are continuous with a bivariate normal distribution. The predictor is, however, usually a test or a test battery and will thus yield a discrete variable. In this paper the predictor is considered a discrete variable. Moreover, the model does not assume linear regression of criterion on predictor scores and normally distributed criterion scores. Linear regression is discussed as a special case of a more general regression function. In many practical applications the assumptions of linear regression and normality are approximately valid or have little effect on the accuracy of utility calculations (Schmidt, Hunter, McKenzy, & Muldrow, 1979). A general model must, however, be preferred to special ones. Nothing is lost using a general model and special cases easily follow from the general model.

*The Linear Utility Model*

Petersen [1976] has pointed out that the threshold utility function can be unrealistic for the culture-fair testing procedure. In this model it is supposed that, for instance, for all accepted subjects with criterion performance above $y^*$, the amount of utility is constant no matter their actual criterion performance. It seems more realistic to suppose that for accepted subjects with performances above $y^*$ the utility is a monotonically increasing function of criterion performance. Novick and Lindley [1978] have described a normal ogive utility function which might be adapted for use with the culture-fair testing problem. In this paper a simpler linear utility function is used. Van der Linden and Mellenbergh [1977] used a linear loss function for determining optimal cutting scores on mastery tests. This function is restated as a utility function for subpopulation $i$ in the present problem:

$$U_i = U_i(Y) = \begin{cases} b_{0i}(y^* - Y) + a_{0i} & \text{for} \quad X < x_i' \\ b_{1i}(Y - y^*) + a_{1i} & \text{for} \quad X \geq x_i' \end{cases} \qquad b_{0i}, \quad b_{1i} > 0, i = 1, 2, \ldots, g, \quad (2)$$

where $x_i'$ is the integer valued cutting score on the predictor variable in subpopulation $i$. The condition $b_{0i}$, $b_{1i} > 0$ is needed for the mathematical derivations given below; it is not really restrictive in applications. Examples of this function are given in Figure 1.

The parameter $a_{0i}$ is a constant for all rejected subjects from subpopulation $i$; it can, for example, represent the utility of testing, which will be mostly negative because costs of testing are involved. The parameter $a_{1i}$ is a constant for all accepted subjects from subpopulation $i$; it can, for example, represent the costs of testing and the cost of an educational program. Both $b_{0i}(y^* - Y)$ and $b_{1i}(Y - y^*)$ represent amounts of utility dependent on the criterion performance. These are proportional to the difference between the criterion

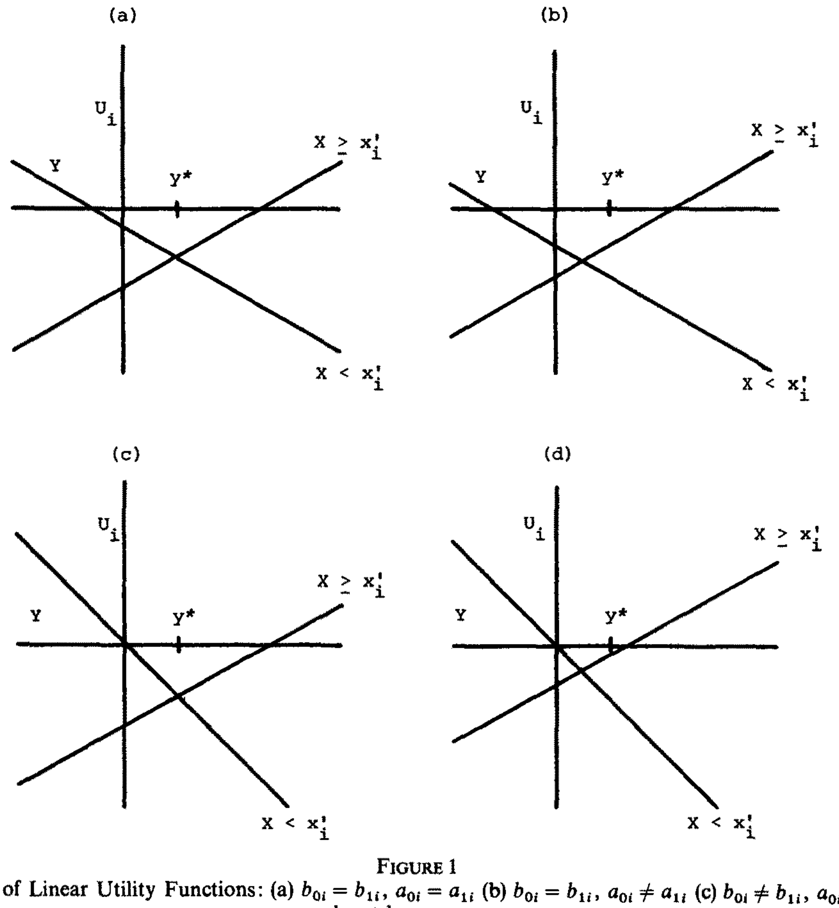


FIGURE 1
Examples of Linear Utility Functions: (a) $b_{0i} = b_{1i}$, $a_{0i} = a_{1i}$ (b) $b_{0i} = b_{1i}$, $a_{0i} \neq a_{1i}$ (c) $b_{0i} \neq b_{1i}$, $a_{0i} = a_{1i}$ (d) $b_{0i} \neq b_{1i}$, $a_{0i} \neq a_{1i}$

performance of a subject and the minimum level of satisfactory criterion performance. The parameters $b_{0i}$ and $b_{1i}$ are constants of proportionality in subpopulation $i$. The values of $a_{0i}$ and $a_{1i}$ should be chosen relative to each other and to $b_{0i}(y^* - Y)$ and $b_{1i}(Y - y^*)$, in such a manner that the resulting utility function represents the psychological, social and economic consequences of the decisions for subpopulation $i$.

The parameter values of the linear utility function Formula 2 should also be chosen such that the resulting utility structure is "fair" to each subpopulation involved. Suppose, for example, that subpopulation $j$ is considered more advantaged than $i$. In choosing values of the parameters of the linear utility function this can be taken into account by requiring that incorrect decisions (wrongly accept and wrongly reject) are considered worse for subpopulation $i$ than for $j$, while correct decisions (rightly accept and rightly reject) are considered more valuable for $i$ than for $j$. This amounts to choosing values of the slope parameters of Formula 2 under the restriction $b_{0i} > b_{0j}$ and $b_{1i} > b_{1j}$. Such consequences should be realized. Utility models require utility statements for each subpopulation, and these statements should result from a public discussion with all interested parties participating in the debate [Petersen & Novick, 1976].

Since the predictor variable for the optimal selection problems considered in this paper is mostly a test scored according to the number right rule, it is realistic to assume that the possible scores on the predictor variable are integers, ranging from 0 to $n$. The expected utility of a randomly selected applicant from the $i$th subpopulation for the linear utility function Formula 2 is:

$$E(U_i) = \sum_{X=0}^{x_i'-1} \int_{-\infty}^{\infty} \{b_{0i}(y^* - Y) + a_{0i}\} k_i(X, Y)\, dY + \sum_{X=x_i'}^{n} \int_{-\infty}^{\infty} \{b_{1i}(Y - y^*) + a_{1i}\} k_i(X, Y)\, dY, \tag{3}$$

where $k_i(X, Y)$ is the joint probability density of the predictor and criterion variable in subpopulation $i$. Although the criterion is conceived as a continuous variable it can also be considered a discrete variable. Substituting the summation sign for the integral sign in Formula 3 will not alter the derivations given below. In Formula 3, it is assumed that the optimal selection rule for subpopulation $i$ takes the form of a cutting score on the test, or, in other words, that the decision problem is monotone. It should be noticed that for the utility function Formula 2 this entails the condition of monotone likelihood ratio of $Y$ given $X$ for subpopulation $i$ [cf. Ferguson, 1967, chap. 6]. From now on, it will be assumed that this (rather mild) condition is fulfilled for all subpopulations.

Using the properties that $k_i(X, Y) = q_i(Y|X)h_i(X)$, $\int_{-\infty}^{\infty} q_i(Y|X)\, dY = 1$, and $\int_{-\infty}^{\infty} Yq_i(Y|X)\, dY = E_i(Y|X)$, ($q_i(Y|X)$ is the conditional density of the criterion variable given the predictor variable, $h_i(X)$ the probability density of the predictor variable, and $E_i(Y|X)$ the regression function of the criterion variable on the predictor variable in subpopulations $i$), it follows that the expected utility of selection for a random applicant from subpopulation $i$ is:

$$E(U_i) = \sum_{X=0}^{x_i'-1} [b_{0i}\{y^* - E_i(Y|X)\} + a_{0i}]h_i(X) - \sum_{X=x_i'}^{n} [b_{1i}\{y^* - E_i(Y|X)\} - a_{1i}]h_i(X). \tag{4}$$

The selection process is viewed as a series of separate decisions, each of which involves one random applicant from the total population, and it is assumed that the overall expected utility of the selection process is the sum of the expected utilities of the applicants. Thus, the overall expected utility of the selection process is:

$$E(U) = \sum_{i=1}^{g} p_i E(U_i), \tag{5}$$

where $p_i$, $\sum_{i=1}^{g} p_i = 1$, is the proportion of applicants from subpopulation $i$ in the total population of applicants. The problem is to find integer values for the predictor cutting scores $x_i'$ that maximize the overall expected utility of Formula 5.

*Optimal Cutting Scores in Quota-Free Selection*

In quota-free selection there is no restriction on the number of applicants that can be accepted. Therefore, Formula 5 is maximized if the expected utility of a random applicant is maximized. This is done by maximizing Formula 4 for every subpopulation separately. Formula 4 is equivalent to:

$$E(U_i) = \sum_{X=0}^{n} [b_{0i}\{y^* - E_i(Y \mid X)\} + a_{0i}]h_i(X)$$

$$- \sum_{X=x_i'}^{n} [(b_{0i} + b_{1i})\{y^* - E_i(Y \mid X)\} + (a_{0i} - a_{1i})]h_i(X). \tag{6}$$

Since $(b_{0i} + b_{1i}) > 0$, and eliminating the constant term from Formula 6, $E(U_i)$ is maximal for the cutting score that minimizes:

$$E(U_i') = \sum_{X=x_i'}^{n} [(b_{0i} + b_{1i})\{y^* - E_i(Y \mid X)\} + (a_{0i} - a_{1i})]h_i(X). \tag{7}$$

The density $h_i(X)$ is equal to or greater than zero for all values of $X$. If the sign of the term

$$(b_{0i} + b_{1i})\{y^* - E_i(Y \mid X)\} + (a_{0i} - a_{1i}) \tag{8}$$

from Formula 7 changes only once from positive to negative in the sequence $x_i' = 0, 1, \ldots, n$, then $E(U_i)$ is maximal for the cutting score for which Formula 8 is negative for the first time. If the sign of Formula 8 changes more than once in the sequence $x_i' = 0, 1, \ldots, n$, it is necessary to compute $E(U_i')$ for all values of $x_i'$; the optimal cutting score is the value of $x_i'$ for which $E(U_i')$ is minimal. In these ways optimal cutting scores can always be determined for all subpopulations.

An interesting special case of the linear utility model is $a_{0i} = a_{1i} = a_i$ $(i = 1, 2, \ldots, g)$. The utility function is:

$$U_i = \begin{cases} b_{0i}(y^* - Y) + a_i & \text{for} \quad X < x_i' \\ b_{1i}(Y - y^*) + a_i & \text{for} \quad X \geq x_i' \end{cases} \qquad b_{0i}, b_{1i} > 0, \quad i = 1, 2, \ldots, g. \tag{9}$$

For this function the last term of Formula 8 vanishes. If the sign of Formula 8 changes only once from positive to negative then Formula 8 is negative for the first time that $E_i(Y \mid X)$ is greater than $y^*$. Therefore, the optimal cutting score is that value of $x_i'$ for which $E_i(Y \mid X)$ is greater than $y^*$ for the first time. If the sign of Formula 8 changes more than once, the optimal cutting score is the value of $x_i'$ for which $E(U_i')$ is minimal. Because $(b_{0i} + b_{1i})$ is a positive constant $E(U_i')$ is minimal for the value of $x_i'$ for which

$$\sum_{X=x_i'}^{n} \{y^* - E_i(Y \mid X)\}h_i(X) \tag{10}$$

is minimal. In both cases $b_{0i}$ and $b_{1i}$ are not necessary for determining the optimal cutting score for subpopulation $i$. If the amount of constant utility is equal for an accepted and a rejected applicant, then there is no need to choose values for $b_{0i}$ and $b_{1i}$. It should be noted, however, that although the optimal cutting score for population $i$ is the same for all values of $b_{0i}$ and $b_{1i}$ when $a_{0i} = a_{1i}$, this does not imply that the cutting scores for all *sub*populations are equal. The cutting scores also depend on the distributions of $(X, Y)$ and these are in general not the same for all subpopulations.

A special case of the regression function is the linear regression function of the criterion variable on the predictor variable:

$$E_i(Y \mid X) = \alpha_i + \beta_i X. \tag{11}$$

The condition that Formula 8 changes sign only once is fulfilled for this regression function. Substituting Formula 11 into Formula 8, setting the result equal to 0, and solving for $X$, gives:

$$x_i' = \frac{y^* - \alpha_i}{\beta_i} + \frac{a_{0i} - a_{1i}}{\beta_i(b_{0i} + b_{1i})}. \tag{12}$$

As indicated earlier, the cutting score is assumed to be an integer: for the first integer smaller than $x_i'$ Formula 8 is positive, and for the first integer greater than $x_i'$ it is negative. Therefore, the optimal cutting score for subpopulation $i$ is the first integer greater than $x_i'$. This value of $x_i'$ will henceforth be indicated by $x_i^*$.

From Formula 12, it follows that for utility function Formula 9 and the linear regression function Formula 11 the optimal cutting score is the first integer greater than:

$$x_i' = \frac{y^* - \alpha_i}{\beta_i}. \tag{13}$$

Formula 13 is the solution to the Formula 1. This shows that the regression model for selection can be considered a special case of the linear utility model. For a linear regression function of the criterion variable on the predictor variable and equal constant amounts of utility for an accepted and a rejected applicant, the linear utility model reduces to the regression model for selection.

### *Optimal Cutting Scores in Quota-Restricted Selection*

The situation is considered where only a fixed number of applicants can be accepted. For a given population of applicants, it is usual to replace this number by a fixed proportion $p$ of all applicants that can be accepted. Therefore, the overall expected utility of Formula 5 is maximized under restriction that

$$\sum_{i=1}^{g} p_i \sum_{X=x_i'}^{n} h_i(X) = p \tag{14}$$

is a fixed constant. As the predictor should be considered a discrete variable, this condition cannot in general be fulfilled exactly. Suppose, however, that the fixed constant $p$ can be replaced by an upper bound $(p_u)$ and a lower bound $(p_l)$. The proportion of accepted applicants from the total population should then be within these bounds. The restriction of Formula 14 becomes:

$$p_l < \sum_{i=1}^{g} p_i \sum_{X=x_i'}^{n} h_i(X) < p_u. \tag{15}$$

Using the linear utility theory given in the previous section, optimal cutting scores can be found for the linear utility function Formula 2 and the interval restriction Formula 15. In principle, the procedure is a search routine based on the simple idea of looking for all possible sets of cutting scores $x_i'$ $(i = 1, 2, \ldots, g)$ fulfilling the restriction of Formula 15. Once these sets are found, the task is to choose the set that gives the maximum of the expected linear utility of Formula 5. More specifically, the following should be done: First, the total number of $s$ sets of cutting scores $x_i'$ ($i = 1, 2, \ldots, g$) fulfilling the restriction of Formula 15 are determined. Second, from empirical data the regression functions $E_i(Y \mid X)$ and the probability densities $h_i(X)$ are estimated. Third, using Formula 4, the linear expected utility is estimated for each subpopulation and for all cutting scores found in this population in the first step. Fourth, using Formula 5, the overall expected utility is computed for each set of cutting scores $x_i'$ ($i = 1, 2, \ldots, g$) found in the first step. Fifth, the maximum value of the overall expected utilities computed in the previous step is determined. The set yielding this maximum value contains the optimal cutting scores $x_i^*$ ($i = 1, 2, \ldots, g$). A numerical example illustrating this procedure is given in the next section.

### *Example*

Van der Flier and Drenth [1977] administered some ability tests to a group of primary school children in Surinam. Examination results were also available for 169 Creoles and 124 Asiatics. For illustrative purpose only the data of the Differences test and the language examination are used. In each item of the test six figures are presented; two figures must be found that deviate from an exemplary figure. The language examination performance was evaluated with school marks.

A mark 5 is considered insufficient, whereas a mark 6 is considered sufficient. Therefore, the minimum level of satisfactory criterion performance is fixed at 5.5. Because the costs for testing are equal for accepted and rejected subjects, the constants in the utility function Formula 2 are set equal to each other in both groups: $a_{0i} = a_{1i}$ ($i = 1, 2$).

First, the quota-free situation is considered. From Table 2 it is seen that for the Creoles the estimated function $\{y^* - \hat{E}_1(Y|X)\} = \{5.5 - \hat{E}_1(Y|X)\}$ changes sign for the cutting score 10 on the test. Consequently, the optimal cutting score for the Creoles is 10. For the Asiatics the function $\{5.5 - \hat{E}_2(Y \mid X)\}$ changes sign more than once. Therefore, the optimal cutting score must be found using Formula 10. These estimated values are reported in Table 2. It is seen that the function has a minimum value for the test score 11. Consequently, the optimal cutting score for the Asiatics is 11.

Second, the quota-restricted situation is considered. Suppose that, based on the test scores, about five pupils can get a scholarship. It is decided that at least four and at most six pupils will get a scholarship. The frequency distributions in Table 2 show that the following pairs of cutting scores fulfil the restriction Formula 15: (30, 36), (30, 34), (31, 36), (31, 34), (31, 31), (32, 34), (32, 31), (32, 28), (34, 28), and (35, 27). Using Formula 6, the expected utility for a given cutting score and utility function can be estimated, and substituting the estimated proportions Creoles and Asiatics in Formula 5, the overall expected utility can be estimated.

The procedure is demonstrated using the pair (31, 31) and the utility function $b_{01} = b_{11} = b_{02} = b_{12} = 1, a_{01} = a_{11} = a_{02} = a_{12} = 0$:

(i) From Table 2 it is seen that for this pair of cutting scores three Creoles and three Asiatics will get a scholarship.

(ii) The specification of Formula 6 for this utility function is:

$$E(U_i) = \sum_{X=0}^{36} \{5.5 - E_i(Y \mid X)\}h_i(X) - 2 \sum_{X=31}^{36} \{5.5 - E_i(Y \mid X)\}h_i(X).$$

From Table 2 it is seen that for the Asiatics the estimated utility is: $\hat{E}(U_2) = -.468 - 2(-.044) = -.380$. From this table it can be computed that the corresponding value for the Creoles is $\hat{E}(U_1) = -.655$.

(iii) The estimated proportion Creoles is $\hat{p} = .5768$ and the estimated proportion Asiatics $\hat{p} = .4232$. Using Formula 5, the estimated overall expected utility is: $\hat{E}(U) = .5768 \times (-.655) + .4232 \times (-.380) = -.539$.

TABLE 2

Optimal Cutting Scores Using a Linear Utility Function Quota-Free Selection

| | Creoles (N = 169) | | Asiatics (N = 124) | | |
|---|---|---|---|---|---|
| X | freq(X) | $\{5.5.-\hat{E}_1(Y\|X)\}$ | freq(X) | $\{5.5-\hat{E}_2(Y\|X)\}$ | $\sum_{X=x'}^{36}\{5.5-\hat{E}_2(Y\|X)\hat{h}_2(X)\}$ |
| 0 | 1 | +0.5000 | 0 | | -.468 |
| 1 | 0 | | 0 | | -.468 |
| 2 | 0 | | 0 | | -.468 |
| 3 | 0 | | 0 | | -.468 |
| 4 | 0 | | 0 | | -.468 |
| 5 | 1 | +1.5000 | 0 | | -.468 |
| 6 | 0 | | 0 | | -.468 |
| 7 | 0 | | 1 | +1.5000 | -.468 |
| 8 | 4 | +0.2500 | 1 | -0.5000 | -.480 |
| 9 | 4 | +0.5000 | 1 | +1.5000 | -.476 |
| 10 | 4 | -0.7500* | 1 | +2.5000 | -.488 |
| 11 | 5 | -0.5000 | 1 | -0.5000 | -.508* |
| 12 | 6 | -0.5000 | 2 | -1.0000 | -.504 |
| 13 | 8 | -0.8750 | 3 | -0.1667 | -.488 |
| 14 | 7 | -0.0714 | 6 | -2.0000 | -.484 |
| 15 | 6 | -0.8333 | 7 | +0.0714 | -.387 |
| 16 | 11 | -0.5909 | 10 | +0.2000 | -.391 |
| 17 | 14 | -0.0714 | 11 | -0.5000 | -.407 |
| 18 | 11 | -0.5000 | 12 | -0.0833 | -.363 |
| 19 | 13 | -0.8077 | 13 | -0.8077 | -.355 |
| 20 | 16 | -0.8125 | 15 | -0.5667 | -.270 |
| 21 | 20 | -1.4000 | 12 | -0.2500 | -.201 |
| 22 | 6 | -1.3333 | 8 | +0.2500 | -.177 |
| 23 | 11 | -0.8636 | 7 | -1.0714 | -.193 |
| 24 | 5 | -0.7000 | 4 | +0.2500 | -.132 |
| 25 | 5 | -0.9000 | 1 | -2.5000 | -.140 |
| 26 | 2 | -1.0000 | 2 | -1.5000 | -.120 |
| 27 | 5 | -1.1000 | 2 | -2.5000 | -.096 |
| 28 | 0 | | 1 | -1.5000 | -.056 |
| 29 | 0 | | 0 | | -.044 |
| 30 | 1 | -2.5000 | 0 | | -.044 |
| 31 | 1 | -2.5000 | 1 | -2.5000 | -.044 |
| 32 | 1 | -2.5000 | 0 | | -.024 |
| 33 | 0 | | 0 | | -.024 |
| 34 | 1 | -0.5000 | 1 | -1.5000 | -.024 |
| 35 | 0 | | 0 | | -.012 |
| 36 | 0 | | 1 | -1.5000 | -.012 |

*The estimated value for which the cutting score is optimal.

The estimated overall expected utility for the above mentioned pairs of cutting scores and three different utility functions is reported in Table 3. The optimal pair of cutting scores for the utility function $b_{01} = b_{11} = b_{02} = b_{12} = 1$ is (35, 27). The consequence of choosing $b_{02} < b_{01}$, and $b_{12} < b_{11}$ is to lower the cutting score for the Creoles and to raise the cutting score for the Asiatics: (30, 34).

Two remarks are appropriate. One, the example was used only to illustrate the procedure. The sample size was rather small, and the estimates of the expected utility functions can be inaccurate. The cutting score(s) are chosen such that the estimated expected utility function has reached its maximum. If in the population some values of the function are near the maximum, a large sample is necessary to determine accurately the maximum of the function, and with that the optimal cutting score(s). It therefore seems wise to use this procedure with large samples. Two, in the quota-restricted situation the calculations were

Table 3

Optimal Cutting Scores Quota-Restricted Selection

| Cutting Score x' | | Number Accepted | | Estimated Overall Expected Utility $\hat{E}(U)$ | | |
|---|---|---|---|---|---|---|
| Creoles | Asiatics | Creoles | Asiatics | Utility Function ($a_{hi}=0$; h=0,1; i=1,2) | | |
| | | | | $b_{h1} = 1$ $b_{h2} = 1$ | $b_{h1} = 1$ $b_{h2} = 2$ | $b_{h1} = 1$ $b_{h2} = .5$ |
| 30 | 36 | 4 | 1 | -.548 | -.736 | -.454 |
| | 34 | 4 | 2 | -.538 | -.716 | -.449* |
| 31 | 36 | 3 | 1 | -.566 | -.754 | -.472 |
| | 34 | 3 | 2 | -.556 | -.733 | -.467 |
| | 31 | 3 | 3 | -.539 | -.699 | -.458 |
| 32 | 34 | 2 | 2 | -.573 | -.751 | -.484 |
| | 31 | 2 | 3 | -.556 | -.717 | -.476 |
| | 28 | 2 | 4 | -.546 | -.696 | -.470 |
| 34 | 28 | 1 | 4 | -.563 | -.714 | -.488 |
| 35 | 27 | 0 | 6 | -.533* | -.649* | -.474 |

*The maximal value of the estimated overall expected utility.

done using a desk calculator. For other applications the calculations can be laborious, but a computer can easily do the job.

## *Discussion*

In the linear utility model it is not assumed that the predictor and criterion variables are continuous with a bivariate normal distribution. These assumptions are also not necessary for the threshold utility model. For this model the expected utility of an applicant from subpopulation $i$ is:

$$E(U_i) = \sum_{j=0}^{1} \sum_{k=0}^{1} u_{ijk} P_{ijk}, \qquad i = 1, 2, \ldots, g \tag{16}$$

where $P_{ijk}$ ($j, k = 0, 1$) are the probabilities of belonging to the cells of Table 1 for subpopulation $i$. For a fixed cutting score on the predictor, the probabilities $P_{ijk}$ can be estimated from empirical data; using Formula 16, the expected utility can be estimated. In quota-free selection the optimal cutting score for subpopulation $i$ is found by computing the expected utility of Formula 16 for all possible cutting scores, setting $x_i'$ equal to 0, 1, ..., $n$; the optimal cutting score is the cutting score for which the expected utility is maximal. In quota-restricted selection all possible sets of cutting scores fulfilling the restriction of Formula 15 are determined. Using Formula 5, the set of cutting scores that has the maximum expected utility is chosen. These procedures were applied to the data of the example from the previous section. The optimal cutting scores for the quota-free situation and the optimal pairs of cutting scores for the quota-restricted situation with different utility functions are reported in Table 4, respectively, 5. The tables show that sometimes two or more cutting scores or pairs of cutting scores have equal estimated maximal utilities. A sensible solution for the quota-free situation is to choose the lowest of these cutting scores implying that the

Table 4

Optimal Cutting Scores Quota-Free Selection

| Utility Function ($u_{i01} = u_{i10} = 0$) | Creoles (i=1) | Asiatics (i=2) |
|---|---|---|
| $u_{i00} = u_{i11} = 1$ | 10 | 11 |
| $u_{i00} = 1$, $u_{i11} = 2$ | 6, 7, 8 | 8, 11 |
| $u_{i00} = 2$, $u_{i11} = 1$ | 10, 12, 13, 15 | 19 |

number of accepted applicants is as high as possible. A fair solution in the quota-restricted situation is to select one set of cutting scores randomly from the sets with equal estimated maximal utilities.

The same problem can arise for the linear utility model when Formula 5 or 6 has more than one absolute maximum. In applications with carefully constructed predictor tests and large samples, this is not likely. Should it occur, however, one may use the same procedure as that described in the threshold utility example.

Instead of assuming a linear regression line, as has occasionally been done in the foregoing, the regression function $E_i(Y \mid X)$ can also be estimated from empirical data, for example, by computing the mean criterion score for every value of the predictor score in subpopulation $i$. $E_i(Y \mid X)$ can also be estimated using polynomial regression functions. A special case of polynomial regression is the regression line. Using a polynomial regression function of a fixed degree, no assumptions regarding the distribution of the residual term are needed. If, however, the degree of the polynomial should also be determined from the data, then it is necessary to assume a normal distributed residual term [Bock, 1975, chap. 4].

The assumptions made in the linear utility model are rather weak. Using estimates of the regression functions and the probability densities of the predictor variable, the optimal cutting scores can be computed. When the degree of the polynomial regression function is estimated from the data, it is also assumed that the residual term is distributed normally. Note that the assumption of normality of the criterion variable is weaker than the assumption of a bivariate normal distribution of the predictor and criterion variable.

Table 5

Optimal Cutting Scores Quota-Restricted Selection

| Utility Function ($u_{i01} = u_{i10} = 0$, i=1,2) | | | | Pair(s) of Cutting Scores |
|---|---|---|---|---|
| $u_{100}$ | $u_{111}$ | $u_{200}$ | $u_{211}$ | |
| 1 | 1 | 1 | 1 | (30,34), (31,31)<br>(32,28), (35,27) |
| 1 | 2 | 1 | 2 | (31,31) |
| 2 | 1 | 2 | 1 | (31,31) |
| 1 | 1 | 2 | 2 | (35,27) |
| 1 | 1 | .5 | .5 | (30,34) |

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