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LETTER TO THE EDITOR

The Liouville–Arnold–Nekhoroshev theorem for non-compact invariant manifolds

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Abstract
Under certain conditions, generalized action–angle coordinates can be introduced near non-compact invariant manifolds of completely and partially integrable Hamiltonian systems.

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1. Introduction

Let us recall that an autonomous Hamiltonian system on a $2n$-dimensional symplectic manifold is said to be completely integrable if there exist n independent integrals of motion in involution. By virtue of the classical Liouville–Arnold theorem [1, 6], such a system admits action–angle coordinates around a connected regular compact invariant manifold. In a more general setting, one considers Hamiltonian systems having partial integrability, i.e. $k \leqslant n$ independent integrals of motion in involution. The Nekhoroshev theorem for these systems [3, 8] generalizes both the Poincaré–Lyapunov theorem ($k = 1$) and the above-mentioned Liouville–Arnold theorem ($k = n$). The Nekhoroshev theorem, in fact, falls into two parts. The first part states the sufficient conditions for an open neighbourhood of an invariant torus T^k to be a trivial fibre bundle (see [3] for a detailed exposition). The second one provides this bundle with partial action–angle coordinates similar to the case of complete integrability.

The present work addresses completely and partially integrable Hamiltonian systems whose invariant manifolds need not be compact. This is the case for any autonomous Hamiltonian system because its Hamiltonian, by definition, is an integral of motion. In the preceding papers, we have shown that if an open neighbourhood of a non-compact invariant manifold of a completely integrable Hamiltonian system is a trivial bundle, it can be equipped with the generalized action–angle coordinates which bring a symplectic form into the canonical

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form [2, 4]. Here, we prove that, under certain conditions, an open neighbourhood of a regular non-compact invariant manifold of a completely integrable system is a trivial bundle (see parts (A)–(C) in the proof of theorem 1) and, consequently, it can be equipped with the generalized action–angle coordinates (see part (D) of theorem 1). Then, this result is extended to partially integrable Hamiltonian systems (see theorem 3). The proof of theorem 3 mainly follows that of theorem 1. Note that part (D) in the proof of theorem 1 can be simplified by the choice of a Lagrangian section σ, but this is not the case for partially integrable systems. This proof also shows that, from the beginning, one can separate integrals of motion whose trajectories live in tori.

It should be emphasized that the results of theorem 3 are not limited by the scope of autonomous mechanics. Any time-dependent Hamiltonian system of n degrees of freedom can be extended to an autonomous Hamiltonian system of $n + 1$ degrees of freedom which has at least one integral of motion, namely, its Hamiltonian [2]. Thus, any time-dependent Hamiltonian system can be seen as a partially integrable autonomous Hamiltonian system whose invariant manifolds are never compact because of the time axis. Just the time is a generalized angle coordinate corresponding to a Hamiltonian of this autonomous system.

One also finds reasons in quantum theory in order to introduce generalized action–angle variables. In particular, quantization with respect to these variables enables one to include a Hamiltonian in the quantum algebra [2, 5].

2. Completely integrable systems

Let (Z, Ω) be a $2n$-dimensional symplectic manifold, and let it admit n real smooth functions $\{F_\lambda\}$, which are pairwise in involution and independent almost everywhere on Z. The latter implies that the set of non-regular points, where the morphism

$$\pi = \overset{\lambda}{\times} F_\lambda : Z \to \mathbb{R}^n \tag{1}$$

fails to be a submersion, is nowhere dense. Bearing in mind physical applications, we agree to think of one of the functions F_λ as being a Hamiltonian and of the others as first integrals of motion. Accordingly, their common level surfaces are called invariant surfaces.

Let M be a regular invariant surface, i.e. the morphism π (1) is a submersion at all points of M or, equivalently, the n-form $\overset{\lambda}{\wedge}\, \mathrm{d}F_\lambda$ vanishes nowhere on M. Hence, M is a closed imbedded submanifold of Z. There exists its open neighbourhood U such that the morphism π is a submersion on U, i.e.

$$\pi : U \to N = \pi(U) \tag{2}$$

is a fibred manifold over an open subset $N \subset \mathbb{R}^n$. The vertical tangent bundle VU of $U \to N$ coincides with the n-dimensional distribution on U spanned by the Hamiltonian vector fields ϑ_λ of the functions F_λ. Integral manifolds of this distribution are components of the fibres of π. They are Lagrangian submanifolds of Z. Let U be connected. Then N is a domain. Without loss of generality, one can suppose that there exists a section of $U \to N$.

If M is connected and compact, we come to the conditions of the Liouville–Arnold theorem. If M need not be compact, one should require something more.

Theorem 1. *Let M be a connected regular invariant manifold of a completely integrable Hamiltonian system $\{F_\lambda\}$ and let U be an open neighbourhood as above. Let us additionally assume that: (i) all fibres of the fibred manifold $U \to N$ (2) are mutually diffeomorphic,* *(ii) the Hamiltonian vector fields ϑ_λ on U are complete. Then, there exists a domain N so that $U \to N$ is a trivial bundle*

$$U = N \times (\mathbb{R}^{n-m} \times T^m) \tag{3}$$

provided with the generalized action–angle coordinates $(I_\lambda; x^a; \phi^i)$ such that the integrals of motion F_λ depend only on the action coordinates I_α and the symplectic form Ω on U reads

$$\Omega = \mathrm{d}I_a \wedge \mathrm{d}x^a + \mathrm{d}I_i \wedge \mathrm{d}\phi^i. \tag{4}$$

Proof. (A) Since Hamiltonian vector fields ϑ_α on U are complete and mutually commutative, their flows assemble into the additive Lie group $\mathbb{R}^n$. This group is naturally identified with its Lie algebra, and its group space is a vector space coordinated by parameters (s^λ) of the flows with respect to the basis $\{e_\lambda\}$ for its Lie algebra. This group acts in U so that its generators e_λ are represented by the Hamiltonian vector fields ϑ_λ and its orbits are fibres of the fibred manifold $U \to N$. Given a point $r \in N$, the action of $\mathbb{R}^n$ in the fibre $M_r = \pi^{-1}(r)$ factorizes as

$$\mathbb{R}^n \times M_r \to G_r \times M_r \to M_r \tag{5}$$

through the free transitive action in M_r of the factor group $G_r = \mathbb{R}^n/K_r$, where K_r is the isotropy group of an arbitrary point of M_r. It is the same group for all points because $\mathbb{R}^n$ is an Abelian group. Since the fibres M_r are mutually diffeomorphic, all isotropy groups K_r are isomorphic to the group $\mathbb{Z}^m$ for some fixed m, $0 \leqslant m \leqslant n$, and the groups G_r are isomorphic to the additive group $\mathbb{R}^{n-m} \times T^m$. Let us show that the fibred manifold $U \to N$ (2) is a principal bundle with the structure group G_0, where we denote $\{0\} = \pi(M)$. For this purpose, let us determine isomorphisms $\rho_r : G_0 \to G_r$ of the group G_0 to the groups G_r, $r \in N$. Then, a desired fibrewise action of G_0 in U is given by the law

$$G_0 \times M_r \to \rho_r(G_0) \times M_r \to M_r. \tag{6}$$

(B) Generators of each isotropy subgroup K_r of $\mathbb{R}^n$ are given by m linearly independent vectors of the group space $\mathbb{R}^n$. One can show that there are ordered collections of generators $(v_1(r), \ldots, v_m(r))$ of the groups K_r such that $r \mapsto v_i(r)$ are smooth $\mathbb{R}^n$-valued fields on N. Indeed, given a vector $v_i(0)$ and a section σ of the fibred manifold $U \to N$, each field $v_i(r) = (s^\alpha(r))$ is the unique smooth solution of the equation

$$g(s^\alpha)\sigma(r) = \sigma(r) \qquad (s^\alpha(0)) = v_i(0) \tag{7}$$

on an open neighbourhood of $\{0\}$. Without loss of generality, one can assume that this neighbourhood is N. Let us consider the decomposition

$$v_i(0) = B_i^a(0)e_a + C_i^k(0)e_k \qquad a = 1, \ldots, n-m \quad k = 1, \ldots, m$$

where $C_i^k(0)$ is a non-degenerate matrix. Since the fields $v_i(r)$ are smooth, there exists an open neighbourhood of $\{0\}$, say N again, where the matrices $C_i^k(r)$ remain non-degenerate. Then, there is a unique linear morphism

$$A_r = \begin{pmatrix} \mathrm{Id} & (B(r) - B(0))C^{-1}(0) \\ 0 & C(r)C^{-1}(0) \end{pmatrix} \tag{8}$$

of the vector space $\mathbb{R}^n$ which transforms its frame $v_\alpha(0) = \{e_a, v_i(0)\}$ into the frame $v_\alpha(r) = \{e_a, v_i(r)\}$. Since it is also an automorphism of the group $\mathbb{R}^n$ sending K_0 onto K_r, we obtain a desired isomorphism ρ_r of the group G_0 to the group G_r. Let an element g of the group G_0 be the coset of an element $g(s^\lambda)$ of the group $\mathbb{R}^n$. Then, it acts in M_r by rule (6) just as the element $g\big(\big(A_r^{-1}\big)^\lambda_\beta s^\beta\big)$ of the group $\mathbb{R}^n$ does. Since entries of the matrix A (8) are smooth functions on N, this action of the group G_0 in U is smooth. It is free and $U/G_0 = N$.

Then, the fibred manifold $U \to N$ is a principal bundle with the structure group G_0 which is trivial because N is a domain.

(C) Given a section σ of the principal bundle $U \to N$, its trivialization $U = N \times G_0$ is defined by assigning the points $\rho^{-1}(g_r)$ of the group space G_0 to points $g_r\sigma(r)$, $g_r \in G_r$, of a fibre M_r. Let us endow G_0 with the standard coordinate atlas $(y^\lambda) = (t^a; \varphi^i)$ of the group $\mathbb{R}^{n-m} \times T^m$. We provide U with a desired trivialization (3) with respect to the coordinates $(J_\lambda; t^a; \varphi^i)$, where $J_\lambda(u) = F_\lambda(u)$, $u \in U$, are coordinates on the base N. The Hamiltonian vector fields ϑ_λ on U relative to these coordinates read

$$\vartheta_a = \partial_a \qquad \vartheta_i = -(BC^{-1})^a_i\partial_a + (C^{-1})^k_i\partial_k. \tag{9}$$

In particular, the Hamilton equation takes the form

$$\dot{J}_\lambda = 0 \qquad \dot{y}^\lambda = f^\lambda(J_\alpha).$$

(D) Since fibres of $U \to N$ are Lagrangian manifolds, the symplectic form Ω on U is given by the coordinate expression

$$\Omega = \Omega^{\alpha\beta}\,\mathrm{d}J_\alpha \wedge \mathrm{d}J_\beta + \Omega^\alpha_\beta\,\mathrm{d}J_\alpha \wedge \mathrm{d}y^\beta. \tag{10}$$

Let us bring it into the canonical form (4). The Hamiltonian vector fields ϑ_λ obey the relations $\vartheta_\lambda \rfloor \Omega = -\mathrm{d}J_\lambda$, which take the coordinate form

$$\Omega^\alpha_\beta\vartheta^\beta_\lambda = \delta^\alpha_\lambda. \tag{11}$$

It follows that Ω^α_β is a non-degenerate matrix whose entries are independent of coordinates y^λ. By virtue of the well-known Künneth formula for the de Rham cohomology of manifold product, the closed form Ω (10) on U (3) is exact, i.e. $\Omega = \mathrm{d}\Xi$ where Ξ reads

$$\Xi = \Xi^\alpha(J_\lambda, y^\lambda)\,\mathrm{d}J_\alpha + \Xi_i(J_\lambda)\,\mathrm{d}\varphi^i. \tag{12}$$

Because entries of $\mathrm{d}\Xi = \Omega$ are independent of y^λ, we obtain the following:

(i) $\Omega^\lambda_i = \partial^\lambda\Xi_i - \partial_i\Xi^\lambda$. Consequently, $\partial_i\Xi^\lambda$ are independent of φ^i, i.e. Ξ^λ are, at most, affine in φ^i and, therefore, are independent of φ^i since these are cyclic coordinates. Hence, $\Omega^\lambda_i = \partial^\lambda\Xi_i$ and $\partial_i\rfloor\Omega = -\mathrm{d}\Xi_i$. A glance at the last equality shows that ∂_i are Hamiltonian vector fields. It follows that we can substitute m integrals of motion among F_λ with the functions Ξ_i, which we continue to denote F_i. The Hamiltonian vector fields of these new F_i are tangent to invariant tori. In this case, the matrix B in expressions (8) and (9) is the zero one, and the Hamiltonian vector fields ϑ_λ read

$$\vartheta_a = \partial_a \qquad \vartheta_i = (C^{-1})^k_i\partial_k. \tag{13}$$

Moreover, the coordinates t^a are exactly the flow parameters s^a. Substituting expressions (13) into conditions (11), we obtain

$$\Omega = \Omega^{\alpha\beta}\,\mathrm{d}J_\alpha \wedge \mathrm{d}J_\beta + \mathrm{d}J_a \wedge \mathrm{d}s^a + C^i_k\,\mathrm{d}J_i \wedge \mathrm{d}\varphi^k.$$

It follows that Ξ_i are independent of J_a, and so are $C^k_i = \partial^k\Xi_i$.

(ii) $\Omega^\lambda_a = -\partial_a\Xi^\lambda = \delta^\lambda_a$. Hence, $\Xi^a = -s^a + E^a(J_\lambda)$ and Ξ^i are independent of s^a.

In view of items (i) and (ii), the Liouville form Ξ (12) reads

$$\Xi = (-s^a + E^a(J_\lambda))\,\mathrm{d}J_a + E^i(J_\lambda)\,\mathrm{d}J_i + \Xi_i(J_j)\,\mathrm{d}\varphi^i.$$

Since the matrix $\partial^k\Xi_i$ is non-degenerate, we perform the coordinate transformation $I_a = J_a$, $I_i = \Xi_i(J_j)$ and obtain

$$\Xi = (-s^a + E'^a(I_\lambda))\,\mathrm{d}I_a + E'^i(I_\lambda)\,\mathrm{d}I_i + I_i\,\mathrm{d}\varphi^i.$$

Finally, put

$$x^a = s^a - E'^a(I_\lambda) \qquad \phi^i = \varphi^i - E'^i(I_\lambda) \tag{14}$$

in order to obtain the desired action–angle coordinates

$$I_a = J_a, \quad I_i(J_j), \quad x^a(J_\lambda, s^a), \quad \phi^i(J_\lambda, \varphi^k).$$

The shifts (14) correspond to the choice of a Lagrangian section σ. □

Let us remark that the generalized action–angle coordinates in theorem 1 are by no means unique. For instance, the canonical coordinate transformations

$$I_a = f_a(I'_\lambda) \qquad I_i = I'_i \qquad x'^a = \frac{\partial f_b}{\partial I'_a} x^b \qquad \phi'^i = \phi^i + \frac{\partial f_a}{\partial I'_i} x^a \tag{15}$$

give new generalized action–angle coordinates on U.

3. Partially integrable systems

Let a $2n$-dimensional symplectic manifold (Z, Ω) admit $k < n$ smooth real functions $\{F_\lambda\}$, which are pairwise in involution and independent almost everywhere on Z. Let us consider the morphism

$$\pi = \overset{\lambda}{\times} F_\lambda : Z \to \mathbb{R}^k \tag{16}$$

and its regular connected common level surface W. There exists an open connected neighbourhood U_W of W such that

$$\pi : U_W \to V_W = \pi(U_W) \tag{17}$$

is a fibred manifold over a domain V_W in $\mathbb{R}^k$. Restricted to U_W, the Hamiltonian vector fields ϑ_λ of functions F_λ define a k-dimensional distribution and the corresponding regular foliation $\mathcal{F}$ of U_W. Its leaves are isotropic. They are located in fibres of the fibred manifold $U_W \to V_W$ and, moreover, make up regular foliations of these fibres.

Let us assume that the foliation $\mathcal{F}$ has a total transversal manifold S and its holonomy pseudogroup on S is trivial. Then, $\mathcal{F}$ is a fibred manifold

$$\pi_1 : U_W \to S' \tag{18}$$

and $S = \sigma(S')$ is its section [7]. Thereby, the fibration π (17) factorizes as

$$\pi : U_W \xrightarrow{\pi_1} S' \xrightarrow{\pi_2} V_W$$

through the fibration π_1 (18). The map π_2 reads $\pi_2 = \pi \circ \sigma$ and, consequently, it is also a fibred manifold.

Proposition 2. *Let us assume that there exists a domain $N \subset S'$ such that: (i) the fibres of the fibred manifold π_1 (18) over N are mutually diffeomorphic, (ii) the Hamiltonian vector fields ϑ_λ on $U = \pi_1^{-1}(N)$ are complete. Then, there exists a domain in S', say N again, such that $U \to N$ is a trivial principal bundle with the structure group $\mathbb{R}^{k-m} \times T^m$.*

Proof. The proof is a straightforward repetition of parts (A) and (B) in the proof of theorem 1. □

Furthermore, one can always choose the domain N in proposition 2 as the domain of a fibred chart of π_2. Following part (C) in the proof of theorem 1, we can provide $U \to N$ with the trivialization

$$U = N \times \mathbb{R}^{k-m} \times T^m \tag{19}$$

coordinated by $(J_\lambda; z^A; y^\lambda)$ where: (i) $J_\lambda(u) = F_\lambda(u)$, $u \in U$, are coordinates on the base V, (ii) $(J_\lambda; z^A)$ are coordinates on N and (iii) $(y^\lambda) = (t^a; \varphi^i)$ are coordinates on $\mathbb{R}^{k-m} \times T^m$. The Hamiltonian vector fields ϑ_α on U with respect to these coordinates read

$$\vartheta_a = \partial_a \qquad \vartheta_i = \vartheta_i^a(J_\lambda, z^A)\partial_a + \vartheta_i^k(J_\lambda, z^A)\partial_k. \tag{20}$$

Since fibres of $U \to N$ are isotropic, the symplectic form Ω on U relative to the coordinates $(J_\lambda; z^A; y^\lambda)$ is given by the expression

$$\Omega = \Omega^{\alpha\beta}\, \mathrm{d}J_\alpha \wedge \mathrm{d}J_\beta + \Omega^\alpha_\beta\, \mathrm{d}J_\alpha \wedge \mathrm{d}y^\beta + \Omega_{AB}\, \mathrm{d}z^A \wedge \mathrm{d}z^B + \Omega^\lambda_A\, \mathrm{d}J_\lambda \wedge \mathrm{d}z^A + \Omega_{A\beta}\, \mathrm{d}z^A \wedge \mathrm{d}y^\beta. \tag{21}$$

The Hamiltonian vector fields ϑ_λ obey the relations $\vartheta_\lambda \rfloor \Omega = -\mathrm{d}J_\lambda$, which give the conditions

$$\Omega^\alpha_\beta \vartheta^\beta_\lambda = \delta^\alpha_\lambda \qquad \Omega_{A\beta}\vartheta^\beta_\lambda = 0.$$

The first of them shows that Ω^α_β is a non-degenerate matrix independent of coordinates y^λ. Then, the second one implies $\Omega_{A\beta} = 0$. The rest is a minor modification of part (D) in the proof of theorem 1.

The symplectic form Ω (21) on U is exact, and the Liouville form is

$$\Xi = \Xi^\alpha(J_\lambda, z^B, y^\lambda)\, \mathrm{d}J_\alpha + \Xi_i(J_\lambda, z^B)\, \mathrm{d}\varphi^i + \Xi_A(J_\lambda, z^B, y^\lambda)\, \mathrm{d}z^A.$$

Since $\Xi_a = 0$ and Ξ_i are independent of φ^i, one easily obtains from the relations $\Omega_{A\beta} = \partial_A \Xi_\beta - \partial_\beta \Xi_A = 0$ that Ξ_i are independent of coordinates z^A, while Ξ_A are independent of coordinates y^λ. Hence, the Liouville form reads

$$\Xi = \Xi^\alpha(J_\lambda, z^B, y^\lambda)\, \mathrm{d}J_\alpha + \Xi_i(J_\lambda)\, \mathrm{d}\varphi^i + \Xi_A(J_\lambda, z^B)\, \mathrm{d}z^A$$

(cf (12)). Running through item (i), we observe that, in the case of a partially integrable system, one can also separate integrals of motion F_i whose Hamiltonian vector fields are tangent to invariant tori. Then, the Hamiltonian vector fields (20) take the form

$$\vartheta_a = \partial_a \qquad \vartheta_i = \vartheta_i^k(J_\lambda, z^A)\partial_k.$$

Following items (i) and (ii) of part (D), we obtain

$$\Xi = (-s^a + E^a(J_\lambda, z^B))\, \mathrm{d}J_a + E^i(J_\lambda, z^B)\, \mathrm{d}J_i + \Xi_i(J_j)\, \mathrm{d}\varphi^i + \Xi_A(J_\lambda, z^B)\, \mathrm{d}z^A.$$

Finally, the coordinates

$$x^a = -s^a + E^a(J_\lambda, z^B) \qquad I_i = \Xi_i(J_j) \qquad I_a = J_a \qquad \phi^i = \varphi^i - E^j(J_\lambda, z^B)\frac{\partial J_j}{\partial I_i}$$

bring Ω into the form

$$\Omega = \mathrm{d}I_a \wedge \mathrm{d}x^a + \mathrm{d}I_i \wedge \mathrm{d}\phi^i + \Omega_{AB}(I_\lambda, z^B)\, \mathrm{d}z^A \wedge \mathrm{d}z^B + \Omega^\lambda_A(I_\lambda, z^B)\, \mathrm{d}I_\lambda \wedge \mathrm{d}z^A. \tag{22}$$

Therefore, one can think of these coordinates as being partial generalized action–angle coordinates. The Hamiltonian vector fields of integral of motions with respect to these coordinates read

$$\vartheta_a = \frac{\partial}{\partial x^a} \qquad \vartheta_i = \frac{\partial J_i}{\partial I_j}\frac{\partial}{\partial \phi^j}.$$

Thus, we have proved the following.

Theorem 3. *Given a partially integrable Hamiltonian system $\{F_\lambda\}$ on a symplectic manifold (Z, Ω), let W be its regular connected level surface, and let $M \subset W$ be a leaf of the characteristic foliation $\mathcal{F}$ of the distribution generated by the Hamiltonian vector fields ϑ_λ of F_λ. Let M have an open satured neighbourhood $U \subset Z$ such that: (i) the foliation $\mathcal{F}$ of U admits a transversal manifold S and its holonomy pseudogroup on S is trivial, (ii) the leaves of this foliation are mutually diffeomorphic, (iii) Hamiltonian vector fields ϑ_λ on U are complete.*

Then, there exists an open satured neighbourhood of M, say U again, which is a trivial bundle (19), provided with the particular coordinates $(I_\lambda; z^A, x^a; \phi^i)$ *such that the integrals of motion* F_λ *depend only on the coordinates* I_α *and the symplectic form* Ω *on U is brought into the form (22).*

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