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THE LINEARIZED UNIFORM ASYMPTOTIC STABILITY OF EVOLUTION DIFFERENTIAL EQUATIONS

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INTRODUCTION

The study of stability of solutions of differential equations often leads to the following situation: We are able to prove that the zero solution of some linear evolution differential equation of the type

$$\frac{dV}{dt} = L(t)\, V \tag{0.1}$$

is stable (or asymptotically stable, exponentially stable, etc.) and we need to find out whether also the zero solution of a differential equation

$$\frac{dU}{dt} = L(t)\, U + N(t)\, U \tag{0.2}$$

(where $N(t)$ is a nonlinear operator) has the same property. If we study stability of the zero solution of the equation (0.2), we are interested in the behaviour of only those solutions of the equation (0.2) which are "near" to the zero solution and, especially if $N(t)\, U$ contains U in some sense only in powers greater than one, it can be expected that this term in the equation (0.2) should have only a minor influence on the behaviour of a solution U "near zero", because its magnitude is very small as compared with the magnitude of the term $L(t)\, U$. This idea is the starting point of a method of investigation of stability of the zero solution of the equation (0.2), based on the neglecting of the nonlinear term $N(t)\, U$ in the equation (0.2). This procedure is usually called the *linearization* and the equation (0.1), resulting from (0.2) by the linearization, is called the *linearized equation*.

Correctness of the method of linearization has been extensively studied for many types of differential equations and various types of stability. Many results concern the case when the linearized equation (0.1) is autonomous. Then in many special cases of the equation (0.1) the zero solution of this equation is uniformly exponentially stable if the spectrum of the operator L lies in a subset of the form $\{z \mid \operatorname{Re} z \leqq k < 0\}$ of the complex plane and this condition (together with some assumptions about the operator $N(t)$ and some other conditions which are usually more or less of technical character) is often sufficient also for the uniform exponential stability of the zero solution of the equation (0.2). From works containing results of a similar type, we mention for example papers of G. Prodi [18], D. H. Sattinger [19] and H. Kielhöfer [11], [12]. The question of linearization in the case that the linearized equation is nonautonomous is studied, besides other problems, for various types of stability for example in articles of G. Iooss [7], O. A. Ladyzhenskaya and V. A.Solonnikov [14], A. Strauss and J. A. Yorke [23] (in this paper the authors deal with ordinary differential equations and their equation, corresponding to our equation (0.1), need not be even linear), J. Barták [1], [2] and J. Neustupa [15], [16].

In this paper we examine the correctness of the method of linearization in the case of uniform asymptotic stability. Under certain conditions we show that the uniform asymptotic stability of the zero solution of the equation (0.1) implies the same property of the zero solution of the equation (0.2). This main result is proved in Section 2. The restriction to the question of stability of the zero solution does not represent a great loss of generality, because the problem of stability of a generally "nonzero" solution can be mostly transformed quite simply to a similar problem concerning the zero solution. In Section 1, we list all necessary assumptions about the equations (0.1) and (0.2). Nonetheless, these equations remain very general throughout Sections 1 and 2. Sections 3, 4 and 5 are devoted to applications of the results derived to some more special classes of differential equations, containing as particular cases for instance the Navier-Stokes equations, the wave equation, the equation of oscillations of a beam and the Timoshenko-type equation.

1. SOME FUNDAMENTAL ASSUMPTIONS

In order to prove the main theorem about the uniform asymptotic stability (see Section 2), we do not need any special properties of the operators $L(t)$ and $N(t)$, appearing in the equations (0.1) and (0.2). What we need are some properties of solutions of these equations. This is why we make assumptions not about the operators $L(t)$ and $N(t)$, but about the solutions of the equations (0.1) and (0.2) in this section. However, in Sections 3, 4 and 5, where in several examples we shall deal with some special differential equations, the operators $L(t)$ and $N(t)$ will be specified in each example quite in detail.

Let X be a Banach space with the norm denoted by $\|\cdot\|$. (Sometimes, if there is a danger of confusion, we use the notation $\|\cdot\|_X$ instead of $\|\cdot\|$.) If $J \subseteq E_1$ and $M \subseteq X$ then $\mathscr{C}^n(J; M)$ (for $n = 0, 1, \ldots$) will be the set of n-times continuously differentiable mappings from J into M and $C^n(J; M)$ (for $n = 0, 1, \ldots$) will be the Banach space of all elements f from $\mathscr{C}^n(J; M)$ with a finite norm

$$\|f\|_{C^n(J;M)} = \sum_{i=0}^{n} \sup_{t\in J} \|f^{(i)}(t)\| .$$

We suppose that every solution of any differential equation under investigation satisfies the differential equation in some sense on some interval $I \subseteq \langle 0, +\infty)$ and its value at each point $t \in I$ is an element of the space X. If $\|U(t)\|$ is bounded on the interval I, we denote

$$|||U|||_I = \sup_{t \in I} \|U(t)\| .$$

It will be sometimes useful to denote the interval I, where U is a solution, by $\mathscr{D}(U)$. We request neither a special smoothness of solutions, nor any special sense in which they have to satisfy the corresponding differential equations. We only suppose that the norm $\|U(t)\|$ of each solution U is the "right upper semicontinuous" function of the variable t (i.e. if $t' \in \mathscr{D}(U)$ and $\varepsilon > 0$ are given, there exists $\xi > 0$ so that the inequality $\|U(t)\| < \|U(t')\| + \varepsilon$ holds for all $t \in \langle t', t' + \xi) \cap \mathscr{D}(U)$) and the "left lower semicontinuous" function of the variable t (i.e. given $t' \in \mathscr{D}(U)$ and $\varepsilon > 0$, there exists $\xi > 0$ so that $\|U(t)\| > \|U(t')\| - \varepsilon$ holds for all $t \in (t' - \xi, t' \rangle \cap \mathscr{D}(U)$).

As we have already mentioned, we suppose that the equation (0.1) is linear. The equation (0.2) need not (and in all applications, described in Sections 3, 4 and 5, will not) be linear, but we assume that it has the zero solution, i.e. the mapping $U(t) = 0_X$ (where 0_X is the zero element of X) is its solution on the interval $\langle 0, +\infty)$.

Further, we shall use the following conditions:

(i) There exists a dense subset X_1 of the space X so that if $\tau \geqq 0$ and $x_1 \in X_1$ then there exists a solution V of the equation (0.1) on the interval $\langle \tau, +\infty)$, satisfying the initial condition $V(\tau) = x_1$.

(ii) There exists $R_1 > 0$ so that if U and V are solutions of the equation (0.2) (and (0.1), respectively) on an interval $\langle \tau, t \rangle \subset \langle 0, +\infty)$, $|||U|||_{\langle \tau, t \rangle} \leqq R_1$ and $|||V|||_{\langle \tau, t \rangle} \leqq R_1$, then

$$\|U(t) - V(t)\| \leqq G(\|U(\tau) - V(\tau)\|, \; t - \tau, \; |||U|||_{\langle \tau, t \rangle}) , \tag{1.1}$$

where $G : \langle 0, 2R_1 \rangle \times \langle 0, +\infty) \times \langle 0, R_1 \rangle \to \langle 0, +\infty)$ is a function with the following properties:

$(\text{ii})_1$ G is nondecreasing in the second and the third variable,

$(\text{ii})_2$ $\lim_{\alpha \to 0+} G(\alpha, \beta, \gamma) = G(0. \beta, \gamma)$ locally uniformly with respect to $\beta \in \langle 0, +\infty)$ and $\gamma \in \langle 0, R_1 \rangle$,

$(\text{ii})_3$ $G(0, \beta, \gamma) = o(\gamma)$ for all $\beta \in \langle 0, +\infty)$.

(iii) Given $\tau \geqq 0$ and $x \in X$, there exists a solution V of the equation (0.1) on the interval $\langle \tau, +\infty)$, satisfying the initial condition $V(\tau) = x$.

(iv) There exists $R_1 > 0$ so that if U and V are solutions of the equation (0.2) and (0.1), respectively, on an interval $\langle \tau, t \rangle \subset \langle 0, +\infty)$, $U(\tau) = V(\tau)$, $|||U|||_{\langle \tau, t \rangle} \leqq R_1$ and $|||V|||_{\langle \tau, t \rangle} \leqq R_1$, then

$$\|U(t) - V(t)\| \leqq \hat{G}(t - \tau, \; |||U|||_{\langle \tau, t \rangle}) , \tag{1.2}$$

where $\hat{G} : \langle 0, +\infty) \times \langle 0, R_1 \rangle \to \langle 0, +\infty)$ is a function with the following properties:

$(\mathrm{iv})_1$ $\hat{G}$ is nondecreasing in both variables,

$(\mathrm{iv})_2$ $\hat{G}(\beta, \gamma) = o(\gamma)$ for all $\beta \in \langle 0, +\infty)$.

(Observe that (iii) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (iv).)

2. THE UNIFORM ASYMPTOTIC STABILITY AND THE LINEARIZATION THEOREM

Definition 2.1. We say that the zero solution of the equation (0.1) is *uniformly asymptotically stable with respect to the norm* $\|\cdot\|$ if there exist $\delta > 0$ and functions $\Phi : \langle 0, \delta\rangle \to \langle 0, +\infty)$ and $\varphi : \langle 0, +\infty) \to \langle 0, +\infty)$ with the following properties:

(a) Φ is a nondecreasing and φ a nonincreasing function,

(b) $\lim_{t \to +\infty} \varphi(t) = 0$,

(c) $\Phi(0) = 0$

so that if V is any solution of the equation (0.1), $\tau \in \mathscr{D}(V)$ and $\|V(\tau\| \leqq \delta$, then

$$\|V(t)\| \leqq \Phi(\|V(\tau)\|)\ \varphi(t - \tau) \tag{2.1}$$

for all $t \in \langle \tau, +\infty) \cap \mathscr{D}(V)$.

We recall that $\mathscr{D}(V) \subseteq \langle 0, +\infty)$ is the interval where V is a solution of the differential equation (0.1).

The uniform asymptotic stability of the zero solution of the differential equation (0.2) or of other differential equations would be defined quite analogously.

Throughout Section 2, we shall deal only with stability with respect to the norm $\|\cdot\|$, and therefore we shall mention it no more.

The uniform asymptotic stability is sometimes defined also in such a way that for each $x \in X$ such that $\|x\| \leqq \delta$ and for each $\tau \geqq 0$, the existence of such a solution V of the differential equation considered is required that $V(\tau) = x$, V is the solution on the whole interval $\langle \tau, +\infty)$ and V satisfies an estimate of the type (2.1) for all $t \in \langle \tau, +\infty)$. However, in what follows, we investigate the uniform asymptotic stability of the zero solution of the equation (0.2) and this equation is so general that we cannot prove anything concerning the existence of its solutions without additional assumptions. This is why we use the above definition of the uniform asymptotic stability in this paper. Observe that the uniform asymptotic stability is here a property of an "a priori" character, because we must first assume that there exists a solution V of the differential equation considered on a time-interval $\mathscr{D}(V)$, this solution is "sufficiently small" at a certain instant $\tau \in \mathscr{D}(V)$ (i.e. $\|V(\tau)\| \leqq \delta$) and then we may conclude that the estimate (2.1) is valid for all $t \in \langle \tau, +\infty) \cap \mathscr{D}(V)$.

The main result of this section is the following theorem:

Theorem 2.1. *Let the conditions* (i) *and* (ii) *or the conditions* (iii) *and* (iv) *be fulfilled and let the zero solution of the linear equation* (0.1) *be uniformly asympto-*

tically stable. Then the zero solution of the equation (0.2) *is uniformly asymptotically stable as well.*

Proof. We confine ourselves to the case when the conditions (i) and (ii) are fulfilled. The proof in the other case would be quite analogous and in several points even less complicated.

Let the symbols δ, Φ and φ have the same meaning as in Definition 2.1. First we prove several lemmas. In all of them, we suppose the conditions (i) and (ii) to be fulfilled and the zero solution of the equation (0.1) to be uniformly asymptotically stable.

Lemma 2.1. $\Phi(\alpha) > 0$ *for* $\alpha \in (0, \delta\rangle$.

Proof. Suppose the contrary. Then there exists $\alpha_1 \in (0, \delta\rangle$ so that $\Phi(\alpha_1) = 0$ and according to the condition (i) there exists a solution V of the equation (0.1) so that $\|V(0)\| \in (0, \alpha_1\rangle$. Due to (2.1) we have $\|V(0)\| \leqq \Phi(\|V(0)\|)$, and using also the property (a) from Definition 2.1, we get

$$0 < \|V(0)\| \leqq \Phi(\|V(0)\|) \leqq \Phi(\alpha_1) = 0\,,$$

which is not possible.

Lemma 2.2. *There exists* $K \geqq \frac{1}{2}$ *and a function* $\psi : \langle 0, +\infty) \to (0, 1\rangle$ *with the properties*

(d) ψ *is a nonincreasing function,*

(e) $\psi(0) = \frac{1}{2}$,

(f) $\lim\limits_{t \to +\infty} \psi(t) = 0$,

(g) $\psi(t_1)\,\psi(t_2) \leqq \psi(t_1 + t_2)$ (*for all* $t_1 \geqq 0$ *and* $t_2 \geqq 0$), *so that if V is any solution of the equation* (0.1) *and* $\tau \in \mathscr{D}(V)$, *then*

$$\|V(t)\| \leqq K\|V(\tau)\|\,\psi(t - \tau) \tag{2.2}$$

for all $t \in \langle \tau, +\infty) \cap \mathscr{D}(V)$.

Proof. Let V be a solution of the equation (0.1), $\tau \in \mathscr{D}(V)$ and let $\|V(\tau)\| > 0$. Since the equation (0.1) is linear, the function $W(t) = \delta\, V(t)/\|V(\tau)\|$ is also a solution of the equation (0.1) on $\mathscr{D}(V)$. We have $\|W(\tau)\| = \delta$ and so it follows from (2.1) that

$$\|W(t)\| \leqq \Phi(\|W(\tau)\|)\,\varphi(t - \tau) = \Phi(\delta)\,\varphi(t - \tau)\,, \tag{2.3}$$

$$\|V(t)\| \leqq \|V(\tau)\|\,\Phi(\delta)\,\varphi(t - \tau)/\delta$$

for $t \in \langle \tau, +\infty) \cap \mathscr{D}(V)$.

Let us define $\psi(0) = \frac{1}{2}$ and

$$\psi(t) = \left[\int_0^t \varphi(\sigma)\,\mathrm{d}\sigma\right] \Big/ (2t\,\varphi(0)) \quad (\text{for } t > 0)\,.$$

It can be easily proved that the function ψ has the properties (d), (e), (f), (g) and that

$$\psi(t) \geqq \varphi(t)/(2\,\varphi(0)) \quad (\text{for } t \geqq 0). \tag{2.4}$$

We prove for instance (g): Let $t_1 \geqq 0$ and $t_2 \geqq 0$. Suppose that $t_1 > 0$ and $t_2 > 0$, otherwise the inequality (g) is evident. One of the numbers t_1 and t_2 is greater than or equal to $(t_1 + t_2)/2$. We can assume without loss of generality that it is the number t_1. Then we have

$$\psi(t_1)\,\psi(t_2) = \frac{1}{2t_1\,\varphi(0)} \int_0^{t_1} \varphi(\sigma)\,\mathrm{d}\sigma\,\psi(t_2) \leqq$$

$$\leqq \frac{t_1 + t_2}{t_1} \frac{1}{2(t_1 + t_2)\,\varphi(0)} \int_0^{t_1 + t_2} \varphi(\sigma)\,\mathrm{d}\sigma\,\tfrac{1}{2} \leqq \psi(t_1 + t_2).$$

Put

$$K = 2\,\Phi(\delta)\,\varphi(0)/\delta. \tag{2.5}$$

Due to (2.3) we have $1 \leqq \Phi(\delta)\,\varphi(0)/\delta$ and thus, $K \geqq 2$. The estimate (2.2) follows immediately from (2.3), (2.1), (2.4) and (2.5).

Lemma 2.3. *If U is a solution of the equation* (0.2) *and there exists $\vartheta \geqq 0$ so that $\|U(\vartheta)\| = 0$, then $\|U(t)\| = 0$ for all $t \geqq \vartheta$ (hence U is also a solution of the equation* (0.2) *on the whole interval $\langle\vartheta, +\infty)$).*

Proof. Suppose that there exists $\vartheta_0 > \vartheta$ so that $\|U(\vartheta_0)\| > 0$.

It is a consequence of the property $(\mathrm{ii})_3$ of the function G that there exists $\gamma_0 \in (0, R_1\rangle$ so that

$$G(0, \vartheta_0 - \vartheta, \gamma)/\gamma \leqq \tfrac{1}{2} \quad (\text{for all } \gamma \in (0, \gamma_0\rangle). \tag{2.6}$$

Using the right upper semicontinuity and the left lower semicontinuity of the function $\|U(t)\|$, we may show that there exists $\zeta \in \langle\vartheta, \vartheta_0\rangle$ so that

$$0 < \|U(\zeta)\| = |||U|||_{\langle\vartheta,\zeta\rangle} \leqq \gamma_0. \tag{2.7}$$

It follows from the condition (i) that for a given $\varepsilon > 0$ there exists a solution V of the equation (0.1) on $\langle\vartheta, +\infty)$ such that $\|V(\vartheta)\| \leqq \varepsilon$. If ε is chosen so small that $\varepsilon \leqq R_1/K$ then $|||V|||_{\langle\vartheta,\zeta\rangle} \leqq R_1$, and using also the condition (ii), we get:

$$\|U(\zeta)\| \leqq \|V(\zeta)\| + G(\|U(\vartheta) - V(\vartheta)\|, \zeta - \vartheta, |||U|||_{\langle\vartheta,\zeta\rangle}) \leqq$$

$$\leqq K\|V(\vartheta)\|\,\psi(\zeta - \vartheta) + G(\|V(\vartheta)\|, \zeta - \vartheta, |||U|||_{\langle\vartheta,\zeta\rangle}) \leqq$$

$$\leqq K\varepsilon\psi(\zeta - \vartheta) + G(\varepsilon, \vartheta_0 - \vartheta, |||U|||_{\langle\vartheta,\zeta\rangle}).$$

Since this is valid for all $\varepsilon \in (0, R_1/K\rangle$, we also have

$$\|U(\zeta)\| \leqq G(0, \vartheta_0 - \vartheta, |||U|||_{\langle\vartheta,\zeta\rangle}).$$

Due to (2.6) and (2.7), this yields

$$\|U(\zeta)\| \leqq [G(0, \vartheta_0 - \vartheta, |||U|||_{\langle\vartheta,\zeta\rangle})/|||U|||_{\langle\vartheta,\zeta\rangle}] \cdot |||U|||_{\langle\vartheta,\zeta\rangle} \leqq$$
$$\leqq \tfrac{1}{2}|||U|||_{\langle\vartheta,\zeta\rangle} = \tfrac{1}{2}\|U(\zeta)\| ,$$

which is a contradiction.

Now, we can turn our attention to the uniform asymptotic stability of the zero solution of the equation (0.2). We shall prove that

(2.8) there exists $\Delta > 0$ so that if U is a solution of the equation (0.2), $\tau \in \mathscr{D}(U)$ and $0 < \|U(\tau)\| \leqq \Delta$, then

$$\|U(t)\| < 2K\|U(\tau)\| \sqrt{(\psi(t - \tau))}$$

(for $t \in \langle\tau, +\infty) \cap \mathscr{D}(U)$).

Suppose that (2.8) is false. Then for each $\Delta > 0$ there exist a solution U of the equation (0.2), $\tau \in \mathscr{D}(U)$ and $t_0 \in \langle\tau, +\infty) \cap \mathscr{D}(U)$ so that $0 < \|U(\tau)\| \leqq \Delta$,

$$\|U(t_0)\| = 2K\|U(\tau)\| \sqrt{(\psi(t_0 - \tau))}, \tag{2.9}$$

$$\|U(t)\| < 2K\|U(\tau)\| \sqrt{(\psi(t - \tau))} \quad (\text{for } t \in \langle\tau, t_0)). \tag{2.10}$$

(The existence of such a t_0 that we can write "=" instead of "$\geqq$" in (2.9) follows from the left lower semicontinuity of the function $\|U(t)\|$.)

In what follows, it will be useful to work only with $\Delta > 0$ so small that $2K\Delta \leqq R_1$, i.e. $\Delta \leqq R_1/(2K)$. Then we have

$$\|U(t)\| \leqq 2K\|U(\tau)\| \sqrt{(\psi(t - \tau))} \leqq 2K\Delta \leqq R_1 \tag{2.11}$$

for all $t \in \langle\tau, t_0\rangle$.

Let β_0 be such a positive number that

$$K\sqrt{(\psi(\beta_0))} \leqq \tfrac{1}{2}. \tag{2.12}$$

Firstly, suppose that $t_0 \in (\tau, \tau + 2\beta_0)$. Let $\varepsilon > 0$ be given. It is a consequence of $(\text{ii})_2$ that there exists $\alpha_1 > 0$ so that if $\alpha \in \langle 0, \alpha_1\rangle$, $\beta \in \langle 0, 2\beta_0\rangle$ and $\gamma \in \langle 0, R_1\rangle$, then $G(\alpha, \beta, \gamma) \leqq \varepsilon + G(0, \beta, \gamma)$. According to the assumption (i), there exists a solution V of the equation (0.1) on $\langle\tau, +\infty)$ such that $\|U(\tau) - V(\tau)\| \leqq \alpha_1$ and $\|V(\tau)\| \leqq \|U(\tau)\|$. Using the inequality (1.1), we have

$$\|U(t_0) - V(t_0)\| \leqq G(\|U(\tau) - V(\tau)\|, t_0 - \tau, |||U|||_{\langle\tau,t_0\rangle}),$$
$$\|U(t_0)\| = 2K\|U(\tau)\| \sqrt{(\psi(t_0 - \tau))} \leqq \|V(t_0)\| +$$
$$+ G(\|U(\tau) - V(\tau)\|, t_0 - \tau, |||U|||_{\langle\tau,t_0\rangle}) \leqq$$
$$\leqq K\|V(\tau)\| \psi(t_0 - \tau) + \varepsilon + G(0, t_0 - \tau, |||U|||_{\langle\tau,t_0\rangle}) \leqq$$
$$\leqq K\|U(\tau)\| \psi(t_0 - \tau) + \varepsilon + G(0, t_0 - \tau, |||U|||_{\langle\tau,t_0\rangle}).$$

Since ε was an arbitrary positive number, we conclude

$$2K\|U(\tau)\| \sqrt{(\psi(t_0 - \tau))} \leqq K\|U(\tau)\| \psi(t_0 - \tau) +$$
$$+ G(0, t_0 - \tau, |||U|||_{\langle\tau,t_0\rangle}) \leqq K\|U(\tau)\| \psi(t_0 - \tau) +$$
$$+ G(0, 2\beta_0, 2K\|U(\tau)\|) \leqq K\|U(\tau)\| \sqrt{(\psi(t_0 - \tau))} + G(0, 2\beta_0, 2K\|U(\tau)\|),$$
$$K\|U(\tau)\| \sqrt{(\psi(2\beta_0))} \leqq K\|U(\tau)\| \sqrt{(\psi(t_0 - \tau))} \leqq G(0, 2\beta_0, 2K\|U(\tau)\|),$$

$$\tfrac{1}{2}\sqrt{(\psi(2\beta_0))} \leqq \frac{G(0, 2\beta_0, 2K\|U(\tau)\|)}{2K\|U(\tau)\|}. \tag{2.13}$$

Let us denote $g(\beta, \gamma) = \sup_{\sigma\in(0,\gamma\rangle} [G(0, \beta, \sigma)/\sigma]$. It follows immediately from $(\mathrm{ii})_1$ and $(\mathrm{ii})_3$ that g is a nondecreasing function in both variables and $\lim_{\gamma\to0+} g(\beta, \gamma) = 0$ for all $\beta \geqq 0$. (2.13) implies that

$$\tfrac{1}{2}\sqrt{(\psi(2\beta_0))} \leqq g(2\beta_0, 2K\Delta). \tag{2.14}$$

Secondly, let $t_0 \geqq \tau + 2\beta_0$. Then there exist a natural number n and a real number $h \in \langle\beta_0, 2\beta_0)$ such that t_0 can be expressed in the form $t_0 = \tau + \beta_0 n + h$. Let $\varepsilon > 0$ be given. Denote again by α_1 such a positive number that $G(\alpha, \beta, \gamma) \leqq \varepsilon + G(0, \beta, \gamma)$ for all $\alpha \in \langle 0, \alpha_1\rangle$, $\beta \in \langle 0, 2\beta_0\rangle$ and $\gamma \in \langle 0, R_1\rangle$. The condition (i) implies that there exists a solution V_n of the equation (0.1) on the interval $\langle\tau + \beta_0 n, t_0\rangle$ such that $\|U(\tau + \beta_0 n) - V_n(\tau + \beta_0 n)\| \leqq \alpha_1$ and $\|V_n(\tau + \beta_0 n)\| \leqq \|U(\tau + \beta_0 n)\|$. (It follows from Lemma 2.3, (2.8) and (2.9) that $\|U(\tau + \beta_0 n)\| > 0$.) Using the inequality (1.1), we obtain

$$\|U(t_0) - V_n(t_0)\| \leqq G(\|U(\tau + \beta_0 n) - V_n(\tau + \beta_0 n)\|, h, |||U|||_{\langle\tau+\beta_0 n,t_0\rangle}),$$
$$\|U(t_0)\| \leqq \|V_n(t_0)\| + \varepsilon + G(0, h, |||U|||_{\langle\tau+\beta_0 n,t_0\rangle}),$$
$$\|U(t_0)\| \leqq K\|V_n(\tau + \beta_0 n)\| \psi(h) + \varepsilon + G(0, h, |||U|||_{\langle\tau+\beta_0 n,t_0\rangle}) \leqq$$
$$\leqq K\|U(\tau + \beta_0 n)\| \psi(h) + \varepsilon + G(0, h, 2K\|U(\tau)\| \sqrt{(\psi(\beta_0 n))}).$$

Since ε was an arbitrary positive number, we have

$$\|U(t_0)\| \leqq K\|U(\tau + \beta_0 n)\| \psi(h) + G(0, h, 2K\| U(\tau)\| \sqrt{(\psi(\beta_0 n))}).$$

Similarly, we can derive the inequality

$$\|U(\tau + \beta_0 i)\| \leqq K\|U(\tau + \beta_0(i - 1))\| \psi(\beta_0) +$$
$$+ G(0, \beta_0, 2K\|U(\tau)\| \sqrt{(\psi(\beta_0(i - 1)))}) \quad (\text{for } i = 1, 2, \ldots, n).$$

Thus, using (2.12) and the property (g) of the function ψ from Lemma 2.2, we also have

$$\|U(t_0)\| = 2K\|U(\tau)\| \sqrt{(\psi(t_0 - \tau))} \leqq K\|U(\tau + \beta_0 n)\| \psi(h) +$$
$$+ G(0, h, 2K\|U(\tau)\| \sqrt{(\psi(\beta_0 n))}) \leqq \tfrac{1}{2}\sqrt{(\psi(h))}\, \|U(\tau + \beta_0 n)\| +$$

$$+ G(0, h, 2K\|U(\tau)\| \sqrt{(\psi(\beta_0 n))}) \leqq \tfrac{1}{2}\sqrt{(\psi(h))}\,\{K\|U(\tau + \beta_0(n-1))\|\,\psi(\beta_0) +$$
$$+ G(0, \beta_0, 2K\|U(\tau)\| \sqrt{(\psi(\beta_0(n-1)))})\} + G(0, h, 2K\|U(\tau)\| \sqrt{(\psi(\beta_0 n))}) \leqq$$
$$\leqq (\tfrac{1}{2})^2 \sqrt{(\psi(h))}\sqrt{(\psi(\beta_0))}\,\|U(\tau + \beta_0(n-1))\| +$$
$$+ \tfrac{1}{2}\sqrt{(\psi(\beta_0))}\,G(0, \beta_0, 2K\|U(\tau)\| \sqrt{(\psi(\beta_0(n-1)))}) +$$
$$+ G(0, h, 2K\|U(\tau)\| \sqrt{(\psi(\beta_0 n))}) \leqq \ldots \leqq (\tfrac{1}{2})^{n+1}\sqrt{(\psi(h))}\,(\sqrt{(\psi(\beta_0))})^n\,\|U(\tau)\| +$$
$$+ \sum_{j=1}^{n} (\tfrac{1}{2})^j \sqrt{(\psi(h))}\,(\sqrt{(\psi(\beta_0))})^{j-1}\,G(0, \beta_0, 2K\|U(\tau)\| \sqrt{(\psi(\beta_0(n-j)))}) +$$
$$+ G(0, h, 2K\|U(\tau)\| \sqrt{(\psi(\beta_0 n))}) \leqq (\tfrac{1}{2})^{n+1}\sqrt{(\psi(\beta_0 n + h))}\,\|U(\tau)\| +$$
$$+ \sum_{j=1}^{n} (\tfrac{1}{2})^j \sqrt{(\psi(\beta_0(j-1) + h))}\,G(0, \beta_0, 2K\|U(\tau)\| \sqrt{(\psi(\beta_0(n-j)))}) +$$
$$+ G(0, h, 2K\|U(\tau)\| \sqrt{(\psi(\beta_0 n))}) =$$
$$= (\tfrac{1}{2})^{n+1}\sqrt{(\psi(\beta_0 n + h))}\,\|U(\tau)\| + \sum_{j=1}^{n} (\tfrac{1}{2})^j \sqrt{(\psi(\beta_0(j-1) + h))}\,.$$
$$.\,\frac{G(0, \beta_0, 2K\|U(\tau)\| \sqrt{(\psi(\beta_0(n-j)))})}{2K\|U(\tau)\| \sqrt{(\psi(\beta_0(n-j)))}}\,2K\|U(\tau)\| \sqrt{(\psi(\beta_0(n-j)))} +$$
$$+ \frac{G(0, h, 2K\|U(\tau)\| \sqrt{(\psi(\beta_0 n))})}{2K\|U(\tau)\| \sqrt{(\psi(\beta_0 n))}}\,2K\|U(\tau)\| \sqrt{(\psi(\beta_0 n))} \leqq$$
$$\leqq (\tfrac{1}{2})^{n+1}\sqrt{(\psi(\beta_0 n + h))}\,\|U(\tau)\| + \sum_{j=1}^{n} (\tfrac{1}{2})^j \sqrt{(\psi(\beta_0(n-1) + h))}\,.$$
$$.\,\frac{G(0, \beta_0, 2K\|U(\tau)\| \sqrt{(\psi(\beta_0(n-j)))})}{2K\|U(\tau)\| \sqrt{(\psi(\beta_0(n-j)))}}\,2K\|U(\tau)\| +$$
$$+ \frac{G(0, h, 2K\|U(\tau)\| \sqrt{(\psi(\beta_0 n))})}{2K\|U(\tau)\| \sqrt{(\psi(\beta_0 n))}}\,2K\|U(\tau)\| \sqrt{(\psi(\beta_0 n))} \leqq$$
$$\leqq (\tfrac{1}{2})^{n+1}\sqrt{(\psi(\beta_0 n + h))}\,\|U(\tau)\| + \sum_{j=1}^{n} (\tfrac{1}{2})^j \sqrt{(\psi(\beta_0 n + h))}\,.$$
$$.\,\frac{G(0, \beta_0, 2K\|U(\tau)\| \sqrt{(\psi(\beta_0(n-j)))})}{2K\|U(\tau)\| \sqrt{(\psi(\beta_0(n-j)))}}\,\frac{2K\|U(\tau)\|}{\sqrt{(\psi(\beta_0))}} +$$
$$+ \sqrt{(\psi(\beta_0 n + h))}\,\frac{G(0, h, 2K\|U(\tau)\| \sqrt{(\psi(\beta_0 n))})}{2K\|U(\tau)\| \sqrt{(\psi(\beta_0 n))}\sqrt{(\psi(h))}}\,2K\|U(\tau)\|\,.$$

Hence

$$[2K\sqrt{(\psi(t_0 - \tau))} - (\tfrac{1}{2})^{n+1}\sqrt{(\psi(\beta_0 n + h))}]\,\|U(\tau)\| \leqq$$
$$\leqq \sum_{j=1}^{n} (\tfrac{1}{2})^j \sqrt{(\psi(\beta_0 n + h))}\,\frac{G(0, \beta_0, 2K\|U(\tau)\| \sqrt{(\psi(\beta_0(n-j)))})}{2K\|U(\tau)\| \sqrt{(\psi(\beta_0(n-j)))}\sqrt{\psi(\beta_0)}}\,2K\|U(\tau)\| +$$
$$+ \sqrt{(\psi(\beta_0 n + h))}\,\frac{G(0, h, 2K\|U(\tau)\| \sqrt{(\psi(\beta_0 n))})}{2K\|U(\tau)\| \sqrt{(\psi(\beta_0 n))}\sqrt{(\psi(h))}}\,2K\|U(\tau)\|$$

and since $t_0 - \tau = \beta_0 n + h$, we get

$$2K - (\tfrac{1}{2})^{n+1} \leqq \frac{2K}{\sqrt{(\psi(\beta_0))}} \sum_{j=1}^{n} (\tfrac{1}{2})^j \frac{G(0, \beta_0, 2K\|U(\tau)\| \sqrt{(\psi(\beta_0(n-j)))})}{2K\|U(\tau)\| \sqrt{(\psi(\beta_0(n-j)))}} +$$

$$+ \frac{2K}{\sqrt{(\psi(h))}} \frac{G(0, h, 2K\|U(\tau)\| \sqrt{(\psi(\beta_0 n))})}{2K\|U(\tau)\| \sqrt{(\psi(\beta_0 n))}},$$

$$2K - (\tfrac{1}{2})^{n+1} \leqq \frac{2K}{\sqrt{(\psi(\beta_0))}} \sum_{j=1}^{n} (\tfrac{1}{2})^j g(\beta_0, 2K\|U(\tau)\|) +$$

$$+ \frac{2K}{\sqrt{(\psi(h))}} g(h, 2K\|U(\tau)\| \leqq \frac{2K}{\sqrt{(\psi(2\beta_0))}} \sum_{j=0}^{n} (\tfrac{1}{2})^j g(2\beta_0, 2K\|U(\tau)\| =$$

$$= \frac{4K}{\sqrt{(\psi(2\beta_0))}} [1 - (\tfrac{1}{2})^n] g(2\beta_0, 2K\|U(\tau)\|,$$

$$2K - 1 \leqq \frac{4K}{\sqrt{(\psi(2\beta_0))}} g(2\beta_0, 2K\|U(\tau)\|)$$

and finally, we have

$$2K - 1 \leqq \frac{4K}{\sqrt{(\psi(2\beta_0))}} g(2\beta_0, 2K\Delta). \tag{2.15}$$

Now, it is seen that we can make the right-hand sides of the inequalities (2.14) and (2.15) arbitrarily small by chosing a sufficiently small $\Delta > 0$. Thus none of the inequalities (2.14) and (2.15) can be satisfied for all $\Delta \in (0, R_1/(2K)$. This is he desired contradiction. Hence (2.8) must be true which together with Lemma 2.3 implies the uniform asymptotic stability of the zero solution of the equation (0.2). ■

We have used the linearity of the equation (0.1) only in the proof of Lemma 2.2, in particular when deducing the inequality (2.2). (But later we have used this inequality only for $\|V(\tau)\| \leqq R_1$.) If the function Φ from Definition 1.1 is such that

(2.16) there exist $R_2 > 0$ and $c > 0$ such that $\Phi(\alpha) \leqq c\alpha$ for all $\alpha \in \langle 0, R_2 \rangle$,

then we can derive an inequality of the type (2.2) (which will be valid only for $\|V(\tau)\| \leqq R_2$) also in the case that the equation (0.1) is nonlinear. Thus, if we use $R_1' = \min [R_1, R_2]$ instead of R_1 and if we proceed in the same way as in the proof of Theorem 2.1, we can prove the following proposition:

Let the conditions (i), (ii) and (2.16) or the conditions (iii), (iv) and (2.16) be satisfied and let the zero solution of the equation (0.1) (which need not be linear here) be uniformly asymptotically stable. Then the zero solution of the equation (0.2) is uniformly asymptotically stable as well.

3. APPLICATIONS TO CERTAIN DIFFERENTIAL EQUATIONS IN BANACH SPACE

Let X_0 be a Banach space with the norm $\|\cdot\|_0$. Suppose that the operator $L(t)$ from the equation (0.1) has the form

$$L(t) = A + B(t),$$

where A is a linear operator from X_0 into itself such that it is the generator of a semigroup of operators e^{At} of the class C_0 in X_0.

Let X_1 be the domain of definition of the operator A and let it be equipped with the graph norm

$$\|x_1\| = \|x_0\| + \|Ax\|_0 . \tag{3.1}$$

Since A is the generator of a semigroup of the class C_0, X_1 is dense in X_0 and A is closed in X_0. Hence X_1 is also a Banach space.

We suppose that X is such a Banach space that $X_1 \subseteq X \subseteq X_0$. Assume that there exist constants c_1 and c_2 such that

$$\|e^{At}x\|_X \leqq c_1\, e^{c_2 t}\|x\|_X \tag{3.2}$$

for $t \geqq 0$ and $x \in X_1$. It is known that a similar inequality is valid if we use the norms $\|\cdot\|_0$ or $\|\cdot\|_1$ instead of $\|\cdot\|_X$ (see e.g. [10]).

For $t \geqq 0$ let $B(t)$ be a linear operator from X_0 into itself with the domain of definition $D(B)$ independent of t and containing X_1. Suppose that $e^{At}B(s)\,x \in X_1$ for all $x \in X_1$, $t > 0$, $s \geqq 0$ and that there exist $p > 1$ and functions $k_1(t)$, $k_2(t)$ (defined for $t > 0$) so that $k_1 \in L_1((0, r))$, $k_2 \in L_p((0, r))$ for all $r > 0$ and the inequalities

$$\|e^{At}B(s)\,x\|_1 \leqq k_1(t)\,\|x\|_1 , \tag{3.3}$$

$$\|e^{At}B(s)\,x\|_X \leqq k_2(t)\,\|x\|_X \tag{3.4}$$

hold for all $x \in X_1$, $t > 0$ and $s \geqq 0$.

Suppose that $N(t)$ is a nonlinear operator from X_0 into X_0 with the domain of definition $D(N)$ independent of t and containing X_1. Let $e^{At}N(s)\,x \in X_1$ for all $x \in X_1$, $t > 0$, $s \geqq 0$ and let there exist $\alpha > 0$, $R > 0$ and a function $k_3(t)$ (defined for $t > 0$) so that $k_3 \in L_1((0, r))$ for all $r > 0$ and the inequality

$$\|e^{At}N(s)\,x\|_X \leqq k_3(t)\,\|x\|_X^{1+\alpha} \tag{3.5}$$

holds for all $t > 0$, $s \geqq 0$ and $x \in X_1$ such that $\|x\|_X \leqq R$.

By solutions of the equations (0.1) and (0.2) on any interval $I \subseteq \langle 0, +\infty)$ we shall mean functions from $\mathscr{C}^0(I; X_1) \cap \mathscr{C}^1(I; X_0)$, satisfying the given equation on I.

It is proved for example in [16] that if $\tau \geqq 0$ and $x_1 \in X_1$ are given then there exists a solution V of the equation (0.1) on $\langle \tau, +\infty)$, satisfying the initial condition $V(\tau) = x_1$ (and this solution is even unique). Thus, the assumption (i) from Section 1 is satisfied.

If we want to apply the results from Section 2, we must also verify the condition (ii). Put $R_1 = R$ and let U, V be solutions of the equation (0.2), (0.1), respectively, on $\langle \tau, t \rangle$ such that $|||U|||_{\langle \tau, t\rangle} \leqq R_1$ and $|||V|||_{\langle \tau, t\rangle} \leqq R_1$. According to the above, there exists a solution W of the equation (0.1) on $\langle \tau, +\infty)$ such that $W(\tau) = U(\tau)$. Moreover,

(3.6) $$\|U(t) - V(t)\|_X \leqq \|U(t) - W(t)\|_X + \|W(t) - V(t)\|_X \,.$$

First we shall estimate the term $\|U(t) - W(t)\|_X$. Making use of the semigroup e^{At}, we can write

$$U(t) - W(t) = \int_\tau^t e^{A(t-\sigma)} B(\sigma) [U(\sigma) - W(\sigma)] \, d\sigma + \int_\tau^t e^{A(t-\sigma)} N(\sigma) U(\sigma) \, d\sigma \,,$$

$$\|U(t) - W(t)\|_X \leqq \int_\tau^t k_2(t - \sigma) \|U(\sigma) - W(\sigma)\|_X \, d\sigma + \int_\tau^t + k_3(t - \sigma) \|U(\sigma)\|_X^{1+\alpha} \, d\sigma \,.$$

Choose a natural number m such that $2^m/(2^m - 1) < p$ and set $a = 2^m/(2^m - 1)$, $b = 2^m$. Hence $1/a + 1/b = 1$ and if we use the Hölder inequality, we get

$$\|U(t) - W(t)\|_X \leqq \left[\int_\tau^t k_2^a(t - \sigma) \, d\sigma\right]^{1/a} .$$

$$\cdot \left[\int_\tau^t \|U(\sigma) - W(\sigma)\|_X^b \, d\sigma\right]^{1/b} + \int_\tau^t k_3(t - \sigma) \|U(\sigma)\|_X^{1+\alpha} \, d\sigma \leqq$$

$$\leqq \left[\int_\tau^t k_2^a(t - \sigma) \, d\sigma\right]^{1/a} \left[\int_\tau^t \|U(\sigma) - W(\sigma)\|_X^b \, d\sigma\right]^{1/b} +$$

$$+ |||U|||_{\langle \tau, t\rangle}^{1+\alpha} \int_\tau^t k_3(t - \sigma) \, d\sigma \,.$$

Denote

(3.7) $$f(t) = \|U(t) - W(t)\|_X^b \,,$$

(3.8) $$\varrho(t - \tau) = 2^{b-1} \left[\int_0^{t-\tau} k_2^a(\sigma) \, d\sigma\right]^{b/a} ,$$

(3.9) $$\chi_1(t - \tau) = 2^{b-1} \left[\int_0^{t-\tau} k_3(\sigma) \, d\sigma\right]^b .$$

By virtue of the inequality $(u + v)^b \leqq 2^{b-1}(u^b + v^b)$ we obtain

$$f(t) \leqq \varrho(t - \tau) \int_\tau^t f(\sigma) \, d\sigma + |||U|||_{\langle \tau, t\rangle}^{(1+\alpha)b} \chi_1(t - \tau) \,.$$

Since ϱ is a nondecreasing function, we also have

$$f(\vartheta) \leqq \varrho(t - \tau) \int_\tau^\vartheta f(\sigma) \, d\sigma + |||U|||_{\langle \tau, t\rangle}^{(1+\alpha)b} \chi_1(\vartheta - \tau)$$

for $\vartheta \in \langle \tau, t \rangle$. Thus, if we use the generalized Gronwall-Bellman inequality, we get

$$f(\vartheta) \leqq |||U|||_{\langle \tau, t \rangle}^{(1+\alpha)b} \left\{ \varrho(t - \tau) \int_\tau^\vartheta \chi_1(\sigma - \tau) \, e^{\varrho(t-\tau)(\sigma-\tau)} \, d\sigma + \chi_1(\vartheta - \tau) \right\},$$

$$f(\vartheta) \leqq |||U|||_{\langle \tau, t \rangle}^{(1+\alpha)b} \left\{ \varrho(t - \tau) \int_0^{\vartheta - \tau} \chi_1(\sigma) \, e^{\varrho(t-\tau)\sigma} \, d\sigma + \chi_1(\vartheta - \tau) \right\},$$

and this yields

(3.10)
$$\|U(t) - W(t)\|_X \leqq$$
$$\leqq |||U|||_{\langle \tau, t \rangle}^{(1+\alpha)} \left\{ \varrho(t - \tau) \int_0^{t - \tau} \chi_1(\sigma) \, e^{\varrho(t-\tau)\sigma} \, d\sigma + \chi_1(t - \tau) \right\}^{1/b}.$$

Now we shall estimate the second term on the right-hand side of (3.6). We have

$$W(t) - V(t) = e^{A(t-\tau)}[U(\tau) - V(\tau)] +$$
$$+ \int_\tau^t e^{A(t-\sigma)} B(\sigma) [W(\sigma) - V(\sigma)] \, d\sigma.$$

Using (3.1) and (3.4), we get

$$\|W(t) - V(t)\|_X \leqq c_1 \, e^{c_2(t-\tau)} \|U(\tau) - V(\tau)\|_X +$$
$$+ \int_\tau^t k_2(t - \sigma) \, \|W(\sigma) - V(\sigma)\|_X \, d\sigma.$$

Denoting

(3.11)
$$\chi_2(t - \tau) = 2^{b-1}[c_1 \, e^{c_2(t-\tau)}]^b,$$

we can derive the inequality

(3.12)
$$\|W(t) - V(t)\|_X \leqq$$
$$\leqq \|U(\tau) - V(\tau)\|_X \left\{ \varrho(t - \tau) \int_0^{t - \tau} \chi_2(\sigma) \, e^{\varrho(t-\tau)\sigma} \, d\sigma + \chi_2(t - \tau) \right\}^{1/b}$$

in the same way as the inequality (3.10). (3.10) and (3.11) imply

(3.13)
$$\|U(t) - V(t)\|_X \leqq$$
$$\leqq |||U|||_{\langle \tau, t \rangle}^{(1+\alpha)} \left\{ \varrho(t - \tau) \int_0^{t - \tau} \chi_1(\sigma) \, e^{\varrho(t-\tau)\sigma} \, d\sigma + \chi_1(t - \tau) \right\}^{1/b} +$$
$$+ \|U(\tau) - V(\tau)\|_X \left\{ \varrho(t - \tau) \int_0^{t - \tau} \chi_2(\sigma) \, e^{\varrho(t-\tau)\sigma} \, d\sigma + \chi_2(t - \tau) \right\}^{1/b}.$$

If we set $G(\|U(\tau) - V(\tau)\|_X, t - \tau, |||U|||_{\langle \tau, t \rangle})$ to be equal to the right-hand side of the inequality (3.13), it is not difficult to find out that the function G has the properties $(\text{ii})_1$, $(\text{ii})_2$ and $(\text{ii})_3$. Therefore, the condition (ii) from Section 1 is also fulfilled and hence the following theorem holds:

Theorem 3.1. *Let the zero solution of the equation*

$$\frac{\mathrm{d}V}{\mathrm{d}t} = AV + B(t)\,V \tag{3.14}$$

be uniformly asymptotically stable with respect to the norm $\|\cdot\|_X$. *Then the zero solution of the equation*

$$\frac{\mathrm{d}U}{\mathrm{d}t} = AU + B(t)\,U + N(t)\,U \tag{3.15}$$

has the same property.

We will show that several important equations of mathematical physics may be considered as special cases of the equation (3.15) in the following three paragraphs. We also choose a concrete space X there.

3.1. THE NAVIER-STOKES EQUATIONS

Suppose that Ω is a bounded domain in E_3 with a lipschitzian boundary $\partial\Omega$. Let $\mathscr{D}(\Omega)$ be the set of all infinitely differentiable vector-functions defined in Ω and having the zero divergence and a compact support in Ω. The Banach space X_0 will be the closure of $\mathscr{D}(\Omega)$ in $\mathbf{L}_2(\Omega)$. Further, denote by $\mathbf{J}_1^\circ(\Omega)$ the closure of $\mathscr{D}(\Omega)$ in $\mathbf{W}_2^1(\Omega)$. Let P be the orthogonal projection of $\mathbf{L}_2(\Omega)$ onto X_0 and $A = \nu P * \Delta$ (where ν is the kinematic coefficient of viscosity appearing in the Navier-Stokes equations, and Δ is the Laplace operator). Put $X_1 = \mathbf{J}_1^\circ(\Omega) \cap \mathbf{W}_2^2(\Omega)$ and let the norm in X_1 be given by (3.1). We choose the Banach space X to be identical with X_1 and $\|\cdot\|_X = \|\cdot\|_1$.

Suppose that $\tilde{U} \in C^1(\langle 0, +\infty); X_0) \cap C^0(\langle 0, +\infty); X_1)$ is a solution of the Navier-Stokes boundary value problem:

$$\frac{\partial U}{\partial t} + (U, \nabla)\,U = F - \frac{1}{\varrho}\operatorname{grad} p + \nu\,\Delta U \quad (\text{in } \Omega), \tag{3.1.1}$$

$$\operatorname{div} U = 0 \quad (\text{in } \Omega), \tag{3.1.2}$$

$$U\big|_{\partial\Omega} = U_0 . \tag{3.1.3}$$

It is known that the question of stability of the solution $\tilde{U}$ may be transformed to the question of stability of the zero solution of the problem given by

$$\frac{\partial U}{\partial t} + (\tilde{U}, \nabla)\,U + (U, \nabla)\,\tilde{U} + (U, \nabla)\,U = -\frac{1}{\varrho}\operatorname{grad} q + \nu\,\Delta U \quad (\text{in } \Omega), \tag{3.1.4}$$

(3.1.2), and by the boundary condition

$$U\big|_{\partial\Omega} = 0 . \tag{3.1.5}$$

The system (3.1.4), (3.1.2) and (3.1.5) contains two unknowns: U and q. But the function q will not be important in the sequel. Applying the projection P to the equation

(3.1.4) and denoting

$$B(t)\,U = -P[(\tilde{U}(t), \nabla)\,U + (U, \nabla)\,\tilde{U}(t)]\,,$$
$$NU = -P[(U, \nabla)\,U]\,,$$

we get the equation (3.15). The term P grad q is equal to zero, because grad q is orthogonal to X_0 in $\mathbf{L}_2(\Omega)$ (see e.g. [13]). It may be easily verified that any function $V \in \mathscr{C}^1(\mathscr{D}(V); X_0) \cap \mathscr{C}^0(\mathscr{D}(V); X_1)$ is a solution of the problem (3.1.4), (3.1.2), (3.1.5) for $t \in \mathscr{D}(V)$ if and only if it satisfies (3.15) for $t \in \mathscr{D}(V)$.

It is shown for example in [8] and [9] that the operator A is the generator of a semigroup e^{At} of the class C_0 in X_0, $e^{At}V \in X_1$ for $V \in \mathbf{\mathring{J}}_1(\Omega)$ and there exists a non-decreasing function $\varkappa$ on $(0, +\infty)$ so that

$$\|e^{At}V\|_1 \leqq \frac{\varkappa(T)}{t^{3/4}}\,\|V\|_{\mathring{J}_1(\Omega)} \tag{3.1.6}$$

for all $T > 0$, $t \in (0, T\rangle$ and $V \in \mathbf{\mathring{J}}_1(\Omega)$. Further, it may be shown by means of the Hölder inequality and the Sobolev embedding theorem that there exist constants c_3 and c_4 such that, if $V \in X_1$ and $s \geqq 0$, then

$$\|B(s)\,V\|_{\mathring{J}_1(\Omega)} \leqq c_3 \|\tilde{U}(s)\|_1\, \|V\|_1\,,$$
$$\|NV\|_{\mathring{J}_1(\Omega)} \leqq c_4 \|V\|_1^2\,.$$

These estimates together with (3.1.6) easily imply that the inequalities (3.3), (3.4) and (3.5) are satisfied (the number R concerning the inequality (3.5) may be chosen arbitrarily). Thus we can use Theorem 3.1 and if we take into account the relation between the equation (3.14) (or (3.15)) and the boundary value problem given by

$$\frac{\partial V}{\partial t} + (V, \nabla)\,\tilde{U} + (\tilde{U}, \nabla)\,V = -\frac{1}{\varrho}\,\mathrm{grad}\,q + \nu\,\Delta V \quad (\text{in } \Omega)\,, \tag{3.1.7}$$

(3.1.2) and (3.1.5) (the boundary value problem given by (3.1.4), (3.1.2) and (3.1.5), respectively), we can state

Theorem 3.1.1. *If the zero solution of the linear boundary value problem* (3.1.7), (3.1.2), (3.1.5) *is uniformly asymptotically stable with respect to the norm* $\|\cdot\|_X$, *then the zero solution of the problem* (3.1.4), (3.1.2), (3.1.5) *has the same property (and consequently, the solution U of the Navier-Stokes problem* (3.1.1), (3.1.2), (3.1.3) *is uniformly asymptotically stable with respect to the norm* $\|\cdot\|_X$ *as well).*

3.2. THE WAVE EQUATION

The equation treated in this paragraph has the form

$$u_{tt} - u_{xx} = a(t, x)\,u + b(t, x)\,u_t + c(t, x)\,u_x + \sum_{i,j=1}^{3} d_{ij}(t, x, u, u_t, u_x)\,u_i u_j \tag{3.2.1}$$

(for $t \in \langle 0, +\infty)$ and $x \in \langle 0, \pi\rangle$). The symbols u_1, u_2 and u_3 mean u, u_t and u_x.

We assume that the functions $a(t, x)$, $b(t, x)$ and $c(t, x)$ are defined for $t \in \langle 0, +\infty)$ and $x \in \langle 0, \pi \rangle$ and that they are continuous and bounded together with their second derivatives with respect to x. Further, we assume that the functions $d_{ij}(t, x, u, u_t, u_x)$ $(i, j = 1, 2, 3)$ are defined for $t \in \langle 0, +\infty)$, $x \in \langle 0, \pi \rangle$, $u \in E_1$, $u_t \in E_1$ and $u_x \in E_1$ and that they together with all their second derivatives with respect to the variables x, u, u_t and u_x are continuous and bounded on each set of the type $\langle 0, +\infty) \times \langle 0, \pi \rangle \times \langle -r, r \rangle \times \langle -r, r \rangle \times \langle -r, r \rangle$, r being an arbitrary positive number.

We shall study the equation (3.2.1) with one of the following boundary conditions:

$$u(t, 0) = u(t, \pi) = 0 \quad (\text{for } t \geqq 0), \tag{3.2.2}$$

$$u_x(t, 0) = u(t, \pi) = 0 \quad (\text{for } t \geqq 0), \tag{3.2.3}$$

$$u_x(t, 0) = u_x(t, \pi) = 0 \quad (\text{for } t \geqq 0). \tag{3.2.4}$$

We shall also suppose that $c(t, 0) = c(t, \pi) = d_{33}(t, 0, 0, 0, u_x) = d_{33}(t, \pi, 0, 0, u_x) = 0$ (or $c(t, \pi) = d_{33}(t, \pi, 0, 0, u_x) = 0$) for all $t \in \langle 0, +\infty)$ and $u_x \in E_1$ if we deal with the boundary conditions (3.2.2) (or (3.2.3), respectively)).

Observe that if we have to study the problem given by the equation (3.2.1) and, for example, by the boundary conditions

$$u_x(t, 0) + \alpha_0 u(t, 0) = u_x(t, \pi) + \alpha_\pi u(t, \pi) = 0, \tag{3.2.5}$$

we can easily transform it to the problem given by an equation of the type (3.2.1) and the boundary conditions (3.2.4), substituting $u(t, x) = v(t, x) e^{-\chi(x)}$ (where χ is a sufficiently smooth function defined on $\langle 0, \pi \rangle$ and such that $\chi'(0) = \alpha_0$, $\chi'(\pi) = \alpha_\pi$) into (3.2.1) and (3.2.5).

Set $X_0 = \{[v_1, v_2] \mid v_1 \in C^1(\langle 0, \pi \rangle);\ v_2 \in C^0(\langle 0, \pi \rangle);\ v_1(0) = v_1(\pi) = v_2(0) = v_2(\pi) = 0$ or $v_1'(0) = v_1(\pi) = v_2(\pi) = 0$ or $v_1'(0) = v_1'(\pi) = 0$ if the boundary conditions (3.2.2) or (3.2.3) or (3.2.4), respectively, are considered$\}$,

$$\|[v_1, v_2]\|_0 = \max_{x \in \langle 0, \pi \rangle} |v_1(x)| + \max_{x \in \langle 0, \pi \rangle} |v_1'(x)| + \max_{x \in \langle 0, \pi \rangle} |v_2(x)| \quad (\text{for } [v_1, v_2] \in X_0),$$

$$X = X_0, \quad \|[v_1, v_2]\|_X = \|[v_1, v_2]\|_0,$$

$X_1 \equiv D(A) = X_0 \cap \{[v_1, v_2] \mid v_1 \in C^2(\langle 0, \pi \rangle);\ \ v_2 \in C^1(\langle 0, \pi \rangle);\ \ v_1''(0) = v_1''(\pi) = 0$ or $v_2'(0) = v_1''(\pi) = 0$ or $v_2'(0) = v_2'(\pi) = 0$ in the case of the boundary conditions (3.2.2) or (3.2.3) or (3.2.4), respectively$\}$,

$$A[v_1, v_2] = [v_2, v_1''] \quad (\text{for } [v_1, v_2] \in X_1),$$

$$\|[v_1, v_2]\|_1 = \|[v_1, v_2]\|_0 + \|A[v_1, v_2]\|_0 \quad (\text{for } [v_1, v_2] \in X_1),$$

$$B(t)[v_1, v_2] = [0, a(t, x) v_1 + b(t, x) v_2 + c(t, x) v_1'],$$

$$N(t)[v_1, v_2] = [0, \sum_{i,j=1}^{3} d_{ij}(t, x, v_1, v_2, v_1') v_i v_j]$$

(where $v_3 = v_1'$ and $[v_1, v_2] \in X_0$).

It may be easily verified that $u \in \mathscr{C}^0(\mathscr{D}(u); C^2(\langle 0, \pi \rangle)) \cap \mathscr{C}^1(\mathscr{D}(u); C^1(\langle 0, \pi \rangle))$ is a solution of the equation (3.2.1) (with one of the boundary conditions (3.2.2), (3.2.3) and (3.2.4)) if and only if $U \equiv [u, u_t] \in \mathscr{C}^0(\mathscr{D}(u); X_1) \cap \mathscr{C}^1(\mathscr{D}(u); X_0)$ (where we consider the same boundary conditions in the definition of the spaces X_0 and X_1) is a solution of the equation (3.15).

The operator A is the generator of a semigroup e^{At} of the class C_0 in X_0 such that

(3.2.6)
$$(e^{At}[v_1, v_2])(x) = \tfrac{1}{2}\Big[v_1(x+t) + v_1(x-t) + \int_{x-t}^{x+t} v_2(\sigma)\, d\sigma,\ v_1'(x+t) - v_1'(x-t) + v_2(x+t) + v_2(x-t)\Big],$$

where the symbols v_1 and v_2 mean the functions v_1 and v_2 with an extended domain of definition so that they are odd and 2π-periodic functions on $(-\infty, +\infty)$ in the case of the boundary conditions (3.2.2); even 4π-periodic functions on $(-\infty, +\infty)$ such that $v_1(x) = -v_1(2\pi - x)$ in the case of the boundary conditions (3.2.3); and even, 2π-periodic functions on $(-\infty, +\infty)$ in the case of the boundary conditions (3.2.4).

The validity of the inequality (3.2) follows immediately from the properties of the semigroup e^{At}.

It may be easily verified that $B(s) : X_1 \to X_1$, $N(s) : X_1 \to X_1$ for all $s \geqq 0$ and by virtue of (3.2.6), it is possible to find out that all inequalities (3.3), (3.4) and (3.5) are satisfied (with $\alpha = 1$ in (3.5) and $R > 0$ being arbitrary).

Thus, we can use Theorem 3.1 obtaining

Theorem 3.2.1. *Let the zero solution of the equation*

(3.2.8)
$$u_{tt} - u_{xx} = a(t, x)\, u + b(t, x)\, u_t + c(t, x)\, u_x$$

(with one of the boundary conditions (3.2.2), (3.2.3) *or* (3.2.4)*) be uniformly asymptotically stable with respect to the norm*

$$\|u(t, \cdot)\| = \max_{x \in \langle 0, \pi \rangle} |u(t, x)| + \max_{x \in \langle 0, \pi \rangle} |u_t(t, x)| + \max_{x \in \langle 0, \pi \rangle} |u_x(t, x)|\,.$$

Then the zero solution of the equation (3.2.1) *(with the same boundary conditions as in the case of the equation* (3.2.8)*) has the same property.*

3.3. THE EQUATION OF OSCILLATIONS OF A BEAM

In this paragraph we shall deal with the equation of the form

(3.3.1)
$$u_{tt} + u_{xxxx} = a(t, x)\, u + b(t, x)\, u_t + c(t, x)\, u_x + d(t, x)\, u_{xx} + \sum_{i,j=1}^{4} e_{ij}(t, x, u, u_t, u_x, u_{xx})\, u_i u_j$$

(for $t \in \langle 0, +\infty)$, $x \in \langle 0, \pi\rangle$). The symbols u_1, u_2, u_3 and u_4 in (3.3.1) mean u, u_t, u_x and u_{xx}.

We suppose that the functions $a(t, x)$, $b(t, x)$, $c(t, x)$ and $d(t, x)$ are defined on $\langle 0, +\infty) \times \langle 0, \pi\rangle$, they are continuous and bounded together with their second derivatives with respect to x and $c(t, 0) = c(t, \pi) = 0$ for all $t \geqq 0$. Further, we assume that the functions $e_{ij}(t, x, u, u_t, u_x, u_{xx})$ $(i, j = 1, 2, 3, 4)$ are defined for $t \in \langle 0, +\infty)$, $x \in \langle 0, \pi\rangle$, $u \in E_1$, $u_t \in E_1$, $u_x \in E_1$ and $u_{xx} \in E_1$ and that they and all their second derivatives with respect to the variables x, u, u_t, u_x and u_{xx} are continuous and bounded on each set of the type $\langle 0, +\infty) \times \langle 0, \pi\rangle \times \langle -r, r\rangle^4$, r being an arbitrary positive number. Moreover, we shall need the equalities $e_{33}(t, 0, 0, 0, u_3, 0) = e_{33}(t, \pi, 0, 0, u_3, 0) = 0$ to be satisfied for all $t \geqq 0$ and $u_3 \in E_1$.

We require all solutions of the equation (3.3.1) to satisfy the boundary conditions

(3.3.2) $$u(t, 0) = u_{xx}(t, 0) = u(t, \pi) = u_{xx}(t, \pi) = 0 \quad (\text{for } t \geqq 0)\,.$$

Set

$$X_0 = \{[v_1, v_2] \mid v_1 \in W_2^2((0, \pi)),\ v_2 \in L_2((0, \pi));\ v_1(0) = v_1(\pi) = 0\}\,,$$

$$([v_1, v_2], [w_1, w_2])_0 = \int_0^\pi [v_1''(x)\, w_1''(x) + v_2(x)\, w_2(x)]\, dx\,,$$

$$\|[v_1, v_2]\|_0^2 = ([v_1, v_2], [v_1, v_2])_0 \quad (\text{for } [v_1, v_2], [w_1, w_2] \in X_0)\,,$$

$$X = \{[v_1, v_2] \mid v_1 \in W_2^3((0, \pi)),\ v_2 \in \mathring{W}_2^1((0, \pi))\,;$$
$$v_1(0) = v_1''(0) = v_1(\pi) = v_1''(\pi) = 0\}\,,$$

$$\|[v_1, v_2]\|_X = \left\{\int_0^\pi [v_1^2(x) + v_2^2(x) + v_1'^2(x) + v_2'^2(x) + v_1''^2(x) + v_1'''^2(x)]\, dx\right\}^{1/2}$$
$$(\text{for } [v_1, v_2] \in X)\,,$$

$$X_1 \equiv D(A) = \{[v_1, v_2] \mid v_1 \in W_2^4((0, \pi)),\ v_2 \in W_2^2((0, \pi))\,;$$
$$v_1(0) = v_1''(0) = v_1(\pi) = v_1''(\pi) = v_2(0) = v_2(\pi) = 0\}\,,$$

$$A[v_1, v_2] = [v_2, -v_1^{(4)}] \quad (\text{for } [v_1, v_2] \in X_1)\,,$$

$$\|[v_1, v_2]\|_1 = \|[v_1, v_2]\|_0 + \|A[v_1, v_2]\|_0 \quad (\text{for } [v_1, v_2] \in X_1)\,,$$

$$B(t)\,[v_1, v_2] = [0, a(t, x)\, v_1 + b(t, x)\, v_2 + c(t, x)\, v_1' + d(t, x)\, v_1'']\,,$$

$$N(t)\,[v_1, v_2] = [0, \sum_{i,j=1}^{4} e_{ij}(t, x, v_1, v_2, v_1', v_1'')\, v_i v_j]$$
$$(\text{where } v_3 = v_1',\ v_4 = v_1'' \text{ and } [v_1, v_2] \in X_0)\,.$$

The problem (3.3.1), (3.3.2) is equivalent to the equation (3.15) in the sense that $U \equiv [u_1, u_2] \in \mathscr{C}^0(\mathscr{D}(U); X_1) \cap \mathscr{C}^1(\mathscr{D}(U); X_0)$ is a solution of (3.15) if and only if $u_1 \in \mathscr{C}^0(\mathscr{D}(U);\ W_2^4((0, \pi))) \cap \mathscr{C}^1(\mathscr{D}(U);\ W_2^2((0, \pi)))$ is a solution of the equation (3.3.1), u_1 satisfies (3.3.2) and $\partial u_1/\partial t = u_2$.

It may be verified that $(\cdot, \cdot)_0$ is a scalar product on X_0, X_1 is dense in X_0 and the operator A is a closed operator from X_0 into X_0. Let $\lambda > 0$ and $(\lambda I - A)[v_1, v_2] = [w_1, w_2]$ ($[v_1, v_2] \in X_1$, $[w_1, w_2] \in X_0$). Then

$$((\lambda I - A)[v_1, v_2], [v_1, v_2])_0 = ([w_1, w_2], [v_1, v_2])_0 ,$$

$$\int_0^\pi [\lambda\, v_1''^2(x) - v_2''(x)\, v_1''(x) + \lambda\, v_2^2(x) + v_1^{(4)}(x)\, v_2(x)]\, dx \leqq \|[w_1, w_2]\|_0\, \|[v_1, v_2\|_0 ,$$

$$\lambda\, \|[v_1, v_2]\|_0 \leqq \|[w_1, w_2]\|_0 .$$

This implies that $\|(\lambda I - A)^{-1}\|_{X_0 \to X_0} \leqq 1/\lambda$ and therefore, according to the Hille-Yosida-Phillips theorem, A is the generator of a contractive semigroup e^{At} of the class C_0 in X_0.

In order to show that (3.2) is satisfied, we shall prove the following lemma:

Lemma 3.3.1. *There exists $c_5 > 0$ such that*

$$\|e^{At}[v_1, v_2]\|_X \leqq c_5 \|[v_1, v_2]\|_X \tag{3.3.3}$$

for all $[v_1, v_2] \in X_1$ and $t \geqq 0$.

Proof. Set

$$Y_0 = \{[v_1, v_2] \mid v_1 \in W_2^3((0, \pi)),\ v_2 \in \mathring{W}_2^1((0, \pi)) ;$$
$$v_1(0) = v_1''(0) = v_1(\pi) = v_1''(\pi) = 0\} ,$$

$$([v_1, v_2], [w_1, w_2])_{Y_0} = \int_0^\pi [v_1'''(x)\, w_1'''(x) + v_2'(x)\, w_2'(x)]\, dx ,$$

$$Y_1 = \{[v_1, v_2] \mid v_1 \in W_2^5((0, \pi)),\ v_2 \in W_2^3((0, \pi)) ;$$
$$v_1(0) = v_1''(0) = v_1(\pi) = v_1''(\pi) = v_2(0) = v_2''(0) = v_2(\pi) = v_2''(\pi) = 0\} .$$

Using again the Hille-Yosida-Phillips theorem, we can show that $A|_{Y_1}$ is the generator of a contractive semigroup of operators S_t of the class C_0 in Y_0. If $[\tilde{v}_1, \tilde{v}_2] \in Y_1$ then $S_t[\tilde{v}_1, \tilde{v}_2] = e^{At}[\tilde{v}_1, \tilde{v}_2]$.

Let $[v_1, v_2] \in X_1$ and let $\varepsilon > 0$ be given. There exists $[\tilde{v}_1, \tilde{v}_2] \in Y_1$ so that $\|[v_1, v_2] - [\tilde{v}_1, \tilde{v}_2]\|_1 < \varepsilon$. We have

$$\|e^{At}[v_1, v_2]\|_X \leqq \|e^{At}[v_1, v_2] - e^{At}[\tilde{v}_1, \tilde{v}_2]\|_X +$$
$$+ \|e^{At}[\tilde{v}_1, \tilde{v}_2]\|_X \leqq \text{const.}\, \|e^{At}[v_1, v_2] - e^{At}[\tilde{v}_1, \tilde{v}_2]\|_1 +$$
$$+ \|S_t[\tilde{v}_1, \tilde{v}_2]\|_X \leqq \text{const.}\, \|[v_1, v_2] - [\tilde{v}_1, \tilde{v}_2]\|_1 +$$
$$+ c_5 \|S_t[\tilde{v}_1, \tilde{v}_2]\|_{Y_0} \leqq \text{const.}\, \varepsilon + c_5 \|[\tilde{v}_1, \tilde{v}_2]\|_{Y_0} \leqq$$
$$\leqq \text{cost.}\, \varepsilon + c_5 \|[\tilde{v}_1, \tilde{v}_2] - [v_1, v_2]\|_{Y_0} + c_5 \|[v_1, v_2]\|_{Y_0} \leqq$$
$$\leqq \text{const.}\, \varepsilon + c_5 \|[\tilde{v}_1, \tilde{v}_2] - [v_1, v_2]\|_1 + c_5 \|[v_1, v_2]\|_X \leqq$$
$$\leqq \text{const.}\, \varepsilon + c_5 \varepsilon + c_5 \|[v_1, v_2]\|_X .$$

Since ε was an arbitrary positive number, the inequality (3.3.3) must hold. ∎

The operators $B(s)$ and $N(s)$ transform X_1 into itself (for all $s \geqq 0$) and if $[v_1, v_2] \in \in X_1$ and $t > 0$, then

$$\|e^{At}B(s)[v_1, v_2]\|_1 \leqq \|B(s)[v_1, v_2]\|_1 = \|B(s)[v_1, v_2]\|_0 + \\ + \|A\, B(s)[v_1, v_2]\|_0 = \|[0, av_1 + bv_2 + cv_1' + dv_1'']\|_0 + \\ + \|[av_1 + bv_2 + cv_1' + dv_1'', 0]\|_0 \leqq \text{const.}\ \|[v_1, v_2]\|_1 .$$

Thus, the inequality (3.3) is satisfied. Similarly, by using the ideas from the proof of Lemma 3.3.1, it may be shown that also the inequalities (3.4) and (3.5) are fulfilled (where $\alpha = 1$ and $R > 0$ may be chosen arbitrarily; but note that the constant in the inequality (3.5) depends on this R).

Hence we can apply the Theorem 3.1. Taking into account the relation between the equation (3.14) (the equation (3.15)) and the problem given by the equation

(3.3.4) $$u_{tt} + u_{xxxx} = a(t, x)\, u + b(t, x)\, u_t + c(t, x)\, u_x + d(t, x)\, u_{xx}$$

and the boundary conditions (3.3.2) (or the problem given by the equation (3.3.1) and the boundary conditions (3.3.2), respectively), we conclude

Theorem 3.3.1. *Let the zero solution of the problem* (3.3.4), (3.3.2) *be uniformly asymptotically stable with respect to the norm*

$$\|u(t, \cdot)\| = \left\{\int_0^\pi [u^2(t,x) + u_t^2(t,x) + u_x^2(t,x) + u_{tx}^2(t,x) + u_{xx}^2(t,x) + u_{xxx}^2(t,x)]\, dx\right\}^{1/2}.$$

Then the zero solution of the problem (3.3.1), (3.3.2) *has the same property.*

4. APPLICATIONS TO CERTAIN DIFFERENTIAL EQUATIONS OF N-TH ORDER IN HILBERT SPACE

In this section we shall treat differential equations of the type

(4.1) $$u^{(n)}(t) + \sum_{i=0}^{n-1} a_i(A)\, u^{(i)}(t) + \sum_{i=0}^{n-1} B_i(t)\, u^{(i)}(t) = \\ = F(t, u)(t), u'(t), \ldots, u^{(n-1)}(t)) .$$

Let us remark that the stability and the correctness of equations of a very similar type are investigated in [2].

We suppose that H is a Hilbert space with the scalar product $(\cdot, \cdot)_H$ and the corresponding norm $(\cdot, \cdot)_H$. $A : H \to H$ is a linear selfadjoint operator with the domain of definition $D(A)$ and the spectral resolution of identity $E(s)$. Further, we assume that $\inf \mathrm{Sp}(A) > 0$.

If $f : \mathrm{Sp}(A) \to E_1$ is a continuous function then we can define

$$D(f(A)) = \left\{x \in H \middle| \int_{\mathrm{Sp}(A)} |f(s)|^2\, d\|E(s)\, x\|_H^2 < +\infty\right\}$$

and

$$f(A)\,x = \int_{\mathrm{Sp}(A)} f(s)\,\mathrm{d}E(s)\,x \quad (\text{for } x \in D(f(A)))\,. \tag{4.2}$$

We assume that a_i $(i = 0, 1, \ldots, n-1)$ are continuous real functions defined on $\mathrm{Sp}(A)$ and such that

$$|a_i(s)| \leqq c_6 s^{(n-i)/n} \tag{4.3}$$

for some constant c_6, $i = 0, 1, \ldots, n-1$ and $s \in \mathrm{Sp}(A)$. Hence $a_i(A)$ are operators defined as is indicated in (4.2).

Let us denote by $m_i(t, \tau, s)$ $(i = 0, 1, \ldots, n-1;\ \tau \geqq 0;\ t \geqq \tau;\ s \in \mathrm{Sp}(A))$ the solutions of the ordinary differential equation

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n} m + \sum_{j=0}^{n-1} a_j(s) \frac{\mathrm{d}^j}{\mathrm{d}t^j} m = 0 \quad (\text{for } t \in \langle \tau, +\infty)), \tag{4.4}$$

satisfying the initial conditions

$$\frac{\mathrm{d}^j}{\mathrm{d}t^j} m_i(\tau, \tau, s) = \delta_{i,j} \quad (j = 0, 1, \ldots, n-1)\,. \tag{4.5}$$

Suppose that there exist constants c_7 and ω so that

$$\left|\frac{\mathrm{d}^j}{\mathrm{d}t^j} m_{n-1}(t, 0, s)\right| s^{(n-j-1)/n} \leqq c_7\, \mathrm{e}^{\omega t} \tag{4.6}$$

for $t \in \langle 0, +\infty)$, $j = 0, 1, \ldots, n-1$ and $s \in \mathrm{Sp}(A)$. In [2], the term "the operator

$$\mathscr{L}\, u(t) = u^{(n)}(t) + \sum_{i=0}^{n-1} a_i(A)\, u^{(i)}(t)$$

is of the type ω" has been used if the inequality (4.6) is satisfied.

Further, let $B_i(t)$ $(i = 0, 1, \ldots, n-1;\ t \in \langle 0, +\infty))$ be linear operators from $D(A^{(n-i)/n})$ into $D(A^{1/n})$ such that if $v \in D(A^{(n-i)/n})$ is given then the mapping $t \mapsto B_i(t)\, v$ belongs to $\mathscr{C}^0(\langle 0, +\infty); H)$ and

$$\|A^{1/n} B_i(t)\, w\|_H \leqq c_8 \|A^{(n-i)/n} w\|_H \tag{4.7}$$

for some constant c_8, $i = 0, 1, \ldots, n-1$, $t \in \langle 0, +\infty)$ and $w \in D(A^{(n-i)/n})$.

The nonlinear operator F on the right-hand side of the equation (4.1) is assumed to satisfy

$$F : \langle 0, +\infty) \times D(A) \times D(A^{(n-1)/n}) \times \ldots \times D(A^{1/n}) \to D(A^{1/n})\,,$$

there exist $R > 0$, $c_9 \geqq 0$ and $\alpha > 0$ so that

$$\|A^{1/n} F(t, v_0, \ldots, v_n\rangle_1)\|_H \leqq c_9 [\sum_{i=0}^{n-1} \|A^{(n-i)/n} v_i\|_H]^{1+\alpha} \tag{4.8}$$

for all $t \geqq 0$ and $[v_0, \ldots, v_{n-1}] \in D(A) \times \ldots \times D(A^{1/n})$ satisfying

$$\sum_{i=0}^{n-1} \|A^{(n-i)/n} v_i\|_H \leqq R\,.$$

Put $X = D(A) \times D(A^{(n-1)/n}) \times \ldots \times D(A^{1/n})$, and if $U \equiv [u_0, u_1, \ldots, u_{n-1}] \in X$, put

$$\|U\|_X = \sum_{i=0}^{n-1} \|A^{(n-i)/n} u_i\|_H .$$

We shall also suppose that F is a continuous mapping from $\langle 0, +\infty) \times \{U \in X \mid \|U\|_X \leqq R\}$ into $D(A^{1/n})$ (where we consider the norm $\|A^{1/n} \cdot\|_H$ in $D(A^{1/n})$).

As solutions of the equation (4.1) on an interval $I \subseteq \langle 0, +\infty)$ we shall regard only functions from $\bigcap_{i=0}^{n} \mathscr{C}^i(I; D(A^{(n-i)/n}))$, satisfying (4.1) on I. Sometimes we shall use the symbol $\mathscr{D}(u)$ again in order to denote the interval where u is a solution of (4.1).

It is obvious that if we denote

$$\begin{aligned} U(t) &= [u(t), u'(t), \ldots, u^{(n-1)}(t)] , \\ L(t)\, V &\equiv L(t) [v_0, \ldots, v_{n-2}, v_{n-1}] = \\ &= [v_1, \ldots, v_{n-1}, -\sum_{i=0}^{n-1} a_i(A)\, v_i - \sum_{i=0}^{n-1} B_i(t)\, v_i] , \\ N(t)\, V &\equiv N(t) [v_0, \ldots, v_{n-2}, v_{n-1}] = \\ &= [0, \ldots, 0, F(t, v_0, v_1, \ldots, v_{n-1})] \\ &\quad (\text{for } V \equiv [v_0, v_1, \ldots, v_{n-1}] \in X) , \end{aligned}$$

then the equation (4.1) can be rewritten in the form (0.2). Also, due to (4.8),

$$\|N(t)\, U\|_X \leqq c_9 \|U\|_X^{1+\alpha} \tag{4.9}$$

and hence the equation (0.2) has the zero solution.

Lemma 4.1. *Let $\tau \geqq 0$ and $x \equiv [x_0, x_1, \ldots, x_{n-1}] \in X$. Then there exists a solution v of the linear equation*

$$v^{(n)}(t) + \sum_{i=0}^{n-1} a_i(A)\, u^{(i)}(t) + \sum_{i=0}^{n-1} B_i(t)\, u^{(i)}(t) = 0 \tag{4.10}$$

on the interval $\langle \tau, +\infty)$, satisfying the initial conditions

$$v^{(j)}(\tau) = x_j \quad (j = 0, 1, \ldots, n-1) . \tag{4.11}$$

Proof. It may be shown that v is a solution of (4.10), (4.11) if and only if it satisfies

$$\begin{aligned} v(t) &= \sum_{i=0}^{n-1} m_i(t, \tau, A)\, x_i - \\ &- \int_\tau^t m_{n-1}(t + \tau - \sigma, \tau, A) \left[\sum_{j=0}^{n-1} B_j(\sigma)\, v^{(j)}(\sigma)\right] \mathrm{d}\sigma . \end{aligned} \tag{4.12}$$

(The implication (4.10), (4.11) $\Rightarrow$ (4.12) follows from Theorem 2.1.1 in [2]).

It will be sufficient even to prove the existence of a function v, satisfying (4.12), only in $\bigcap_{i=0}^{n-1} \mathscr{C}^i(\langle\tau, +\infty); D(A^{(n-i)/n}))$, because using the "good" properties of the functions m_i and the operators $B_j(t)$, we can show that then the right-hand side of (4.12) (and hence also v) belongs to $\mathscr{C}^n(\langle\tau, +\infty); H)$.

Let $T > 0$. Denote

$$Y_T = \bigcap_{i=0}^{n-1} \mathscr{C}^i(\langle\tau, \tau + T\rangle; D(A^{(n-i)/n})),$$

$$\|v\|_{Y_T} = \sum_{i=0}^{n-1} \max_{t\in\langle\tau,\tau+T\rangle} \|A^{(n-i)/n} v^{(i)}(t)\|_H ,$$

$$\mathscr{B}(t)\, v(t) = -\sum_{i=0}^{n-1} B_j(t)\, v^{(j)}(t) ,$$

$$\mathscr{G}(v)\,(t) = \int_\tau^t m_{n-1}(t + \tau - \sigma, \tau, A)\, \mathscr{B}(\sigma)\, v(\sigma)\, \mathrm{d}\sigma ,$$

$$w(t) = \sum_{i=0}^{n-1} m_i(t, \tau, A)\, x_i .$$

Then (4.12) (for $t \in \langle\tau, \tau + T\rangle$) is equivalent to the relation

$$v = w - \mathscr{G}(v) \tag{4.13}$$

in Y_T. First, we show that if T is small enough then (4.13) has a solution in Y_T. We have

$$\|\mathscr{G}v\|_{Y_T} \leqq \sum_{i=0}^{n-1} \max_{t\in\langle\tau,\tau+T\rangle} \left\{\int_\tau^t \left[\int_{\mathrm{Sp}(A)} s^{2(n-i)/n} \left|\frac{\mathrm{d}^i}{\mathrm{d}t^i}\, m_{n-1}(t + \tau - \sigma, \tau, s)\right|^2 . \right.\right.$$

$$\left.\left. .\, \mathrm{d}\|E(s)\, \mathscr{B}(\sigma)\, v(\sigma)\|_H^2\right]^{1/2} \mathrm{d}\sigma\right\} \leqq$$

$$\leqq \sum_{i=0}^{n-1} \max_{t\in\langle\tau,\tau+T\rangle} \left\{\int_\tau^t c_7\, \mathrm{e}^{\omega(t-\sigma)} \left[\int_{\mathrm{Sp}(A)} s^{2/n}\, \mathrm{d}\|E(s)\, \mathscr{B}(\sigma)\, v(\sigma)\|_H^2\right]^{1/2} \mathrm{d}\sigma\right\} =$$

$$= \sum_{i=0}^{n-1} \max_{t\in\langle\tau,\tau+T\rangle} \left\{\int_\tau^t c_7\, \mathrm{e}^{\omega(t-\sigma)} \|A^{1/n}\, \mathscr{B}(\sigma)\, v(\sigma)\|_H\, \mathrm{d}\sigma\right\} \leqq$$

$$\leqq n \max_{t\in\langle\tau,\tau+T\rangle} \int_\tau^t c_7\, \mathrm{e}^{\omega(t-\sigma)} c_8 \left[\sum_{j=0}^{n-1} \|A^{(n-j)/n}\, v^{(j)}(\sigma)\|_H\right] \mathrm{d}\sigma =$$

$$= c_7 c_8 n T \mathrm{e}^{\omega T} \sum_{j=0}^{n-1} \max_{\sigma\in\langle\tau,\tau+T\rangle} \|A^{(n-j)/n}\, v^{(j)}(\sigma)\|_H = c_7 c_8 n T \mathrm{e}^{\omega T} \|v\|_{Y_T} .$$

Choosing T small enough, we may achieve that the operator $\mathscr{G}$ is contractive from Y_T into itself and thus, due to the Banach fixed point theorem, (4.13) (and consequently also (4.10), (4.11)) has a solution v in Y_T.

Similarly, we can prove the existence of a solution $\hat{v}$ of (4.10) with the initial conditions

$$\hat{v}^{(i)}(\tau + T) = v^{(i)}(\tau + T) \quad (i = 0, 1, \ldots, n - 1)$$

also on the interval $\langle \tau + T, \tau + 2T \rangle$, and it may be easily checked that the function equal to v on $\langle \tau, \tau + T \rangle$ and to $\hat{v}$ on $\langle \tau + T, \tau + 2T \rangle$ is a solution of (4.10), (4.11) on $\langle \tau, \tau + 2T \rangle$. We can proceed in this way and prove the existence of a solution of (4.10), (4.11) on the whole interval $\langle \tau, +\infty)$. ∎

It follows from Lemma 4.1 that the condition (iii) from Section 1 is satisfied. In what follows, we show that also the condition (iv) is fulfilled.

Put $R_1 = R$ and let U and V be solutions of the equation (0.2) and (0.1), respectively, on $\langle \tau, t \rangle \subset \langle 0, +\infty)$ so that $U(\tau) = V(\tau)$, $|||U|||_{\langle \tau, t \rangle} \leqq R_1$ and $|||V|||_{\langle \tau, t \rangle} \leqq R_1$. Then it is obvious that U and V have the form $U(t) \equiv [u(t), u'(t), \ldots, u^{(n-1)}(t)]$ and $V(t) \equiv [v(t), v'(t), \ldots, v^{(n-1)}(t)]$, where u and v are solutions of the equations (4.1) and (4.10), respectively, on the interval $\langle \tau, t \rangle$. Using the expressions for the solutions u and v in the form analogous to (4.12) we can write

$$u(t) - v(t) = -\int_\tau^t m_{n-1}(t + \tau - \sigma, \tau, A) \left\{ \sum_{j=0}^{n-1} B_j(\sigma) [u^{(j)}(\sigma) - v^{(j)}(\sigma)] \right\} d\sigma +$$

$$+ \int_\tau^t m_{n-1}(t + \tau - \sigma, \tau, A) F(\sigma, u(\sigma), u'(\sigma), \ldots, u^{(n-1)}(\sigma)) \, d\sigma ,$$

$$\sum_{i=0}^{n-1} \| A^{(n-i)/n} [u^{(i)}(t) - v^{(i)}(t)] \|_H = \| U(t) - V(t) \|_H \leqq$$

$$\leqq \sum_{i=0}^{n-1} \int_\tau^t \left[\int_{\mathrm{Sp}(A)} s^{2(n-i)/n} \left| \frac{d^i}{dt^i} m_{n-1}(t + \tau - \sigma, \tau, s) \right|^2 . \right.$$

$$\left. . \, d \| E(s) \sum_{j=0}^{n-1} B_j(\sigma) [u^{(j)}(\sigma) - v^{(j)}) \sigma)] \|_H^2 \right]^{1/2} d\sigma +$$

$$+ \sum_{i=0}^{n-1} \int_\tau^t \left[\int_{\mathrm{Sp}(A)} s^{2(n-i)/n} \left| \frac{d^i}{dt^i} m_{n-1}(t + \tau - \sigma, \tau, s) \right|^2 . \right.$$

$$\left. . \, d \| E(s) F(\sigma, u(\sigma), u'(\sigma), \ldots, u^{(n-1)}(\sigma)) \|_H^2 \right]^{1/2} d\sigma \leqq$$

$$\leqq \sum_{i=0}^{n-1} \int_\tau^t c_7 \, e^{\omega(t-\sigma)} \left[\int_{\mathrm{Sp}(A)} s^{2/n} \, d \| E(s) \sum_{j=0}^{n-1} B_j(\sigma) [u^{(j)}(\sigma) - v^{(j)}(\sigma)] \|_H^2 \right]^{1/2} d\sigma +$$

$$+ \sum_{j=0}^{n-1} \int_\tau^t c_7 \, e^{\omega(t-\sigma)} \left[\int_{\mathrm{Sp}(A)} s^{2/n} \, d \| E(s) F(\sigma, u(\sigma), u'(\sigma), \ldots, u^{(n-1)}(\sigma)) \|_H^2 \right]^{1/2} d\sigma =$$

$$= n \int_\tau^t c_7 \, e^{\omega(t-\sigma)} \| A^{1/n} \sum_{j=0}^{n-1} B_j(\sigma) [u^{(j)}(\sigma) - v^{(j)}(\sigma)] \|_H \, d\sigma +$$

$$+ n \int_\tau^t c_7 \, e^{\omega(t-\sigma)} \| A^{1/n} F(\sigma, u(\sigma), u'(\sigma), \ldots, u^{(n-1)}(\sigma)) \|_H \, d\sigma \leqq$$

$$\leqq n \int_\tau^t c_7 \, e^{\omega(t-\sigma)} \sum_{j=0}^{n-1} c_8 \|A^{(n-j)/n}[u^{(j)}(\sigma) - v^{(j)}(\sigma)]\|_H \, d\sigma +$$

$$+ n \int_\tau^t c_7 \, e^{\omega(t-\sigma)} c_9 [\sum_{j=0}^{n-1} \|A^{(n-j)/n} u^{(j)}(\sigma)\|_H]^{1+\alpha} \, d\sigma \leqq$$

$$\leqq c_7 c_8 n \, e^{\omega(t-\tau)} \int_\tau^t \|U(\sigma) - V(\sigma)\|_X \, d\sigma + n c_7 c_9 \, e^{\omega(t-\tau)} \int_\tau^t \|U(\sigma)\|_X^{1+\alpha} \, d\sigma .$$

Similarly, we have also for all $\vartheta \in \langle \tau, t \rangle$:

$$\|U(\vartheta) - V(\vartheta)\|_X \leqq c_7 c_8 n \, e^{\omega(t-\tau)} \int_\tau^\vartheta \|U(\sigma) - V(\sigma)\|_X \, d\sigma +$$

$$+ c_7 c_9 n \, e^{\omega(t-\tau)} \int_\tau^t \|U(\sigma)\|_X^{1+\alpha} \, d\sigma .$$

Using the generalized Gronwall-Bellman inequality, we obtain

$$\|U(\vartheta) - V(\vartheta)\|_X \leqq \left[c_7 c_9 n \, e^{\omega(t-\tau)} \int_\tau^t \|U(\sigma)\|_X^{1+\alpha} \, d\sigma \right] +$$

$$+ \left[c_7 c_9 n \, e^{\omega(t-\tau)} \int_\tau^t \|U(\sigma)\|_X^{1+\alpha} \, d\sigma \right] [c_7 c_9 n \, e^{\omega(t-\tau)}] .$$

$$\cdot \int_\tau^\vartheta \exp [c_7 c_8 n \, e^{\omega(t-\tau)} (\xi - \tau)] \, d\xi \leqq$$

$$\leqq c_7 c_9 n \, e^{\omega(t-\tau)} \int_\tau^t \|U(\sigma)\|_X^{1+\alpha} \, d\sigma \exp [c_7 c_8 n \, e^{\omega(t-\tau)} (\vartheta - \tau)] .$$

Since this is valid for all $\vartheta \in \langle \tau, t \rangle$, we also have

$$\|U(t) - V(t)\|_X \leqq c_7 c_9 n \, e^{\omega(t-\tau)} \int_\tau^t \|U(\sigma)\|_X^{1+\alpha} \, d\sigma \exp [c_7 c_8 \, e^{\omega(t-\tau)} (t - \tau)] ,$$

$$\|U(t) - V(t)\|_X \leqq c_7 c_9 n \, e^{\omega(t-\tau)} (t - \tau) \, |||U|||_{\langle \tau, t \rangle}^{1+\alpha} \exp [c_7 c_8 \, e^{\omega(t-\tau)} (t - \tau)] .$$

If we denote by $\hat{G}(t - \tau, |||U|||_{\langle \tau, t \rangle})$ the right-hand side of the last inequality, we can see that it has all the properties required in the condition (iv) and hence the condition (iv) is satisfied. Thus, using Theorem 2.1 and taking into account the relation between the equations (4.10), (4.1) and the equations (0.1) (0.2), respectively, we obtain

Theorem 4.1. *Let the zero solution of the equation* (4.10) *be uniformly asymptotically stable with respect to the norm*

$$\|u(t)\| = \sum_{i=0}^{n-1} \|A^{(n-i)/n} u^{(i)}(t)\|_H .$$

Then the zero solution of the equation (4.1) *has the same property.*

Several special examples of the equation (4.1) are shown in [2]. Observe that among the special cases of the equation (4.1) we can include also for instance the Timoshenko-type equation

$$\begin{gathered} u^{(4)}(t) + \alpha_1 A^{1/2} u''(t) + \alpha_2 A u(t) + b_1(t) u'''(t) + \\ + b_2(t) A^{1/4} u''(t) + b_3(t) u''(t) + \\ + b_4(t) A^{1/2} u'(t) + b_5(t) A^{1/4} u'(t) + \\ + b_6(t) u'(t) + b_7(t) A^{3/4} u(t) + b_8(t) A^{1/2} u(t) + \\ + b_9(t) A^{1/4} u(t) + b_{10}(t) u(t) = F(t, u(t), u'(t), u''(t), u'''(t)), \end{gathered}$$

where $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_1^2 - 4\alpha \geqq 0$ and $b_1, b_2, \ldots, b_{10}$ are bounded continuous functions on $\langle 0, +\infty)$.

5. APPLICATIONS TO THE NAVIER STOKES EQUATIONS IN SPACES OF CONTINUOUS FUNCTIONS

In this section we shall treat again the system (3.1.4), (3.1.2), (3.1.5). We shall suppose that Ω is a bounded or unbounded domain in E_3 with a compact boundary $\partial\Omega$ of the class $C^{2+(\alpha)}$ (for some $\alpha \in (0, 1)$). Assume that the solution $\tilde{U}$ of (3.1.1), (3.1.2), (3.1.3) belongs to $C^{2+(\alpha), 1+(\alpha/2)}(\bar{Q}_T)$ for all $T > 0$ (where $Q_T = \Omega \times (0, T)$) and that there exists $c_{10} > 0$ so that

$$|\tilde{U}|_{Q_T}^{2+(\alpha),1+(\alpha/2)} \leqq c_{10} \quad (\text{for all } T > 0). \tag{5.1}$$

As solutions of the problem (3.1.4), (3.1.2), (3.1.3) and the linearized problem (3.1.7), (3.1.2), (3.1.3) we shall regard only functions U satisfying the corresponding equations and boundary conditions and such that $U \in C^{2+(\alpha),1+(\alpha/2)}(\bar{\Omega} \times I)$ for every compact interval $I \subseteq \mathscr{D}(U)$.

Set

$$\begin{aligned} X &= \{V \in C^{1+(\alpha)}(\bar{\Omega}) \mid \operatorname{div} V = 0 \text{ in } \Omega \text{ and } V|_{\partial\Omega} = 0\}, \\ X_1 &= X \cap C^{2+(\alpha)}(\bar{\Omega}). \end{aligned}$$

Let the operators A, $B(t)$ and $N(t)$ be the same as in 3.1 and let $L(t)$ be again equal to $A + B(t)$.

It follows from Theorem 9.1 in [21] that given $V_0 \in X_1$ and $\tau \geqq 0$, then there exists a unique solution V (for $t \in \langle \tau, +\infty)$) of the linear system (3.1.7), (3.1.2), (3.1.3) (and hence also a unique solution of the equation (0.1)), satisfying the initial condition $V(\tau) = V_0$. This result implies that the condition (i) from Section 1 is fulfilled. Denote by $\mathscr{U}(t, \tau) V_0$ the value of the solution V at the time t. It is also proved in [21] (p. 218) that there exist constants $c_{11} > 0$ and $\gamma \in E_1$ (not depending on V_0) so that

$$\begin{aligned} \|\mathscr{U}(t, \tau) V_0\|_X &= |\mathscr{U}(t, \tau) V_0|_\Omega^{1+(\alpha)} \leqq \\ &\leqq c_{11}(t - \tau)^{-1/2} e^{\gamma(t-\tau)} |V_0|_\Omega^{(\alpha)} \quad (\text{for } t \geqq \tau). \end{aligned} \tag{5.2}$$

Now we shall verify the condition (ii) from Section 1. Let U and V be solutions of the equations (0.2), (0.1), respectively, on an interval $\langle\tau, t\rangle$. Using (5.2) and the expression for the solutions U and V as in [21], p. 182, we can write

$$U(t) - V(t) = \mathscr{U}(t, \tau)\,[U(\tau) - V(\tau)] + \int_\tau^t \mathscr{U}(t, \sigma)\,\mathrm{N}U(\sigma)\,\mathrm{d}\sigma\,,$$

$$\|U(t) - V(t)\|_X \leqq c_{11}(t-\tau)^{-1/2}\,\mathrm{e}^{\gamma(t-\tau)}|U(\tau) - V(\tau)|_\Omega^{(\alpha)} + \\ + \int_\tau^t c_{11}(t-\sigma)^{-1/2}\,\mathrm{e}^{\gamma(t-\sigma)}|\mathrm{N}U(\sigma)|_\Omega^{(\alpha)}\,\mathrm{d}\sigma\,.$$

It may be easily shown that there exists $c_{12} > 0$ so that

$$|\mathrm{N}U|_\Omega^{(\alpha)} \leqq c_{12}(|U|_\Omega^{1+(\alpha)})^2$$

for all $U \in C^{1+(\alpha)}(\bar{\Omega})$. Thus, we have

$$\|U(t) - V(t)\|_X \leqq c_{11}(t-\tau)^{-1/2}\,\mathrm{e}^{\gamma(t-\tau)}\|U(\tau) - V(\tau)\|_X + \\ + c_{11}c_{12}\int_\tau^t (t-\sigma)^{-1/2}\,\mathrm{e}^{\gamma(t-\sigma)}\|U(\sigma)\|_X^2\,\mathrm{d}\sigma\,,$$

$$\|U(t) - V(t)\|_X \leqq c_{11}(t-\tau)^{-1/2}\,\mathrm{e}^{\gamma(t-\tau)}\|U(\tau) - V(\tau)\|_X + \\ + c_{11}c_{12}|||U|||_{\langle\tau,t\rangle}^2\int_0^{t-\tau}\frac{1}{\sqrt{\sigma}}\,\mathrm{e}^{\gamma\sigma}\,\mathrm{d}\sigma\,.$$

We can denote by $G(\|U(\tau) - V(\tau)\|, t-\tau, |||U|||_{\langle\tau,t\rangle})$ the right-hand side of the last inequality. The number $R_1 > 0$, appearing in (ii), may be chosen arbitrarily. The function G has the properties $(\mathrm{ii})_1$, $(\mathrm{ii})_2$ and $(\mathrm{ii})_3$ and therefore, the condition (ii) is satisfied.

Thus, due to Theorem 2.1, Theorem 3.1.1 remains valid also if we consider the norm $\|\cdot\|_X$ to be equal to $|\cdot|_\Omega^{1+(\alpha)}$ and if the function U and the domain Ω have the properties assumed in this section.

Remark 5.1. The space $C^{2+(\alpha),1+(\alpha/2)}(\bar{Q}_T)$, $C^{n+(\alpha)}(\bar{\Omega})$ and the norms $|\cdot|_{Q_T}^{2+(\alpha),1+(\alpha/2)}$, $|\cdot|_\Omega^{n+(\alpha)}$ (n being a natural number and $\alpha \in (0, 1)$) are defined for example in [13].

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