

THE LIPSCHITZ CONTINUITY OF THE DISTANCE FUNCTION TO THE CUT LOCUS

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ABSTRACT. Let N be a closed submanifold of a complete smooth Riemannian manifold M and $U\nu$ the total space of the unit normal bundle of N . For each $v \in U\nu$, let $\rho(v)$ denote the distance from N to the cut point of N on the geodesic γ_v with the velocity vector $\dot{\gamma}_v(0) = v$. The continuity of the function ρ on $U\nu$ is well known. In this paper we prove that ρ is locally Lipschitz on which ρ is bounded; in particular, if M and N are compact, then ρ is globally Lipschitz on $U\nu$. Therefore, the canonical interior metric δ may be introduced on each connected component of the cut locus of N , and this metric space becomes a locally compact and complete length space.

Let N be an immersed submanifold of a complete C^∞ Riemannian manifold M and $\pi : U\nu \rightarrow N$ the unit normal bundle of N . For each positive integer k and vector $v \in U\nu$, let a number $\lambda_k(v)$ denote the parameter value of γ_v , where γ_v denotes the geodesic for which the velocity vector is v at $t = 0$, such that $\gamma_v(\lambda_k(v))$ is the k -th focal point (conjugate point for the case in which N is a point) of N along γ_v , counted with focal (or conjugate) multiplicities. A unit speed geodesic segment $\gamma : [0, a] \rightarrow M$ emanating from N is called an N -segment if $t = d(N, \gamma(t))$ on $[0, a]$. By the first variation formula, an N -segment is orthogonal to N . A point $\gamma_v(t_0)$ on an N -segment $\gamma_v, v \in U\nu$, is called a *cut point* of N if there is no N -segment properly containing $\gamma[0, t_0]$. For each $v \in U\nu$, let $\rho(v)$ denote the distance from N to the cut point on γ_v of N . Whitehead [27] investigated the structure of the conjugate locus and the cut locus of a point on a real analytic Finsler manifold. He determined the structure of the conjugate locus around a conjugate point for which the conjugate multiplicity is locally constant on its neighborhood (cf. also [25]) and proved the continuity of the function ρ . In compact symmetric spaces, T. Sakai [19] and M. Takeuchi [23] determined the detailed structure of the cut locus of a point. The detailed structure of the cut locus of a point in a 2-dimensional Riemannian manifold has been investigated by Poincaré, Myers, and others [7], [11], [13]. Hartman first tried to show the absolute continuity of the function ρ when M is 2-dimensional. He proved in [8] that if ρ is of bounded variation, then ρ is absolutely continuous. Recently, Hebda [11] and the first named author [13] independently proved Ambrose's problem by showing that ρ is absolutely continuous on a closed arc on which ρ is bounded when N is a point in a 2-dimensional Riemannian manifold. Therefore, the cut locus of a point in a compact 2-dimensional

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Riemannian manifold has finite 1-dimensional Hausdorff measure, and any two cut points can be connected by a rectifiable curve in the cut locus.

In the present paper, we prove that the focal locus and the cut locus of a submanifold of a complete C^∞ Riemannian manifold have weak differentiability:

Theorem A. *Let N be an immersed submanifold of a complete C^∞ Riemannian manifold M and $\pi : U\nu \rightarrow N$ the unit normal bundle of N . Then, for each positive integer k and $v \in U\nu$ with $\lambda_k(v) < \infty$, λ_k is locally Lipschitz around v .*

Theorem B. *Let N be an embedded submanifold of a complete C^∞ Riemannian manifold M and $\pi : U\nu \rightarrow N$ the unit normal bundle of N . Then, for each $v \in U\nu$ with $\rho(v) < \infty$, ρ is locally Lipschitz around v . In particular, if M and N are compact, then ρ is globally Lipschitz on $U\nu$ and hence the cut locus has finite $(m - 1)$ -dimensional Hausdorff measure, where m denotes the dimension of M .*

Note that λ_k is not always differentiable (see Example 3.1). If there exists a neighborhood of $\lambda_k(v)v$ in which the focal multiplicity of each focal tangent vector is constant, then λ_k is C^∞ around v , as Warner [25] and Hebda [9] reported. In particular, if M is 2-dimensional, then λ_k is C^∞ on which λ_k is bounded. In fact, the focal multiplicity is 1 at each focal point.

Rademacher's theorem (cf. [16]) states that a Lipschitz map of a domain in R^k into R^l is differentiable almost everywhere. Therefore, as corollaries to Theorems A and B, there exist tangent spaces at almost all points in the tangent focal locus and the tangent cut locus, respectively.

Since a Lipschitz continuous function is absolutely continuous, Theorem B generalizes the previously mentioned result by Hebda and the first named author; therefore, this theorem is new, even for 2-dimensional M . Theorem B has a few corollaries. If a cut point q is not a focal point of the submanifold along an N -segment, then the Hausdorff dimension of the focal locus around q equals $m - 1$ (cf. [14] for the case in which N is a point).

Corollary C. *Suppose N is a closed submanifold of M . Then the canonical interior metric δ may be introduced on each connected component of C_N . Moreover the topology introduced from δ coincides with the relative one of (M, g) , and (C_N, δ) is a locally compact and complete length space.*

Note that the cut locus of a compact subset of an Alexandrov surface admits the canonical interior metric, which is a result given by Shiohama and the second named author [22]. Corollary C raises the following interesting problem:

Does the metric space (C_N, δ) have curvature bounded below (or above) in the sense of Alexandrov?

The answer is no. Counterexamples are given in Section 3.

Refer to [1] or [2] for the geometry on metric spaces. [20] is a good reference on Riemannian geometry and in particular on the Morse index theorem.

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1. THE DISTANCE FUNCTIONS TO THE TANGENT FOCAL LOCUS

Let (M, g) denote a complete, m -dimensional C^∞ Riemannian manifold. We denote by TM the total space of the tangent bundle over M , and by \exp the exponential map defined on TM . The fiber over p is denoted by T_pM . Let N denote a C^∞ n -dimensional submanifold of M and $\pi : \nu \rightarrow N$ the normal bundle of N . The fiber over p is denoted by ν_p . For each $\xi \in \nu_p$, let A_ξ denote the shape operator of N with respect to ξ , which is a symmetric linear transformation on T_pN (see [20] for the definition of the shape operator). Suppose that a unit speed geodesic $\gamma : [0, \infty) \rightarrow M$ is given, for which $\xi := \dot{\gamma}(0) \in \nu_p$. A Jacobi field Y along γ is called an N -Jacobi field if Y satisfies the following two initial conditions:

$$(1.1) \quad Y(0) \in T_pN, \quad Y'(0) - A_\xi Y(0) \in \nu_p,$$

where Y' denotes the covariant derivative of the Jacobi field Y along γ . Note that if N consists of a single point p , then an N -Jacobi field Y is a Jacobi field along γ emanating from p with $Y(0) = 0$ and $Y'(0) \in T_pM$. The following equations, (1.2) and (1.3), are very important in proving Theorems A and B (cf. [4]). For any two Jacobi fields X, Y along a geodesic $\gamma : [0, \infty) \rightarrow M$, there exists a constant c such that

$$(1.2) \quad g(X'(t), Y(t)) - g(X(t), Y'(t)) = c$$

for any $t \geq 0$. In particular, the equality

$$(1.3) \quad g(X'(t), Y(t)) = g(X(t), Y'(t))$$

holds for any N -Jacobi fields X, Y . A point $\gamma(t_0)$, where t_0 is a positive number (respectively $t_0\dot{\gamma}(0)$), is called a *focal point* (respectively *focal tangent vector*) of N along a geodesic γ emanating perpendicularly from N if there exists a non-zero N -Jacobi field Y along γ with $Y(t_0) = 0$. For each geodesic $\gamma : [0, b] \rightarrow M$ emanating perpendicularly from N , let $\text{ind}_N(\gamma)$ denote the index of γ (see [20] for the definition of the index). Let $\pi : U\nu \rightarrow N$ denote the unit sphere normal bundle over N . For each positive integer k and each unit tangent vector $v \in U\nu$ we define a number $\lambda_k(v)$ by

$$(1.4) \quad \lambda_k(v) := \sup\{t; \text{ind}_N(\gamma_v|_{[0,t]}) \leq k - 1\},$$

where γ_v denotes the geodesic $\gamma_v(t) := \exp(tv)$. The differential of the normal exponential map \exp^\perp is singular at $v \in \nu$ if and only if $\exp(v)$ is a focal point of N along γ_v . It is clear that $0 < \lambda_1(v) \leq \lambda_2(v) \leq \lambda_3(v) \leq \dots$ and it follows from the Morse index theorem (cf. [20], also [15] or [16]) that $\gamma_v(\lambda_k(v))$ is the k -th focal point of N along γ_v , counted with focal multiplicities. Here the *focal multiplicity* of a focal point $\gamma_v(t_0)$ is the dimension of the kernel of $d\exp^\perp$ at t_0v , where $d\exp^\perp$ denotes the differential of \exp^\perp . Hence $\lambda_k(v)$ is the distance function to the k -th focal tangent vector of N along γ_v , counting focal multiplicities.

Definition 1.1. For each $v \in U\nu$ and $w \in T_{\pi(v)}M$, let $Y(t; v, w)$ denote the N -Jacobi field $Y(t)$ along the geodesic γ_v with initial conditions $Y(0) = w^T$ and $Y'(0) = A_v w^T + w^\perp$, where w^T and w^\perp denote the images of w under orthogonal projection to $T_{\pi(v)}N$ and $\nu_{\pi(v)}$, respectively.

Definition 1.2. For each positive integer k and $v \in U_p\nu := \nu_p \cap U\nu$ with $\lambda_k(v) < \infty$, let $F(\lambda_k(v)v)$ denote the kernel of the linear map $w \in T_pM \longrightarrow Y(\lambda_k(v); v, w) \in T_{\gamma_v(\lambda_k(v))}M$.

Note that the dimension of $F(\lambda_k(v)v)$ is the same as the focal multiplicity of the focal point $\gamma_v(\lambda_k(v)v)$.

Lemma 1.1. Let $\{v_j\}$ be a sequence of vectors in $U\nu$ convergent to a tangent vector $v \in U_p\nu$. Suppose that there exist positive integers k_1, \dots, k_l such that the sequences $\{\lambda_{k_i}(v_j)\}_j$ converge to a common real number t_0 , and that $\lambda_{k_1}(v_j) < \lambda_{k_2}(v_j) < \dots < \lambda_{k_l}(v_j)$ for each j . If there exists a linear subspace $F_i := \lim_{j \rightarrow \infty} F(\lambda_{k_i}(v_j)v_j)$ of $F(t_0v)$ for each $i = 1, \dots, l$, i.e., there exists a convergent sequence of a basis of $F(\lambda_{k_i}(v_j)v_j)$, then $Y'(t_0; v, x)$ and $Y'(t_0; v, y)$ are orthogonal for any $x \in F_a$ and $y \in F_b$ ($a < b$), and in particular the dimension of $F_1 + \dots + F_l$ equals $\sum_{i=1}^l \dim F_i$.

Proof. Let $\{x_j\}$ and $\{y_j\}$ be sequences of elements of $F(\lambda_{k_a}(v_j)v_j)$ and $F(\lambda_{k_b}(v_j)v_j)$ convergent to x and y respectively. Then, from (1.3) it follows that

$$g(Y'(t; v_j, x_j), Y(t; v_j, y_j)) = g(Y(t; v_j, x_j), Y'(t; v_j, y_j))$$

for any $t \geq 0$. Since $Y(\lambda_{k_a}(v_j); v_j, x_j) = 0$, we get

$$(1.5) \quad g(Y'(\lambda_{k_a}(v_j); v_j, x_j), Y(\lambda_{k_a}(v_j); v_j, y_j)) = 0.$$

Since $Y(t; v_j, y_j) = 0$ at $t = \lambda_{k_b}(v_j)$, there exists a C^∞ vector field $X(t; v_j, y_j)$ along γ_{v_j} that is smoothly dependent on (v_j, y_j) and such that

$$(1.6) \quad Y(t; v_j, y_j) = (t - \lambda_{k_b}(v_j))X(t; v_j, y_j), \quad X(\lambda_{k_b}(v_j); v_j, y_j) = Y'(\lambda_{k_b}(v_j); v_j, y_j).$$

By (1.5) and (1.6), we get

$$(1.7) \quad g(Y'(\lambda_{k_a}(v_j); v_j, x_j), X(\lambda_{k_a}(v_j); v_j, y_j)) = 0.$$

If we take the limit of (1.7), then it follows from (1.6) that

$$(1.8) \quad g(Y'(t_0; v, x), Y'(t_0; v, y)) = 0.$$

Let f denote the linear map of T_pM into $T_{\gamma_v(t_0)}M$ defined by $f(w) = Y'(t_0; v, w)$. Since the $f(F_i)$, $i = 1, \dots, l$, are mutually orthogonal by (1.8), we have

$$\sum_{j=1}^l \dim f(F_i) = \dim(f(F_1) + \dots + f(F_l)) \leq \dim(F_1 + \dots + F_l).$$

Since $f|_{F_i}$ is injective, $\dim f(F_i) = \dim F_i$ for each i . Therefore, the dimension of $F_1 + \dots + F_l$ equals $\sum_{i=1}^l \dim F_i$. \square

Proposition 1.2. For each positive number t , the function

$$v \in U\nu \longrightarrow \text{ind}_N(\gamma_v|_{[0,t]})$$

is locally constant around each tangent vector $v \in U\nu$ if $\gamma_v(t)$ is not a focal point of N along γ_v . Furthermore, the function $\lambda_k : U\nu \longrightarrow (0, \infty]$ is continuous for each k .

Proof. Take a vector $v_0 \in U\nu$ such that $\gamma_{v_0}(t)$ is not a focal point of N along γ_{v_0} . Since the index form depends continuously on the geodesic segment $\gamma_v|_{[0,t]}$, it is clear that

$$(1.9) \quad \text{ind}_N(\gamma_{v_0}|_{[0,t]}) \leq \text{ind}_N(\gamma_v|_{[0,t]})$$

for any $v \in U\nu$ sufficiently close to v_0 . Suppose that there exists a sequence $\{v_j\}$ of elements of $U\nu$ convergent to v_0 such that $\text{ind}_N(\gamma_{v_0}|_{[0,t]}) \neq \text{ind}_N(\gamma_{v_j}|_{[0,t]})$. By taking a subsequence of the sequence, and by (1.9), we may assume that

$$(1.10) \quad \text{ind}_N(\gamma_{v_0}|_{[0,t]}) < \text{ind}_N(\gamma_{v_j}|_{[0,t]})$$

for any j , and that the limit linear space $F_k := \lim_{j \rightarrow \infty} F(\lambda_k(v_j)v_j)$ exists for each k with $\lim_{j \rightarrow \infty} \lambda_k(v_j) < t$. It follows from the Morse index theorem and (1.11) that

$$(1.11) \quad \text{ind}_N(\gamma_{v_j}|_{[0,t]}) = \sum \dim F(\lambda_k(v_j)v_j) = \sum \dim F_k$$

for any sufficiently large j , where the sums are taken over the set $\{\lambda_k(v_j); \lambda_k(v_j) < t\}$. It follows from the Morse index theorem and Lemma 1.1 that

$$(1.12) \quad \text{ind}_N(\gamma_{v_0}|_{[0,t]}) \geq \sum \dim F_k = \text{ind}_N(\gamma_{v_j}|_{[0,t]}).$$

However, a contradiction exists between (1.10) and (1.12). Therefore, the function $v \in U\nu \rightarrow \text{ind}_N(\gamma_v|_{[0,t]})$ is locally constant around each tangent vector $v \in U\nu$ if $\gamma_v(t)$ is not a focal point of N along γ_v . Take any $v_0 \in U\nu$ and any positive number $t > \lambda_k(v_0)$ (respectively $t < \lambda_k(v_0)$) such that $\gamma_{v_0}(t)$ is not a focal point of N along γ_{v_0} . Since $\text{ind}_N(\gamma_v|_{[0,t]})$ is locally constant around v_0 , we get $\lambda_k(v) > t$ (respectively $\lambda_k(v) < t$) for any v sufficiently close to v_0 , implying the continuity of λ_k . \square

Fix any positive integer k and any $v_0 \in U_p\nu$ with $\lambda_k(v_0) < +\infty$. We want to prove the local Lipschitz continuity of λ_k around v_0 . For convenience, introduce a C^∞ Riemannian metric G on $U\nu$. The Riemannian distance function induced from G is denoted by D . For each positive number δ , we denote the open ball centered at v_0 with radius δ by $B_D(v_0; \delta)$.

Definition 1.3. For each $q \in M$, let S_qM denote the set of all unit tangent vectors of T_qM , and for each tangent vector v , let $\|v\|$ denote the length of v , i.e., $\|v\| := \sqrt{g(v, v)}$.

Since λ_k is continuous, there exists a relatively compact convex neighborhood $B_D(v_0; \delta_0(k))$, on which λ_k does not exceed $\lambda_k(v_0) + 1$. Since each Jacobi field $Y(t)$ is uniquely determined by $Y(t_1)$ and $Y'(t_1)$ for some t_1 , the number

$$(1.13) \quad 2C_0(J', k) := \min\{ \|Y'(\lambda_i(v_0); v_0, w)\|^2; 1 \leq i \leq k, w \in S_pM \cap F(\lambda_i(v_0)v_0) \}$$

is positive. Since each λ_i is continuous, there exists a positive number $\delta_1(k)$ ($\leq \delta_0(k)$) such that

$$(1.14) \quad C_0(J', k) \leq \|Y'(\lambda_i(v); v, w)\|^2$$

for any $v \in B_D(v_0; \delta_1(k))$ and any $w \in F(\lambda_i(v)v) \cap S_{\pi(v)}M$, $1 \leq i \leq k$. For each $v \in B_D(v_0; \delta_1(k))$, choose a sufficiently small positive number $\epsilon(v)$ with the following two properties: The closed intervals $[s_i(v), t_i(v)]$, $1 \leq i \leq k$, are mutually disjoint if $\lambda_i(v) \neq \lambda_j(v)$, where $s_i(v) := \lambda_i(v) - \epsilon(v)$, $t_i(v) := \lambda_i(v) + \epsilon(v)$. For each positive integer i ($\leq k$), the geodesic segment $\gamma_v|_{[s_i(v), t_i(v)]}$ lies in a convex ball.

Definition 1.4. For each $v \in B_D(v_0; \delta_1(k))$, $\tau \in (\lambda_i(v), t_i(v)]$, and $w \in F(\lambda_i(v)v)$ ($1 \leq i \leq k$), let $X(t; v, w, \tau)$ denote the broken Jacobi field $X(t)$ along γ_v such that

$$X(t) = \begin{cases} Y(t; v, w) & \text{on } [0, s_i(v)], \\ Y(t; v, w, \tau) & \text{on } [s_i(v), \tau], \\ 0 & \text{on } [\tau, \infty), \end{cases}$$

where $Y(t; v, w, \tau)$ denotes the Jacobi field along γ_v satisfying

$$Y(s_i(v); v, w, \tau) = Y(s_i(v); v, w), \quad Y(\tau; v, w, \tau) = 0.$$

Note that the Jacobi field $Y(t; v, w, \tau)$ is uniquely determined by the property

$$Y(\tau; v, w, \tau) = 0, \quad Y(s_i(v); v, w, \tau) = Y(s_i(v); v, w)$$

for each $\tau \in (s_i(v), t_i(v)]$, since $\gamma_v|_{[s_i(v), t_i(v)]}$ lies in a convex ball. The uniqueness implies that $Y(t; v, \sum_j c_j w_j, \tau) = \sum_j c_j Y(t; v, w_j, \tau)$, and thus

$$X(t; v, \sum_j c_j w_j, \tau) = \sum_j c_j X(t; v, w_j, \tau)$$

for any finitely many real numbers c_j and vectors w_j which are elements in a common $F(\lambda_i(v)v)$. By taking a smaller $\epsilon(v)$, we may assume that the length $\|X(t; v, w, \tau)\|$ of $X(t; v, w, \tau)$ is monotone on $[s_i(v), \tau]$. Therefore, if

$$(1.15) \quad C(J, k) := \sup\{ \|Y(t; v, w)\|^2 ; 0 \leq t \leq \lambda_k(v_0) + 1, \\ v \in B_D(v_0; \delta_1(k)), w \in S_{\pi(v)}M \},$$

then

$$(1.16) \quad C(J, k) \geq \|X(t; v, w, \tau)\|^2$$

on $[0, \infty)$ for each broken Jacobi field $X(t; v, w, \tau)$. Let $\{e_1, \dots, e_m\}$ denote a C^∞ local frame field on a neighborhood V of $p = \pi(v_0)$ such that $\{e_1(q), \dots, e_m(q)\}$ and $\{e_1(q), \dots, e_n(q)\}$ are orthonormal bases of T_qM and T_qN for each $q \in N \cap V$, respectively.

Definition 1.5. For each $v \in U \cap \pi^{-1}(V \cap N)$ let $\{E_1(t; v), \dots, E_m(t; v)\}$ denote the set of parallel vector fields along the geodesic γ_v such that $E_i(0; v) = e_i(\pi(v))$ for each i .

Choose a positive number $\delta_2(k)$ ($\leq \delta_1(k)$) so as to satisfy

$$B_D(v_0; \delta_2(k)) \subset U \cap \pi^{-1}(V \cap N).$$

Let I_0^t denote the index form with respect to a geodesic $\gamma_v|_{[0, t]}$, i.e.,

$$I_0^t(X, Y) = \int_0^t g(X'(t), Y'(t)) - g(R(X(t), \dot{\gamma}_v(t))\dot{\gamma}_v(t), Y(t))dt \\ + g(A_v(X(0)), Y(0))$$

for piecewise C^∞ vector fields X, Y along $\gamma_v|_{[0, t]}$, where R denotes the sectional curvature tensor field of (M, g) . For simplicity, $I_0^t(X, X)$ will be denoted by $I_0^t(X)$. Since

$$R_{ij}(t, v) := g(R(E_i(t; v), \dot{\gamma}_v(t))\dot{\gamma}_v(t), E_j(t; v)), \quad i, j = 1, \dots, m,$$

$$f_{kl}(v) := g(A_v(e_k(\pi(v))), e_l(\pi(v))), \quad k, l = 1, \dots, n,$$

are C^∞ functions, we may choose constants $C(R, k)$ and $C(A)$ such that the inequalities

$$(1.17) \quad \begin{aligned} |R_{ij}(t, v_1) - R_{ij}(t, v_2)| &\leq C(R, k)D(v_1, v_2), \\ |f_{kl}(v_1) - f_{kl}(v_2)| &\leq C(A)D(v_1, v_2) \end{aligned}$$

hold for any $t \in [0, \lambda_k(v_0) + 1]$, $i, j \in \{1, \dots, m\}$, $k, l \in \{1, \dots, n\}$ and $v_1, v_2 \in B_D(v_0; \delta_3(k))$, where $\delta_3(k) := \frac{1}{2}\delta_2(k)$.

Lemma 1.3. *For any $v \in B_D(v_0; \delta_3(k))$, $w \in F(\lambda_i(v)v)$ and $\tau \in (\lambda_i(v), t_i(v)]$ ($1 \leq i \leq k$),*

$$(1.18) \quad I_0^\tau(X(\cdot; v, w, \tau)) = -g(Y(\tau; v, w), Y'(\tau; v, w, \tau)).$$

Moreover, for each $v \in B_D(v_0; \delta_3(k))$ and positive integer $i(\leq k)$, there exists a real number $\tau_i(v) \in (\lambda_i(v), t_i(v))$ such that, for any $\tau \in (\lambda_i(v), \tau_i(v))$ and $w \in F(\lambda_i(v)v)$,

$$(1.19) \quad I_0^\tau(X(\cdot; v, w, \tau)) \leq -\frac{1}{2}C_0(J', k)(\tau - \lambda_i(v))\|w\|^2.$$

Proof. Since $X(t; v, w, \tau)|_{[0, s_i(v)]}$ and $X(t; v, w, \tau)|_{[s_i(v), \tau]}$ are Jacobi fields along γ_v , we get

$$\begin{aligned} I_0^\tau(X(\cdot; v, w, \tau)) &= g(Y'(s_i(v); v, w), Y(s_i(v); v, w, \tau)) \\ &\quad - g(Y(s_i(v); v, w), Y'(s_i(v); v, w, \tau)). \end{aligned}$$

It follows from (1.2) that

$$I_0^\tau(X(\cdot; v, w, \tau)) = g(Y'(\tau; v, w), Y(\tau; v, w, \tau)) - g(Y(\tau; v, w), Y'(\tau; v, w, \tau)).$$

Since $Y(\tau; v, w, \tau) = 0$, equation (1.18) holds. Since $Y(t; v, w) = 0$ at $t = \lambda_i(v)$, there exists a C^∞ vector field $X(t; v, w)$ such that $Y(t; v, w) = (t - \lambda_i(v))X(t; v, w)$. Since

$$\lim_{\tau \rightarrow \lambda_i(v)} Y'(\tau; v, w, \tau) = Y'(\lambda_i(v); v, w, \lambda_i(v)) = Y'(\lambda_i(v); v, w) = \lim_{\tau \rightarrow \lambda_i(v)} X(\tau; v, w),$$

it follows from (1.14) that there exists $\tau_i(v) \in (\lambda_i(v), t_i(v))$ such that

$$\begin{aligned} -g(Y(\tau; v, w), Y'(\tau; v, w, \tau)) &\leq -\frac{1}{2}\|Y'(\lambda_i(v); v, w)\|^2(\tau - \lambda_i(v)) \\ &\leq -\frac{1}{2}C_0(J', k)\|w\|^2(\tau - \lambda_i(v)) \end{aligned}$$

for any $\tau \in (\lambda_i(v), \tau_i(v))$ and $w \in F(\lambda_i(v)v)$, completing the proof of (1.19). \square

Proof of Theorem A. Fix any $v_1 \in B_D(v_0; \delta_3(k))$. We prove that the inequality

$$\lambda_k(v_2) - \lambda_k(v_1) \leq L_k D(v_1, v_2)$$

holds for any $v_2 \in B_D(v_0; \delta_3(k))$ sufficiently close to v_1 , where

$$L_k := \frac{4mkC(J, k)}{C_0(J', k)}(C(A) + (\lambda_k(v_0) + 1)C(R, k)).$$

Thus, the above inequality can be easily proven for any $v_2 \in B_D(v_0; \delta_3(k))$, and λ_k is Lipschitz continuous on $B_D(v_0; \delta_3(k))$ with Lipschitz constant L_k . For each positive integer $i \leq k$, choose a unit vector w_i from $F(\lambda_i(v_1)v_1)$ so as to satisfy the following property: for distinct $i, j \leq k$, w_i and w_j are orthogonal whenever $\lambda_i(v_1) = \lambda_j(v_1)$. Set $a_i := \lambda_i(v_1) + \epsilon$, where $\epsilon (\leq 1)$ is a sufficiently small positive

number satisfying $a_i \in (\lambda_i(v_1), \tau_i(v_1))$ for each $i \leq k$. Let $W(\gamma_{v_1})$ denote the k -dimensional linear space spanned by piecewise C^∞ vector fields $X_i(t; v_1)$, $1 \leq i \leq k$, along γ_{v_1} , where $X_i(t; v_1) := X(t; v_1, w_i, a_i)$. We first prove that the inequality

$$(1.20) \quad I_0^{a_k} \left(\sum_{i=1}^k c_i X_i(\cdot; v_1) \right) \leq -\frac{\epsilon}{2} C_0(J', k) \sum_{i=1}^k c_i^2$$

holds for any real numbers c_i 's. Choose a maximal subset $\{i_1, \dots, i_l\}$ of $\{1, \dots, k\}$ satisfying $\lambda_{i_1}(v_1) < \lambda_{i_2}(v_1) < \dots < \lambda_{i_l}(v_1)$. Set

$$N_s := \{j; \lambda_j(v_1) = \lambda_{i_s}(v_1)\}$$

for each $s \in \{1, \dots, l\}$. The fact that the N_s are mutually disjoint subsets of $\{1, \dots, k\}$ with $N_1 \cup \dots \cup N_l = \{1, \dots, k\}$ is trivial. Since

$$\sum_{i=1}^k c_i X_i(t; v_1) = \sum_{s=1}^l X(t; v_1, \sum_{i \in N_s} c_i w_i, a_{i_s}),$$

it follows that

$$(1.21) \quad I_0^{a_k} \left(\sum_{i=1}^k c_i X_i(\cdot; v_1) \right) = \sum_{s=1}^l I_0^{a_k} \left(X(\cdot; v_1, \sum_{i \in N_s} c_i w_i, a_{i_s}) \right).$$

Note that

$$I_0^{a_k} (X(\cdot; v_1, x_i, a_i), X(\cdot; v_1, x_j, a_j)) = 0$$

for any $x_i \in F(\lambda_i(v_1)v_1)$, $y_i \in F(\lambda_j(v_1)v_1)$ with $\lambda_i(v_1) < \lambda_j(v_1)$. By applying (1.19) to each broken Jacobi field $X(t; v_1, \sum_{i \in N_s} c_i w_i, a_{i_s})$, it follows that (1.21) implies (1.20). Choose $v_2 \in U(v_0; \delta_3(k))$ sufficiently close to v_1 to satisfy

$$\epsilon := L_k D(v_1, v_2) < \min\{\tau_i(v_1) - \lambda_i(v_1); 1 \leq i \leq k\}.$$

By (1.20), the inequality

$$(1.22) \quad I_0^{a_k} \left(\sum_{i=1}^k c_i X_i(\cdot; v_1) \right) \leq -\frac{L_k}{2} C_0(J', k) D(v_1, v_2) \sum_{i=1}^k c_i^2$$

holds for any $v_2 \in B_D(v_0; \delta_3(k))$ sufficiently close to v_1 and any real numbers c_i . For each $X \in W(\gamma_{v_1})$, we construct a piecewise C^∞ vector field $\tilde{X}(t)$ along γ_{v_2} by

$$\tilde{X}(t) := \sum_{i=1}^m g(X(t), E_i(t; v_1)) E_i(t; v_2).$$

For simplicity, set

$$Z(t) := \sum_{i=1}^k c_i X_i(t; v_1).$$

It follows from (1.17) and the Schwarz inequality that

$$I_0^{a_k}(\tilde{Z}) \leq I_0^{a_k}(Z) + mkC(J, k)D(v_1, v_2)(C(A) + (\lambda_k(v_0) + 1)C(R, k)) \sum_{i=1}^k c_i^2.$$

Hence, by (1.22), we get

$$I_0^{a_k}(\tilde{Z}) \leq -\frac{1}{4} L_k C_0(J', k) D(v_1, v_2) \sum_{i=1}^k c_i^2,$$

which holds for any $v_2 \in B_D(v_0; \delta_3(k))$ sufficiently close to v_1 . This inequality implies the index form $I_0^{a_k}$ is negative definite on the k -dimensional linear space $\{\dot{X}(t); X \in W(\gamma_{v_1})\}$, and so $\text{ind}_N(\gamma_{v_2}|_{[0, a_k]})$ is not less than k . Therefore,

$$\lambda_k(v_2) \leq a_k = \lambda_k(v_1) + L_k D(v_1, v_2)$$

for any $v_2 \in D(v_0; \delta_3(k))$ sufficiently close to v_1 , completing the proof of Theorem A. \square

2. THE DISTANCE FUNCTION TO THE CUT LOCUS

Throughout this section N always denotes an embedded submanifold of M . A unit speed geodesic segment $\gamma : [0, a] \rightarrow M$ emanating from N is called an N -segment if $t = d(N, \gamma(t))$ on $[0, a]$. Note that any N -segment is orthogonal to N , a consequence of the first variation formula.

Definition 2.1. For each point $x \in M \setminus N$,

$$\Lambda_N(x) := \{-\dot{\gamma}(d(N, x)) ; \gamma \text{ is an } N\text{-segment reaching } x\}.$$

Definition 2.2. For any distinct points x, y lying in a convex neighborhood around x , we define a unit tangent vector $v_x(y)$ at x by

$$v_x(y) := \dot{\gamma}(0),$$

where $\gamma : [0, b] \rightarrow M$ denotes the unique unit speed minimizing geodesic joining x to y .

Lemma 2.1. Let $\{x_n\}$ be a sequence of points in $M \setminus N$ converging to a point $x \notin N$. For each x_n , choose an element w_n in $\Lambda_N(x_n)$. If $\lim_{n \rightarrow \infty} v_x(x_n) =: v$ and $\lim_{n \rightarrow \infty} w_n =: w_\infty \in \Lambda_N(x)$ exist, then

$$\angle(v, w_\infty) = \min\{\angle(v, w); w \in \Lambda_N(x)\},$$

where $\angle(v, w_\infty)$ denotes the angle made by v and w_∞ . Moreover,

$$\lim_{n \rightarrow \infty} \frac{d(N, x_n) - d(N, x)}{d(x_n, x)} = -\cos \angle(v, w_\infty).$$

Remark. This lemma holds even when N is a point in an Alexandrov space; cf. Lemma 6.3 in [21] and Theorem 3.5 in [18].

Proof. Define N -segments α and α_n by

$$\alpha(t) := \exp((t - d(N, x))w_\infty), \quad \alpha_n(t) := \exp((t - d(N, x))w_n).$$

Fix any N -segment β reaching x and choose a point $y (\neq x)$ on β in a convex neighborhood V_x around x . Let η denote the angle made by v and $w := -\dot{\beta}(d(N, x))$. It follows from the first variation formula that there exists a constant C such that

$$d(y, x_n) - d(y, x) \leq -d(x_n, x) \cos \eta_n + C d(x_n, x)^2$$

for any sufficiently large n , where $\eta_n = \angle(v_x(x_n), w)$. By the triangle inequality,

$$d(N, x_n) - d(N, x) \leq d(y, x_n) - d(y, x)$$

for any n . Thus, we get

$$(2.1) \quad \limsup_{n \rightarrow \infty} \frac{d(N, x_n) - d(N, x)}{d(x_n, x)} \leq -\lim_{n \rightarrow \infty} \cos \eta_n = -\cos \eta.$$

On the other hand, choose a point $z (\neq x)$ on α in the neighborhood V_x . For each n , choose a point y_n lying on α_n satisfying $d(y_n, x_n) = d(x, z)$. Hence, the sequence $\{y_n\}$ converges to z . By the triangle inequality,

$$d(N, x_n) - d(N, x) \geq d(y_n, x_n) - d(y_n, x)$$

for any n . Let θ_n denote the angle made by $v_x(x_n)$ and $v_x(y_n)$. By the hypothesis, the sequence $\{\theta_n\}$ converges to $\angle(v, w_\infty)$. Since the distance function is C^∞ around (x, z) , it follows from the first variation formula that there exists a positive constant C such that

$$d(y_n, x_n) - d(y_n, x) \geq -d(x_n, x) \cos \theta_n - C d(x_n, x)^2$$

for any sufficiently large n . Thus,

$$(2.2) \quad \liminf_{n \rightarrow \infty} \frac{d(N, x_n) - d(N, x)}{d(x_n, x)} \geq - \lim_{n \rightarrow \infty} \cos \theta_n = - \cos \angle(v, w_\infty).$$

By (2.1) and (2.2), we complete the proof. \square

Definition 2.3. We define a function $\rho(v)$, $v \in U\nu$, which is called *the distance function to the cut locus* of N , by

$$\rho(v) := \sup\{t; \gamma|_{[0,t]} \text{ is an } N\text{-segment}\}.$$

The set

$$C_N := \{\gamma_v(\rho(v)); v \in U\nu, \rho(v) < \infty\}$$

is called the *cut locus* of N , and each point of C_N is called a *cut point* of N .

Note that ρ is positive on $U\nu$, since N is an embedded submanifold of M . It is well-known that ρ is continuous and $\rho \leq \lambda_1$ on $U\nu$ (for example, see [20]). Let $v : (a, b) \rightarrow (U\nu, G)$ denote a unit speed geodesic on $U\nu$, where G is a C^∞ Riemannian metric on $U\nu$, assuming that

$$\rho(s) := \rho(v(s)) < \lambda(s) := \lambda_1(v(s))$$

on (a, b) .

Definition 2.4. For each $v(s)$ define an N -Jacobi field $Y_N(t; v(s))$ along $\gamma_{v(s)}$ by

$$Y_N(t; v(s)) := \frac{\partial}{\partial s} \exp(t v(s)).$$

Actually, $Y_N(t; v(s))$ is a Jacobi field satisfying the initial conditions

$$Y_N(0; v(s)) = d\pi(\dot{v}(s)), \quad Y_N'(0; v(s)) = A_{v(s)}(d\pi(\dot{v}(s))) + (v'(s))^\perp.$$

Definition 2.5. For each $s \in (a, b)$ we define the unit tangent vectors $e_1(s)$ and $e_2(s)$ by

$$e_1(s) := -\dot{\gamma}_{v(s)}(\rho(s)), \quad e_2(s) := \frac{1}{\|Y_N(\rho(s); v(s))\|} Y_N(\rho(s); v(s)).$$

Note that $e_1(s)$ and $e_2(s)$ are mutually orthogonal according to (1.2). Since we assumed $\rho < \lambda$ on (a, b) , the continuous curve $c(s) := \exp(\rho(s)v(s))$ lies in an immersed surface

$$S := \{\exp(t v(s)); s \in (a, b), 0 < t < \lambda(s)\}$$

of M . It is clear that $\{e_1(s), e_2(s)\}$ is an orthonormal basis for the tangent space $T_{c(s)}S$ for each $s \in (a, b)$. For each $w \in \Lambda_N(c(s)) \setminus \{e_1(s)\}$, let $H(w)$ denote the

hypersurface of $T_{c(s)}M$ orthogonal to $w - e_1(s)$. The dimension of the linear space $T_{c(s)}S \cap H(w)$ is 1, since $e_1(s)$ is tangent to S , but not to $H(w)$. Therefore, for each $w \in \Lambda_N(c(s)) \setminus \{e_1(s)\}$ there exists a unique unit tangent vector $\eta_+(w)$ (respectively $\eta_-(w)$) in $T_{c(s)}S \cap H(w)$ such that the angle made by $\eta_+(w)$ (respectively $\eta_-(w)$) and $e_2(s)$ is smaller (respectively greater) than $\frac{\pi}{2}$.

Definition 2.6. For each $s \in (a, b)$, let $\xi_+(s)$ (respectively $\xi_-(s)$) denote the unique element $\eta_+(w_+(s))$ (resp. $\eta_-(w_-(s))$) in $\{\eta_+(w); w \in \Lambda_N(c(s)) \setminus \{e_1(s)\}\}$ (resp. $\{\eta_-(w); w \in \Lambda_N(c(s)) \setminus \{e_1(s)\}\}$) such that

$$\angle(\eta_+(w_+(s)), e_1(s)) = \min\{\angle(\eta_+(w), e_1(s)); w \in \Lambda_N(c(s)) \setminus \{e_1(s)\}\}$$

or, respectively,

$$\angle(\eta_-(w_-(s)), e_1(s)) = \min\{\angle(\eta_-(w), e_1(s)); w \in \Lambda_N(c(s)) \setminus \{e_1(s)\}\}.$$

Note that the choices of $w_{\pm}(s)$ may not be unique. Choose one $w_{\pm}(s)$ corresponding to each $s \in (a, b)$, and fix them.

Proposition 2.2. *At each $s_0 \in (a, b)$,*

$$(2.3) \quad \lim_{s \rightarrow s_0+0} v_{c(s_0)}(c(s)) = \xi_+(s_0)$$

and

$$(2.4) \quad \lim_{s \rightarrow s_0-0} v_{c(s_0)}(c(s)) = \xi_-(s_0).$$

Furthermore, the right and left derivatives $D^+\rho(s_0)$ and $D^-\rho(s_0)$ of ρ exist, and

$$(2.5) \quad D^+\rho(s_0) = -\|Y_N(\rho(s_0); v(s_0))\| \cot \theta_+(s_0)$$

and

$$(2.6) \quad D^-\rho(s_0) = -\|Y_N(\rho(s_0); v(s_0))\| \cot \theta_-(s_0),$$

where

$$\theta_+(s_0) := \angle(\xi_+(s_0), e_1(s_0)), \quad \theta_-(s_0) := \angle(\xi_-(s_0), -e_1(s_0)).$$

Proof. Only equations (2.3) and (2.5) are proven, because the other equations may be proven in the same manner. Let $\{s_i\}$ denote a monotone decreasing sequence converging to s_0 such that $\eta(s_0) := \lim_{i \rightarrow \infty} v_{c(s_0)}(c(s_i))$ exists. By applying Lemma 2.1 to the sequences $\{c(s_i)\}_i$ and $\{e_1(s_i)\}_i$, we have

$$(2.7) \quad \angle(e_1(s_0), \eta(s_0)) = \min\{\angle(w, \eta(s_0)); w \in \Lambda_N(c(s_0))\}.$$

On the other hand, there exists a unit tangent vector $w \in \Lambda_N(c(s_0)) \setminus \{e_1(s_0)\}$ that is a limit vector of a sequence $\{w_i\}$, where $w_i \in \Lambda_N(c(s_i)) \setminus \{e_1(s_i)\}$, because $c(s_0)$ is not a focal point of N . Thus it follows from Lemma 2.1 that

$$(2.8) \quad \angle(w, \eta(s_0)) = \min\{\angle(w, \eta(s_0)); w \in \Lambda_N(c(s_0))\}.$$

By (2.7) and (2.8), $\eta(s_0)$ is a unit tangent vector in $H(w) \cap T_{c(s_0)}S$ such that $\angle(\eta(s_0), e_2(s_0))$ is smaller than $\frac{\pi}{2}$, and hence equals $\xi_+(s_0)$. This implies that $\lim_{s \rightarrow s_0+0} v_{c(s_0)}(c(s))$ exists and is equal to $\xi_+(s_0)$. It follows from Lemma 2.1 that

$$(2.9) \quad \lim_{s \rightarrow s_0+0} \frac{\rho(s) - \rho(s_0)}{d(c(s), c(s_0))} = -\cos \theta_+(s_0).$$

For each $s > s_0$ sufficiently close to s_0 , choose the nearest point $\gamma_{v(s)}(a(s))$ on $\gamma_{v(s)}|_{[\rho(s_0)-\delta, \rho(s_0)+\delta]}$ to the point $c(s_0)$, where δ is a sufficiently small positive number such that $\gamma_{v(s)}[\rho(s_0) - \delta, \rho(s_0) + \delta]$ lies in a convex neighborhood around $c(s_0)$.

So we may assume that $\gamma_{v(s)}$ is orthogonal at $\gamma_{v(s)}(a(s))$ to the minimal geodesic joining $\gamma_{v(s)}(a(s))$ to $c(s_0)$ for each $s > s_0$ sufficiently close to s_0 . Thus, we have

$$(2.10) \quad \lim_{s \rightarrow s_0+0} \frac{k(s)}{d(c(s_0), c(s))} = \sin \theta_+(s_0),$$

where $k(s) := d(c(s_0), \gamma_{v(s)}(a(s)))$. Since $\lim_{s \rightarrow s_0+0} v_{c(s_0)}(\gamma_{v(s)}(a(s))) = e_2(s_0)$ is perpendicular to $e_1(s_0)$, it follows from Lemma 2.1 that

$$(2.11) \quad \lim_{s \rightarrow s_0+0} \frac{a(s) - \rho(s_0)}{k(s)} = 0.$$

By the triangle inequality, we have

$$-|a(s) - \rho(s_0)| + m(s) \leq k(s) \leq m(s),$$

which holds for each $s > s_0$ sufficiently close to s_0 , where

$$m(s) := d(c(s_0), \gamma_{v(s)}(\rho(s_0))).$$

Therefore, by (2.11), we get the equality

$$(2.12) \quad \lim_{s \rightarrow s_0+0} \frac{m(s)}{k(s)} = 1.$$

Let $\exp_{c(s_0)}^{-1}$ denote the local inverse of $\exp_{c(s_0)} := \exp|_{T_{c(s_0)}M}$ around $c(s_0)$. Since $d \exp_{c(s_0)}$ is the identity map on $T_{c(s_0)}M$ at the zero vector, it follows that

$$(2.13) \quad \lim_{s \rightarrow s_0+0} \frac{m(s)}{s - s_0} = \left\| \frac{\partial}{\partial s} \Big|_{s=s_0} \exp_{c(s_0)}^{-1} \gamma_{v(s)}(\rho(s_0)) \right\| = \|Y_N(\rho(s_0); v(s_0))\|.$$

It follows from (2.9), (2.10), (2.12) and (2.13) that we get (2.5). \square

Theorem 2.3. *For each cut point q of N which is not a focal point of N along each N -segment reaching q , the space of directions at q coincides with the cut locus of $\Lambda_N(q)$ in the sphere S_qM . Here the space of directions at q is defined to be the set of all limit unit tangent vectors at q of sequences $\{v_q(q_i)\}$ as cut points q_i of N tend to q .*

Proof. By Proposition 2.2, we have proved that any element of the space of directions at q is a cut point of $\Lambda_N(q)$. Suppose that there exists a cut point v of $\Lambda_N(q)$ that is not an element of the space of directions at q . Since v is a cut point of $\Lambda_N(q)$, we may choose two unit speed geodesics $c_i : [0, \theta] \rightarrow S_qM$, $i = 1, 2$, joining v_i to v , none of which meet $\Lambda_N(q)$, except v_i . For each positive ϵ let $\gamma_\epsilon : [0, 2\theta] \rightarrow M$ be a curve joining $\exp(\epsilon v_1)$ to $\exp(\epsilon v_2)$ such that $\gamma_\epsilon(t) = \exp(\epsilon c_1(t))$ and $\gamma_\epsilon(t) = \exp(\epsilon c_2(2\theta - t))$ for $t \in [0, \theta]$ and $t \in [\theta, 2\theta]$, respectively. By definition, for any sufficiently small positive ϵ , the curve γ_ϵ does not meet the cut locus of N . Thus, there exists a unique curve $\tilde{\gamma}_\epsilon : [0, 2\theta] \rightarrow \nu$ in the open subset $\{tv; 0 < t < \rho(v), v \in U\nu\}$ of the normal bundle that satisfies $\exp^\perp(\tilde{\gamma}_\epsilon(t)) = \gamma_\epsilon(t)$. It is clear that a family of curves $\{\tilde{\gamma}_\epsilon(t)\}_\epsilon$ is equicontinuous, since the lengths of the velocity vectors of $\tilde{\gamma}_\epsilon$ are bounded. It follows from the Ascoli-Arzelà theorem that the family has a limit curve $\tilde{\gamma}$, which is continuous, as ϵ goes to zero. Hence, $\exp^\perp(\tilde{\gamma}(t)) = q$ for any $t \in [0, 2\theta]$. If we define a continuous curve $\xi(t)$ in $U\nu$ by

$$\xi(t) := \frac{1}{\|\tilde{\gamma}(t)\|} \tilde{\gamma}(t),$$

then from the construction it follows that $\rho(\xi(t)) \geq \|\tilde{\gamma}(t)\|$. Thus $\|\tilde{\gamma}(t)\| = d(N, q)$ for any $t \in [0, 2\theta]$, since $\exp^\perp(\tilde{\gamma}(t)) = q$. Therefore we get a family of N -segments $\{\gamma_{\xi(t)}[0, d(N, q)]\}_{t \in [0, 2\theta]}$ reaching q such that

$$\dot{\gamma}_{\xi(0)}(d(N, q)) = -v_1 \quad \text{and} \quad \dot{\gamma}_{\xi(2\theta)}(d(N, q)) = -v_2.$$

This implies q is a focal point of N , which contradicts the hypothesis of the theorem. Thus, the proof is complete. \square

To prove the local Lipschitz continuity of ρ at v_0 , fix any $v_0 \in U_p\nu$ with $\rho(v_0) < \infty$. Let $B_D(v_0; \delta_0(v_0))$ denote a relatively compact convex ball in $(U\nu, G)$ centered at v_0 with radius $\delta_0(v_0)$, on which $\rho \leq \rho(v_0) + 1$.

Lemma 2.4. *There exist positive numbers $C_1(v_0)$ and $\delta_1(v_0)$ ($< \delta_0(v_0)$) such that for any $v, w \in B_D(v_0; \delta_1(v_0))$ with $\gamma_v(\rho(v)) = \gamma_w(\rho(w))$ the inequality*

$$(2.14) \quad C_1(v_0)D(v, w) < \angle(\dot{\gamma}_v(\rho(v)), \dot{\gamma}_w(\rho(w)))$$

holds.

Proof. Since $\gamma_{v_0}(t_0)$, where $t_0 = \frac{\rho(v_0)}{2}$, is not a focal point of N along γ_{v_0} , the differential of the normal exponential map has maximal rank at t_0v_0 . Thus, there exist a positive constant C_1 and a convex ball $B_D(v_0; \delta_1(v_0))$ ($\delta_1(v_0) < \delta_0(v_0)$) in $U\nu$ such that

$$(2.15) \quad C_1 D(u, w) < d(\gamma_u(t_0), \gamma_w(t_0))$$

for any $u, w \in B_D(v_0; \delta_1(v_0))$. By taking a smaller $\delta_1(v_0)$, we may assume that

$$\frac{3}{2}t_0 < \rho(v) < \frac{5}{2}t_0$$

on $B_D(v_0; \delta_1(v_0))$. Let K be the closure of the set $\{\exp(\rho(v)v); v \in B_D(v_0; \delta_1(v_0))\}$. Note that K is compact, because $\rho < \frac{5}{2}t_0$ on $B_D(v_0; \delta_1(v_0))$. Thus, there exists a constant C_2 such that

$$\max\{\|Y(t)\|; 0 \leq t \leq 2t_0\} \leq C_2$$

for any Jacobi field Y along a geodesic that emanates from K with initial conditions $Y(0) = 0, \|Y'(0)\| = 1$. Suppose that $v, w \in B_D(v_0; \delta_1(v_0))$ satisfy $\gamma_w(\rho(w)) = \gamma_v(\rho(v)) =: q$. Let $\xi : [0, \phi] \rightarrow S_qM$ denote a unit speed minimal geodesic joining $-\dot{\gamma}_v(\rho(v))$ to $-\dot{\gamma}_w(\rho(w))$, where $\phi = \angle(\dot{\gamma}_v(\rho(v)), \dot{\gamma}_w(\rho(w)))$. The curve $x(\theta) = \exp_q(t_1\xi(\theta)), \theta \in [0, \phi]$, joins $\gamma_v(t_0)$ to $\gamma_w(t_0)$, where $t_1 := \rho(w) - t_0 = \rho(v) - t_0$. By definition,

$$d(\gamma_v(t_0), \gamma_w(t_0)) \leq \int_0^\phi \|\dot{x}(\theta)\| d\theta.$$

Since $\|\dot{x}(\theta)\| \leq C_2$, we get

$$(2.16) \quad d(\gamma_v(t_0), \gamma_w(t_0)) \leq C_2\phi = C_2\angle(\dot{\gamma}_v(\rho(v)), \dot{\gamma}_w(\rho(w))).$$

From (2.15) and (2.16) we get (2.14). \square

Since the differential of the map $(\pi, \exp) : TM \rightarrow M \times M, (q, v) \rightarrow (q, \exp_q(v))$ has maximal rank at each zero vector, it has a C^∞ local inverse Φ on an open set $U_r \supset \{(\gamma_{v_0}(t), \gamma_{v_0}(t)); 0 \leq t \leq r\}$, where $r := \rho(v_0) + 1$. Choose a positive number $\delta_2(v_0)$ ($< \delta_1(v_0)$) such that, for any $v_1, v_2 \in B_D(v_0; \delta_2(v_0))$ and any $t \in [0, r]$, $(\gamma_{v_1}(t), \gamma_{v_2}(t)) \in U_r$.

Definition 2.7. For each distinct $v, \tilde{v} \in B_D(v_0; \delta_2(v_0))$ let $X(t; v, \tilde{v})$ denote the vector field along $\gamma_v|_{[0, r]}$ defined by

$$X(t; v, \tilde{v}) := \frac{1}{\psi} \Phi(\gamma_v(t), \gamma_{\tilde{v}}(t)),$$

where $\psi = D(v, \tilde{v})$.

It is trivial that there exists a positive constant $C_2(v_0)$ such that

$$(2.17) \quad \angle(\dot{\gamma}_v(\rho(v)), \dot{\gamma}_w(\rho(w))) \geq C_2(v_0)$$

for any $v \in B_D(v_0; \delta_3(v_0))$ and $w \in U\nu \setminus B_D(v_0; \delta_1(v_0))$ with $\gamma_v(\rho(v)) = \gamma_w(\rho(w))$, where $\delta_3(v_0) := \frac{\delta_2(v_0)}{2}$.

Lemma 2.5. *There exists a positive number $C_3(v_0)$ such that for any $t \in [0, r]$ and any unit speed minimizing geodesic $\xi(s)$ ($0 \leq s \leq \psi$) in $B_D(v_0; \delta_2(v_0))$*

$$(2.18) \quad \|X(t; \xi(0), \xi(\psi)) - Y_N(t; \xi(0))\| + \|X'(t; \xi(0), \xi(\psi)) - Y'_N(t; \xi(0))\| \leq C_3(v_0)\psi,$$

where $Y_N(t; \xi(0))$ is the N -Jacobi field along $\gamma_{v(0)}$ defined in Definition 2.4.

Proof. Since $(\gamma_{\xi(0)}(t), \gamma_{\xi(s)}(t)) \in U_r$ for any $t \in [0, r]$ and any $s \in [0, \psi]$, the vector field $\Phi(\gamma_{\xi(0)}(t), \gamma_{\xi(s)}(t))$ along $\gamma_{\xi(0)}|_{[0, r]}$ is well-defined for each $s \in [0, \psi]$. Let $f : [0, r] \times B_D(v_0; \delta_2(v_0)) \times B_D(v_0; \delta_2(v_0)) \rightarrow TM$ be a C^∞ map defined by

$$f(t, v_1, v_2) := \Phi(\gamma_{v_1}(t), \gamma_{v_2}(t))$$

and put $h(s) := f(t, \xi(0), \xi(s))$. Since

$$h(\psi) = h'(0)\psi + \psi^2 \int_0^1 u \int_0^1 h''(su\psi) ds du$$

and $h'(0) = Y_N(t; \xi(0))$, we get

$$X(t; \xi(0), \xi(\psi)) = Y_N(t; \xi(0)) + \psi \int_0^1 u \int_0^1 h''(su\psi) ds du.$$

Hence, the inequality (2.18) is trivial. \square

Lemma 2.6. *Let $v : (a, b) \rightarrow B_D(v_0; \delta_3(v_0))$ be a unit speed geodesic such that $\lambda(s) > \rho(s)$ on (a, b) . Then for each $s \in (a, b)$,*

$$(2.19) \quad |D^\pm \rho(s)| \leq C(J_N) \max \left(\cot \frac{C_4(v_0)}{2}, \frac{\pi^2 C_3(v_0) C_1(v_0)^{-2}}{2} \right),$$

where

$$C_4(v_0) = \min(C_2(v_0), C_1(v_0)\delta_3(v_0))$$

$$C(J_N) = \sup\{\|Y_N(t; v(s))\|, \|Y'_N(t; v(s))\|; 0 \leq t \leq r,$$

$$v(s) \text{ is a unit speed geodesic in } B_D(v_0; \delta_3(v_0))\}.$$

Proof. Let $e_3(s)$ denote the unit tangent vector satisfying

$$(2.20) \quad w_+(s) = e_1(s) \cos \phi(s) + e_3(s) \sin \phi(s),$$

where $\phi(s) := \angle(w_+(s), e_1(s))$. Since $\angle(e_1(s), \xi_+(s)) = \theta_+(s)$ and $\angle(\xi_+(s), e_2(s)) < \frac{\pi}{2}$, it follows that

$$(2.21) \quad \xi_+(s) = e_1(s) \cos \theta_+(s) + e_2(s) \sin \theta_+(s).$$

Since $\xi_+(s)$ is orthogonal to $w_+(s) - e_1(s)$, it follows from (2.20) and (2.21) that

$$(2.22) \quad \cot \theta_+(s) = \frac{\sin \phi(s)}{1 - \cos \phi(s)} g(e_2(s), e_3(s)).$$

Hence, by (2.5), we get

$$(2.23) \quad D^+ \rho(s) = -\frac{\sin \phi(s)}{1 - \cos \phi(s)} g(Y_N(\rho(s)), e_3(s)) = -\cot \frac{\phi(s)}{2} g(Y_N(\rho(s)), e_3(s)),$$

where $Y_N(t) := Y_N(t; v(s))$. Let $\tilde{v}(s) \in U\nu$ denote the vector satisfying $-w_+(s) = \hat{\gamma}_{\tilde{v}(s)}(\rho(\tilde{v}(s)))$. If $\phi(s)$ is not less than $C_4(v_0)$, then from (2.23) it is trivial that

$$(2.24) \quad |D^+ \rho(s)| \leq C(J_N) \cot \frac{C_4(v_0)}{2}.$$

If $\phi(s)$ is less than $C_4(v_0)$, then it follows from (2.14) and (2.17) that $D(v(s), \tilde{v}(s)) < \delta_3(v_0)$. Thus, by the triangle inequality, $\tilde{v}(s) \in B_D(v_0; \delta_2(v_0))$. The vector field $X(t) := X(t; v(s), \tilde{v}(s))$ is well-defined by Definition 2.7. Since $X(\rho(s)) = 0$, we get

$$(2.25) \quad X'(\rho(s)) = \frac{1}{\psi(s)} (e_1(s) - w_+(s)) = \frac{1}{\psi(s)} ((1 - \cos \phi(s))e_1(s) - e_3(s) \sin \phi(s)),$$

where $\psi(s) := D(v(s), \tilde{v}(s))$. Let $\xi : [0, \psi(s)] \rightarrow B_D(v_0; \delta_2(v_0))$ denote the unit speed minimal geodesic joining $v(s)$ to $\tilde{v}(s)$. It follows from (2.23) and (2.25) that

$$(2.26) \quad D^+ \rho(s) = \cot \frac{\phi(s)}{2} \frac{\psi(s)}{\sin \phi(s)} g(Y_N(\rho(s)), X'(\rho(s))).$$

It follows from (1.3) that

$$g(Y_N(\rho(s)), X'(\rho(s))) = g(Y_N(\rho(s)), X'(\rho(s)) - X'_N(\rho(s))) \\ + g(Y'_N(\rho(s)), X_N(\rho(s))),$$

where $X_N(t) := Y_N(t; \xi(0))$. Hence, by (2.26), we have

$$(2.27) \quad |D^+ \rho(s)| \leq \cot \frac{\phi(s)}{2} \frac{\psi(s)}{\sin \phi(s)} C(J_N) (\|X'(\rho(s)) - X'_N(\rho(s))\| + \|X_N(\rho(s))\|).$$

Since $X(\rho(s)) = 0$, by (2.14), (2.18) and (2.27), we get

$$(2.28) \quad |D^+ \rho(s)| \leq \cot \frac{\phi(s)}{2} \frac{\psi(s)^2}{\sin \phi(s)} C(J_N) C_3(v_0) \leq \frac{\pi^2}{2} C_1(v_0)^{-2} C_3(v_0) C(J_N).$$

By (2.24) and (2.28), we get (2.19). The estimate for $D^- \rho(s)$ is the same as the one for $D^+ \rho(s)$. \square

Proof of Theorem B. Let $v_0 \in U\nu$ be any vector with $\rho(v_0) < \infty$. Choose a small convex ball $B_D(v_0; \delta_4(v_0))$, $\delta_4(v_0) < \delta_3(v_0)$, on which $\rho < \lambda_1$ or λ_1 is Lipschitz continuous with Lipschitz constant $L(\lambda_1)$. Let $v_1, v_2 \in B_D(v_0; \delta_4(v_0))$ be any distinct vectors with $\rho(v_1) \leq \rho(v_2)$. Let $\xi : [0, \psi] \rightarrow B_D(v_0; \delta_4(v_0))$ be the unit speed geodesic joining v_1 to v_2 , so that $\psi = D(v_1, v_2)$. If $\lambda_1(v_1) = \rho(v_1)$, then

$$(2.29) \quad |\rho(v_1) - \rho(v_2)| = \rho(v_2) - \rho(v_1) \leq \lambda_1(v_2) - \lambda_1(v_1) \leq L(\lambda_1) D(v_1, v_2).$$

Suppose that $\lambda_1(v_1) > \rho(v_1)$. Let $(0, a)$ be the maximal open subinterval of $[0, \psi]$ on which $\lambda_1 > \rho$. By Lemma 2.5,

$$|D^\pm \rho(s)| \leq C_5(J_N, v_0)$$

on $(0, a)$, where

$$C_5(J_N, v_0) := C(J_N) \max \left(\cot \frac{C_4(v_0)}{2}, \frac{\pi^2 C_3(v_0) C_1(v_0)^{-2}}{2} \right).$$

Hence, $\rho \circ \xi$ is Lipschitz continuous with Lipschitz constant $C_5(J_N, v_0)$ on $[0, a]$. In particular,

$$(2.30) \quad |\rho(v_1) - \rho(\xi(a))| \leq C_5(J_N, v_0)a.$$

If $a < \psi$, then $\lambda_1(\xi(a)) = \rho(\xi(a))$. Thus by (2.30), we get

$$(2.31) \quad |\rho(v_1) - \rho(v_2)| \leq \lambda_1(v_2) - \lambda_1(\xi(a)) + |\rho(\xi(a)) - \rho(v_1)| \leq L(\rho)D(v_1, v_2),$$

where $L(\rho) := \max(L(\lambda_1), C_5(J_N, v_0))$. If $a = \psi$, then (2.31) is trivial by (2.30). Therefore, by (2.29) and (2.31),

$$|\rho(v_1) - \rho(v_2)| \leq L(\rho)D(v_1, v_2)$$

for any $v_1, v_2 \in B_D(v_0; \delta_4(v_0))$. □

The length $L(c)$ of a continuous curve $c : [a, b] \rightarrow M$ is defined as

$$L(c) := \sup \sum_{i=1}^k d(c(t_{i-1}), c(t_i)),$$

where the supremum is taken over all subdivisions

$$a = t_0 < t_1 < \cdots < t_k = b$$

of $[a, b]$. Note that any absolutely continuous curve has finite length (cf. [28] for the definition of an absolutely continuous curve). We omit the proof of the following lemma, since it is standard (cf. [28]).

Lemma 2.7. *For any absolutely continuous curve $c : [a, b] \rightarrow M$,*

$$L(c) = \int_a^b \|\dot{c}(t)\| dt.$$

We introduce an interior metric δ on a component C_N^0 of C_N by

$$\delta(p, q) := \inf \{L(c); c \text{ is a continuous curve on } C_N^0 \text{ joining } p \text{ to } q\}.$$

By Theorem B, $\delta(p, q)$ is finite for any $p, q \in C_N^0$. Any two points $p, q \in C_N^0$ can be connected by a minimal curve c ; that is, there exists a continuous curve c joining p to q such that $\delta(p, q) = L(c)$ (for example, cf. Theorem 5.18 in [3]). It follows from Lemma 2.7 that δ coincides with the usual definition of the Riemannian distance function, or, in other words,

$$\delta(p, q) = \inf \left\{ \int_0^1 \|\dot{c}(t)\| dt; c \text{ is an absolutely continuous curve on } C_N^0 \text{ joining } p \text{ to } q \right\}.$$

Proof of Corollary C. Let $\{p_n\}$ be a sequence of points in C_N^0 such that

$$\lim_{n \rightarrow \infty} d(p, p_n) = 0.$$

Since the cut locus is closed, p is a cut point of N . For each p_n choose a vector $v_n \in U\nu$ with $\exp(\rho(v_n)v_n) = p_n$. Let $v \in U\nu$ be a limit vector of the sequence $\{v_n\}$. Let $\xi_n : [0, D(v, v_n)] \rightarrow U\nu$ be a minimizing geodesic joining v to v_n , and put $\bar{\xi}_n(t) := \exp(\rho(\xi_n(t))\xi_n(t))$. Since $\bar{\xi}_n$ is a (Lipschitz) continuous curve in C_N^0 joining p to p_n , we get

$$(2.32) \quad \delta(p, p_n) \leq L(\bar{\xi}_n).$$

Since ρ is locally Lipschitz, the map $w \in U\nu \rightarrow \exp(\rho(w)w) \in M$ is also locally Lipschitz. Thus, there exist a positive constant C and a neighborhood V around v such that

$$(2.33) \quad L(\bar{\xi}_n) \leq CL(\xi_n) = CD(v, v_n)$$

for any n with $v_n \in V$. By (2.35) and (2.36), we get $\lim_{n \rightarrow \infty} \delta(p, p_n) = 0$. Thus, the topology introduced from δ coincides with the relative topology of (M, g) . The other claims are clear from this property. \square

3. OPEN PROBLEMS AND EXAMPLES

The functions λ_k are not always differentiable, except when M is of dimension 2. The following example shows that λ_1 need not be differentiable.

Example 3.1. Let M denote the Riemannian product of two 2-dimensional unit spheres S^2 . Choose a unit tangent vector v_1 to S^2 at a point p_1 . For each $\theta \in [0, \frac{\pi}{2}]$, we define a geodesic γ_θ on M by

$$\gamma_\theta(t) := (\exp(tv_1 \cos \theta), \exp(tv_1 \sin \theta)).$$

Let λ_1 denote the distance function to the first conjugate tangent vectors of the point $p := (p_1, p_1) \in M$. Thus

$$\lambda_1(\dot{\gamma}_\theta(0)) = \min\left(\frac{\pi}{\cos \theta}, \frac{\pi}{\sin \theta}\right).$$

Hence $\lambda_1(\dot{\gamma}_\theta(0))$ is not differentiable at $\theta = \frac{\pi}{4}$, that is, λ_1 is not differentiable at $(v_1/\sqrt{2}, v_1/\sqrt{2})$.

There exist many surfaces admitting a cut locus with branch points (for example cf. [7] or the following example). This implies such a cut locus need not have curvature bounded below in the sense of Alexandrov.

Example 3.2. Let N be a smooth convex Jordan curve in the 2-dimensional Euclidean plane \mathbf{R}^2 which contains a regular triangle T , except around its three vertices. Then the cut locus of N contains three line segments emanating from the center of T .

The following example shows that there is a cut locus containing a neighborhood of the vertex of a flat cone. This implies this cut locus cannot have curvature bounded above in the sense of Alexandrov.

Example 3.3. Take a C^∞ Jordan arc \mathcal{C} in the yz plane in the 3-dimensional Euclidean space \mathbf{R}^3 with endpoints $(0, 0, \pm 1)$ as follows:

(1) \mathcal{C} contains three arcs

$$C_1 := \{(0, \cos \theta, \sin \theta); -\frac{\pi}{2} \leq \theta \leq -\frac{\pi}{2} + \delta\},$$

$$C_2 := \{(0, \cos \theta, \sin \theta); -\frac{\pi}{4} + \delta \leq \theta \leq \frac{\pi}{2}\}$$

and

$$C_3 := \{(0, \frac{1}{\sqrt{2}} + \frac{\delta}{10} \cos \phi, -\frac{1}{\sqrt{2}} + \frac{\delta}{10} \sin \phi); -\frac{\pi}{4} - \delta \leq \phi \leq -\frac{\pi}{4} + \delta\},$$

where δ is a sufficiently small positive constant.

(2) $\mathcal{C} \setminus (C_1 \cup C_2 \cup C_3)$ consists of two Jordan subarcs which are mutually symmetric with respect to the line through $(0, 0, 0)$ and $(0, 1, -1)$.

(3) The cut locus of $\mathcal{C} \setminus \{(0, 0, \pm 1)\}$ in the yz plane is the line segment with endpoints $(0, 0, 0)$ and $(0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.

Let N be the surface of revolution obtained by rotating \mathcal{C} about the z axis. Then the cut locus of N coincides with a cone

$$\{(x, y, z); x^2 + y^2 = z^2, -\frac{1}{\sqrt{2}} \leq z \leq 0\}.$$

The cut loci constructed in Examples 3.2 and 3.3 are those of a submanifold that is not a single point. By making use of Weinstein's technique ([26]), we may regard these cut loci as being of a single point.

Finally, we state five interesting open problems, some of which might be proved using the local Lipschitz continuity of the function ρ .

J. Hebda and J. Itoh affirmatively solved Ambrose's problem in the 2-dimensional case (cf. [11], [13]). They solved it by proving that the cut locus of a point on a 2-dimensional Riemannian manifold has finite 1-dimensional Hausdorff measure. Hebda had pointed out in [10] that it is sufficient to prove the property above to solve the problem in the 2-dimensional case. Theorem B generalizes this property for any dimensional compact Riemannian manifolds. Thus we might be able to solve Ambrose's problem using this property.

Problem 3.1. *Solve Ambrose's problem for any dimensional Riemannian manifold.*

The authors proved in [14] that for each cut point q of a point p on M , there exists a nonnegative integer k such that the cut locus of p is locally k -dimensional around q . We call the integer k the *local dimension* of the cut locus at q .

Problem 3.2. *Let q denote a cut point of a point p on M at which the local dimension of the cut locus is k . Is the cut locus locally a k -dimensional submanifold of M around q , except for a k -null subset of M ? Here a subset of M is said to be k -null if it is of k -dimensional Hausdorff measure zero.*

Hereafter N denotes an embedded submanifold of a complete Riemannian manifold M . A point $q \in M \setminus N$ is called a *critical point* of the distance function from N if for each unit tangent vector v at q there exists a unit tangent vector w in $\Lambda_N(q)$ such that the angle made by v and w is not greater than $\frac{\pi}{2}$. A real number c is called a *critical value* of the distance function from N if there exists a critical point

q whose distance is c from N . It is well-known that for each positive number c the set of all points whose distances are c is a topological hypersurface in M , if c is not a critical value of the distance function (cf. [5]). In [22], it was proved that the set of all critical values of the distance function from a compact subset in an Alexandrov surface is of Lebesgue measure zero. Does what we call a ‘‘Sard Theorem for the distance function’’ hold for the distance function from N ? Namely,

Problem 3.3. *Is the set of all critical values of the distance function from N of Lebesgue measure zero?*

We showed in Examples 3.2 and 3.3 that the cut locus is not always an Alexandrov space. How about the tangent cut locus?

Problem 3.4. *Is the tangent cut locus of N an Alexandrov space?*

We proved in Theorem 2.3 that the space of directions at a non-focal cut point q of N coincides with the cut locus of $\Lambda_N(q)$ in S_qM . Here a *non-focal* cut point q is a cut point that is not a focal point along each N -segment reaching q . Therefore, the following problem is an interesting investigation into the structure of a cut locus.

Problem 3.5. *Let q be a non-focal cut point of N . Then, is $S(q; \delta) \cap C_N$ homeomorphic to the cut locus of $\Lambda_N(q)$ in S_qM for any sufficiently small positive δ ? Here $S(q; \delta)$ denotes a geodesic sphere in M centered at q with radius δ .*

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