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THE LOCAL ASYMPTOTIC NORMALITY OF A FAMILY OF MEASURES GENERATED BY SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS WITH A SMALL FRACTIONAL BROWNIAN MOTION

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Abstract. A formula for the likelihood ratio of measures generated by solutions of a stochastic differential equation with a fractional Brownian motion is established in the paper. We find sufficient conditions that the family of measures generated by solutions of such an equation is locally asymptotically normal.

Introduction

We consider the stochastic differential equation

$$X_t = x_0 + \int_0^t S(\theta, u, X_u)\,du + \varepsilon B_t, \qquad t \in [0,T], \tag{1}$$

where $x_0 \in \mathbb{R}$, $\varepsilon \in (0,1)$; $S(\theta,t,x)\colon \mathbb{R}^d \times [0,T] \times \mathbb{R} \to \mathbb{R}$ is a nonrandom function of drift; $\theta \in \Theta \subset \mathbb{R}^d$ is an unknown parameter of the system; $B_t = B_t^H$ is a fractional Brownian motion with the Hurst parameter $H \in (\frac{1}{2}, 1)$.

Along with equation (1) we consider the deterministic equation

$$x_t = x_0 + \int_0^t S(\theta, u, x_u)\,du, \qquad t \in [0,T], \tag{2}$$

whose solution is $x = x(\theta)$.

Equation (1) describes the evolution of a dynamic system with a small noise being a fractional Brownian motion. The problem of the statistical estimation is well studied for systems with a small noise being a standard Brownian process (see [1]). In particular, the consistency and asymptotic normality of the maximum likelihood estimator of the parameter θ is proved under certain assumptions for systems with Brownian noise. As shown in the monograph [2, Chapter II], several important properties of statistical estimators follow from the local asymptotic normality of a system of measures generated by the random element $X_\theta^{(\varepsilon)}$. Thus the proof of the local asymptotic normality is a necessary step to obtain results similar to the Kutoyants results [1] in the case of a fractional Brownian motion. In this paper, we obtain some conditions under which the family of probability measures $\{P_\theta^{(\varepsilon)}, \theta \in \Theta\}$ generated by solutions of equation (1) that correspond to different parameters θ in the measurable space $\big(C[0,T], \mathcal{B}_T\big)$ is locally asymptotically normal as $\varepsilon \to 0$.

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1. Notation, definitions, and conditions for the existence and uniqueness of a solution

For $\lambda \in (0,1]$, denote by $\{C^\lambda[0,T], \|\cdot\|_{C^\lambda}\}$ the space of Hölder functions $f\colon [0,T] \to \mathbb{R}$. The parameter λ determines the norm $\|f\|_{C^\lambda}$ defined by

$$\|f\|_{C^\lambda} = \max_{x\in[0,T]} |f(x)| + \sup_{\substack{x_1,x_2\in[0,T]\\ x_1\neq x_2}} \frac{|f(x_1)-f(x_2)|}{|x_1-x_2|^\lambda}.$$

Set $C^{\mu-}[0,T] = \bigcap_{\lambda<\mu} C^\lambda[0,T]$. In what follows we use the same symbol C for all constants whose precise value is not important for our consideration.

Definition 1. A continuous Gaussian process with stationary increments and such that

(1) $B_0 = 0$;
(2) $\mathsf{E}\, B_t = 0$ for all $t \geq 0$;
(3) $\mathsf{E}\, B_s B_t = \frac{1}{2}\big(s^{2H} + t^{2H} - |t-s|^{2H}\big)$ for all $s,t \geq 0$

is called a fractional Brownian motion $B = B^H$ with Hurst parameter $H \in (0,1)$.

The trajectories of the process $B^H = \big(B^H_t, t\in[0,T]\big)$ belong with probability one to the space $C^{H-}[0,T]$. Since the process B^H is not a semimartingale for $H \neq \frac{1}{2}$, one can define the integral $\int_0^T f(t)\,dB_t$ as the limit of integral sums neither in probability nor in the mean square sense. The integral $\int_0^T f(t)\,dB_t$ is constructed pathwise in [3] with the help of fractional integro-differential calculus. It is shown in [3] that this integral exists with probability one and coincides with the Stieltjes–Riemann type integral for

$$f(\omega) \in \bigcup_{\lambda>1-H} C^\lambda[0,T].$$

Set

$$C_0 = \frac{1}{2}\big((H-1/2)\cdot H\cdot(1-H)\cdot B(3/2-H,3/2-H)\cdot B(H-1/2,3/2-H)\big)^{-1/2},$$

$$C_1 = B(3/2-H,3/2-H)\cdot C_0, \tag{3}$$

where

$$B(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1}\,du$$

is the Euler beta function. We also set

$$z(t,u) = C_0 u^{1/2-H}(t-u)^{1/2-H}, \qquad w(t,u) = C_0 u^{3/2-H}(t-u)^{1/2-H}. \tag{4}$$

It is shown in [4] that a Wiener process can be constructed from a fractional Brownian motion and vice versa. The construction uses two steps. First, it is proved that

$$M_t := \int_0^t z(t,u)\,dB_u \tag{5}$$

is well defined as a pathwise integral with respect to the flow of σ-algebras

$$(\mathcal{F}_t) = (\mathcal{F}\{B_u, u\leq t\})$$

with the quadratic characteristics

$$[M]_t = \frac{t^{2-2H}}{2-2H}.$$

Then

$$W_t := \int_0^t u^{H-1/2}\,dM_u \tag{6}$$

is a Wiener process with respect to the same flow $(\mathcal{F}_t)$.

The following assertion is a special case of the Nualart and Răscanu [5] result containing sufficient conditions for the existence and uniqueness of a solution of a system of stochastic differential equations with a fractional Brownian motion.

Proposition 1. *Let a Borel function $S\colon [0,T]\times\mathbb{R}\to\mathbb{R}$ be such that*

$$\begin{aligned}&\text{for all } N\ge 0 \quad \text{there exists } L_N>0 \quad \text{such that}\\ &|S(t,x)-S(t,y)|\le L_N\,|x-y| \quad \text{for all } |x|,\ |y|\le N \text{ and all } t\in[0,T];\end{aligned} \tag{7}$$

$$\text{there exists } M>0 \text{ such that} \quad |S(t,x)|\le M(1+|x|) \quad \text{for all } x\in\mathbb{R} \text{ and } t\in[0,T]. \tag{8}$$

Then the equation

$$X_t = x_0 + \int_0^t S(u,X_u)\,du + \varepsilon B_t, \qquad t\in[0,T],$$

has a unique solution X; this solution belongs with probability one to the class $C^{H-}[0,T]$.

As in the case of stochastic differential equations with a standard Wiener process, the following result holds for the stochastic differential equations with a fractional Brownian motion.

Proposition 2. *Let $\theta\in\Theta$ be fixed and let a Borel function $S(t,x)=S(\theta,t,x)$ satisfy conditions* (7) *and* (8)*. Moreover, assume that L_N in condition* (7) *does not depend on N, that is, $L_N=L$ for some L and all N. If X_t and x_t are solutions of equations* (1) *and* (2)*, respectively, then*

$$\sup_{t\in[0,T]}|X_t-x_t|\le \varepsilon C\sup_{t\in[0,T]}|B_t|\,, \tag{9}$$

where $C=\exp\{LT\}$.

Proof. This is a corollary of the Gronwall lemma. □

Following [2] we use the notion of the local asymptotic normality of a system of measures. Let $\{\mathcal{X}^{(\varepsilon)},U^{(\varepsilon)}\}$ be a family of measurable spaces and let $\Theta\subset\mathbb{R}^d$ be an open set. Let $\mathcal{E}_\varepsilon=\left\{\mathcal{X}^{(\varepsilon)},U^{(\varepsilon)},P_\theta^{(\varepsilon)},\theta\in\Theta\right\}$ be a collection of statistical experiments and $X^{(\varepsilon)}$ be the corresponding observation. The derivative

$$\frac{dP_{\theta_2}^{(\varepsilon)}}{dP_{\theta_1}^{(\varepsilon)}}\left(X^{(\varepsilon)}\right)$$

of the absolutely continuous component of the measure $P_{\theta_2}^{(\varepsilon)}$ with respect to the measure $P_{\theta_1}^{(\varepsilon)}$ at the observation $X^{(\varepsilon)}$ is called the likelihood ratio.

Definition 2. A family $\{P_\theta^{(\varepsilon)},\theta\in\Theta\}$ is called *locally asymptotically normal* at a point $\theta_0\in\Theta$ as $\varepsilon\to 0$ if

$$Z_{\varepsilon,\theta_0}(u)=\frac{dP_{\theta_0+\phi_\varepsilon u}^{(\varepsilon)}}{dP_{\theta_0}^{(\varepsilon)}}\left(X^{(\varepsilon)}\right)=\exp\left\{u^\top\Delta_{\varepsilon,\theta_0}-\frac12|u|^2+\psi_\varepsilon(u,\theta_0)\right\}$$

and $\mathcal{L}\big(\Delta_{\varepsilon,\theta_0}\mid P_{\theta_0}^{(\varepsilon)}\big)\to\mathcal{N}(0,J)$ as $\varepsilon\to 0$ for all $u\in\mathbb{R}^d$ and some nonsingular $d\times d$ matrix $\phi_\varepsilon=\phi_\varepsilon(\theta_0)$, where J is the unit $d\times d$ matrix and ψ is such that

$$\psi_\varepsilon(u,\theta_0)\to 0 \quad \text{in probability } P_{\theta_0}^{(\varepsilon)} \quad \text{as } \varepsilon\to 0$$

for all $u\in\mathbb{R}^d$.

2. The absolute continuity of measures

Consider two equations

$$X_t = x_0 + \int_0^t S_i(u, X_u)\,du + \varepsilon B_t, \qquad t \in [0,T],\ i = 1,2. \tag{10}$$

Let X^i be a solution of the equation involving S_i and let $P_{X^i}(dx)$ be the measure on $(C[0,T], \mathcal{B}_T)$ generated by the solution X^i, $i = 1,2$. The likelihood ratio $\frac{dP_{X^2}}{dP_{X^1}}\left(X^1\right)$ is established in the following theorem.

Theorem 1. *Let the functions $S_1, S_2\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:*

(1) *$S_i \in C^1([0,T] \times \mathbb{R})$, $i = 1,2$;*
(2) *there exists a constant $M > 0$ such that $|S_i(t,x)| \le M(1 + |x|)$ for all $x \in \mathbb{R}$ and $t \in [0,T]$, $i = 1,2$.*

Then each of equations (10) *has a solution. Moreover, the solution of each equation is unique and belongs almost surely to the class $C^{H-}[0,T]$. In addition, $P_{X^1} \sim P_{X^2}$ and*

$$\frac{dP_{X^2}}{dP_{X^1}}\left(X^1\right) = \exp\left\{\frac{1}{\varepsilon} L_T - \frac{1}{2\varepsilon^2}\langle L\rangle_T\right\}, \tag{11}$$

where

$$\begin{aligned} L_T &= \int_0^T \Bigg[(2-2H)t^{1/2-H} \\ &\qquad \times \left(C_1 \Delta S(0,x_0) + \int_0^t u^{2H-3} \int_0^u w(u,v)\, d\left(\Delta S\left(v, X_v^1\right)\right) du\right) \\ &\qquad\qquad + t^{H-3/2} \int_0^t w(t,u)\, d\left(\Delta S\left(u, X_u^1\right)\right)\Bigg] dW_t \\ &= \int_0^T \Bigg[(2-2H)t^{1/2-H}\left(C_1 \Delta S(0,x_0) + \int_0^t u^{2H-3} R_1(u)\,du\right) \\ &\qquad\qquad + t^{H-3/2} R_1(t)\Bigg] dW_t \end{aligned} \tag{12}$$

and

$$\begin{aligned} R_1(t) = \int_0^t w(t,v) \Bigg\{ &\left(\frac{\partial}{\partial v} \Delta S\left(v, X_v^1\right) + \frac{\partial}{\partial x} \Delta S\left(v, X_v^1\right) S\left(\theta_1, v, X_v^1\right)\right) dv \\ &+ \varepsilon \frac{\partial}{\partial x} \Delta S\left(v, X_v^1\right) dB_v \Bigg\}. \end{aligned}$$

Here we set $\Delta S(t,x) = S_2(t,x) - S_1(t,x)$, the constant C_1 is defined in (3)*, the function $w(t,u)$ is given by* (4)*, and the Wiener process W_t is constructed from B_t in the way described in Section* 1.

Proof. Note that conditions (7) and (8) hold if conditions 1) and 2) are satisfied. Thus, according to Proposition 1, a solution of each of equations (10) exists, is unique, and belongs to the class $C^{H-}[0,T]$ if 1) and 2) are satisfied.

For $X = X^i$, consider the stochastic process

$$\tilde{X}_t := \int_0^t z(t,u)\, dX_u = \int_0^t z(t,u) S(u, X_u)\, du + \varepsilon \int_0^t z(t,u)\, dB_u, \tag{13}$$

where the function $z(t,u)$ is defined by (4). The process $\tilde{X}_t$ is well defined for all $t \in [0,T]$, since both terms on the right-hand side of (13) are well defined. Now we prove that the function

$$I(t)=\int_0^t z(t,u)S(u,X_u)\,du = C_1S(0,x_0)t^{2-2H}+\int_0^t z(t,u)\int_0^u dS(v,X_v)\,du$$

is differentiable, where the constant C_1 is defined in (3).

Fix $\lambda\in\left(\frac12,H\right)$. Since condition 1) holds, we have

$$s(v):=S(v,X_v)\in C^{H-}[0,T]\subset C^{\lambda}[0,T]\quad \mathsf{P}\text{-a.s.}$$

Similarly to [6] one can obtain the following result.

Lemma 1. *Let $H,\beta,\lambda\in\left(\frac12,1\right)$ and let $f,s\colon\mathbb{R}\to\mathbb{R}$ be Hölder functions with exponents β and λ, respectively. Then the function*

$$J(t)=\int_0^t u^{1/2-H}(t-u)^{1/2-H}\left(\int_0^u f(v)\,ds(v)\right)du$$

is represented as follows:

$$J(t)=t^{2-2H}\int_0^t \delta_u\,du,$$

where

$$\delta_t=t^{2H-3}\int_0^t u^{3/2-H}(t-u)^{1/2-H}f(u)\,ds(u)$$

belongs to the class $L_1(0,T)$, that is,

$$\int_0^T |\delta_u|\;du<\infty.$$

We follow the method of [6] to prove Lemma 1. In doing so, we obtain the estimate

$$\left|\int_0^t u^{3/2-H}(t-u)^{1/2-H}f(u)\,ds(u)\right|\le K(f,s)Ht^{2-H}, \tag{14}$$

where $K(f,s)=C_{T,\beta,\lambda}\left\|f\right\|_{C^\beta}\left\|s\right\|_{C^\lambda}$. The latter estimate implies Lemma 1 (see [6]).

Applying Lemma 1 to $I(t)$, we get

$$I(t)=t^{2-2H}\left(C_1S(0,x_0)+\int_0^t \alpha(u)\,du\right),$$

where

$$\alpha(u)=u^{2H-3}\int_0^u w(u,v)\,dS(v,X_v)$$

and $w(t,u)=C_0u^{3/2-H}(t-u)^{1/2-H}$.

Turning to the proof of equality (13) we use notation (5) and write

$$\tilde X_t=\int_0^t \gamma(u)\,du+\varepsilon M_t,$$

where

$$\gamma(u)=(2-2H)u^{1-2H}\left(C_1S(0,x_0)+\int_0^u \alpha(v)\,dv\right)+u^{2-2H}\alpha(u).$$

Consider the process

$$\tilde{\tilde X}_t:=\frac1\varepsilon\int_0^t u^{H-1/2}\,d\tilde X_u=\frac1\varepsilon\int_0^t u^{H-1/2}\gamma(u)\,du+\int_0^t u^{H-1/2}\,dM_u.$$

Relation (6) implies that $\tilde{\tilde X}$ is an Itô process with the differential

$$d\tilde{\tilde X}_t=\delta(t,X)\,dt+dW_t, \tag{15}$$

where

$$\delta(t,X)=\frac{1}{\varepsilon}\Bigg[(2-2H)t^{1/2-H}\left(C_1S(0,x_0)+\int_0^t u^{2H-3}\int_0^u w(u,v)\,dS(v,X_v)\,du\right) + t^{H-3/2}\int_0^t w(t,u)\,dS(u,X_u)\Bigg]. \tag{16}$$

Note that the mapping

$$\mathsf{A}\colon C^{H-}[0,T]\ni X\to\tilde{\tilde{X}}\in C^{1/2-}[0,T],$$

defined as superposition of the mappings

$$X\to\tilde{X}_t=\int_0^t z(t,u)\,dX_u,\qquad t\in[0,T],$$
$$\tilde{X}\to\tilde{\tilde{X}}_t=\frac{1}{\varepsilon}\int_0^t u^{H-1/2}\,d\tilde{X}_u,\qquad t\in[0,T],$$

has the inverse (see [4]). The inverse mapping A^{-1} is given by

$$\left(\mathsf{A}^{-1}\tilde{\tilde{X}}\right)_t=\varepsilon\int_0^t\psi(t,u)\,d\tilde{\tilde{X}}_u,$$

where

$$\psi(t,u)=C_2u^{1/2-H}\int_u^t v^{H-1/2}(v-u)^{H-3/2}\,dv,$$
$$C_2=(H\cdot(2H-1))^{1/2}B(H-1/2,2-2H)^{-1/2}.$$

Substituting $X=\mathsf{A}^{-1}\tilde{\tilde{X}}$ in (15) we obtain

$$d\tilde{\tilde{X}}_t=\delta_t\left(\mathsf{A}^{-1}\tilde{\tilde{X}}\right)dt+dW_t.$$

Since $\delta_t(\mathsf{A}^{-1}\,\boldsymbol{\cdot})$ is a nonanticipating functional, we conclude that $\tilde{\tilde{X}}$ is a diffusion type process. According to Theorem 7.7 in [7], the measures $P_{\tilde{\tilde{X}}}$ and P_W are equivalent if and only if

$$\mathsf{P}\left\{\int_0^T\delta(t,\mathsf{A}^{-1}\tilde{\tilde{X}})^2\,dt<\infty\right\}=\mathsf{P}\left\{\int_0^T\delta(t,X)^2\,dt<\infty\right\}=1, \tag{17}$$

$$\mathsf{P}\left\{\int_0^T\delta(t,\mathsf{A}^{-1}W)^2\,dt<\infty\right\}=\mathsf{P}\left\{\int_0^T\delta(t,B)^2\,dt<\infty\right\}=1. \tag{18}$$

The ratios $\frac{dP_{\tilde{\tilde{X}}}}{dP_W}(W)$ and $\frac{dP_W}{dP_{\tilde{\tilde{X}}}}(\tilde{\tilde{X}})$ are given by

$$\frac{dP_{\tilde{\tilde{X}}}}{dP_W}(W)=\exp\left\{\int_0^T\delta(t,B)\,dW_t-\frac{1}{2}\int_0^T\delta(t,B)^2\,dt\right\}, \tag{19}$$

$$\frac{dP_W}{dP_{\tilde{\tilde{X}}}}(\tilde{\tilde{X}})=\exp\left\{-\int_0^T\delta(t,X)\,dW_t+\frac{1}{2}\int_0^T\delta(t,X)^2\,dt\right\}, \tag{20}$$

respectively.

Now we prove that equality (17) holds. Indeed,

$$\begin{aligned}\int_0^T \delta(t,X)^2\,dt &\le C\biggl[S(0,x_0)^2+\int_0^T t^{1-2H}\biggl(\int_0^t u^{2H-3}\int_0^u w(u,v)\,dS(v,X_v)\,du\biggr)^2 dt\\ &\qquad+\int_0^T t^{2H-3}\biggl(\int_0^t w(t,u)\,dS(u,X_u)\biggr)^2 dt\biggr]\\ &=C[I_1+I_2+I_3].\end{aligned}$$

Using estimate (14), we get

$$\begin{aligned}\int_0^t u^{2H-3}\int_0^u w(u,v)\,dS(v,X_v)\,du &\le C_0\;K\bigl(1,S(\cdot,X.)\bigr)t^H \qquad \mathsf{P}\text{-a.s.},\\ \int_0^t w(t,u)\,dS(u,X_u) &\le C_0K\bigl(1,S(\cdot,X.)\bigr)Ht^{2-H} \qquad \mathsf{P}\text{-a.s.}\end{aligned}$$

It follows from the above inequalities that $I_2<\infty$ and $I_3<\infty$. Equality (18) can be proved in a similar way.

Now we come back to solutions X^1 and X^2 of equations (10). Let $\delta_i(t,X)$ be defined by equality (16) for $S=S_i$, $i=1,2$. Write equalities (19) and (20) for $\tilde{\tilde{X}}^1$ and $\tilde{\tilde{X}}^2$ instead of $\tilde{\tilde{X}}$ and then use them to get

$$\begin{aligned}&\frac{dP_{\tilde{\tilde{X}}^2}}{dP_{\tilde{\tilde{X}}^1}}\left(\tilde{\tilde{X}}^1\right)\\ &\quad=\exp\biggl\{\int_0^T\left(\delta_2\left(t,X^1\right)-\delta_1\left(t,X^1\right)\right)d\tilde{\tilde{X}}^1_t-\frac12\int_0^T\left(\delta_2\left(t,X^1\right)^2-\delta_1\left(t,X^1\right)^2\right)dt\biggr\}\end{aligned}$$

by the chain differentiation rule. Substituting the differential of the process $\tilde{\tilde{X}}^1$ to the latter relation we obtain

$$\begin{aligned}&\frac{dP_{\tilde{\tilde{X}}^2}}{dP_{\tilde{\tilde{X}}^1}}\left(\tilde{\tilde{X}}^1\right)\\ &\quad=\exp\left\{\int_0^T\left(\delta_2\left(t,X^1\right)-\delta_1\left(t,X^1\right)\right)dW_t-\frac12\int_0^T\left(\delta_2\left(t,X^1\right)-\delta_1\left(t,X^1\right)\right)^2 dt\right\}.\end{aligned}$$

Note that

$$\frac{dP_{\tilde{\tilde{X}}^2}}{dP_{\tilde{\tilde{X}}^1}}\left(\tilde{\tilde{X}}^1\right)=\frac{d\left(\mathsf{A}^{-1}P_{X^2}\right)}{d\left(\mathsf{A}^{-1}P_{X^1}\right)}\left(AX^1\right)=\frac{dP_{X^2}}{dP_{X^1}}\left(X^1\right),$$

since the mappings A and A^{-1} are measurable. Thus

$$\begin{aligned}&\frac{dP_{X^2}}{dP_{X^1}}\left(X^1\right)\\ &\quad=\exp\biggl\{\int_0^T\left(\delta_2\left(t,X^1\right)-\delta_1\left(t,X^1\right)\right)dW_t-\frac12\int_0^T\left(\delta_2\left(t,X^1\right)-\delta_1(t,X^1)\right)^2 dt\biggr\}.\end{aligned}$$

Now relations (11) and (12) follow by substituting $\delta_i(t,X)$ defined by (16) into the latter equality. Equality (12) is obtained by applying the chain differentiation rule for a superposition of a smooth function and a Hölder function (see [3]). □

3. Local asymptotic normality of a system of measures generated by solutions of an equation

Theorems 2 and 3 below contain sufficient conditions that a system of probability measures $\{P_\theta^{(\varepsilon)}, \theta \in \Theta\}$ generated by solutions of equation (1) is locally asymptotically normal as $\varepsilon \to 0$. Theorem 2 is an analog of Theorem 2.1 in [1] where the case of a Wiener process is considered. Like Theorem 2.1 of [1], conditions of Theorem 2 are given in terms of the process X. Note however that conditions of this type are not easy to check. Theorem 3 contains sufficient conditions for the local asymptotic normality posed on the function S; thus we avoid the process X in the corresponding assumption.

Let $\Theta \subset \mathbb{R}^d$ be an open set, $P_\theta^{(\varepsilon)}$ be a measure in the measurable space

$$\big(C[0,T], \mathcal{B}_T\big)$$

that corresponds to the solution of equation (1).

Theorem 2. *Let, for every $\theta \in \Theta$, the following conditions be satisfied:*

1) $S(\theta, \cdot, \cdot) \in C^1([0,T] \times \mathbb{R})$.
2) *There exists $M(\theta) > 0$ such that $|S(\theta, t, x)| \le M(\theta)(1 + |x|)$ for all $x \in \mathbb{R}$ and $t \in [0,T]$.*
3) *The derivative $\frac{\partial}{\partial\theta} S(\theta, 0, x_0)$ exists.*
4) *There exist d-dimensional functions $q, r\colon \Theta \times [0,T] \to \mathbb{R}^d$ such that the limits*

$$\lim_{\varepsilon\to 0}\bigg\| \frac{1}{\varepsilon}\left(\frac{\partial}{\partial t} S(\theta + \varepsilon Y, t, X_t) - \frac{\partial}{\partial t} S(\theta, t, X_t)\right) - \big(Y, q(\theta, t)\big)\bigg\|_{L_2([0,T], |\ln(t)| \vee 1)} = 0, \tag{21}$$

$$\lim_{\varepsilon\to 0}\bigg\| \frac{1}{\varepsilon}\left(\frac{\partial}{\partial x} S(\theta + \varepsilon Y, t, X_t) - \frac{\partial}{\partial x} S(\theta, t, X_t)\right) - \big(Y, r(\theta, t)\big)\bigg\|_{L_2([0,T], |\ln(t)| \vee 1)} = 0 \tag{22}$$

exist in probability $P_\theta^{(\varepsilon)}$ for all $Y \in \mathbb{R}^d$, where the norm in $L_2\big([0,T], \phi(t)\big)$ is defined by

$$\|f\|^2_{L_2([0,T], \phi(t))} = \int_0^T f(t)^2 \phi(t)\, dt.$$

5) *The limit*

$$\lim_{\varepsilon\to 0}\left\| \frac{\partial}{\partial x} S(\theta + \varepsilon Y, t, X_t) - \frac{\partial}{\partial x} S(\theta, t, X_t)\right\|_{C^\lambda} = 0 \tag{23}$$

exists in probability $P_\theta^{(\varepsilon)}$ for some $\lambda \in \big(\frac{1}{2}, H\big)$ and all $Y \in \mathbb{R}^d$.

6) *The matrix*

$$I(\theta, x) = \int_0^T Q(\theta, t, x(\theta)) \times Q(\theta, t, x(\theta))^T\, dt$$

is positive definite, where

$$\begin{aligned} Q(\theta, t, x) &= (2 - 2H) t^{1/2 - H} \\ &\quad\times \left(C_1 \frac{\partial}{\partial\theta} S(\theta, 0, x_0) + \int_0^t u^{2H-3} \int_0^u w(u, v)\big(q(\theta, v) + r(\theta, v) S(\theta, v, x_v)\big)\, dv\, du\right) \\ &\quad+ t^{H - 3/2} \int_0^t w(t, u)\big(q(\theta, u) + r(\theta, u) S(\theta, u, x_u)\big)\, du. \end{aligned}$$

Then the family of measures $\{P_\theta^{(\varepsilon)}, \theta \in \Theta\}$ *is locally asymptotically normal in* Θ *as* $\varepsilon \to 0$ *and the normalizing matrix is*

$$\phi_\varepsilon(\theta) = \varepsilon I(\theta, x)^{-1/2}.$$

Proof. Put

(24)
$$\begin{aligned}\delta_{\theta_1}^{(\varepsilon)}&(\theta_2, t, x)\\ &= \frac{1}{\varepsilon}\left[(2-2H)t^{1/2-H}\left(C_1 S(\theta_2, 0, x_0) + \int_0^t u^{2H-3} R_2(u)\, du\right) + t^{H-3/2} R_2(t)\right],\end{aligned}$$

where

$$\begin{aligned}R_2(t) = \int_0^t w(t,v)\Bigg\{\left(\frac{\partial}{\partial v} S(\theta_2, v, x_v) + \frac{\partial}{\partial x} S(\theta_2, v, x_v) S(\theta_1, v, x_v)\right) dv\\ + \varepsilon \frac{\partial}{\partial x} S(\theta_2, v, x_v)\, dB_v\Bigg\}.\end{aligned}$$

According to Theorem 1, the likelihood ratio $\frac{P_{\theta+\varepsilon Y}^{(\varepsilon)}}{P_\theta^{(\varepsilon)}}(X)$ is given by

$$\begin{aligned}\frac{P_{\theta+\varepsilon Y}^{(\varepsilon)}}{P_\theta^{(\varepsilon)}}(X) = \exp\Bigg\{\int_0^T \left(\delta_\theta^{(\varepsilon)}(\theta+\varepsilon Y, t, X) - \delta_\theta^{(\varepsilon)}(\theta, t, X)\right) dW_t\\ - \frac{1}{2}\int_0^T \left(\delta_\theta^{(\varepsilon)}(\theta+\varepsilon Y, t, X) - \delta_\theta^{(\varepsilon)}(\theta, t, X)\right)^2 dt\Bigg\}.\end{aligned}$$

If

(25)
$$\lim_{\varepsilon\to 0}\left\|\left(\delta_\theta^{(\varepsilon)}(\theta+\varepsilon Y, t, X) - \delta_\theta^{(\varepsilon)}(\theta, t, X)\right) - \big(Y, Q(\theta, t, x)\big)\right\|_{L_2} = 0$$

in probability $P_\theta^{(\varepsilon)}$, then we complete the proof of Theorem 2 by following the lines of that of Theorem 2.1 in [1].

Now we are going to prove (25). First,

$$\begin{aligned}\int_0^T &\left[\left(\delta_\theta^{(\varepsilon)}(\theta+\varepsilon Y, t, X) - \delta_\theta^{(\varepsilon)}(\theta, t, X)\right) - \big(Y, Q(\theta, t, x)\big)\right]^2 dt\\ &\le C\Bigg[\int_0^T t^{1-2H}\xi_0(\varepsilon)^2 dt\\ &\qquad + \int_0^T t^{1-2H}\left(\int_0^t u^{2H-3}\int_0^u w(u,v)\Big[(\xi_1(\varepsilon, v) + \xi_2(\varepsilon, v) S(\theta, v, X_v))\, dv\right.\\ &\qquad\qquad \left.+ \xi_3(\varepsilon, v) dB_v\Big] du\right)^2 dt\\ &\qquad + \int_0^T t^{2H-3}\left(\int_0^t w(t,u)\big[\big(\xi_1(\varepsilon, u) + \xi_2(\varepsilon, u) S(\theta, u, X_u)\big)\, du\right.\\ &\qquad\qquad \left.+ \xi_3(\varepsilon, u)\, dB_u\big]\, du\right)^2 dt\Bigg],\end{aligned}$$

where

$$\xi_0(\varepsilon)=\frac{1}{\varepsilon}\big(S(\theta+\varepsilon Y,0,x_0)-S(\theta,0,x_0)\big)-\left(Y,\frac{\partial}{\partial\theta}S(\theta,0,x_0)\right),$$
$$\xi_1(\varepsilon,t)=\frac{1}{\varepsilon}\left(\frac{\partial}{\partial t}S(\theta+\varepsilon Y,t,X_t)-\frac{\partial}{\partial t}S(\theta,t,X_t)\right)-\big(Y,q(\theta,t,x_t)\big),$$
$$\xi_2(\varepsilon,t)=\frac{1}{\varepsilon}\left(\frac{\partial}{\partial x}S(\theta+\varepsilon Y,t,X_t)-\frac{\partial}{\partial x}S(\theta,t,X_t)\right)-\big(Y,r(\theta,t,x_t)\big),$$
$$\xi_3(\varepsilon,t)=\frac{\partial}{\partial x}S(\theta+\varepsilon Y,t,X_t)-\frac{\partial}{\partial x}S(\theta,t,X_t).$$

Then

$$\int_0^T\Big[\Big(\delta_\theta^{(\varepsilon)}(\theta+\varepsilon Y,t,X)-\delta_\theta^{(\varepsilon)}(\theta,t,X)\Big)-\big(Y,Q(\theta,t,x)\big)\Big]^2\,dt\le C{\xi_0}^2+C\sum_{k=2}^{7}I_k(\varepsilon),$$

where

$$I_2(\varepsilon)=\int_0^T t^{1-2H}\left(\int_0^t u^{2H-3}\int_0^u w(u,v)\xi_1(\varepsilon,v)\,dv\,du\right)^2dt,$$
$$I_3(\varepsilon)=\sup_{t\in[0,T]}S(\theta,t,X_t)^2\int_0^T t^{1-2H}\left(\int_0^t u^{2H-3}\int_0^u w(u,v)\xi_2(\varepsilon,v)\,dv\,du\right)^2dt,$$
$$I_4(\varepsilon)=\int_0^T t^{1-2H}\left(\int_0^t u^{2H-3}\int_0^u w(u,v)\xi_3(\varepsilon,v)\,dB_v\,du\right)^2dt,$$
$$I_5(\varepsilon)=\int_0^T t^{2H-3}\left(\int_0^t w(t,u)\xi_1(\varepsilon,u)\,du\right)^2dt,$$
$$I_6(\varepsilon)=\sup_{t\in[0,T]}S(\theta,t,X_t)^2\int_0^T t^{2H-3}\left(\int_0^t w(t,u)\xi_2(\varepsilon,u)\,du\right)^2dt,$$
$$I_7(\varepsilon)=\int_0^T t^{2H-3}\left(\int_0^t w(t,u)\xi_3(\varepsilon,u)\,dB_u\right)^2dt.$$

It follows from condition 3) that $\xi_0(\varepsilon)^2\to 0$ as $\varepsilon\to 0$. Now we estimate $I_2(\varepsilon)$:

$$\begin{aligned}I_2(\varepsilon)&=\int_0^T t^{1-2H}\left(\int_0^t u^{2H-3}\int_0^u w(u,v)\xi_1(\varepsilon,v)\,dv\,du\right)^2dt\\&=\int_0^T t^{1-2H}\left(\int_0^t\int_0^1 w(1,v)\xi_1(\varepsilon,u\cdot v)\,dv\,du\right)^2dt\\&\le C\int_0^T t^{2-2H}\int_0^t\int_0^1\xi_1(\varepsilon,u\cdot v)^2\,dv\,du\,dt\\&=C\int_0^T t^{3-2H}\int_0^1\int_0^1\xi_1(\varepsilon,t\cdot u\cdot v)^2\,dv\,du\,dt.\end{aligned}$$

We need the following auxiliary result.

Lemma 2. *Let $\psi\ge 0$ and $\psi\in L_1\big([0,1],|\ln(u)|\big)$. Then*

$$\int_0^1\int_0^1\psi(u\cdot v)\,du\,dv\le\int_0^1\psi(u)\left|\ln(u)\right|\,du.$$

Proof.

$$\int_0^1 \int_0^1 \psi(u\cdot v)\,du\,dv = \lim_{n\to\infty} \sum_{k=0}^{n-1} \psi\left(\frac{k}{n}\right) \mu\left(\left\{(u,v)\colon 0\le u,v\le 1, \frac{k}{n}\le u\cdot v < \frac{k+1}{n}\right\}\right),$$

where μ is the Lebesgue measure in $\mathbb{R}^2$. The measures of the above sets can be estimated as follows:

$$\begin{aligned}\mu\left(\left\{0\le u,v\le 1, \frac{k}{n}\le u\cdot v < \frac{k+1}{n}\right\}\right) &\le \mu\left(\left\{\frac{k}{n}\le u\le 1, \frac{k}{n}\le u\cdot v < \frac{k+1}{n}\right\}\right)\\ &= \int_{k/n}^1 \left(\frac{k+1}{n}-\frac{k}{n}\right)\frac{du}{u} = \frac{1}{n}\left(-\ln\left(\frac{k}{n}\right)\right).\end{aligned}$$

Thus

$$\int_0^1 \int_0^1 \psi(u\cdot v)\,du\,dv \le \lim_{n\to\infty} \sum_{k=0}^{n-1} \psi\left(\frac{k}{n}\right)\frac{1}{n}\left(-\ln\left(\frac{k}{n}\right)\right) = \int_0^1 \psi(u)\,|\ln(u)|\,du. \qquad \square$$

We turn back to the estimation of $I_2(\varepsilon)$ and use Lemma 2:

$$\begin{aligned} I_2(\varepsilon) &\le C\int_0^T t^{3-2H}\int_0^1 \xi_1(\varepsilon, t\cdot u)^2\,|\ln(u)|\,du\,dt\\ &= C\int_0^T t^{2-2H}\int_0^t \xi_1(\varepsilon,u)^2\left|\ln\left(\frac{u}{t}\right)\right|\,du\,dt\\ &\le C\left(\int_0^T t^{2-2H}(1+|\ln(t)|)\,dt\right)\cdot \|\xi_1(\varepsilon)\|^2_{L_2([0,T],|\ln(t)|\vee 1)}\\ &= C\,\|\xi_1(\varepsilon)\|^2_{L_2([0,T],|\ln(t)|\vee 1)}\,. \end{aligned}$$

Considering (21), we obtain $I_2(\varepsilon)\to 0$ in probability $P_\theta^{(\varepsilon)}$ as $\varepsilon\to 0$.

Similarly we prove that the integral involved in the definition of $I_3(\varepsilon)$ tends to zero; that is, we prove that

$$\int_0^T t^{1-2H}\left(\int_0^t u^{2H-3}\int_0^u w(u,v)\xi_2(\varepsilon,v)\,dv\,du\right)^2 dt \to 0 \quad \text{in probability } P_\theta^{(\varepsilon)}. \tag{26}$$

Note that

$$\text{for all } \delta>0, \text{ there exists } N>0 \text{ such that for all } \varepsilon\in(0,1),$$
$$P_\theta^{(\varepsilon)}\left(\sup_{t\in[0,T]} S(\theta,t,X_t)^2 > N\right) < \delta, \tag{27}$$

since $S(\theta,t,x)$ is continuous, the distribution of B_t does not depend on ε, and since

$$\sup_{t\in[0,T]}|X_t| \le \varepsilon C_1 \sup_{t\in[0,T]}|B_t| + C_2,$$

where

$$C_i = C_i\big(L_{N_0}, M(\theta), T, \sup|x_t|\big), \qquad i=1,2,$$

and $N_0>0$ is a fixed number. The latter result follows from inequality (8) of [8], since conditions (7) and (8) hold. The convergence $I_3(\varepsilon)\to 0$ in probability $P_\theta^{(\varepsilon)}$ follows from (26), (27), and from the inequality

$$P_\theta^{(\varepsilon)}\big(\xi\cdot\eta^{(\varepsilon)}>\lambda\big) \le P_\theta^{(\varepsilon)}\big(\xi > N\sqrt{\lambda}\big) + P_\theta^{(\varepsilon)}\big(\eta^{(\varepsilon)} > \sqrt{\lambda}/N\big) \quad \text{for all } \lambda>0, N>0, \tag{28}$$

where we set

$$\xi = \sup_{t\in[0,T]} S(\theta,t,X_t)^2,$$

$$\eta^{(\varepsilon)} = \int_0^T t^{1-2H}\left(\int_0^t u^{2H-3}\int_0^u w(u,v)\xi_2(\varepsilon,v)\,dv\,du\right)^2 dt.$$

Now we estimate $I_4(\varepsilon)$. Fix $\lambda \in \left(\frac{1}{2}, H\right)$ such that condition 5) holds. According to (14) we have

$$\begin{aligned} I_4(\varepsilon) &= \int_0^T t^{1-2H}\left(\int_0^t u^{2H-3}\int_0^u w(u,v)\xi_3(\varepsilon,v)\,dB_v\,du\right)^2 dt \\ &\le \int_0^T t^{1-2H}\left(\int_0^t C_0 K\big(\xi_3(\varepsilon),B\big)Hu^{H-1}\,du\right)^2 dt \\ &= C\left\|\xi_3(\varepsilon)\right\|^2_{C^\lambda[0,T]}\left\|B\right\|^2_{C^\lambda[0,T]}. \end{aligned} \tag{29}$$

The convergence $I_4(\varepsilon)\to 0$ in probability $P_\theta^{(\varepsilon)}$ follows from condition 5) and inequality (28).

The terms I_5, I_6, and I_7 are estimated similarly to the terms I_2, I_3, and I_4, respectively. □

The following result contains conditions placed upon the function $S(\theta,t,x)$ that yield relations 4) and 5) of Theorem 2. Note that conditions 4) and 5) are expressed in terms of the process X.

Theorem 3. *Let a function $S(\theta,t,x)$ be such that*

1) *for any $\theta\in\Theta$ there exists $L(\theta)>0$ such that*

$$|S(\theta,t,x)-S(\theta,t,y)| \le L(\theta)\,|x-y|$$

for all $x,y\in\mathbb{R}$, $t\in[0,T]$;

2) *the derivatives*

$$\frac{\partial^2}{\partial\theta\,\partial t}S(\theta,t,x), \qquad \frac{\partial^2}{\partial\theta\,\partial x}S(\theta,t,x)$$

exist and are continuous for all $\theta\in\Theta$, $t\in[0,T]$, and $x\in\mathbb{R}$;

3) *for every compact set $B\subset\mathbb{R}$ and for every point $\theta_0\in\Theta$, the functions*

$$\frac{\partial^2}{\partial\theta\partial t}S(\theta,t,x) \quad \text{and} \quad \frac{\partial^2}{\partial\theta\partial x}S(\theta,t,x)$$

are continuous in θ at the point θ_0 uniformly in t and x belonging to the set $[0,T]\times B$;

4) *$\alpha_B,\beta_B\in L_1([\theta,\theta+Y])$ for all $Y\in\mathbb{R}^d$ with $|Y|=\delta$ for some $\delta>0$, all $\theta\in\Theta$, all compact sets $B\subset\mathbb{R}$, and some $\lambda\in\left(\frac{1}{2},H\right)$, where*

$$\alpha_B(\theta) = \sup_{t\in[0,T]}\left\|\left|\frac{\partial^2}{\partial\theta\partial x}S(\theta,t,\cdot)\right|\right\|_{C^1(B)},$$

$$\beta_B(\theta) = \sup_{x\in B}\left\|\left|\frac{\partial^2}{\partial\theta\partial x}S(\theta,\cdot,x)\right|\right\|_{C^\lambda[0,T]}.$$

Then assumptions 4) and 5) of Theorem 2 hold with

$$q(\theta,t) = \frac{\partial^2}{\partial\theta\partial t}S(\theta,t,x_t), \qquad r(\theta,t) = \frac{\partial^2}{\partial\theta\partial x}S(\theta,t,x_t). \tag{30}$$

Proof. Set

$$f(\theta,t,x)=\frac{\partial}{\partial t}S(\theta,t,x).$$

Now we prove that the limit

$$\lim_{\varepsilon\to 0}\left\|\frac{f(\theta_0+\varepsilon Y,t,X_t)-f(\theta_0,t,X_t)}{\varepsilon}-\left(Y,\frac{\partial}{\partial\theta}f(\theta_0,t,x_t)\right)\right\|_{L_2([0,T],|\ln(t)|\vee 1)}=0$$

exists in probability $P_{\theta_0}^{(\varepsilon)}$ for all $Y\in\mathbb{R}^d$.

Indeed, we obtain from the equality

$$f(\theta_0+Y,t,x)-f(\theta_0,t,x)=\int_0^1\left(Y,\frac{\partial}{\partial\theta}f(\theta_0+sY,t,x)\right)ds$$

that

$$\begin{aligned}
&\left\|\frac{f(\theta+\varepsilon Y,t,X_t)-f(\theta,t,X_t)}{\varepsilon}-\left(Y,\frac{\partial}{\partial\theta}f(\theta,t,x_t)\right)\right\|\\
&\quad=\left\|\int_0^1\left(Y,\frac{\partial}{\partial\theta}f(\theta+\varepsilon sY,t,X_t)-\frac{\partial}{\partial\theta}f(\theta,t,x_t)\right)ds\right\|\\
&\quad\le|Y|\left(\left\|\int_0^1\left|\frac{\partial}{\partial\theta}f(\theta+\varepsilon sY,t,X_t)-\frac{\partial}{\partial\theta}f(\theta,t,X_t)\right|ds\right\|\right.\\
&\qquad\qquad\left.+\left\|\left|\frac{\partial}{\partial\theta}f(\theta,t,X_t)-\frac{\partial}{\partial\theta}f(\theta,t,x_t)\right|\right\|\right)\\
&\quad\le|Y|^2\,C\left(\int_0^1\sup_{t\in[0,T]}\left|\frac{\partial}{\partial\theta}f(\theta+\varepsilon sY,t,X_t)-\frac{\partial}{\partial\theta}f(\theta,t,X_t)\right|ds\right.\\
&\qquad\qquad\left.+\sup_{t\in[0,T]}\left|\frac{\partial}{\partial\theta}f(\theta,t,X_t)-\frac{\partial}{\partial\theta}f(\theta,t,x_t)\right|\right).
\end{aligned}$$

Condition 1) of Theorem 3 and Proposition 2 imply that

$$\sup_{t\in[0,T]}|X_t-x_t|\le\varepsilon C(\theta)\sup_{t\in[0,T]}|B_t|, \tag{31}$$

whence we derive that

$$\sup_{t\in[0,T]}\left|\frac{\partial}{\partial\theta}f(\theta,t,X_t)-\frac{\partial}{\partial\theta}f(\theta,t,x_t)\right|\to 0\quad\text{in probability } P_\theta^{(\varepsilon)}$$

in view of condition 2). Furthermore, relation (31) implies (27) and this together with condition 3) proves that

$$\begin{aligned}
&\int_0^1\sup_{t\in[0,T]}\left|\frac{\partial}{\partial\theta}f(\theta+\varepsilon sY,t,X_t)-\frac{\partial}{\partial\theta}f(\theta,t,X_t)\right|ds\\
&\quad=\sup_{t\in[0,T]}\left|\frac{\partial}{\partial\theta}f(\theta+\varepsilon\tilde{s}Y,t,X_t)-\frac{\partial}{\partial\theta}f(\theta,t,X_t)\right|\to 0\quad\text{in probability } P_\theta^{(\varepsilon)}
\end{aligned}$$

for some $\tilde{s}\in[0,1]$. Therefore relation (21) is proved.

Relation (22) is proved in the same way.

Now we prove that condition 5) holds. Put

$$
\begin{aligned}
I(\theta,\varepsilon) &= \left\| \frac{\partial}{\partial x} S(\theta_0+\varepsilon Y, t, X_t) - \frac{\partial}{\partial x} S(\theta_0, t, X_t) \right\|_{C^\lambda} \\
&= \left\| \int_0^1 \left(\varepsilon Y, \frac{\partial}{\partial\theta\partial x} S(\theta_0+\varepsilon s Y, t, X_t) \right) ds \right\|_{C^\lambda} \\
&\le \varepsilon\,|Y| \int_0^1 \left\| \left| \frac{\partial}{\partial\theta\partial x} S(\theta_0+\varepsilon s Y, t, X_t) \right| \right\|_{C^\lambda} ds.
\end{aligned} \tag{32}
$$

It is straightforward from the definition of the Hölder norm that

$$
\|G(\cdot, X.)\|_{C^\lambda[0,T]} \le \sup_{t\in[0,T]} \|G(t,\cdot)\|_{C^1(X[0,T])} \cdot \|X\|_{C^\lambda[0,T]} + \sup_{x\in X[0,T]} \|G(\cdot,x)\|_{C^\lambda[0,T]},
$$

where $X[0,T]=\{X_t, t\in[0,T]\}$. Using this bound in (32) we get

$$
I(\theta,\varepsilon) \le \varepsilon\,|Y| \left(\int_0^1 \alpha_{X[0,T]}(\theta_0+\varepsilon s Y)\,ds \cdot \|X\|_{C^\lambda[0,T]} + \int_0^1 \beta_{X[0,T]}(\theta_0+\varepsilon s Y)\,ds \right). \tag{33}
$$

Relation (27) together with representation (1) for the process X_t implies that

$$
\begin{gathered}
\text{for all } \delta>0 \quad \text{there exists } N_\delta>0 \quad \text{such that for all } \varepsilon\in(0,1), \\
P_\theta^{(\varepsilon)}\left(\|X\|_{C^\lambda[0,T]} > N_\delta \right) < \delta,
\end{gathered} \tag{34}
$$

since X_t is a sum of a smooth process

$$
x_0 + \int_0^t S(\theta, s, X_s)\,ds
$$

and εB_t. Using relation (33) and conditions (27), (34), and assumption 4) of Theorem 3 we get $I(\theta,\varepsilon)\to 0$ in probability $P_\theta^{(\varepsilon)}$ as $\varepsilon\to 0$. □

Example 1. Let $S(\theta,t,x)=\theta\cdot t\cdot x$. Then the solution of the deterministic equation (2) is given by

$$
x_t = (x_0-1) + \exp\left(\frac{\theta}{2} t^2\right), \qquad t\in[0,T].
$$

It is clear that the function S satisfies conditions 1)–3) of Theorem 2 in this case. It is also obvious that assumptions of Theorem 3 hold, whence conditions 4) and 5) of Theorem 2 follow. The functions $q(\theta,t)$ and $r(\theta,t)$ defined by equalities (30) and the function $Q(\theta,t,x)=Q(\theta,t)$ defined in assumption 6) of Theorem 2 are such that

$$
q(\theta,t) = x_t = (x_0-1) + \exp\left(\frac{\theta}{2}\, t^2\right), \qquad r(\theta,t)=t,
$$

and

$$
\begin{aligned}
Q(\theta,t) = (2-2H) t^{1/2-H} &\int_0^t u^{2H-3} \int_0^u w(u,v) \left(1+\theta v^2\right) \left((x_0-1) + \exp\left(\theta/2v^2\right)\right) dv\,du \\
&+ t^{H-3/2} \int_0^t w(t,u) \big(1+\theta u^2\big) \big((x_0-1) + \exp\left(\theta/2u^2\right)\big)\,du,
\end{aligned}
$$

respectively. Note that the function $Q(\theta,t,x)=Q(\theta,t)$ can be expressed explicitly in terms of generalized hypergeometric functions.

Therefore all the assumptions of Theorem 2 hold, so that the family of probability measures $\{P_\theta^{(\varepsilon)}, \theta\in\Theta\}$ generated by the solution of equation (1) with $S(\theta,t,x)=\theta\cdot t\cdot x$

is locally asymptotically normal in Θ as $\varepsilon \to 0$. In this case, the normalizing factor is

$$\phi_\varepsilon(\theta) = \varepsilon \cdot \left(\int_0^T Q(\theta, t)^2 \, dt \right)^{-1/2}.$$

4. Conclusion

A formula for the likelihood ratio of measures generated by solutions of a stochastic differential equation with fractional Brownian motion is obtained in this paper. Sufficient conditions that a family of probability measures $\{P_\theta^{(\varepsilon)}, \theta \in \Theta\}$ be locally asymptotically normal as $\varepsilon \to 0$ are given for the case where the measures are generated by solutions of a stochastic differential equation that depends on a parameter θ and involves fractional Brownian noise.

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