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# THE LINKAGE OF A GRAPH*

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**Abstract.** The linkage of a graph is defined to be the maximum min-degree of any of its subgraphs. It is known that the linkage of a graph is equal to its width: for an arbitrary linear ordering of the vertices of the graph, consider the maximum, with respect to any vertex $v$, of the number of vertices connected with $v$ and preceding it in the ordering; the width of the graph is the minimum of these maxima over all possible linear orderings. Width has been used in artificial intelligence in the context of constraint satisfaction problems (CSPs). A more general notion is defined by considering not the number of vertices preceding and connected with $v$ but rather the least number of vertices preceding and connected with any cluster of at most $j$ consecutive vertices extending to the right up to $v$ ($j$ is a given integer). The graph parameter thus defined is called $j$-width. No efficient algorithm was known for computing the $j$-width. In this paper, we introduce a graph parameter depending on $j$ that refers to the subgraphs of the graph and generalizes the notion of linkage. We prove the min–max theorem that this graph parameter, which we call $j$-linkage, is equal to $j$-width, and we then give a polynomial-time algorithm for computing it (for constant $j$). We also find *tight* lower and upper bounds for the $j$-linkage (equivalently, the $j$-width) of graphs with given numbers of vertices and edges. It is interesting to note that a lower bound for the width of a graph had been found by Erdös; as we show, however, that bound is not tight. Moreover, we prove that our lower bound for width is also a *tight* lower bound for treewidth, pathwidth, and bandwidth, graph parameters that may be arbitrarily larger than width. Finally, we show that computing the $j$-linkage is a P-complete problem, whereas we prove that approximating it is a threshold problem: it is in NC for approximation factors $< 1/(2j)$, and it is P-complete for approximation factors $> 1/2$.

**Key words.** width parameters of a graph, linkage of a graph, backtrack-free search, extremal graph properties, algorithms in NC, P-complete problems

**AMS subject classifications.** 68Q15, 68Q20, 68Q22, 68Q25, 68R10, 05C35, 05C85

## 1. Introduction.

Let $G = (V, E)$ be an undirected graph without multiple edges or loops. Let $n = |V|$ and $e = |E|$.

The linkage of $G$ is defined to be the maximum min-degree of any of the subgraphs of $G$ (the min-degree of a subgraph is the least degree of any of its vertices; the degree of a vertex is taken relative to the subgraph).

The width of $G$ is defined to be the the minimum, over all linear orderings of the vertices of $G$, of the maximum, with respect to any vertex $v$, of the number of vertices connected with $v$ and preceding it in the linear ordering. In [14], it was proved that the width of a graph is equal to its linkage (see also [7]). The term linkage was introduced by Freuder in [7]; however, the corresponding notion dates back to Szekeres and Wilf [19] and Matula [12]. Using the aforementioned equality of the two parameters, it can be proved that they can be computed in polynomial time.

Erdös [6] proved that every graph $G$ has a subgraph with min-degree at least equal to the density $\lceil e/n \rceil$ of $G$. Therefore, the density is a lower bound for the linkage (equivalently, width) of graphs with given numbers of edges and vertices.

Given a positive integer $j$, if in the definition of width we consider not the number of vertices preceding and connected with $v$ but rather the least number of vertices preceding and connected with any cluster of at most $j$ consecutive vertices extending

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to the right up to $v$, we get a graph parameter known as $j$-width (see next section for formal definitions). This notion is due to Freuder [8].

Freuder [7] related width to backtrack-free search for a solution of a constraint satisfaction problem (CSP). Roughly, by Freuder's result, when the level of local consistency (strong consistency) of the constraint graph of a CSP exceeds its width, there is a backtrack-free search for a globally consistent solution. Corresponding results that concern backtrack-bounded search hold in relation to $j$-width [8]. However, the algorithm given by Freuder for computing the $j$-width has worst-case complexity of the order of $n!$ (for constant $j$). To our knowledge, no polynomial-time algorithm for computing the $j$-width was known.

In this paper, we introduce a graph parameter that we call $j$-linkage. Given a subgraph $H = (V_H, E_H)$ of $G$, consider all nonempty sets $S \subseteq V_H$ with cardinality at most $j$ and find the minimum, over all such subsets $S$, of the number of vertices in $V_H - S$ that are adjacent to vertices in $S$; we then find the maximum, over all subgraphs $H$ of $G$, of the corresponding minima. This maximum is by definition the $j$-linkage of the graph (see next section for formal definitions). Notice that the linkage of $G$ is equal to its 1-linkage. We prove the min–max result that the $j$-width of a graph is equal to its $j$-linkage. We also give a polynomial-time sequential algorithm for computing the $j$-linkage when $j$ is a fixed constant. We thus provide an efficient way for computing the $j$-width.

Furthermore, we find a *tight* lower bound for the $j$-linkage of graphs with given numbers of vertices and edges (for any $j \geq 1$). In contrast, we show that the aforementioned lower bound by Erdös (for $j = 1$) is not tight. Moreover, we prove that our lower bound for width is also a *tight* lower bound for treewidth, pathwidth and bandwidth, graph parameters that may be arbitrarily larger than width. There is extensive literature on lower bounds for these latter parameters [15–17, 20], but no result referring to tightness was known. To obtain the lower bound for $j$-linkage, we consider graphs that have the extremal property that their $j$-width increases by the addition of even one arbitrary edge. We then find the maximum number of edges that such an extremal graph may have, when its $j$-linkage and its number of vertices is $k$ and $n$, respectively. Call this maximum max-$E(n, j, k)$. We prove that for an arbitrary graph with $e$ edges and $n$ vertices, respectively, the least $k$ for which max-$E(n, j, k) \geq e$ is the required lower bound for the $j$-linkage. We show that this bound is tight by constructing the corresponding extremal graphs. We believe that the method of using extremal graph properties to compute tight bounds is interesting in itself.

Using a method similar to the one described above, we obtain a *tight upper* bound for the $j$-linkage. It should be mentioned that one can find a nontight upper bound with simpler computations. However, for certain values of $j$, the difference of this upper bound from the tight one that we provide is $\Theta(e^{1/2})$.

Finally, we study the parallel complexity of computing the $j$-linkage. For constant $j$, we show that the problem of determining whether $j$-linkage is no more than $k$ is P-complete for a any fixed $k \geq 2$, whereas it is in NC for $k \leq 1$. We also consider the parallel complexity of approximating the $j$-linkage. We show that this problem is of the threshold type: it is in NC for approximation factors $< 1/(2j)$, and it is P-complete for approximation factors $> 1/2$.

In summary: (1) we study the sequential and parallel complexity of computing the $j$-width (alternatively, $j$-linkage), we resolve the question of efficiency in the sequential case, and we prove positive and negative results for the parallel case; (2) we provide optimal upper and lower estimates for the $j$-width that, when $j = 1$, improve previous results, and we give connections with other graph parameters.

**2. Graph-theoretic properties and the elimination algorithm.** In this section, we first give formal definitions for the concepts we use, and we then prove that the $j$-width of a graph is equal to its $j$-linkage. Finally, we give a polynomial-time sequential algorithm for computing the $j$-linkage.

**2.1. Definitions.** Let $l$ be a *layout* of a graph $G = (V, E)$, i.e., a linear ordering $v_1, \ldots, v_n$ of its vertices. The *width* with respect to $l$ of a set $S = \{v_{i-k+1}, \ldots, v_i\}$ of $k$ consecutive vertices (notationally $\text{width}_l(S)$) is defined to be the number of vertices in the set $\{v_1, \ldots, v_{i-k}\}$ adjacent to vertices in $S$ $(1 \leq k \leq i \leq n)$.

Informally, the width of $S$ with respect to $l$ is the number of vertices preceding $S$ and adjacent to elements of $S$.

Now, let $j$ be an integer such that $1 \leq j \leq n$.

The *$j$-width* of a vertex $v_i$ with respect to $l$ is the least width of any set of at most $\min(i, j)$ consecutive vertices extending to the right up to $v_i$, i.e.,

$$j\text{-width}_l(v_i) = \min\{\text{width}_l(v_{i-k+1}, \ldots, v_i) \colon k = 1, \ldots, \min(i, j)\}.$$

The *$j$-width* of $G$ with respect to $l$ is defined to be the maximum of $j\text{-width}_l(v_i)$ over all vertices $v_i$ of $G$.

Finally, the $j$-width of $G$ (not dependent on a specific layout) is defined to be the minimum of the $j$-widths of $G$ with respect to any of the $n!$ layouts of $G$.

The 1-width of $G$ is simply called the width of $G$.

We now introduce the notion of the $j$-linkage of a graph. The related definitions follow.

*External degree* of a nonempty set $S$ of vertices of a subgraph $H$ of $G$ (notationally $\text{ext-degree}_H(S)$) is the number of vertices of $H$ that do not belong to $S$ and are adjacent to an element of $S$.

The $j$-min-*degree* of a subgraph $H$ of $G$ is the minimum $\text{ext-degree}_H(S)$ over all sets $S$ of vertices of $H$ with $1 \leq |S| \leq j$. Obviously, the 1-min-degree of a subgraph is the least degree of its vertices.

The *$j$-linkage* of $G$ is the maximum $j$-min-degree of any subgraph of $G$.

Obviously, the 1-linkage of $G$ is the maximum min-degree of any of its subgraphs. The 1-linkage of $G$ is simply called the linkage of $G$.

**2.2. $j$-width equals $j$-linkage.**

THEOREM 2.1. *For any graph $G = (V, E)$, the $j$-width of $G$ is equal to its $j$-linkage.*

*Proof.* Let $j\text{-linkage}(G) = \lambda$. We first show that $j\text{-width}(G) \geq \lambda$. From the definition of linkage, it follows that there is an induced subgraph $H = (V_H, E_H)$ of $G$ such that for any set $S$ of vertices of $V_H$ with $1 \leq |S| \leq j$, $\text{ext-degree}_H(S) \geq \lambda$. Now, consider an arbitrary layout $l$ of $G$. Let $v_i$ be the last vertex in this layout that belongs to $V_H$. We claim that the $j$-width of $v_i$ with respect to $l$ is at least $\lambda$. Indeed, let $S_0$ be the intersection $V_H$ with an arbitrary set $\{v_{i-k+1}, \ldots, v_i\}$ of $k$ consecutive vertices extending to the right up to $v_i$, where $k$ is an arbitrary integer less than or equal to $\min(i, j)$. Obviously, $S_0$ is a nonempty subset of $V_H$ of cardinality at most $j$. Moreover, all vertices of $V_H - S_0$ that are adjacent to vertices in $S_0$ must belong to $\{v_1, \ldots, v_{i-k}\}$ because $v_i$ was chosen to be the last with respect to $l$ vertex of $V_H$. Therefore, because $\text{ext-degree}_H(S_0) \geq \lambda$, we get, as we claimed, that the $j$-width of $v_i$ with respect to $l$ is at least $\lambda$. Because $l$ was arbitrary, it follows that the $j$-width$(G) \geq \lambda = j$-linkage$(G)$.

We will now show that $j$-width$(G) \leq \lambda$. From the definition of linkage, it also follows that for any induced subgraph $H = (V_H, E_H)$ of $G$, there is a set $S$ of vertices of $V_H$ with $1 \leq |S| \leq j$ and ext-degree$_H(S) \leq \lambda$. In the next paragraph, we give a construction of a layout $l$ of $G$ such that for any vertex $v$ of $G$, $j$-width$_l(v) \leq \lambda$. Once such an $l$ is constructed, the required inequality easily follows.

The construction of $l$ will be given in a last-first fashion, i.e., we proceed from defining the last vertices in $l$ towards defining the first elements in it. Because $j$-linkage$(G) = \lambda$ and since $G$ is a subgraph of itself, we get that $j$-min-degree$(G) \leq \lambda$. Therefore, there exists a set $S_0$ of vertices of $G$ such that $1 \leq |S_0| \leq j$ and ext-degree$_G(S_0) \leq \lambda$. We place the vertices of $S_0$ (in any arbitrary order) as the last vertices in the layout under construction. Notice that every subset of $S_0$ is adjacent with at most $\lambda$ vertices among the ones that will be later placed in the layout. Therefore, any vertex in $S_0$ has $j$-width with respect to the layout $l$ at most $\lambda$. We now consider the subgraph $H$ induced by the set of vertices $V - S_0$. Again because $j$-min-degree$(H) \leq \lambda$, there must exist a subset $S_1$ of vertices of $H$ such that $1 \leq |S_1| \leq j$ and ext-degree$_H(S_1) \leq \lambda$. As a next step towards defining $l$, we place the elements of $S_1$ (in any arbitrary order) to the left of the vertices of $S_0$. By the same argument as above, any vertex in $S_1$ must have $j$-width (with respect to the layout under construction) at most $\lambda$. Continuing recursively in the same way until the vertices of $G$ are exhausted, we obtain the required layout of $G$. $\square$

In the rest of the paper, we use the terms $j$-linkage and $j$-width interchangeably.

**2.3. The elimination algorithm.** In this subsection, we describe an algorithm that given a $k$ $(0 \leq k \leq n-j-1)$ finds the unique maximal vertex-induced subgraph of $G = (V, E)$ with $j$-min-degree at least $k+1$, in the case that there is such a subgraph; otherwise it returns the empty set. The complexity of the algorithm, for fixed $j$, is polynomial in $n$. This algorithm can be used to compute the $j$-linkage of a graph in time polynomial in $n$. We call this algorithm the $(k, j)$-Elimination Algorithm or just $k$-elimination when no confusion may arise. The graph that the algorithm returns on input $G$ will be denoted by $G^{(k,j)}$.

$(k, j)$-ELIMINATION ALGORITHM
**do while** $\exists S \subseteq V$ such that $1 \leq |S| \leq j$ and ext-degree$_G(S) \leq k$
  **begin**
  Let $S$ be a set of vertices with $1 \leq |S| \leq j$ and ext-degree$_G(S) \leq k$
  $V := V - S$
  **if** $V \neq \emptyset$ **then** $G :=$ the graph induced by $V$
      **else** $G := \emptyset$
  **end**
**return** $G$

THEOREM 2.2. *The $(k, j)$-Elimination Algorithm finds a maximal vertex-induced subgraph of $G$ that has $j$-min-degree at least $k+1$ in the case that there is such a subgraph; otherwise, it returns the empty set. Moreover, such a maximal subgraph is unique.*

*Proof.* Suppose that the algorithm outputs a nonempty subgraph $H = (V_H, E_H)$. Because $V_H$ does not contain any nonempty set $S$ with ext-degree$_H(S) \leq k$, we have that $j$-min-degree$(H) > k$. Now suppose, towards a contradiction, that there is a proper supergraph $R = (V_R, E_R)$ of $H$, with $j$-min-degree$(R) > k$.

Let $S$ be the first set of vertices containing vertices of $R$ that is removed, according to the algorithm, from the set of vertices of $G$ (such a set exists because $R$ is a proper supergraph of $H$). Notice that by the definition of $S$, no vertex of $R$ is removed before the set $S$ is discarded.

Notice now that because $j$-min-degree$(R) > k$, ext-degree$_R(S \cap V_R) > k$. But then it easily follows that the external degree of $S$ with respect to any supergraph of $R$ whose vertices contain $S$ is strictly greater than $k$. This contradicts the fact that $S$ is discarded by the algorithm at a stage when all the vertices of $R$ are still present.

To show the uniqueness of such a maximal subgraph, repeat the above argument taking $R$ not to be a proper supergraph of $H$ but rather a graph with $V_R - V_H \neq \emptyset$. $\square$

It is easy to see that in order to compute the $j$-linkage of $G$, we have simply to find the smallest $k$ for which the $k$-elimination returns the empty graph. The complexity of this algorithm is $O(n^{j+2})$.

**3. Extremal graph properties and applications in computing tight bounds.** In this section, we find tight lower and upper bounds for the $j$-linkage of graphs with given numbers of vertices and edges.

**3.1. Lower bound.** To compute a lower bound for $j$-linkage, we consider the graphs whose $j$-linkage increases by the addition of even one arbitrary edge. We then find the maximum number of edges that such an extremal graph may have when its $j$-linkage and its number of vertices are $k$ and $n$, respectively (see also [11] for a similar result for the case $j = 1$). Call this maximum max-$E(n, j, k)$. Next, we prove that for an arbitrary graph with $e$ edges and $n$ vertices, respectively, the unique $k$ for which max-$E(n, j, k-1) < e \leq$ max-$E(n, j, k)$ is the required lower bound for $j$-linkage.

LEMMA 3.1. *Let $G$ be a graph with $j$-linkage at most $k$. Suppose that $G$ has the property that any graph obtained from $G$ by adding one more edge (on the same set of vertices) has $j$-linkage equal to $k+1$. Then the number of edges of $G$ is at most*

$$\text{max-}E(n,j,k) = \binom{k}{2} + (n-k)k - \frac{1}{2}(n-k) + \frac{1}{2}(\lfloor (n-k)/j \rfloor j^2 + ((n-k) \bmod j)^2). \tag{1}$$

*Moreover, for any $n \geq 1$, for any $j = 1, \dots, n$ and any $k$ such that $0 \leq k \leq n-j$, there exists at least one graph with $n$ vertices,* max-$E(n,j,k)$ *edges, and $j$-linkage equal to $k$ and such that if one more edge is added to it (on the same set of vertices), its $j$-linkage increases to $k+1$.*

*Proof.* To prove the above lemma, we use a technical result about extreme points.

TECHNICAL LEMMA. *Given real numbers $Z$ and $D$, among all sequences $A_1$, $A_2, \dots, A_r$ of real numbers such that*

- *$r$ is an arbitrary integer $\geq 1$,*
- *$A_1 + \cdots + A_r = Z$, and*
- *$A_i \leq D$, for all $i = 1, \dots, r$,*

*the maximum value that the quantity $\sum_{i=1}^{r} A_i^2$ attains is $\lfloor Z/D \rfloor D^2 + (Z - \lfloor Z/D \rfloor D)^2$. This value is obtained for $r = \lceil Z/D \rceil$, $A_i = D$ (for all $i = 1, \dots, \lfloor Z/D \rfloor$), and (only in case $Z/D$ is not an integer) $A_{\lceil Z/D \rceil} = Z - \lfloor Z/D \rfloor D$.*

The Technical Lemma follows from the following claim applied for $W = Z/D$ and $y_i = A_i/D$ (for notational convenience, we set $\{W\} = W - \lfloor W \rfloor$).

CLAIM. *For any integer $n > 0$, real number $W \geq 0$ and sequence of reals $y_i$, $i = 1, \dots, n$, such that $0 \leq y_i \leq 1$, if $\sum_{i=1}^{n} y_i = W$, then $\sum_{i=1}^{n} y_i^2 \leq \lfloor W \rfloor + \{W\}^2$.*

*Proof.* We use induction on $n$. For $n=1$, we can see that

- if $y_1 = 1$, the claim holds because $1^2 = \lfloor 1 \rfloor + \{1\}^2$;
- if $0 \le y_1 < 1$, the claim holds because $y_1^2 = \lfloor y_1 \rfloor + \{y_1\}^2$.

Suppose that the proposition holds for $n = k$. We are going to prove that it also holds for $n = k+1$. So we have to prove that for $n = k+1$, any real number $W > 0$, and any sequence $y_i$, $i = 1, \ldots, k+1$, such that $0 \le y_i \le 1$, if $\sum_{i=1}^{k+1} y_i = W$, then $\sum_{i=1}^{k+1} y_i^2 \le \lfloor W \rfloor + \{W\}^2$.

Indeed, because $\sum_{i=1}^{k+1} y_i = W$, we have that $\sum_{i=1}^{k} y_i = W - y_{k+1}$. By applying the induction hypothesis, we have that

$$\sum_{i=1}^{k} y_i^2 \le \lfloor W - y_{k+1} \rfloor + \{W - y_{k+1}\}^2.$$

Therefore,

$$\sum_{i=1}^{k} y_i^2 + y_{k+1}^2 \le \lfloor W - y_{k+1} \rfloor + \{W - y_{k+1}\}^2 + y_{k+1}^2. \tag{2}$$

We distinguish two cases:

1. $y_{k+1} \le \{W\}$ and
2. $y_{k+1} > \{W\}$.

In the first case, we have that

$$\lfloor W - y_{k+1} \rfloor = \lfloor W \rfloor, \text{ and therefore} \tag{3}$$

$$\{W - y_{k+1}\} = \{W\} - y_{k+1}. \tag{4}$$

From relations (2)–(4), we have that

$$\begin{aligned}\sum_{i=1}^{k+1} y_i^2 &\le \lfloor W \rfloor + \{W\}^2 + y_{k+1}^2 - 2\{W\}^2 y_{k+1} + y_{k+1}^2 \\ &= \lfloor W \rfloor + \{W\}^2 + 2y_{k+1}(y_{k+1} - \{W\}) \\ &\le \lfloor W \rfloor + \{W\}^2.\end{aligned}$$

In the second case, we have that

$$\lfloor W - y_{k+1} \rfloor = \lfloor W \rfloor - 1, \text{ and therefore} \tag{5}$$

$$\{W - y_{k+1}\} = 1 + \{W\} - y_{k+1}. \tag{6}$$

From relations (2), (5), and (6), we have that

$$\begin{aligned}\sum_{i=1}^{k+1} y_i^2 &\le \lfloor W \rfloor - 1 + 1 + \{W\}^2 + y_{k+1}^2 + 2\{W\} - 2y_{k+1} - 2\{W\}y_{k+1} + y_{k+1}^2 \\ &= \lfloor W \rfloor + \{W\}^2 + 2y_{k+1}(y_{k+1} - 1) - 2\{W\}(y_{k+1} - 1) \\ &= \lfloor W \rfloor + \{W\}^2 + 2(y_{k+1} - \{W\})(y_{k+1} - 1) \\ &\le \lfloor W \rfloor + \{W\}^2.\end{aligned}$$

Therefore, the Claim holds for $n = k+1$. That proves the Claim and concludes the proof of the Technical Lemma. $\square$

We now proceed with the proof of Lemma 3.1.

*Proof of Lemma* 3.1. By hypothesis, we have that $G$ satisfies the following two properties:

1. $j$-linkage$(G) = k$ and
2. $j$-linkage$(G') = k + 1$ where $G'$ is any graph obtained from $G$ by adding one more edge (on the same set of vertices).

Property 1 above implies that if we apply the $k$-elimination to $G$, all vertices of $G$ will be removed, whereas by property 2, the $k$-elimination does not remove all vertices of a graph $G'$ obtained by adding to $G$ at least one more edge (on the same set of vertices).

Now suppose that the $k$-elimination is applied to $G$ and let $S_0, \dots, S_l$ be the sets of vertices successively removed from $V$ by the algorithm. Let $V_i$ $(i = 1, \dots, l)$ be the set of vertices of $G$ remaining after the removal of $S_{i-1}$. For notational convenience, let $V_0$ be the initial set of vertices of $G$. Let also $G_i$ be the graph induced by $V_i$. Let $q$ be the first step such that $|V_{q+1}| \le k + j$ (notice that then $|V_{q+1}| > k$).

Observe first that for all $i = 0, \dots, q$, ext-degree$_{G_i}(S_i) = k$. Indeed, otherwise, we could add one more edge to the graph $G$ without increasing its $j$-linkage. What's more, observe that for the same reason *all* vertices of each $S_i$ are connected with the same set of exactly $k$ vertices in $V_{i+1}$. Moreover, again for the same reason, the graphs induced by $S_i$ $(i = 1, \dots, q)$ and $V_{q+1}$ are cliques. Now, for notational convenience, let $|V_{q+1}| = k + \rho$, where $1 \le \rho \le j$. From the above observations, it follows that the number of edges of $G$ is

$$\sum_{i=0}^{q} \left( |S_i| k + \binom{|S_i|}{2} \right) + \binom{k+\rho}{2}.$$

Now observe that $\binom{k+\rho}{2} = \binom{k}{2} + \binom{\rho}{2} + \rho k$. Again, for notational convenience, let $A_i = |S_i|$ (for $i = 0, \dots, q$) and $A_{q+1} = \rho$. We can easily see that $A_i \le j$ (for $1 \le i \le q+1$) and that $\sum_{i=0}^{q+1} A_i = n - k$.

So, finally,

$$\begin{aligned}
&\sum_{i=0}^{q} \left( |S_i| k + \binom{|S_i|}{2} \right) + \binom{k+\rho}{2} \\
&= \sum_{i=0}^{q} \left( A_i k + \binom{A_i}{2} \right) + A_{q+1} k + \binom{k}{2} + \binom{A_{q+1}}{2} \\
&= \sum_{i=0}^{q+1} A_i k + \frac{1}{2} \sum_{i=0}^{q+1} (A_i^2 - A_i) + \binom{k}{2} \\
&= (n-k)k - \frac{1}{2}(n-k) + \frac{1}{2} \sum_{i=0}^{q+1} A_i^2 + \binom{k}{2}.
\end{aligned}$$

By applying the Technical Lemma for $Z = n - k$ and $D = j$, we obtain that the quantity $\sum_{i=1}^{q+1} A_i^2$ has maximum value $\lfloor (n-k)/j \rfloor j^2 + ((n-k) \bmod j)^2$. Therefore, we conclude that a graph with properties 1 and 2 has at most

$$\binom{k}{2} + (n-k)k - \frac{1}{2}(n-k) + \frac{1}{2}((\lfloor (n-k)/j \rfloor) j^2 + ((n-k) \bmod j)^2)$$

edges. This completes the proof of the first statement of the lemma.

To prove the second statement of the lemma, given $n, j$, and $k$ $(n \ge 1, j = 1, \dots, n,$ $0 \le k \le n-j)$, we construct a graph $G$ with max-$E(n, j, k)$ edges, $n$ vertices, $j$-linkage

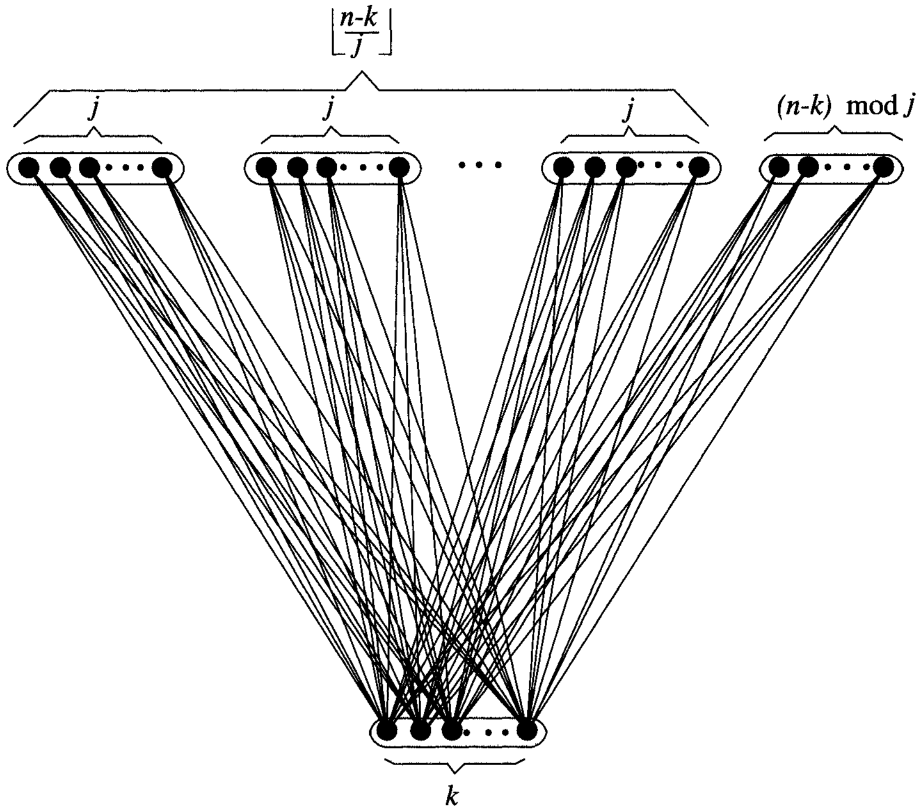



FIG. 1. *The construction of the second part of Lemma* 3.1.

equal to $k$, and such that with one more edge the $j$-linkage of $G$ becomes $k+1$. Towards this, consider one clique with $k$ vertices, $\lfloor (n-k)/j \rfloor$ cliques with $j$ vertices each, and one clique with $(n-k) \bmod j$ vertices. Connect all the vertices of the $k$-clique with all the vertices of all other cliques. Let $G$ be the graph thus constructed (see Fig. 1). Observe that $G$ has $k + \lfloor (n-j)/j \rfloor j + ((n-k) \bmod j) = n$ vertices and max-$E(n,j,k)$ edges. Also, observe that there is no nonempty set $S$ with at most $j$ vertices and with ext-degree$_G(S) < k$. Moreover, a $k$-elimination discards all vertices of $G$. Therefore, $j$-linkage$(G) = k$. Also, it easy to see that if we add one more edge to $G$, then a $k$-elimination returns a nonempty set. Therefore, we also proved the second statement of the lemma. □

We now state the following two easy lemmas.

LEMMA 3.2. *A graph $G$ with $j$-linkage$(G) \leq k$ has at most* max-$E(n,j,k)$ *edges.*

LEMMA 3.3. *The function* max-$E(n,j,k)$ *is strictly increasing with respect to $k$.*

*Proofs.* The proofs of both the above lemmas are straightforward. Indeed, to prove the first, proceed recursively as follows: as long as a new edge can be added to $G$ without increasing its $j$-linkage strictly above $k$, add such an edge. The graph thus obtained has at most max-$E(n,j,k)$ edges. For the second lemma, consider a graph with max-$E(n,j,k)$ edges. As long as a new edge can be added to this graph without increasing its $j$-linkage strictly above $k+1$, add such an edge; notice that the resulting graph has strictly more edges than the original. □

Now, given natural numbers $n \geq 1$, $j = 1, \ldots, n$, and $e$ such that max-$E(n,j,0) <$

$e \leq \binom{n}{2}$ let $L(e,n,j)$ (or just $L$ when no confusion may arise) be the unique integer that satisfies the following relation:

$$\text{max-}E(n,j,L(e,n,j)-1) < e \leq \text{max-}E(n,j,L(e,n,j)). \tag{7}$$

The uniqueness of the integer $L(e,n,j)$ follows from Lemma 3.3. Moreover, for completeness, we set $L(e,n,j) = 0$, for any $n \geq 1, j = 1,\dots,n$, and $e$ such that $0 \leq e \leq \text{max-}E(n,j,0)$.

THEOREM 3.4. *$L(e,n,j)$ is a lower bound for the $j$-linkage of graphs with $e$ edges and $n$ vertices. Moreover, for arbitrary $n$ and $e$ such that $n \geq 1$ and $0 \leq e \leq \binom{n}{2}$, there is a graph $G$ with $e$ edges and $n$ vertices such that $j$-linkage$(G) = L(e,n,j)$ (i.e., $L(e,n,j)$ is a* tight *lower bound for the $j$-linkage).*

*Proof.* We give the proof only if $\text{max-}E(n,j,0) < e \leq \binom{n}{2}$. The case where $e \leq \text{max-}E(n,j,0)$ is similar and easier. The fact that $L(e,n,j)$ is a lower bound follows easily from Lemma 3.2 and relation (7). In order to prove that $L(e,n,j)$ is also a tight bound, we construct, for given $e,n$, and $j$, a graph $G$ with $e$ edges, $n$ vertices, and $j$-linkage equal to $L(e,n,j)$.

First, by applying the construction in the proof of the second statement of Lemma 3.1 for $k = L-1$, we obtain a graph—call it $H$—such that $j$-linkage$(H) = L-1$ and such that any graph obtained from $H$ by adding one more edge (on the same set of vertices) has $j$-linkage equal to $L$. By construction, the graph $H$ has $\text{max-}E(n,j,L-1)$ edges. Let $d = e - \text{max-}E(n,j,L-1)$. From relations (1) and (7), we have that $1 \leq d \leq n - L - ((n-L) \bmod j)$. We will construct $G$ by adding to $H$ $d$ new edges. We distinguish the following two cases:

- *Case* 1. $(n-(L-1)) \bmod j = 0$.
- *Case* 2. $(n-(L-1)) \bmod j \neq 0$.

In the first case, among the $j$-cliques that were used to construct $H$, we take an arbitrary one and connect one arbitrary vertex of it with $d$ vertices of the remaining $j$-cliques (these $d$ vertices are again arbitrarily chosen). Such a selection of $d$ vertices is possible because (i) $d \leq n - L - ((n-L) \bmod j)$, (ii) $(n-L) \bmod j = j-1$ (because $(n-(L-1)) \bmod j = 0$), and (iii) the remaining $j$-cliques have exactly $n-(L-1)-j = n-L-(j-1)$ vertices. It is now easy to see that the graph $G$ has $j$-linkage equal to $L$.

In the second case (see Fig. 2), we construct $G$ by connecting an arbitrary vertex of the $((n-(L-1)) \bmod j)$-clique of $H$ with $d$ vertices chosen from the remaining $j$-cliques of $H$. The number of vertices of the $j$-cliques of $H$ are $n-(L-1)-((n-(L-1)) \bmod j)$. The selection of $d$ vertices is possible because (i) as we noticed before, $d \leq n-L-((n-L) \bmod j)$ and (ii) obviously, $n-L-((n-L) \bmod j) \leq n-(L-1)-((n-(L-1)) \bmod j)$. Finally, it is again easy to see that the graph $G$ has $j$-linkage equal to $L$. □

Since $\text{max-}E(n,j,L)$ is given by a closed formula, the number $L(e,n,j)$ can be efficiently computed. Moreover, as the following theorem shows, we can find an almost closed formula for computing $L(e,n,j)$. Indeed for $j=1$, the two bounds provided by the theorem below lead to an exact value for $L$, whereas, if $j$ is arbitrary, the difference between the two bounds is $O(j)$.

THEOREM 3.5. *For any $e$, $n$, and $j$ such that $n \geq 1$, $1 \leq j \leq n$, and $0 \leq e \leq \binom{n}{2}$, we have that*

$$L(e,n,j) \geq \frac{1}{2}\left(2n - j - \sqrt{(2n-j)^2 + 4n(j-1) - 8e}\right) \quad \textit{and}$$
$$L(e,n,j) < \frac{1}{2}\left(2n - j + 2 - \sqrt{(2n-j)^2 + 4n(j-1) - 8e + j^2 - 1}\right).$$

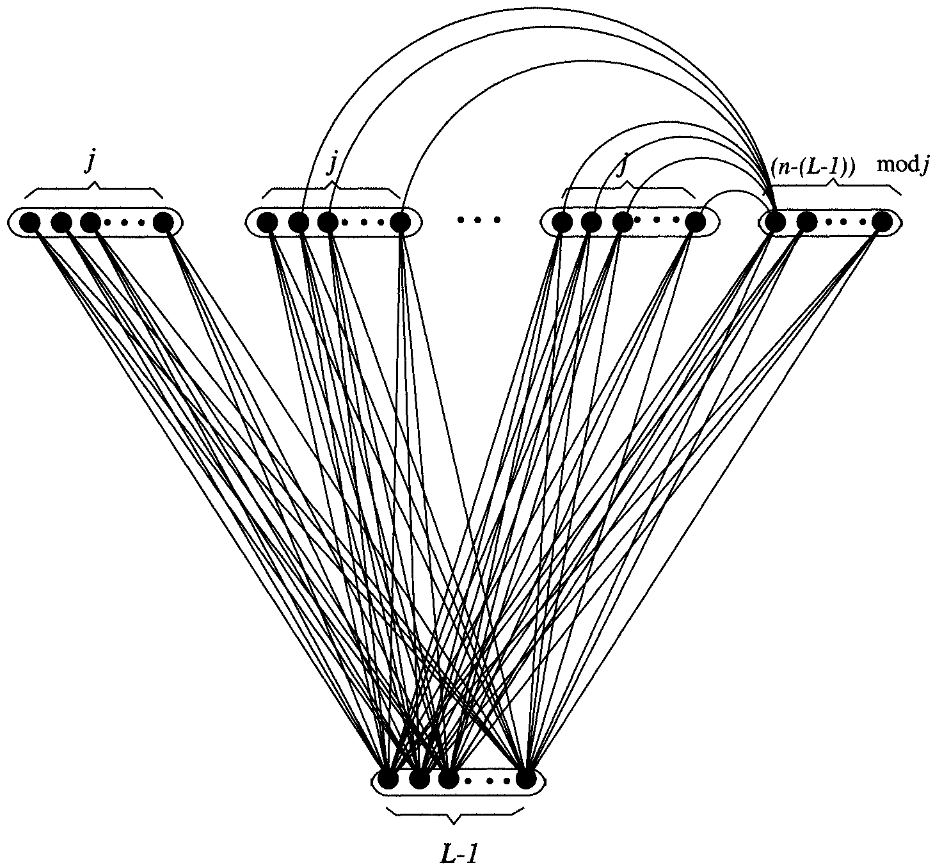



FIG. 2. *The construction of the second part of Theorem* 3.4.

*Proof.* For notational convenience, define

$$\begin{aligned}\sigma_L &= (n-L) \bmod j \text{ and}\\ Q_L &= \lfloor (n-L)/j \rfloor j^2 + ((n-L) \bmod j)^2.\end{aligned}$$

Observe that

$$\text{max-}E(n,j,L) = \binom{L}{2} + (n-L)L - \frac{1}{2}(n-L) + \frac{1}{2}Q_L. \tag{8}$$

Also observe that $0 \le \sigma_L \le j-1$. We now obtain

$$\begin{aligned}Q_L &= \lfloor (n-L)/j \rfloor j^2 + \sigma_L^2\\ &\le \lfloor (n-L)/j \rfloor j^2 + \sigma_L j\\ &= (\lfloor (n-L)/j \rfloor j + \sigma_L)j\\ &= (n-L)j. \qquad (9)\end{aligned}$$

Moreover,

$$Q_L = \lfloor (n-L)/j \rfloor j^2 + \sigma_L^2$$

$$
\begin{aligned}
&= (n-L)j - \sigma_L j + \sigma_L^2 \\
&= (n-L)j + \sigma_L^2 - \sigma_L(j-1) - \sigma_L \\
&= (n-L)j + \sigma_L^2 - \sigma_L(j-1) + \frac{1}{4}(j-1)^2 - \frac{1}{4}(j-1)^2 - \sigma_L \\
&= (n-L)j + (\sigma_L - \frac{1}{2}(j-1))^2 - \frac{1}{4}(j-1)^2 - \sigma_L \\
&\geq (n-L)j - \frac{1}{4}(j-1)^2 - \sigma_L \\
&\geq (n-L)j - \frac{1}{4}(j-1)^2 - (j-1). \qquad (10)
\end{aligned}
$$

So by relations (7)–(9), the value $L$ satisfies

$$e \leq \binom{L}{2} + L(n-L) - \frac{1}{2}(n-L) + \frac{1}{2}(n-L)j. \qquad (11)$$

Also, by relations (7), (8), and (10), the value $L$ satisfies

$$
\begin{aligned}
e > \binom{L-1}{2} + (L-1)(n-L+1) - \frac{1}{2}(n-L+1) + \\
\frac{1}{2}(n-L+1)j - \frac{1}{8}(j-1)^2 - \frac{1}{2}(j-1). \qquad (12)
\end{aligned}
$$

From relation (11), by finding the roots of the corresponding second-order (in $L$) equation, we have that $L$ must satisfy

$$\frac{1}{2}\left(2n - j - \sqrt{(2n-j)^2 + 4n(j-1) - 8e}\right) \leq L \text{ and} \qquad (13)$$

$$\frac{1}{2}\left(2n - j + \sqrt{(2n-j)^2 + 4n(j-1) - 8e}\right) \geq L. \qquad (14)$$

Similarly, from relation (12), we have that either

$$L < \frac{1}{2}\left(2n - j + 2 - \sqrt{(2n-j)^2 + 4n(j-1) - 8e + j^2 - 1}\right) \text{ or} \qquad (15)$$

$$L > \frac{1}{2}\left(2n - j + 2 + \sqrt{(2n-j)^2 + 4n(j-1) - 8e + j^2 - 1}\right). \qquad (16)$$

Finally, from inequalities (13)–(16), we obtain that

$$
\begin{aligned}
L &\geq \frac{1}{2}\left(2n - j - \sqrt{(2n-j)^2 + 4n(j-1) - 8e}\right) \text{ and} \\
L &< \frac{1}{2}\left(2n - j + 2 - \sqrt{(2n-j)^2 + 4n(j-1) - 8e + j^2 - 1}\right).
\end{aligned}
$$

This concludes the proof of the theorem. □

The following corollary is now immediate.

COROLLARY 3.6. *For a graph $G$ with $n$ vertices and $e$ edges,*

$$j\text{-}linkage(G) \geq \frac{1}{2}\left(2n - j - \sqrt{(2n-j)^2 + 4n(j-1) - 8e}\right).$$

For the case $j = 1$, not only is the lower bound for the linkage of a graph given in this paper greater than or equal to the lower bound given by Erdös in [6] (this is an immediate conclusion of the fact that we give a tight lower bound) but, moreover, as the following proposition shows, when the number of edges exceeds a low threshold, the inequality is proper.

PROPOSITION 3.7. *Let* $L^{ER}(n,e) = \lceil e/n \rceil$. *Then* $L(e,n,1)$ *is strictly greater than* $L^{ER}(e,n)$ *for any* $e \geq \text{max-}E(n,1,i-1)+1$, *where* $i = \lceil(-1+\sqrt{1+8n})/2\rceil$.

*Proof.* Let $i = \lceil(-1+\sqrt{1+8n})/2\rceil$. Also, let $e$ be such that $e \geq \text{max-}E(n,1,i-1)+1$. First, observe that

$$\forall k \geq i, \quad \text{max-}E(n,1,k) \leq (k-1)n. \tag{17}$$

Now recall (see relation (7)) that $L(e,n,1)$ ($L$ for short) is the unique integer for which

$$\text{max-}E(n,1,L-1) < e \leq \text{max-}E(n,1,L). \tag{18}$$

Also, $\text{max-}E(n,1,k)$ is a strictly increasing function of $k$ and therefore its values delimit on the real line a succession of consecutive intervals. By relation (18), $\text{max-}E(n,1,L-1)$ and $\text{max-}E(n,1,L)$ are the two endpoints of the one among these intervals that contains $e$.

On the other hand, it is trivial to see that $L^{ER}(e,n)$ ($L^{ER}$ for short) is the unique integer for which

$$(L^{ER}-1)n < e \leq L^{ER}n. \tag{19}$$

Also, $kn$ is a strictly increasing function of $k$ and therefore its values delimit on the real line a succesion of consecutive intervals. By relation (19), $(L^{ER}-1)n$ and $L^{ER}n$ are the two endpoints of the one among these intervals that contains $e$.

By relation (17), for all $k \geq i$, the intervals delimited by the values of the function $\text{max-}E(n,1,k)$ lie completely to the left of the intervals delimited by the corresponding values of the function $kn$. Finally, for any $e > \text{max-}E(n,1,i-1), L(e,n,1) \geq i$. Therefore, we conclude that $\forall e > \text{max-}E(n,1,i-1)$, $L^{ER}(n,e) < L(e,n,1)$ because otherwise, $e$ would have to belong to two disjoint intervals. $\square$

**3.2. A lower bound for treewidth, pathwidth, and bandwidth.** In this subsection, we show that $L(e,n,1)$, which is equal to $\frac{1}{2}(2n-1-\sqrt{(2n-1)^2-8e})$, is a *tight* lower bound for treewidth, pathwidth, and bandwidth. (Recall that $L(e,n,1)$ is a tight lower bound for width.) A proof that $L(e,n,1)$ is a lower bound for bandwidth can be found in [15]. In [20], X. Yan independently proved that $L(e,n,1)$ is a lower bound for treewidth and pathwidth (see also [16]). However, in none of these papers is it shown that $L(e,n,1)$ is a *tight* lower bound.

First, for completeness, we give some formal definitions (see also [4, 10, 17]).

DEFINITION 1. *A* tree-decomposition *of* $G = (V,E)$ *is defined to be a pair* $(\{X_i : i \in I\}, T)$, *where* $\{X_i : i \in I\}$ *is a collection of subsets of* $V$ *and* $T = (I,F)$ *is a tree having the index set* $I$ *as set of vertices, such that the following conditions are satisfied:*

1. $\bigcup_{i \in I} X_i = V$.
2. $\forall \{u,w\} \in E, \ \exists i \in I : u,w \in X_i$.
3. $\forall i,j,k \in I$ : *if* $j$ *is on a path in* $T$ *from* $i$ *to* $k$, *then* $X_i \cap X_k \subseteq X_j$.

*The* treewidth of a tree-decomposition $(\{X_i : i \in I\}, T)$ *is defined to be equal to* $\max_{i \in I} |X_i| - 1$. *The* treewidth of $G$ *is defined to be the minimum treewidth of any tree-decomposition of* $G$.

DEFINITION 2. *A* path-decomposition *of* $G = (V, E)$ *is defined to be a class* $\{X_i : i = 1, \ldots, r\}$ *of subsets of* $V$ *such that the following conditions are satisfied:*

1. $\bigcup_{i=1}^{r} X_i = V$.
2. $\forall \{u, w\} \in E,\ \exists i : u, w \in X_i$.
3. $\forall i, j, k$, *if* $1 \leq i \leq j \leq k \leq r$, *then* $X_i \cap X_k \subseteq X_j$.

*The* pathwidth of a path-decomposition $\{X_i : i = 1, \ldots, r\}$ *is defined to be equal to* $\max_{1 \leq i \leq r} |X_i| - 1$. *The* pathwidth of $G$ *is defined to be the minimum pathwidth of any path-decomposition of* $G$.

DEFINITION 3. *The* bandwidth of a layout $l = (v_1, \ldots, v_n)$ *of* $G = (V, E)$ *is defined to be equal to* $\max\{|i - j| : \{v_i, v_j\} \in E\}$. *The* bandwidth of $G$, *is defined to be the minimum bandwidth of any layout of* $G$.

Treewidth has many equivalent characterizations (see, e.g., [2, 4]). We are going to use the one expressed in terms of elimination orderings of graphs [2]. An *elimination ordering* of a graph $G = (V, E)$ is an ordering $\pi = (v_1, \ldots, v_n)$ of the vertices of $G$. The graphs generated during an elimination of the vertices of $G$ according to $\pi$ are defined as follows: $G_1 = G$; $G_{i+1}$ is equal to the graph obtained from $G_i$ by deleting the vertex $v_i$ and adding new edges (if necessary) so that all pairs of neighbors of $v_i$ in $G_i$ are adjacent in $G_{i+1}$. Obviously, $G_{n+1}$ is equal to the empty graph. The *dimension of* $v_i$ with respect to $\pi$ is defined to be the degree of $v_i$ in $G_i$. The *dimension of* $\pi$ is the maximum dimension of any of the $v_i$'s. Finally, the *elimination dimension* of $G$ is the minimum dimension of any elimination ordering of $G$. The following result can be found in [2].

THEOREM 3.8 (Arnborg [2]). *The treewidth of a graph is equal to its elimination dimension.*

We now show the following.

THEOREM 3.9. *The quantity* $L(e, n, 1)$ *is a lower bound for the treewidth, pathwidth, and bandwidth of a graph with* $e$ *vertices and* $n$ *edges. Moreover, for arbitrary* $n$ *and* $e$, *with* $n \geq 1$ *and* $0 \leq e \leq \binom{n}{2}$, *there is a graph* $G$ *with* $e$ *edges and* $n$ *vertices such that its pathwidth, treewidth, and bandwidth are all equal to* $L(e, n, 1)$. *(i.e.,* $L(e, n, 1)$ *is a tight lower bound for the aforementioned parameters).*

*Proof.* From the characterization of treewidth in terms of elimination orderings, it is easy to see that linkage$(G) \leq$ treewidth$(G)$. Also, obviously, treewidth$(G) \leq$ pathwidth $(G)$. Finally, treewidth$(G) \leq$ bandwidth $(G)$ (see [3]). Therefore, $L(e, n, 1)$ is a lower bound of all three parameters, as the theorem requires.

In order to prove that $L(e, n, 1)$ is also a tight lower bound, it suffices to construct, for given $e$ and $n$, a graph $G$ with $e$ edges and $n$ vertices such that pathwidth$(G) =$ treewidth$(G) =$ bandwidth$(G) = L(e, n, 1)$. First, we arrange $n$ vertices on a layout $l = \{v_1, \ldots, v_n\}$ and connect two vertices $v_i, v_j$ $(1 \leq i, j \leq n)$ with an edge iff $|i - j| \leq L - 1$ (this graph is described in [15]; see also [3, 17]). Call this graph $H$ and note that it has $\binom{L-1}{2} + (n - (L-1))(L-1)$ edges. The graph $G$ is constructed from $H$ by adding to the latter the edges $\{v_1, v_{L+1}\}, \ldots, \{v_d, v_{L+d}\}$, where $d = e - (\binom{L-1}{2} + (n - (L-1))(L-1))$ (see Fig. 3). It is easy to see that the graph $G$ has linkage, treewidth, pathwidth, and bandwidth all equal to $L(e, n, 1)$. This completes the proof of the theorem. $\square$

Notice that for the cases of treewidth and pathwidth, the extremal graph construction of Theorem 3.4 for the case $j = 1$ is also sufficient to prove the tightness of

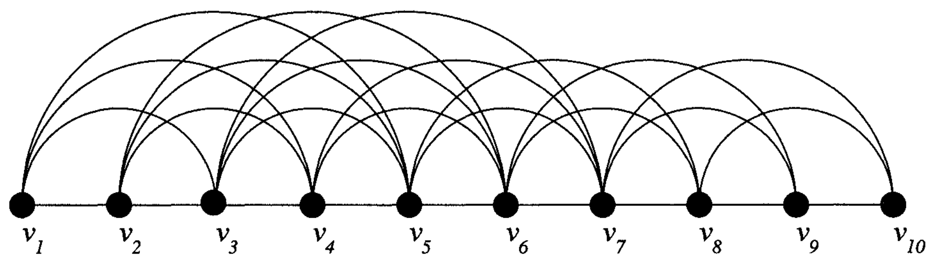



FIG. 3. *The construction of Theorem* 3.9 *for* $n = 10$, $e = 27$, *and* $L = 4$.

the lower bound.

**3.3. Upper bound.** In this subsection, we derive a tight upper bound for the $j$-linkage of graphs with given numbers of vertices and edges. In analogy to the previous subsection, we consider the graphs whose $j$-linkage decreases by the deletion of even one edge. We then consider the minimum number of edges that such a graph may have, given its $j$-linkage and its number of edges. To proceed, define the following function:

$$\text{min-}E(n,j,k) = \begin{cases} \lceil (k+j)k/2 \rceil & \text{if } 1 < k \leq n-j, \\ j & \text{if } k = 1, \\ 0 & \text{if } k = 0. \end{cases}$$

LEMMA 3.10. *let $G$ be a graph with $j$-linkage at least $k$. Suppose that $G$ has the property that any graph obtained from $G$ by deleting one of its edges has $j$-linkage equal to $k-1$. Then the number of edges of $G$ is at least* min-$E(n,j,k)$. *Moreover, for any $n \geq 1$, for any $j = 1, \ldots, n$ and any $k$ such that $0 \leq k \leq n-j$, there exist graphs with $n$ vertices,* min-$E(n,j,k)$ *edges, and $j$-linkage equal to $k$ and such that if we subtract one edge, the $j$-linkage decreases to $k-1$.*

*Proof.* Assume first that $k > 1$. By hypothesis, we have the following:

1. $j$-linkage$(G) = k$;
2. $j$-linkage$(G') = k-1$, where $G'$ is any graph obtained from $G$ by deleting one edge.

Property 1 above implies that the $(k-1)$-elimination cannot remove all vertices of $G$, and property 2 implies that the $(k-1)$-elimination removes all vertices of any graph $G'$ obtained from $G$ by removing an arbitrary edge.

Observe that the $(k-1)$-elimination applied to $G$ not only returns a nonempty graph but, moreover, does not remove any nonisolated vertex of $G$ (i.e., a vertex with degree in $G \geq 1$). Indeed, otherwise, the $(k-1)$-elimination would also remove at least one edge—say $\{v,u\}$—of $G$ and therefore this $(k-1)$-elimination applied to $G - \{u,v\}$ would return a nonempty subgraph—a contradiction.

Let the number of nonisolated vertices in $G$ be $n'$. Since $k \geq 1$, $n' > 0$. Observe that a nonisolated vertex must have at least $k$ incident edges (otherwise, it would be removed by the $(k-1)$-elimination). Therefore, the number of edges of $G$ is $\geq n'k/2$.

Now, observe that $n' \geq k+j$ because if $0 < n' < k+j$, then nonisolated vertices of $G$ would be removed by the $(k-1)$-elimination.

Therefore, from the last two facts, we have that

$$\text{min-}E(n,j,k) = \lceil (k+j)k/2 \rceil$$

is a lower bound for $e$.

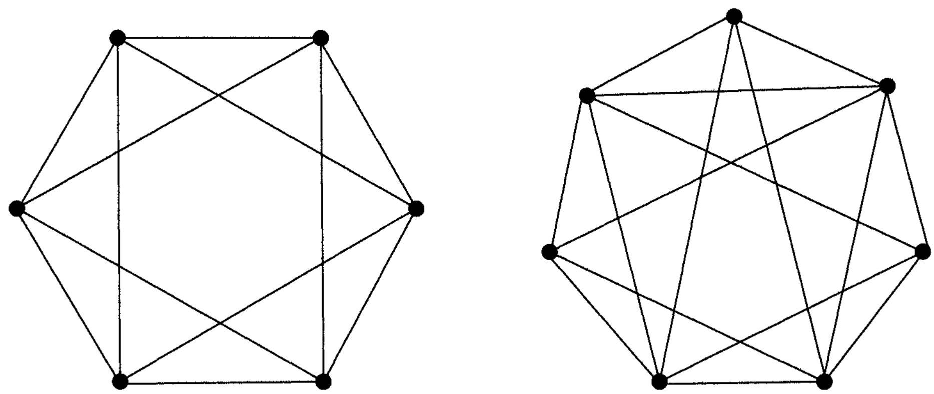


FIG. 4. *The construction of Lemma* 3.10 *for* $j = 2$, $k = 4$, *and* $k = 5$.

Finally, it is easy to see that for $k = 1$, a graph with the given properties should have at least $j = \text{min-}E(n, j, 1)$ edges; the case $k = 0$ is trivial since $\text{min-}E(n, j, 0) = 0$. This completes the prooof of the first statement of the lemma.

To show the second statement, given $n \geq 1$, $j = 1, \ldots n$, and any $2 \leq k \leq n - j$, we construct a graph with $n$ vertices, $\text{min-}E(n, j, k)$ edges, and $j$-linkage equal to $k$ and such that the $j$-linkage becomes $k - 1$ if one edge is deleted. For this construction, consider $k + j$ vertices with originally no edges connecting them. It is convenient to think of these vertices as cyclically lying on the plane. Connect with an edge any pair of these vertices whose cyclic distance is at most $\lfloor k/2 \rfloor$. Also, only in the case where $k$ is odd, arbitrarily choose $\lceil (k + j)/2 \rceil$ successive vertices on the circle and connect each of them with the vertex at clockwise distance $\lfloor (k + j)/2 \rfloor$ (see Fig. 4). Finally, add to this graph $n - k - j$ isolated vertices. It is easy to see that this graph satisfies the requirements. For the case $k = 1$, consider an arbitrary tree with $j + 1$ vertices and add $n - j - 1$ isolated vertices. This graph again satisfies the requirements. For the case where $k = 0$, consider a graph with no edges. □

The following two lemmas can be easily proved analogously to Lemmas 3.2 and 3.3, respectively. We omit the proofs.

LEMMA 3.11. *A graph $G$ with $j$-linkage$(G) \geq k$ has at least* $\text{min-}E(n, j, k)$ *edges.*

LEMMA 3.12. *The function* $\text{min-}E(n, j, k)$ *is strictly increasing with respect to $k$.*

Given natural numbers $n \geq 1$, $j = 1, \ldots, n$, and $e$ such that $0 \leq e < \text{min-}E(n, j, n - j)$, let $U(e, n, j)$ (or just $U$ when no confusion may arise) be the unique integer that satisfies the following relation:

$$\text{min-}E(n, j, U(e, n, j)) \leq e < \text{min-}E(n, j, U(e, n, j) + 1). \tag{20}$$

Notice that the uniqueness of $U$ follows from Lemma 3.12. Moreover, for completeness, we set $U(e, n, j) = n - j$ for any $n \geq 1$, $j = 1, \ldots, n$, and $e$ such that $\text{min-}E(n, j, n - j) \leq e \leq \binom{n}{2}$.

We now prove the following theorem.

THEOREM 3.13. *The function $U(e, n, j)$ is an upper bound for the $j$-linkage of graphs with $n$ vertices and $e$ edges. Moreover, for arbitrary $n$ and $e$ such that $n \geq 1$ and $0 \leq e \leq \binom{n}{2}$, there is a graph $G$ with $e$ edges and $n$ vertices such that $j$-linkage$(G) = U(e, n, j)$ (i.e, $U(e, n, j)$ is a* tight *upper bound for the $j$-linkage).*

*Proof.* We give the proof only for $0 \leq e < \text{min-}E(n, j, n - j)$. The case where $\text{min-}E(n, j, n - j) \leq e \leq \binom{n}{2}$ is similar and easier. The fact that $U(e, n, j)$ is an upper bound follows easily from Lemma 3.11 and relation (20). In order to prove that $U(e,n,j)$ is a tight upper bound, if suffices to construct for given $e, n$, and $j$ a graph with $e$ edges, $n$ vertices, and $j$-linkage equal to $U$. We distinguish the following three cases:

1. $\text{min-}E(n,j,0) = 0 \leq e < j = \text{min-}E(n,j,1)$,
2. $\text{min-}E(n,j,1) = j \leq e < j+2 = \text{min-}E(n,j,2)$,
3. $\text{min-}E(n,j,2) \leq e < \text{min-}E(n,j,n-j)$.

For the first case, we have only to notice that for any graph $G$ with $n$ vertices and $e$ edges, if $e < j$, then $j$-linkage$(G) = 0$; thus the construction for this case is obvious..

For the second case, we can see that any tree with $j+1$ vertices and $j$ edges has $j$-linkage equal to 1. Also, any circle with $j+1$ vertices and $j+1$ edges has $j$-linkage equal to 1.

For the third case, by applying the second part of Lemma 3.10 for $k = U$, we obtain a graph—call it $H$—with $\text{min-}E(n,j,U)$ edges such that $j$-linkage$(H) = U$. We will describe a way to add $d = e - \text{min-}E(n,j,U)$ new edges to $H$ so that the the resulting graph has the required properties. Observe that by relation (20), $0 \leq d < \text{min-}E(n,j,U+1) - \text{min-}E(n,j,U)$. We first claim that also $d < U + j$. Indeed, we have the following:

1. If $j$ is odd, then $\text{min-}E(n,j,U+0,n,j) - \text{min-}E(n,j,U) = U + \frac{j+1}{2}$.
2. If $j$ is even, then
   - if $U$ is odd, then $\text{min-}E(n,j,U+1) - \text{min-}E(n,j,U) = U + \frac{j}{2}$;
   - if $U$ is even, then $\text{min-}E(n,j,U+1) - \text{min-}E(n,j,U) = U + 1 + \frac{j}{2}$.

From the analysis above, we observe that in any case,

$$\text{min-}E(n,j,U+1) - \text{min-}E(n,j,U) \leq U + j.$$

Now, from its construction, we see that $H$ has at least one isolated vertex and $U+j$ nonisolated ones. We construct the graph $G$ by connecting the isolated vertex of $H$ with $d$ nonisolated vertices of it. This is possible since, as we showed above, $d < U+j$. It is now easy to see that the graph $G$ has $j$-linkage equal to $U$. $\square$

Since $\text{min-}E(n,j,U)$ is given by a closed formula, $U(e,n,j)$ can be efficiently computed. Moreover, as the following theorem shows, we can find an almost closed formula for computing $U(e,n,j)$.

THEOREM 3.14. *For any $n \geq 1$, $j = 1, \ldots, n$, $j+2 \leq e \leq \binom{n}{2}$,*

$$U(e,n,j) > \frac{1}{2}\left(-j + \sqrt{j^2 - 4 + 8e}\right) - 1 \quad \textit{and}$$
$$U(e,n,j) \leq \frac{1}{2}\left(-j + \sqrt{j^2 + 8e}\right).$$

*Proof.* For notational convenience, we define

$$\delta_j = (U+j)U \bmod 2.$$

We first see that for $e \geq j+2$,

$$\text{(21)} \qquad \text{min-}E(n,j,U) = \lceil (U+j)U/2 \rceil = \begin{cases} \frac{1}{2}(U+j)U & \text{if } \delta_j = 0, \\ \frac{1}{2}((U+j)U+1) & \text{if } \delta_j = 1. \end{cases}$$

We distinguish the following two cases:

1. $j$ is odd. Then $\delta_j = 0$.
2. $j$ is even. Then
   - $\delta_j = 1$, if $U$ is odd;
   - $\delta_j = 0$, if $U$ is even.

For the first case, we have from relations (20) and (21) that

$$\frac{1}{2}(U+j)U \leq e < \frac{1}{2}(U+j+1)(U+1). \tag{22}$$

Now, from the inequalities in (22), by computing the roots of the second-order (in $U$) equations, we obtain that

$$U(e,n,j) = \left\lfloor \frac{1}{2}\left(-j+\sqrt{j^2+8e}\right)\right\rfloor. \tag{23}$$

For the second case, again by relations (20) and (21), we have that

$$\frac{1}{2}(U+j)U \leq e < \frac{1}{2}((U+j+1)(U+1)+1). \tag{24}$$

Now, as in Theorem 3.14, computing the roots corresponding the second-order equations (in U), we obtain that inequality (24) is satisfied for

$$U(e,n,j) > \frac{1}{2}\left(-j+\sqrt{j^2-4+8e}\right)-1 \quad \text{and} \tag{25}$$

$$U(e,n,j) \leq \frac{1}{2}\left(-j+\sqrt{j^2+8e}\right). \tag{26}$$

From relations (23), (25), and (26), we conclude that in all cases the requirement holds. □

The following corollary is now immediate.

COROLLARY 3.15. *For a graph $G$ with $n$ vertices and $e$ edges,*

$$j\text{-}linkage(G) \leq \frac{1}{2}\left(-j+\sqrt{j^2+8e}\right).$$

By easier calculations, it can be proved that $\min(2e/j, (2e)^{1/2})$ is also an upper bound for the $j$-linkage. However, when $j = \Theta((2e)^{1/2})$, the difference of this bound from the optimal one that we provide is $\Theta(e^{1/2})$.

CONCLUSION. *From the above results, we conclude that for a graph $G$ with $n$ vertices and $e \geq j+2$ edges, an estimation for the $j$-linkage$(G)$(equivalently, the $j$-width$(G)$) is given by the following two inequalities:*

$$\left\lceil \frac{1}{2}\left(2n-j-\sqrt{(2n-j)^2+4n(j-1)-8e}\right)\right\rceil \leq j\text{-}linkage(G),$$

$$\left\lfloor \frac{1}{2}\left(-j+\sqrt{j^2+8e}\right)\right\rfloor \geq j\text{-}linkage(G).$$

*The estimation from below is tight within an additive term $O(j)$, whereas the estimation from above is tight within an additive constant. (If $e \leq j+1$, $j$-linkage$(G)$ is equal to 0 or 1.)*

**4. Parallel complexity and approximations.** The problem of computing of the $j$-linkage of a graph $G$ can be formulated as a decision problem: Given a graph $G$ and an integer $k$, determine whether $G^{(k,j)}$ (the graph that the $(k,j)$-Elimination Algorithm outputs on input $G$) is not empty. A stronger decision problem is: Given a graph $G$, an integer $k \geq 0$ and a vertex $u$ of $G$, determine whether $u \in G^{(k,j)}$. For $j = 1$, these problems are examined by Anderson and Mayr in [1]. However not all their proofs generalize to arbitrary $j$. A related problem, which exhibits a threshold-type complexity, can be found in [9] (see also [18]).

**4.1. An NC algorithm and P-completeness.**

THEOREM 4.1. *The problem of deciding whether $u \in G^{(1,j)}$ is in* NC *(with $j$ as part of the input).*

*Proof.* We give an algorithm in NC that on input $G$ returns $G^{(1,j)}$:

Find all biconnected components of $G$. It is easy to see that all vertices of such a component with strictly more than $j+1$ vertices are contained in $G^{(1,j)}$. So we have to face the problem of which vertices in biconnected components with $\leq j+1$ vertices remain in $G^{(1,j)}$. To solve this problem, construct a new graph $H$ that has three groups of vertices: (i) one vertex for each biconnected component of $G$ with at most $j+1$ vertices, (ii) one vertex for each articulation point of $G$ that belongs only to biconnected components of vertex cardinality at most $j+1$, and (iii) as many vertices as the vertices of $G$ that belong to biconnected components with strictly more than $j+1$ vertex cardinality. Now, connect two vertices in group (iii) iff the they are connected in $G$ as well. Also connect a vertex in group (i) with a vertex in group (iii) (respectively, group (ii)) iff the vertex of group (iii) (respectively, (ii)) corresponds to an articulation point of the biconnected component represented by the vertex of group (i).

In the graph thus constructed, recursively delete all vertices that are elements of chains (a chain is defined to be a path of vertices that starts with a vertex of degree 1 and contains no vertex of degree strictly greater than 2). The recursion is repeated until no chains appear in the current graph. Then reconstruct the part of $G$ that corresponds to the part of $H$ that has remained. Since the creation of a new vertex of degree 1 in the current $H$ requires the removal of at least two chains, the number of chains removed decreases by at least half at each phase of the recursion. Therefore, it is not hard to see that this is an algorithm in NC that outputs $G^{(1,j)}$. $\square$

THEOREM 4.2. *If $j$ and $k > 1$ are fixed, it is* P*-complete for a graph $G$ to determine whether a given vertex $u$ lies in $G^{(k,j)}$ or even to determine whether $G^{(k,j)}$ is nonempty.*

*Proof.* The proof of this result is a straightforward generalization of the corresponding result in [1], where the P-completeness for $j = 1$ is proved by a reduction from the fanout-2 monotone circuit value problem. Therefore, we omit the proof; however, for the sake of completeness, in Fig. 5 we give the modified gadgets for arbitrary $j$ (for the problem of determining whether $G^{(k,j)}$ is nonempty). $\square$

**4.2. Parallel approximations: A threshold behavior.** In this subsection, we consider the problem of approximating the $j$-linkage of a graph $G$. An algorithm with approximation factor a constant $c > 0$ is one that on input $G$ returns an integer $j\text{-linkage}_{\mathrm{ap}}(G)$ such that

$$j\text{-linkage}(G) \geq j\text{-linkage}_{\mathrm{ap}}(G) \geq c(j\text{-linkage}(G)).$$

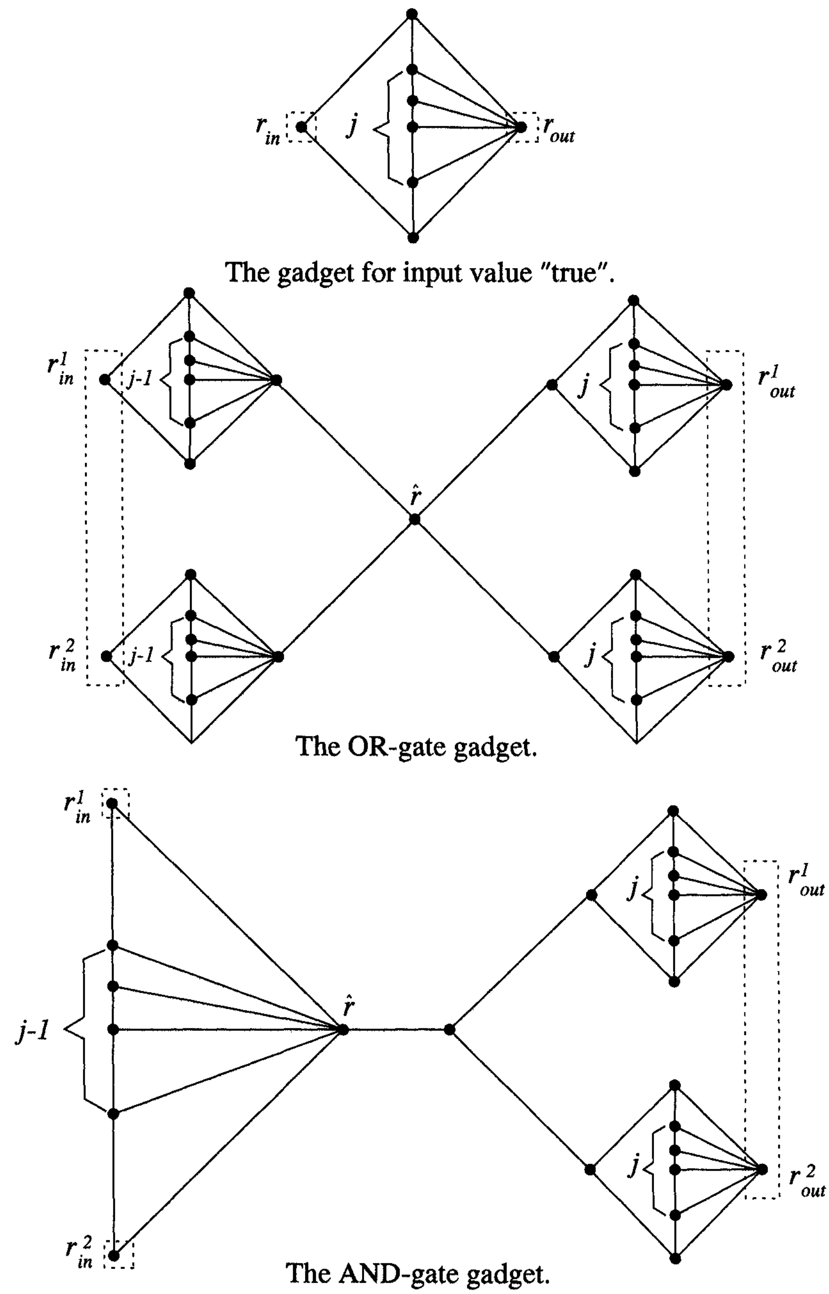
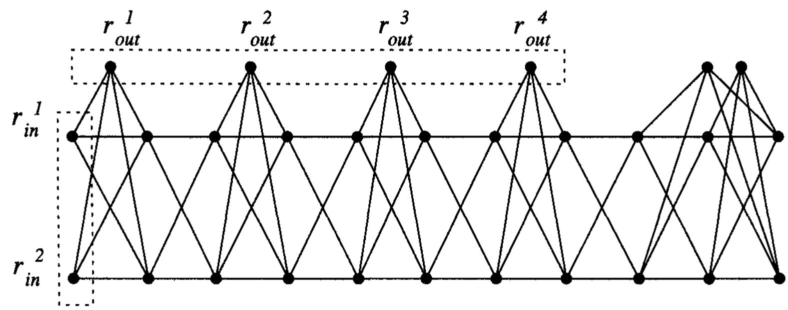


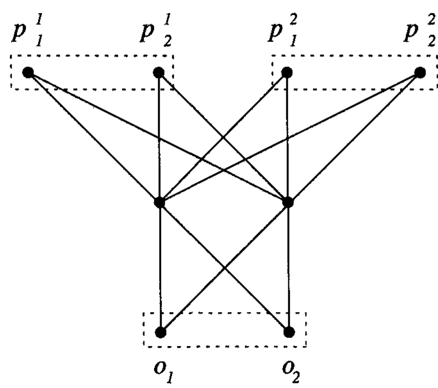

FIG. 5. *The gadgets of Theorem* 4.2.

THEOREM 4.3. *For any constant* $0 < c < 1/(2j)$, *for fixed* $j$, *the problem of approximating the* $j$*-linkage of a graph* $G$ *with an approximation factor equal to* $c$ *is in NC.*

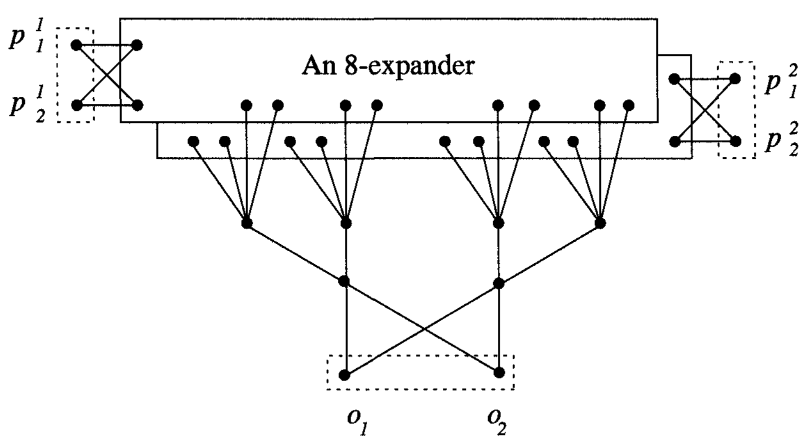
*Proof.* First, observe that $\lceil e/(nj) \rceil \leq L(e,n,j)$. Following Anderson and Mayr in [1], we define a procedure Test$(k)$ that returns either that the graph has no subgraph of $j$-min-degree at least $k$ or that the graph has $j$-linkage at least $\frac{1-\epsilon}{2j}k$. The procedure




A 4-expander.




The OR-gate gadget.




The AND-gate gadget.

FIG. 6. *The gadgets of Theorem* 4.4.

deletes in parallel all sets of vertices $S$ whose cardinality is at most $j$ and whose external degree with respect to the current graph $G'$ is strictly less than $k$ until the ratio of the vertices of $G'$ that are contained in at least one such set $S$ becomes $\leq \epsilon$ ($\epsilon$ is to be determined). This procedure is in NC. If it returns the empty graph, then $G$ has no subgraph of $j$-min-degree at least $k$. If the procedure returns a subgraph $G'$ with $n' > 0$ vertices, then the number of vertices with degree strictly less than $k$ is at most

$\epsilon n'$, so $G'$ has at least $\frac{(1-\epsilon)n'k}{2}$ edges, and so it follows that $j$-linkage$(G) \geq \lceil \frac{1-\epsilon}{2j} k \rceil$.

We then apply the procedure Test$(k)$ in parallel for $k = 0, \ldots, n$, and we thus find a value $k_0$ such that $G$ has no subgraph with $j$-min-degree $> k_0$ but it has a $j$-linkage at least $\lceil \frac{1-\epsilon}{2j} k_0 \rceil$. So the $j$-linkage of the graph will be a value between $\lceil \frac{1-\epsilon}{2j} k_0 \rceil$ and $k_0$. The algorithm finally returns $\lceil \frac{1-\epsilon}{2j} k_0 \rceil$ as an aproximation value for the $j$-linkage. Given any $c < \frac{1}{2j}$, choose a suitable $\epsilon$ so that the value returned as an approximation satisfies

$$j\text{-linkage}(G) \geq j\text{-linkage}_{\text{ap}}(G) \geq c(j\text{-linkage}(G)).$$

This completes the proof. □

THEOREM 4.4. *If* P $\neq$ NC, *then for any fixed $j$, it is not possible to approximate the $j$-linkage$(G)$ by a factor strictly greater than $\frac{1}{2}$ in* NC.

*Proof.* We follow the methodology used in the proof of the case where $j = 1$. We give a logspace transformation of an instance of the monotone circuit value problem to a graph $G$ such that $G$ has $j$-linkage equal to $k+1$ if the output of the circuit is "false", whereas it has $j$-linkage equal to $2k$ if the output of the circuit is "true".

For the construction of the gadgets that simulate the circuit, we need an expander that propagates the values of the circuit. This expander is shown in Fig. 6. The $k$ leftmost vertices $r_{in}^1, r_{in}^2, \ldots, r_{in}^k$ of this gadget are called the in-vertices of the expander (in Fig. 6, $k = 2$). The $m$ top vertices $r_{out}^1, r_{out}^2, \ldots, r_{out}^m$ are called the out-vertices of the expander (we call such an expander an $m$-expander; $m$ is chosen as needed; for the first expander in Fig. 6, $m = 4$). Observe that for large enough $m$, any set $S, |S| \leq j$, of vertices in an $m$-expander has external degree strictly more than $k$.

The gadgets for OR-gates and AND-gates of the circuit are given in Fig. 6. Observe that in the AND-gate gadget, we use two copies of a $k^3$ expander. The two clusters of in-vertices of these expanders are the two clusters of vertices corresponding to the input of the AND-gate. The vertices corresponding to the output of the AND-gate are connected as in Fig. 6 with the out-vertices of the expanders. We then identify the output clusters of each gate with the corresponding, according to the circuit structure, input clusters of other gates.

Let now $r$ be the number of clusters of vertices of $G$ that correspond to gates of the circuit that are not connected to previous gates, but originally have the value "true" as input. We identify these $r$ clusters of vertices of $G$ (containing $k$ vertices each) with the $kr$ out-vertices of a $kr$-expander. The in-vertices of this $kr$-expander are identified with the vertices corresponding to the final output gate of the circuit. This completes the construction of $G$.

Now it can be proved that if the circuit outputs the value "true", then $j$-linkage$(G) = 2k$ and if the circuit outputs the value "false", then $j$-linkage$(G) = k+1$. Indeed, if the circuit outputs the value "true", then it is easy to see that a $(2k, j)$-elimination removes all vertices of $G$, whereas a $(2k-1, j)$-elimination leaves a nonempty subgraph; moreover, if the circuit outputs the value "false", then a $(k+1, j)$-elimination removes the vertices of $G$, whereas a $(k, j)$-elimination leaves a nonempty subgraph (because the expander construction has a subgraph with $j$-min-degree equal to $k+1$). □

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us that—contrary to a common assumption—approximations need not be inherently iterative. They may well be done in parallel.

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