The local homeomorphism property of spatial quasiregular mappings with distortion close to one

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Abstract

We give a quantitative proof for a theorem of Martio, Rickman and Väisälä [14] on the rigidity of the local homeomorphism property of spatial quasiregular mappings with distortion close to one. The proof is based on a distortion theory established by using two main tools. First, we use a conformal invariant between sphere families and components of their preimages under quasiregular mappings. Secondly, we use Hall's quantitative isoperimetric inequality result [9] to relate two different types of distortion.

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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a domain, $n \geq 2$. We call a mapping $f : \Omega \to \mathbb{R}^n$ quasiregular, if $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, and if there exists $1 \leq K < \infty$ so that

$$|Df(x)|^n \leq K J_f(x)$$

for almost all $x \in \Omega$ (the notation will be explained below). The theory of quasiregular mappings, initiated by the results of Reshetnyak and Martio, Rickman and Väisälä, shows that these mappings generalize analytic functions to higher dimensions in a natural way. In particular, Reshetnyak proved that non-constant quasiregular mappings are discrete, open and locally Hölder continuous, and map sets of measure zero to sets of measure zero. The basic references for the theory of quasiregular mappings are [16], [17], [19] and [12].

One of the most interesting aspects of higher dimensional (spatial) quasiregular mappings is the fact that they share certain topological properties that planar analytic functions do not possess. The most classical result of this kind is Liouville's theorem from 1850, later generalized by Gehring and Reshetnyak (see [16] Theorem 5.10), stating that 1-quasiregular mappings in dimensions higher than two are in fact restrictions of Möbius transformations, and thus homeomorphisms in particular. In the study of spatial quasiregular mappings it has turned out that there are also other rigidity phenomena, particularly when the local homeomorphism property is studied.

Already in the early stages of the theory of higher dimensional quasiregular mappings, Zorich [24] showed that spatial quasiregular local homeomorphisms from $\mathbb{R}^n$ are in fact global homeomorphisms i.e. quasiconformal mappings. This result was generalized by Martio, Rickman and Väisälä [14]. Zorich's result fails in dimension two, which is shown by the mapping $z \to \exp(z)$. Also, there are several sufficient conditions for a spatial quasiregular mapping to be a local homeomorphism. For instance, when f is sufficiently smooth, it cannot have branch points, see [3], [17] page 12. The current paper is concerned with the following result of Martio, Rickman and Väisälä [14]: There exists a constant $\epsilon(n) > 0$ so that every non-constant K-quasiregular mapping in dimension $n \geq 3$ is a local homeomorphism when $K < 1 + \epsilon(n)$. This result is proved in [14] indirectly, using Liouville's theorem and a normal family method. In particular, this proof does not give any estimates for $\epsilon(n)$. The main purpose of this paper is to give a quantitative proof of this result. As far as we know, this is the first proof of such kind. Also, this is the first geometric proof, so that it does not depend on Liouville's theorem.

Theorem 1.1. *Suppose that $\Omega \subset \mathbb{R}^n$, $n \geq 3$ is a domain, and $f : \Omega \to \mathbb{R}^n$ a non-constant quasiregular mapping. If $K < (1+\epsilon)^{\frac{1}{3}}$, where ϵ is as in* (6.30), *then f is a local homeomorphism.*

In [14] it is conjectured that the conclusion of Theorem 1.1 is true for $K_I < 2$, where K_I is the inner distortion of f. Thus, conjecturally, the winding mapping $(r, \varphi, y) \to (r, 2\varphi, y)$ in cylindrical coordinates, $y \in \mathbb{R}^{n-2}$, is extremal for this property. The conjecture has been shown to hold true in some special cases, namely when the branch set is assumed to be geometrically well-behaved, cf. [8]. In particular, the conjecture is true under the assumption that the branch set contains a rectifiable curve. The general problem of describing the branching of quasiregular mappings is discussed in [11].

The main task in proving Theorem 1.1 is to verify certain 'inverse' distortion properties of quasiregular mappings, so that these properties do not depend on multiplicity. These properties hold also in the plane; the assumption $n \geq 3$ is used only at the end of the proof, in the effect of having a sufficiently large set of branch points. We believe that the distortion results given in this paper are independently interesting, also in the theory of quasiconformal mappings.

The main part of the proof consists of combining the well-known path family method with the method of 'surface' families, which are families of $(n-1)$-dimensional spheres, and components of their preimages under the quasiregular mapping f. Moduli of surface families, as well as more general sets, were already studied by Fuglede [5] in 1957. Although there are some previous results concerning quasiregular mappings and surface families, cf. [1], [2], it seems that this theory has not had applications so far. Here the conformal modulus of such families, combined with a symmetrization method, allows us to estimate the isoperimetric defect of components of preimages of balls centered at a fixed point. The idea of using symmetrization when studying conformal invariants goes back to Gehring [6]. A crucial result that allows for a more useful interpretation of these estimates (in dimensions higher than two, in the plane this is easy) is a theorem by Hall [9] (see also [10]), which implies that when a component of a preimage of a ball under a quasiregular mapping looks like a ball in the sense of the isoperimetric inequality, then it is almost a ball also in the sense of metric distortion.

This paper is organized as follows. In Section 2 we introduce the notation and some preliminary results. In Section 3 we establish surface family inequalities for quasiregular mappings. These inequalities correspond to the path family inequalities for quasiregular mappings, see [17] Chapter II, and they are proved by closely following the proofs in the path family case. In Section 4 we use Hall's result in order to show that small isoperimetric defect of certain kinds of domains implies small metric distortion. In Section 5 we give an estimate showing that when the distortion of a quasiregular mapping is small, then we have efficient estimates for the isoperimetric defects of certain components of preimages of balls under the mapping. Finally, the results of the previous sections are used in Section 6 to prove Theorem 1.1.

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2 Preliminaries

We will mainly use the notation of [17]. Open euclidean balls with center x and radius r are denoted by $B(x,r)$, while the corresponding $(n-1)$-dimensional spheres are denoted by $S(x,r)$. Corresponding closed balls are denoted by $\overline{B}(x,r)$. In the case $x=0$ the notations B_r and S_r are often used. The boundary of a general set E is denoted by ∂E.

Let f be a quasiregular mapping and $Df(x)$ the differential matrix of f. Set

$$|Df(x)| = \sup_{\{y\in\mathbb{R}^n:|y|=1\}} |Df(x)y| \quad \text{and} \quad l(Df(x)) = \inf_{\{y\in\mathbb{R}^n:|y|=1\}} |Df(x)y|.$$

We denote the adjoint matrix of $Df(x)$ by $D^{\#}f(x)$, and $J_f(x)$ stands for the Jacobian determinant of $Df(x)$. The inner, outer and maximal distortion functions will be used:

$$\begin{aligned}
K_I(x) &= \frac{J_f(x)}{l(Df(x))^n} = \frac{|D^{\#}f(x)|^n}{J_f(x)^{n-1}}, \\
K_O(x) &= \frac{|Df(x)|^n}{J_f(x)} = \frac{J_f(x)^{n-1}}{l(D^{\#}f(x))^n}, \\
K(x) &= \max\{K_I(x), K_O(x)\},
\end{aligned}$$

assuming that $J_f(x) > 0$, otherwise set each distortion function to be zero. We have

$$K_I(x)K_O(x) \leq K(x)^2.$$

In what follows, K_I, K_O and K will denote the essential supremums of the inner, outer and maximal distortions of f, respectively. The Lebesgue n-measure of a measurable set A is denoted by $|A|$. The Lebesgue measure of the unit n-ball is denoted by α_n. The $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ of a measurable set A is defined as

$$\mathcal{H}^{n-1}(A) = \lim_{\delta\to 0} \inf \Big\{ \sum_{i=1}^{\infty} \lambda_{n-1} \operatorname{diam}(A_i)^{n-1} : A \subset \bigcup_{i=1}^{\infty} A_i, \operatorname{diam}(A_i) < \delta \Big\},$$

where

$$\lambda_{n-1} = \frac{\Gamma(\frac{1}{2})^{n-1}}{2^{n-1}\Gamma(\frac{1}{2}(n+1))},$$

and Γ is the familiar gamma function. Furthermore, we denote $\mathcal{H}^{n-1}(S_1) = \omega_{n-1}$ when S_1 is $(n-1)$-dimensional.

Let $f : \Omega \to \mathbb{R}^n$ be a continuous, sense-preserving, discrete and open mapping. A domain $D \subset \Omega$ is called a normal domain (of f), if $f(\partial D) = \partial f(D)$. Furthermore, if D is a normal domain and $x \in D$ so that $f^{-1}(x) \cap D = \{x\}$, then D is called a normal neighborhood of x. For a domain $U \subset \Omega$ and a point $y \in \mathbb{R}^n \setminus f(\partial U)$, the topological degree of f at y with respect to U is denoted by $\mu(y, f, U)$, see [17] page 16 for the definition. For a set $E \subset \Omega$, we set $N(y, f, E) = \operatorname{card}\{x \in E : f(x) = y\}$. When U is a normal domain, $\mu(y, f, U)$ equals a constant $\mu(f, U)$ for each $y \in f(U)$, and $\mu(f, U) = \sup_{y\in f(U)} N(y, f, U)$. The x-component of the preimage of the ball $B(f(x), r)$ under f is denoted by $U(x, f, r)$. By [17] II Lemma 4.1, for each $x \in \Omega$ there exists $\sigma_x > 0$ so that for each $s < \sigma_x$ the following properties hold:

1. $U(x, f, s)$ is a normal neighborhood of x,
2. $U(x, f, s) = U(x, f, \sigma_x) \cap f^{-1}(B(f(x), s))$,
3. $\partial U(x, f, s) = U(x, f, \sigma_x) \cap f^{-1}(S(f(x), s))$,

4. $\mathbb{R}^n \setminus U(x,f,s)$ and $\mathbb{R}^n \setminus \overline{U}(x,f,s)$ are connected.

The local index $i(x,f)$ of f at a point $x \in \Omega$ can be defined by setting $i(x,f) = \mu(f,U)$, where U is any normal neighborhood of x; the definition does not depend on the normal neighborhood U. The set of points in Ω where $|i(x,f)| \geq 2$, i.e. where f is not a local homeomorphism, is called the branch set of f and denoted by B_f. We shall use the fact that for quasiregular mappings, $|B_f| = |fB_f| = 0$. See [17] I 4 for further properties of discrete and open mappings.

The following dilatation functions will be used:

$$
\begin{aligned}
L(x,f,r) &= \sup_{|x-y|=r} |f(y)-f(x)|, \\
l(x,f,r) &= \inf_{|x-y|=r} |f(y)-f(x)|, \\
L^*(x,f,r) &= \sup_{z\in\partial U(x,f,r)} |x-z|, \\
l^*(x,f,r) &= \inf_{z\in\partial U(x,f,r)} |x-z|.
\end{aligned}
$$

In this paper we will abuse the terminology by calling general Hausdorff $(n-1)$-dimensional sets, as well as their images and components of their preimages under a quasiregular mapping, surfaces. When U is a normal domain of f and $B_r \subset f(U)$, we denote $B_r' := f^{-1}(B_r) \cap U$. Also, a similar notation for components of preimages of spheres will be used.

In what follows, we shall use the facts that quasiregular mappings are differentiable almost everywhere, and that $J_f(x) > 0$ almost everywhere, see [17] I 2.4, I 4.11 and I 4.14. Let μ and ν be measures in sets X and Y, respectively, and $f : X \to Y$. Then, condition N is defined by requiring that if $E \subset X$ is a set of μ-measure zero, then also $\nu(f(E)) = 0$. Conversely, condition co-N is satisfied if for each set $F \subset Y$ with $\nu(F) = 0$ also $\mu(f^{-1}(F)) = 0$ holds.

3 Modulus of surface families

In this section we give inequalities between the conformal moduli of certain surface families related to quasiregular mappings. This is done by following the arguments of the proofs of the corresponding inequalities for path families, see [17] Chapter II. For a family Λ of Borel-measurable subsets of $\mathbb{R}^n$, set

$$
\begin{aligned}
M_S(\Lambda) \;=\; \inf \Bigg\{ & \int_{\mathbb{R}^n} \rho(x)^{\frac{n}{n-1}}\, dx : \rho : \mathbb{R}^n \to [0,\infty] \text{ is Borel measurable}, \\
& \int_S \rho(x)\, d\mathcal{H}^{n-1}(x) \geq 1 \Bigg\}.
\end{aligned}
$$

We will also use a modification of the definition; we set

$$
\begin{aligned}
M_S^m(\Lambda) &= \inf\left\{\int_{\mathbb{R}^n} \rho(x)^{\frac{n}{n-1}}\,dx : \rho:\mathbb{R}^n \to [0,\infty] \text{ is Borel measurable,}\right.\\
&\qquad \left.\int_S \rho(x)\,d\mathcal{H}^{n-1}(x) \geq m\right\}.
\end{aligned}
$$

We shall need a change of variables -type formula for subsets of spheres. This is given in the following lemma, which follows easily from [13] Theorem 9.2.

Lemma 3.1. *Let $S(x,r) \subset \Omega \subset \mathbb{R}^n$ be a sphere, and suppose that $f:\Omega \to \mathbb{R}^n$ is a continuous and weakly differentiable mapping. If the restriction of f to $S(x,r)$ is a weakly differentiable mapping satisfying condition N (with respect to $\mathcal{H}^{n-1}$), then for all measurable subsets E of $S(x,r)$ and for all measurable functions $u: E \to \mathbb{R}$ we have*

$$
\int_E u(x)|D^{\#}f(x)|\,d\mathcal{H}^{n-1}(x) \geq \int_{\mathbb{R}^n}\Big(\sum_{x\in E\cap f^{-1}(y)} u(x)\Big)\,d\mathcal{H}^{n-1}(y).
$$

We now have the following inequality that corresponds to the K_O-inequality of quasiregular mappings. This result will not be used later in this paper, but we state it as it is easy to verify.

Theorem 3.2. *Let $f: B_1 \to \mathbb{R}^n$ be a quasiregular mapping, and suppose that $I \subset (0,1)$ is a Borel set. Let*

$$
\Lambda := \{S_t : t \in I\}
$$

be a family of spheres inside B_1. If $\rho:\mathbb{R}^n \to [0,\infty]$ is a Borel function with the property

$$
\int_{f(S_t)} N(y,f,S_t)\rho(y)\,d\mathcal{H}^{n-1}(y) \geq 1 \quad \textit{for all } t\in I,
$$

then

$$
M_S(\Lambda) \leq K_I^{\frac{1}{n-1}} \int_{\mathbb{R}^n} N(y,f,B_1)\rho(y)^{\frac{n}{n-1}}\,dy.
$$

Proof. Let ρ be a test function as in the theorem. Define a Borel function $\rho' : B_1 \to [0,\infty]$ by setting

$$
\rho'(x) = \rho(f(x))|D^{\#}f(x)|
$$

when f is differentiable and $\rho'(x) = 0$ otherwise (recall that f is differentiable almost everywhere). By the Sobolev embedding thereom the restriction of

f to the sphere S_t satisfies condition N with respect to $\mathcal{H}^{n-1}$ for almost all $t \in (0,1)$. Thus Lemma 3.1 shows that

$$\begin{aligned} \int_{S_t} \rho'(x)\, d\mathcal{H}^{n-1}(x) &= \int_{S_t} \rho(f(x))|D^{\#} f(x)|\, d\mathcal{H}^{n-1}(x) \\ \geq \int_{\mathbb{R}^n} N(y,f,S_t)\rho(y)\, d\mathcal{H}^{n-1}(y) \geq 1 \end{aligned}$$

for almost all $t \in I$. On the other hand, by quasiregularity of f we have

$$\begin{aligned} &\int_{B_1} \rho'(x)^{\frac{n}{n-1}}\, dx = \int_{B_1} \rho(f(x))^{\frac{n}{n-1}} |D^{\#} f(x)|^{\frac{n}{n-1}}\, dx \\ \leq\ & K_I^{\frac{1}{n-1}} \int_{B_1} \rho(f(x))^{\frac{n}{n-1}} J_f(x)\, dx = K_I^{\frac{1}{n-1}} \int_{\mathbb{R}^n} N(y,f,B_1)\rho(y)^{\frac{n}{n-1}}\, dx \end{aligned}$$

by the change of variables formula (see [13], Theorem 9.2). The proof is complete.

□

We will need an inequality similar to the Väisälä inequality [22] for path families. Here we shall consider the following situation. Let $\Omega \subset \mathbb{R}^n$ be a domain, $0 \in \Omega$, and suppose $f : \Omega \to \mathbb{R}^n$ is a quasiregular mapping. Furthermore, assume that $f(0) = 0$ and that $1 < \sigma_0$, where σ_x is as in [17] II Lemma 4.1, so that the 0-component $U = U(0,1)$ of $f^{-1}(B_1)$ is a normal neighborhood of 0 with $\mu(f,U) = i(0,f) = m$. Since we will only consider surfaces inside a normal domain, the proofs of the following results are easier than the proofs of the classical Poletsky and Väisälä inequalities. Also, we will only concentrate on sphere families and their preimages, whereas the classical results are stated for general path families. For any normal domain $V \subset U$ with $\mu(f,V) = k$, define the 'inverse' mapping $g_V : f(V) \to \mathbb{R}^n$ by

$$g_V(y) = \frac{1}{k} \sum_{x \in f^{-1}(y) \cap V} i(x,f)x.$$

Then, by [17] II Lemma 7.1, $g_V \in W^{1,n}(f(V), \mathbb{R}^n)$. We have the following absolute continuity result that corresponds to the Poletsky lemma for path families [15].

Lemma 3.3. *In the situation as above, for almost all $r \in (0,1)$, the restriction of the mapping f to S'_r satisfies condition co-N (with respect to $\mathcal{H}^{n-1}$).*

Proof. Consider the sets $A_k := \{x \in U : i(x,f) = k\}$ for $k = 1,\dots,m$. Choose, for each $x \in A_k$, a ball B_x so that the x-component U_x of $f^{-1}(B_x)$ is a normal neighborhood of x. We can further choose a countable subcollection $\{U_{x_{jk}} : j \in \mathbb{N}\}$ of $\{U_x : x \in A_k\}$, covering A_k. Denote $U_{x_{jk}}$ by U_{jk} and

$B_{x_{jk}}$ by B_{jk}. Since $\mu(f, U_{jk}) = k$ for each $j \in \mathbb{N}$, U_{jk} is a normal neighborhood of every $z \in A_k \cap U_{jk}$. Now, since $g_{U_{jk}} \in W^{1,n}(B_{jk}, \mathbb{R}^n)$, the Sobolev embedding theorem implies that the restriction of $g_{U_{jk}}$ to $S_r \cap B_{jk}$ satisfies condition N for all $r \in (0,1)$, $S_r \cap B_{jk} \neq \emptyset$, outside a set $E_{jk} \subset (0,1)$ of linear measure zero. Notice that, since $\mu(f, U_{jk}) = k$, for each $y \in f(A_k) \cap B_{jk}$, $g_{U_{jk}}(y)$ is the unique point $x \in A_k \cap U_{jk}$ for which $f(x) = y$. Hence

$$h_{jk} := f_{|U_{jk} \cap A_k} : U_{jk} \cap A_k \to B_{jk} \cap f(A_k)$$

is bijective, and the inverse mapping is the restriction of $g_{U_{jk}}$ to $f(A_k)$. In particular, condition co-N holds true for $h_{jk\,|S'_r}$ for all $r \in (0,1) \setminus E_{jk}$ for which $S_r \cap B_{jk} \neq \emptyset$.

Now denote

$$E = \bigcup_{k=1}^{m} \bigcup_{j=1}^{\infty} E_{jk} \subset (0,1).$$

Then E is of linear measure zero. Furthermore, choose $r \in (0,1) \setminus E$, and a set $F \subset S'_r$ so that $\mathcal{H}^{n-1}(f(F)) = 0$. If $\mathcal{H}^{n-1}(F) > 0$, then $\mathcal{H}^{n-1}(F \cap A_k) > 0$ for some $k \in \{1, \dots, m\}$. Moreover, then $\mathcal{H}^{n-1}(F \cap A_k \cap U_{jk}) > 0$ for some $j \in \mathbb{N}$. However, this is a contradiction, since condition co-N is satisfied by the restriction of h_{jk} to S'_r. The proof is complete. □

We are now in a position to prove a Väisälä-type inequality between the moduli of sphere families and families of components of their preimages.

Theorem 3.4. *Suppose the situation is as above. Let $I \subset (0,1)$ be a Borel set. If $\Lambda := \{S_t : t \in I\}$ and $\Lambda' = \{S'_t : t \in I\}$, then*

$$M_S \Lambda \leq \frac{K_O^{\frac{1}{n-1}}}{m} M_S^m \Lambda'.$$

Proof. Let $\rho : U \to [0, \infty]$ be a Borel function such that

$$\int_{S'_t} \rho(x)\, d\mathcal{H}^{n-1}(x) \geq m \quad \text{for all } t \in I.$$

Set $F = B_f \cup E$, where E is the set of all points $x \in U$ so that either f is not differentiable at x or $J_f(x) = 0$. As noted in the Section 2, we have $|F| = 0$. We define a function $\rho' : B_1 \to [0, \infty]$ by setting

$$\rho'(y) = \frac{1}{m} \sum_{x \in f^{-1}(y) \cap U} \sigma(x),$$

where

$$\sigma(x) = \begin{cases} \frac{\rho(x)}{l(D^{\#} f(x))}, & x \in U \setminus F \\ 0 & \text{otherwise.} \end{cases}$$

Then ρ' is a Borel function, which is seen as in the proof of [17] II Theorem 9.1. We need to show that for almost all spheres S_t in Λ,

$$\int_{S_t} \rho'(y)\, d\mathcal{H}^{n-1}(y) \geq 1.$$

Fix a sphere S_t on which the property co-N holds for the restriction of f to S'_t. By Lemma 3.3 almost all spheres in Λ have this property. We also assume, without loss of generality, that $\mathcal{H}^{n-1}(F \cap S_t) = 0$. We consider the family of all balls $B(x,s)$ with the following properties:

1. $x \in S_t$
2. $B(x,s) \cap S_t \cap f(B_f) = \emptyset$
3. each component of $f^{-1}(B(x,s)) \cap U$ is a normal neighborhood of x.

By Vitali's covering theorem we find a countable subfamily $\{V_i\}$ of disjoint balls that almost covers S_t. For each V_i there are m quasiconformal homeomorphisms $h_j : V_i \to h_j(V_i)$ so that $f \circ h_j = \mathrm{Id}$ for each j. By the property co-N of f on S'_t, the property N holds for the restriction of each h_j to S_t. Hence we can use Lemma 3.1 in order to have

$$\begin{aligned}
&\int_{V_i \cap S_t} \rho'(y)\, d\mathcal{H}^{n-1}(y) = \frac{1}{m} \int_{V_i \cap S_t} \sum_{x \in f^{-1}(y) \cap U} \sigma(x)\, d\mathcal{H}^{n-1}(y) \\
= \quad & \frac{1}{m} \sum_j \int_{V_i \cap S_t} \rho(h_j(y)) |D^{\#} h_j(y)|\, d\mathcal{H}^{n-1}(y) \\
\geq \quad & \frac{1}{m} \sum_j \int_{h_j(V_i \cap S_t)} \rho(x)\, d\mathcal{H}^{n-1}(x) = \frac{1}{m} \int_{f^{-1}(V_i \cap S_t) \cap U} \rho(x)\, d\mathcal{H}^{n-1}(x).
\end{aligned}$$

Summing over i yields

$$\begin{aligned}
&\int_{S_t} \rho'(y)\, d\mathcal{H}^{n-1}(y) = \sum_i \int_{(V_i \cap S_t)} \rho'(y)\, d\mathcal{H}^{n-1}(y) \\
\geq \quad & \frac{1}{m} \sum_i \int_{f^{-1}(V_i \cap S_t) \cap U} \rho(x)\, d\mathcal{H}^{n-1}(x) \geq \frac{1}{m} \int_{S'_t} \rho(x)\, d\mathcal{H}^{n-1}(x) \geq 1,
\end{aligned}$$

as desired. In order to estimate the integral $\int_{B(0,1)} \rho'(y)^{\frac{n}{n-1}}\, dy$, we use a method similar to the one used above. We find a countable family of disjoint balls $V_i \subset B_1$ so that $V_i \cap f(B_f) = \emptyset$, $f^{-1}(V_i) \cap U = \cup h_j(V_i)$, where the mappings h_j are the inverse mappings of the restrictions of f to different components of $f^{-1}(V_i) \cap U$, and $\cup V_i$ almost covers B_1. By Hölder's inequality, we have

$$\begin{aligned}
&\int_{V_i} \rho'(y)^{\frac{n}{n-1}}\, dy = \int_{V_i} \Big(\frac{1}{m} \sum_{x \in f^{-1}(y) \cap U} \sigma(x) \Big)^{\frac{n}{n-1}} dy \\
\leq \quad & \int_{V_i} \frac{1}{m} \sum_{x \in f^{-1}(y) \cap U} \sigma(x)^{\frac{n}{n-1}}\, dy.
\end{aligned}$$

By using the definition of $\sigma(x)$, quasiconformality of the h_j:s and the change of variables formula, we can estimate the last term, so that

$$\begin{aligned}
&\frac{1}{m}\sum_j \int_{V_i} \rho(h_j(y))^{\frac{n}{n-1}} |D^{\#} h_j(y)|^{\frac{n}{n-1}}\, dy \\
\leq\ & \frac{K_O^{\frac{1}{n-1}}}{m} \sum_j \int_{V_i} \rho(h_j(y))^{\frac{n}{n-1}} J_{h_j}(y)\, dy \\
=\ & \frac{K_O^{\frac{1}{n-1}}}{m} \int_{f^{-1}(V_i)\cap U} \rho(x)^{\frac{n}{n-1}}\, dx.
\end{aligned}$$

By summation over i we have

$$\int_{B_1} \rho'(y)^{\frac{n}{n-1}}\, dy \leq \frac{K_O^{\frac{1}{n-1}}}{m} \int_U \rho(x)^{\frac{n}{n-1}}\, dx.$$

The proof is complete.

□

By using polar coordinates and Hölder's inequality, it is easy to see that for a sphere family Λ as in Theorem 3.4,

$$M_S(\Lambda) = \omega_{n-1}^{\frac{-1}{n-1}} \int_I \frac{dt}{t}.$$

In particular, when $I = (r_1, r_2)$, we have

$$M_S(\Lambda) = \omega_{n-1}^{\frac{-1}{n-1}} \log \frac{r_2}{r_1}.$$

Also, clearly $M_S^m(\Lambda') = m^{\frac{n}{n-1}} M_S(\Lambda')$. In the next two sections we develop a method to give efficient upper bounds for the moduli $M_S(\Lambda')$ in the case where the distortion of f is close to one.

4 Quantitative isoperimetry

In this section we apply a quantitative isoperimetric inequality result by Hall [9] in order to give 'inverse' distortion estimates for quasiregular mappings with small isoperimetric defect. The results presented in this section are easy to prove in the planar case, and thus we shall restrict ourselves to dimensions higher than two. Recall that the isoperimetric inequality says that for a bounded domain (or a more general set) $\Omega \subset \mathbb{R}^n$, we have

$$|\Omega| \leq C_I \mathcal{H}^{n-1}(\partial\Omega)^{\frac{n}{n-1}}, \tag{4.1}$$

where $C_I = n^{\frac{-n}{n-1}} \alpha_n^{1-n}$. For domains equality occurs in (4.1) if and only if Ω is a ball. We define the isoperimetric defect $\delta(\Omega)$ of Ω by setting

$$\delta(\Omega) = 1 - \frac{|\Omega|}{C_I \mathcal{H}^{n-1}(\partial\Omega)^{\frac{n}{n-1}}}.$$

Moreover, the Fraenkel asymmetry $\lambda(\Omega)$ of Ω is defined as

$$\lambda(\Omega) = \min_{x \in \mathbb{R}^n} \frac{|B(x,R) \setminus \Omega|}{|\Omega|},$$

where R is chosen so that $|B_R| = |\Omega|$. Here the minimizing point x always exists, but may not be unique. The following result is a modification of a theorem by Hall [9] Theorem 1, see also [10] Theorem 6.3. It shows that for domains small isoperimetric defect implies small Fraenkel asymmetry. In [9] it is assumed that the boundaries are smooth and that the Fraenkel asymmetry is small, whereas here we need the result for all domains, assuming apriori smallness from the isoperimetric defect.

Theorem 4.1. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain and $n \geq 3$. If*

$$\delta(\Omega) \leq 1 - (1 + 400^{-1} n^{\frac{-13}{2}})^{\frac{-n}{n-1}} =: C_1, \tag{4.2}$$

then

$$\lambda(\Omega) \leq \Big(C(n) \Big(\frac{1}{(1-\delta(\Omega))^{\frac{n-1}{n}}} - 1 \Big) \Big)^{\frac{1}{4}} =: C_2(\delta(\Omega), n), \tag{4.3}$$

i.e.

$$\delta(\Omega) \geq 1 - \left(\frac{1}{1 + \frac{\lambda(\Omega)^4}{C(n)}} \right)^{\frac{n}{n-1}},$$

where $C(n) = 175000 n^{2n+17/2}$.

Proof. Only small modifications to [9] Theorem 1 are needed. Let us first prove the claim for smooth domains. We Schwarz symmetrize Ω with respect to the coordinate axis x_i for each $i = 1, \dots, n$. The symmetrized domains are denoted by $\tilde{\Omega}_i$. It is well known that

$$\mathcal{H}^{n-1}(\partial\Omega) \geq \mathcal{H}^{n-1}(\partial\tilde{\Omega}_i) \quad \text{for each } i = 1, \dots, n.$$

Now [9] Theorem 2 implies that if $\delta(\tilde{\Omega}_i) \leq C_1$, then $\lambda(\tilde{\Omega}_i) < \frac{1}{2n}$. Moreover, [10] Theorem 6.3 says that if $\max_i \lambda(\tilde{\Omega}_i) < \frac{1}{2n}$, then

$$\lambda(\Omega) < 1 - (2n)^{-n}. \tag{4.4}$$

On the other hand, when (4.4) holds, we have $\lambda(\Omega)^2 < (2n)^n(1 - \lambda(\Omega))$. Now the claim for smooth domains is deduced as in [9] page 163.

Let us then verify the claim for general domains. We first notice that in [9] the smoothness assumption is only needed for Schwarz symmetrized sets. Since Schwarz symmetrizations of domains are also domains, it is sufficient by [9] Theorem 2 to approximate a domain $\tilde{\Omega}$, having an axis of symmetry, by smooth domains Ω_j so that $\partial\Omega_j$ lies in the $1/j$-neighborhood of $\partial\tilde{\Omega}$ and $\mathcal{H}^{n-1}(\partial\Omega_j) < \mathcal{H}^{n-1}(\partial\tilde{\Omega}) + 1/j$. By the symmetry property of $\tilde{\Omega}$ we can further assume that $\tilde{\Omega}$ is a plane domain whose boundary is a closed path with finite length. As the boundaries of such domains can easily be approximated by smooth plane domains with the required properties, the proof is complete. $\square$

Remark 4.2. As noted in [9], the result holds true also without the assumption (4.2), possibly with weaker constant $C(n)$ in (4.3).

Before applying Theorem 4.1, we state, for future reference, a distortion lemma for the inverse dilatation of quasiregular mappings. Results similar to this one are standard in the theory of quasiregular mappings, cf. [17] Lemma 4.1.

Lemma 4.3. *Let $f : \Omega \to \mathbb{R}^n$ be a quasiregular mapping and $B(f(x), r) \subset f(\Omega)$. Suppose that $U(x, f, 2L(x, f, L^*(x, f, r))$ is a normal domain of f. Then*

$$\frac{L^*(x,f,r)}{l^*(x,f,r)} \le \exp\Big(\Big(\frac{\omega_{n-1}K_I}{\sqrt{3}C_n}\Big)^{\frac{1}{n-1}}\Big) =: C^*(n,K), \tag{4.5}$$

where C_n is a constant only depending on n, given in [21] (10.11).

Proof. Since $U(x, f, r)$ is a normal domain, the closures of the sets $V_1 := f(B(x, l^*(x, f, r)))$ and $V_2 := f(B(x, L^*(x, f, r)))$ both intersect $S(x, r)$. By [21] Theorem 10.12, we have

$$M\Gamma \ge \sqrt{3}C_n,$$

where C_n is as in [21] (10.11) and Γ is the family of all paths joining V_1 and $\mathbb{R}^n \setminus V_2$. Consider the lifts of all paths of Γ starting at $B(x, l^*(x, f, r))$. Since $U(x, f, 2L(x, f, L^*(x, f, r))$ is a normal domain, then so is the x-component U_2 of $f^{-1}(V_2)$. Since

$$B(x, L^*(x, f, r)) \subset U_2 \quad \text{and} \quad f\partial U_2 = \partial V_2,$$

we see that for each path of Γ there exists a lift starting at $B(x, l^*(x, f, r))$ and terminating at $\mathbb{R}^n \setminus B(x, L^*(x, f, r))$. Denote the family of all these lifts by Γ'. Then $f\Gamma' = \Gamma$, and we can use the Poletsky inequality in order to have

$$\sqrt{3}C_n \le M\Gamma \le K_I M\Gamma' \le \omega_{n-1}K_I \log^{1-n}\frac{L^*(x,f,r)}{l^*(x,f,r)},$$

which yields the claim. $\square$

We will now return to the situation of the end of Section 3; $f : \Omega \to \mathbb{R}^n$, $n \geq 2$, is a quasiregular mapping, $f(0) = 0$, $U \subset\subset \Omega$ is a normal neighborhood of 0 such that $f(U) = B_1$ and $i(0, f) = m$. Also, we let the assumptions of Lemma 4.3 be valid for $x = 0$ and $r = 1$ (and thus also for any $r \in (0, 1)$). Recall that for each point x there exists a radius r satisfying this assumption, so that no generality is lost in this sense.

For $t \in (0, 1)$, set

$$\alpha(B_t') = \inf\left\{\frac{R}{r} : S_t' \subset B(x, R) \setminus \overline{B}(x, r), x \in \mathbb{R}^n\right\}.$$

We will next show that if the isoperimetric defect $\delta(B_t')$ is small, then $\alpha(B_t')$ is close to one, with quantitative bounds depending only on K_I and n.

Proposition 4.4. *Let the situation be as above. Suppose that*

$$\delta(B_t') < \min\left\{C_1, 1 - \left(1 + \frac{1}{C^*(n,K)^{4n^2}(n+1)^{4n}C(n)}\right)^{\frac{-n}{n-1}}\right\} =: C_3, \tag{4.6}$$

where C_1 and $C(n)$ are as in (4.2) *and* (4.3)*, respectively, and $C^*(n, K)$ is as in Lemma 4.3. Then*

$$\alpha(B_t') \leq \frac{1 + A}{1 - A},$$

and

$$A = 2(n+1)^{\frac{1}{n}} C^*(n,K) C_2(\delta(B_t'), n)^{\frac{1}{n^2}}, \tag{4.7}$$

where $C_2(\delta(B_t'), n)$ is as in (4.3)*.*

Proof. Suppose that

$$\lambda(B_t') = \frac{|B(y, s) \setminus B_t'|}{|B_t'|}.$$

Now, if $B(y, d) \cap B_t' = \emptyset$, then by the definition of Fraenkel asymmetry

$$\alpha_n d^n \leq \lambda(B_t')|B_t'| = \alpha_n \lambda(B_t') s^n.$$

Hence we find a point $x \in B_t'$ so that $d = |x - y| < \lambda(B_t')^{\frac{1}{n}} s$. Since $d < s$, we have

$$\begin{aligned} \frac{|B(x,s) \setminus B_t'|}{|B_t'|} &\leq \lambda(B_t') + \frac{\alpha_n((d+s)^n - s^n)}{|B_t'|} \leq \lambda(B_t') \\ &+ \frac{\alpha_n n d s^{n-1}}{|B_t'|} \leq \lambda(B_t') + n\lambda(B_t')^{\frac{1}{n}} \leq (n+1)\lambda(B_t')^{\frac{1}{n}}. \end{aligned} \tag{4.8}$$

Set $R = \min\{h : B_t' \subset B(x, h)\}$ and $r = \max\{h : B(x, h) \subset B_t'\}$. Recall that since U is a normal neighborhood of 0, $\{f^{-1}(0)\} \cap B_t' = \{0\}$. We will first give an upper bound for R with respect to s. In order to do that we will need to estimate $\lambda(B_t')$.

We next show that there exists a ball inside B_t', not intersecting $B(x,s)$, whose size has a lower bound depending on $R-s$, K and n. We split the proof of this fact into two cases.
Case 1: $0 \notin B(x, R - \frac{R-s}{2})$.

Then by Lemma 4.3 there exists a ball

$$B(0,h) \subset B_t' \setminus B(x,s),$$

so that

$$h > \min\left\{C^*(n,K)^{-1}s, \frac{(R-s)}{2}\right\}.$$

In view of (4.8) this yields

$$(n+1)\lambda(B_t')^{\frac{1}{n}} > \min\left\{C^*(n,K)^{-n}, \left(\frac{R-s}{2s}\right)^n\right\}.$$

Case 2: $0 \in B(x, R - \frac{R-s}{2})$.

Fix a point $z \in S_t'$ such that $|z-x| = R$. Now one may choose a lift γ' of the path $\gamma : [0,1] \to \mathbb{R}^n$; $\gamma(t) = tf(z)$, so that $\gamma'(0) = 0$ and $\gamma'(1) = z$. By our assumption there exists a point $w = \gamma'(t_0)$ so that $|w-x| = R - \frac{R-s}{2}$. Consider the w-component $U_1 := U(w, f, |f(z) - f(y)|)$ of the preimage of the ball $B(f(w), |f(w) - f(z)|)$. Since $z \in \overline{U_1}$,

$$L^*(w, f, |f(w) - f(z)|) \geq \frac{R-s}{2}.$$

Application of Lemma 4.3 gives

$$l^*(w, f, |f(w) - f(z)|) \quad \geq \quad \frac{L^*(w, f, |f(w) - f(z)|)}{C^*(n,K)} \geq \frac{R-s}{2C^*(n,K)},$$

and so there exists a ball

$$B(w,h) \subset B_t' \setminus B(x,s),$$

so that

$$h > \frac{R-s}{2C^*(n,K)}.$$

Thus we have a lower bound also in this case:

$$(n+1)\lambda(B_t')^{\frac{1}{n}} > \left(\frac{R-s}{2C(n,K)^*s}\right)^n.$$

By combining the two cases we see that

$$(n+1)\lambda(B_t')^{\frac{1}{n}} \geq \min\left\{C^*(n,K)^{-n}, \left(\frac{R-s}{2C^*(n,K)s}\right)^n\right\} =: C_R.$$

In view of Theorem 4.1, we have

$$C_R \le (n+1)C_2(\delta(B_t'), n)^{\frac{1}{n}}$$

whenever $\delta(B_t') < C_1$. Hence, when

$$\delta(B_t') < \min\Big\{C_1, 1 - \Big(1 + \frac{1}{C^*(n,K)^{4n^2}(n+1)^{4n}C(n)}\Big)^{\frac{-n}{n-1}}\Big\}$$

where C_1 and $C(n)$ are as in Theorem 4.1, the definition of C_R implies that

$$\Big(\frac{R-s}{2C^*(n,K)s}\Big)^n \le (n+1)\lambda(B_t')^{\frac{1}{n}} \le (n+1)C_2(\delta(B_t'), n)^{\frac{1}{n}},$$

i.e.

$$R \le s\Big(1 + 2(n+1)^{\frac{1}{n}}C^*(n,K)C_2(\delta(B_t'), n)^{\frac{1}{n^2}}\Big). \tag{4.9}$$

Now we will give a lower bound for r with respect to s. Let $v \in S_t'$ be a point such that $|v - x| = r$. Now $\gamma : [1,\infty) \to \mathbb{R}^n$; $\gamma(t) = tf(v)$ has a lift γ' starting at v so that $\gamma'([1,\infty)) \subset \mathbb{R}^n \setminus B_t'$ and so that there exists a point $q = \gamma'(t_0)$ for which $|q - x| = \frac{s-r}{2}$. As in the case 2 above, we can deduce that there exists a ball

$$B\Big(q, \frac{s-r}{2C^*(n,K)}\Big) \subset B(x,s) \setminus B_t'.$$

Hence

$$\Big(\frac{s-r}{2C^*(n,K)s}\Big)^n \le (n+1)\lambda(B_t')^{\frac{1}{n}} \le (n+1)C_2(\delta(B_t'), n)^{\frac{1}{n}},$$

so that

$$r \ge s\Big(1 - 2(n+1)^{\frac{1}{n}}C^*(n,K)C_2(\delta(B_t'), n)^{\frac{1}{n^2}}\Big). \tag{4.10}$$

The claim follows by combining estimates (4.9) and (4.10). □

Remark 4.5. (i) In the setting of Proposition 4.4, the planar case differs essentially from the higher dimensional case. In particular, the result is true for any bounded planar domain, with bounds sharper than in (4.7), cf. [4] and the references therein. In higher dimensions results like this do not hold for general domains, which is seen by gluing thin 'needles' to a ball.

(ii) As far as we know, Proposition 4.4 is new also in the case of quasiconformal mappings.

5 Modulus inequalities and isoperimetric defect

Let f be a quasiregular mapping as in the situation described before Proposition 4.4. We will normalize the set U by setting $l^*(0, f, 1) = 1$. In this section we show that, in a certain quantitative and asymptotically sharp manner, the isoperimetric defects of the sets B'_t can be controlled by the distortion of f. One essential point in these estimates is the fact that they do not depend on multiplicity, so that f behaves like a quasiconformal mapping in this sense. For the linear dilatation $H(x, f) = \limsup_{r \to 0} L(x, f, r)/l(x, f, r)$ of quasiconformal mappings, good estimates have been obtained by using estimates related to the Grötzsch condenser, cf. [20], [18].

We first recall a continuity estimate from [17] III Lemma 4.7.

Lemma 5.1. *In the above situation, for all $x \in B_1$, we have*

$$\log \frac{1}{C^*(n, K)|x|} \leq \left(\frac{K_I}{m} \right)^{\frac{1}{n-1}} \log \frac{1}{|f(x)|}, \tag{5.1}$$

i.e.

$$|f(x)| \leq (C^*(n, K)|x|)^{\mu},$$

where $C^*(n, K)$ is as in Lemma 4.3 and $\mu = (\frac{m}{K_I})^{\frac{1}{n-1}}$.

Now we want to give an upper bound for the surface modulus of a family of surfaces S'_t with respect to the isoperimetric defects of the sets B'_t. For this we use a point symmetrization method. See [6], [7] and [23] for results related to the setting considered here.

For each $t \in (0, 1)$, the point symmetrizations of the set B'_t and its closure will be the open ball $B(0, \alpha^{\frac{-1}{n}} |B'_t|^{\frac{1}{n}})$ and its closure, respectively. Thus the symmetrization of each S'_t will be a sphere enclosing a ball with the same volume as the set enclosed by S'_t. We define a function $p : (0, 1) \to (0, \infty)$ by setting $p(t) = \alpha^{\frac{-1}{n}} |B'_t|^{\frac{1}{n}}$. Thus the image of the set S'_t under point symmetrization is the sphere $S_{p(t)}$. Note that p is strictly increasing. The following properties of the point symmetrization are direct consequences of the definition:

$$\mathcal{H}^{n-1}(S_{p(t)}) = (1 - \delta(B'_t))^{\frac{n-1}{n}} \mathcal{H}^{n-1}(S'_t), \tag{5.2}$$

if $I \subset (0, 1)$ is a measurable set, $P = \cup_{t \in I} S'_t$ and $\hat{P} = \cup_{t \in I} S_{p(t)}$ is the corresponding point symmetrized set, then

$$|P| = |\hat{P}|. \tag{5.3}$$

Properties (5.2) and (5.3) now give a useful estimate for the surface modulus under consideration. In what follows, we shall integrate over symmetrized sets, and so it is convenient to use the following notation: if $s = p(t)$, we denote the isoperimetric defect of B'_t by

$$\delta_s := \delta(B'_t).$$

Lemma 5.2. *Suppose that $I \subset (0,1)$ is a Borel measurable set and*

$$\Lambda = \{S'_t : t \in I\}.$$

Then

$$M_S^m(\Lambda) \le \Big(\frac{m^n}{\omega_{n-1}}\Big)^{\frac{1}{n-1}} \int_{p(I)} \frac{1-\delta_s}{s}\, ds.$$

Proof. We define $\rho : U \to [0,\infty]$ by setting

$$\rho(x) = \begin{cases} m\mathcal{H}^{n-1}(S'_t)^{-1}, & x \in S'_t \text{ for some } t \in I \\ 0, & \text{otherwise.} \end{cases}$$

Then ρ is a Borel function and

$$\int_{S'_t} \rho(x)\, \mathcal{H}^{n-1}(x) \ge m$$

for each $t \in I$. By properties (5.2) and (5.3), the use of polar coordinates yields (here $\hat{P}$ is as in (5.3))

$$\begin{aligned} \int_U \rho(x)^{\frac{n}{n-1}}\, dx &= m^{\frac{n}{n-1}} \int_{\hat{P}} \frac{1-\delta_{|x|}}{\omega_{n-1}^{\frac{n}{n-1}} |x|^n}\, dx \\ &= \Big(\frac{m^n}{\omega_{n-1}}\Big)^{\frac{1}{n-1}} \int_{p(I)} \frac{1-\delta_s}{s}\, ds, \end{aligned}$$

as desired. □

We are now in a position where we can efficiently apply the surface modulus inequality, Theorem 3.4. This is done by combining the inequality with Lemmas 5.1 and 5.2. Recall that the logarithmic measure $\mu(E)$ of a measurable set $E \subset (0,\infty)$ is defined as the number

$$\mu(E) = \int_E \frac{dr}{r}.$$

Now for $r \in (0,1)$ and for a positive number h, consider the set

$$E_h^r := \{s : \delta_s > h, s = p(t), t \in (r,1)\}.$$

Recall that in our situation $l^*(0,f,1) = 1$.

Proposition 5.3. *Suppose the situation is as above, and that $K < (1+\epsilon)^{\frac{1}{3}}$, $\epsilon < 10^{-5}$. Then, for a small enough $r \in (0,1)$, depending only on n,*

$$\mu(E^r_{\epsilon^{1/4}}) \le 4\epsilon^{\frac{1}{4}} \log \frac{1}{p(r)}.$$

Proof. By the distortion Lemma 4.3 and our normalization, we have $p(1) \in (1, C^*(n,K))$. Since

$$\frac{\mu\big((1, C^*(n,K))\big)}{\log \frac{1}{p(r)}} \to 0 \quad \text{as} \quad r \to 0,$$

we may assume that $p(1) = 1$. Consider the sphere family $\Lambda = \{S_t : t \in (r,1)\}$, and denote $\Lambda' = \{S'_t : t \in (r,1)\}$. Suppose that r is so small that $l^*(0,f,r) \leq C^*(n,K)^{-j-1}$. By Lemma 4.3,

$$\log \frac{1}{p(r)} \leq \log\Big(\frac{1}{l^*(0,f,r)}\Big) \leq \frac{\log C^*(n,K)^{j+1}}{\log C^*(n,K)^j} \log\Big(\frac{1}{C^*(n,K)|x|}\Big)$$

for some $x \in S'_r$. For the right hand side we can use Lemma 5.1, so that

$$\begin{aligned}\Big(1+\frac{1}{j}\Big)\log\Big(\frac{1}{C^*(n,K)|x|}\Big) &\leq \Big(1+\frac{1}{j}\Big)\Big(\frac{K_I}{m}\Big)^{\frac{1}{n-1}} \log\frac{1}{r} \\ &= \Big(1+\frac{1}{j}\Big)\Big(\frac{\omega_{n-1}K_I}{m}\Big)^{\frac{1}{n-1}} M_S(\Lambda).\end{aligned}$$

On the other hand, Theorem 3.4, combined with Lemma 5.2, yields

$$\begin{aligned}\Big(\frac{\omega_{n-1}K_I}{m}\Big)^{\frac{1}{n-1}} M_S(\Lambda) &\leq \Big(\frac{\omega_{n-1}K_IK_O}{m^n}\Big)^{\frac{1}{n-1}} M_S^m(\Lambda') \\ &\leq (K_IK_O)^{\frac{1}{n-1}} \int_{p(r)}^1 \frac{1-\delta_s}{s}\,ds,\end{aligned}$$

By combining the estimates we have

$$\log\frac{1}{p(r)} \leq \Big(1+\frac{1}{j}\Big)(K_IK_O)^{\frac{1}{n-1}} \int_{p(r)}^1 \frac{1-\delta_s}{s}\,ds. \tag{5.4}$$

Note that this estimate does not depend on multiplicity m.

Let β be a small positive constant, so that

$$\log\frac{1}{p(r)} = k\log(1+\beta),$$

where k is an integer. We divide the interval $(p(r),1)$ to intervals

$$A_i := ((1+\beta)^{-i-1}, (1+\beta)^{-i}), \quad i \in L = \{0,1,\ldots,k-1\}.$$

Set

$$J = \left\{ i \in L : \int_{A_i} \frac{1-\delta_s}{s}\,ds \leq (1-\epsilon^{\frac{1}{2}})\log(1+\beta)\right\},$$

$J' = L \setminus J$ and $N = \#J$. Then

$$\int_{p(r)}^{1} \frac{1-\delta_s}{s}\,ds = \sum_{i=0}^{k-1} \int_{A_i} \frac{1-\delta_s}{s}\,dt = \sum_{J} \int_{A_i} \frac{1-\delta_s}{s}\,ds$$

$$+ \quad \sum_{J'} \int_{A_i} \frac{1-\delta_s}{s}\,dt \le N(1-\epsilon^{\frac{1}{2}})\log(1+\beta) + (k-N)\log(1+\beta)$$

$$= \quad \Big(1 - \frac{N\epsilon^{\frac{1}{2}}}{k}\Big)\log\frac{1}{p(r)},$$

and combining with (5.4) yields

$$\frac{N}{k} \le \frac{\Big(1+\frac{1}{j}\Big)(K_I K_O)^{\frac{1}{n-1}} - 1}{\Big(1+\frac{1}{j}\Big)(K_I K_O)^{\frac{1}{n-1}}\epsilon^{\frac{1}{2}}}. \tag{5.5}$$

By choosing r to be so small that

$$\Big(1+\frac{1}{j}\Big)K^{\frac{2}{n-1}} \le K^3,$$

(recall that $K_I K_O \le K^2$), and using our assumption on K, we can estimate the right hand side in (5.5) in order to have

$$\frac{N}{k} \le \epsilon^{\frac{1}{2}}. \tag{5.6}$$

This estimate tells us that in most of the annuli A_i, we have

$$(1-\epsilon^{\frac{1}{2}})\log(1+\beta) \le \int_{A_i} \frac{1-\delta_s}{s}\,dt.$$

Fix such an annulus A_i, and consider the sets

$$G = \{s \in A_i : \delta_s \le \epsilon^{\frac{1}{4}}\}, \quad F = A_i \setminus G.$$

Now there exists a constant $C(\beta)$, $C(\beta) \to 1$ as $\beta \to 0$, so that on A_i we have

$$C(\beta)\beta(1-\epsilon^{\frac{1}{2}}) \le (1-\epsilon^{\frac{1}{2}})\log(1+\beta) \le \int_{A_i} \frac{1-\delta_s}{s}\,ds \tag{5.7}$$

$$= \int_G \frac{1-\delta_s}{s}\,ds + \int_F \frac{1-\delta_s}{s}\,ds$$

$$\le \int_{(1+\beta)^{-i-1}}^{(1+\beta)^{-i-1}+|G|} \frac{ds}{s} + \int_{(1+\beta)^{-i-1}}^{(1+\beta)^{-i-1}+|F|} \frac{1-\epsilon^{\frac{1}{4}}}{s}\,ds$$

We denote by $W \in [0,1]$ the number for which

$$W((1+\beta)^{-i} - (1+\beta)^{-i-1}) = |F|,$$

and estimate the right hand side in (5.7) in order to have

$$\begin{aligned}
&\int_{(1+\beta)^{-i-1}}^{(1+\beta)^{-i-1}+|G|} \frac{ds}{s} + \int_{(1+\beta)^{-i-1}}^{(1+\beta)^{-i-1}+|F|} \frac{1-\epsilon^{\frac{1}{4}}}{s}\, ds \\
= \ & \log\left(1 + \frac{|G|}{(1+\beta)^{-i-1}}\right) + (1-\epsilon^{\frac{1}{4}}) \log\left(1 + \frac{|F|}{(1+\beta)^{-i-1}}\right) \\
= \ & \log(1 + (1-W)\beta) + (1-\epsilon^{\frac{1}{4}}) \log(1 + W\beta) \\
\leq \ & (1-W)\beta + (1-\epsilon^{\frac{1}{4}}) W\beta = (1 - W\epsilon^{\frac{1}{4}})\beta,
\end{aligned}$$

where the inequality $\log(1+x) \leq x$ is used. Hence

$$C(\beta)\beta(1-\epsilon^{\frac{1}{2}}) \leq (1 - W\epsilon^{\frac{1}{4}})\beta,$$

i.e.

$$W\epsilon^{\frac{1}{4}} \leq C(\beta)\epsilon^{\frac{1}{2}} + 1 - C(\beta).$$

We require β to be so small that $W \leq \frac{3}{2}\epsilon^{\frac{1}{4}}$, so that

$$\mu(F) \leq 2\epsilon^{\frac{1}{4}} \log(1+\beta) \tag{5.8}$$

when $\epsilon < 10^{-5}$. By combining the estimates (5.6) and (5.8), we see that

$$\mu(E^r_{\epsilon^{\frac{1}{4}}}) \leq 2(\epsilon^{\frac{1}{2}} + \epsilon^{\frac{1}{4}}) \log \frac{1}{p(r)} \leq 4\epsilon^{\frac{1}{4}} \log \frac{1}{p(r)},$$

as desired. □

6 Proof of Theorem 1.1

We shall assume that the situation is as in Section 5, so that $K \leq (1+\epsilon)^{\frac{1}{3}}$ holds. An estimate for ϵ will be given at the end of the proof. We will here also assume that $n \geq 3$, although this assumption will only be used at the end of the proof. We assume that 0 is a branch point of f, so that

$$2 \leq m = i(0,f) \leq i(z,f) \text{ for all } z \in B_f.$$

Now the goal is to arrive at a contradiction, showing that f has to be a local homeomorphism at 0. Recall that by [17] III Corollary 5.8,

$$m \leq 9K_I \leq 10 \tag{6.1}$$

when K is small enough.

We first state a lemma that gives an estimate for the surface modulus of the family of all surfaces separating the unit sphere and another sphere inside the unit ball. This result probably appears somewhere in the literature, but we cannot give a direct reference.

Lemma 6.1. *Suppose that $S(x_1, 1-u) \subset B_1$. Denote by Λ the family of all sets separating $S(x_1, 1-u)$ and S_1. Denote $|x_1| = Lu$, $L \in [0,1)$. Then*

$$M_S(\Lambda) = \omega_{n-1}^{\frac{-1}{n-1}} \log \frac{1}{\tanh \frac{\rho(L,u)}{2}},$$

where

$$\rho(L,u) = \frac{1}{2} \log \frac{(2-u)^2 - (Lu)^2}{u^2(1-L^2)}.$$

Proof. Denote by Γ the family of all of all paths joining $S(x_1, 1-u)$ and $S(0,1)$. By [23] (see also [7]), $M(\Gamma) = M_S(\Lambda)^{\frac{1}{1-n}}$. We calculate the hyperbolic radius $\rho(L,u)$ of $B(x_1, 1-u)$. Recall that euclidean balls in $B(0,1)$ are also balls in the hyperbolic metric. Now $2\rho(L,u)$ equals the hyperbolic length of the geodesic segment $B(x_1, 1-u) \cap l$, where l is the geodesic line intersecting 0 and x_1. Hence we get $\rho(L,u)$ by integrating;

$$\begin{aligned} \rho(L,u) &= \frac{1}{2} \int_{Lu-1+u}^{Lu-1+u} \frac{2ds}{1-s^2} = \frac{1}{2} \log \frac{(2+(L-1)u)(2-(L+1)u)}{(1-L^2)u^2} \\ &= \frac{1}{2} \log \frac{(2-u)^2 - (Lu)^2}{u^2(1-L^2)}. \end{aligned}$$

We choose a Möbius transformation T that maps $B(0,1)$ onto itself so that the hyperbolic center of $B(x_1, 1-u)$ gets mapped to 0. Since T is an isometry in the hyperbolic metric, $T(B(x_1, 1-u)) = D(0, \rho(L,u))$ (here D means hyperbolic ball). Since T is conformal and $D(0, \rho(L,u)) = B(0, M)$ for some M, we have

$$M(\Gamma) = M(\Gamma') = \omega_{n-1} \log^{1-n} \frac{1}{M},$$

where Γ' is the family of all paths joining S_1 and S_M. By [19] I 2.22, we have $M = \tanh \frac{\rho(L,u)}{2}$. The proof is complete. $\square$

Remark 6.2. It is essential in Lemma 6.1 that $\tanh \frac{\rho(0,u)}{2} = 1-u$, $\tanh \rho(L,u)$ is increasing with respect to L and decreasing with respect to u, and

$$\tanh \frac{\rho(L,u)}{2} \to 1 \quad \text{as} \quad L \to 1.$$

We now proceed with the proof of Theorem 1.1. Let r be small enough, so that Proposition 5.3 holds true. Fix a constant $\beta > 0$ so that

$$-\log p(r) = k \log(1+\beta) \tag{6.2}$$

where k is an integer. We will define β more accurately later. As in the proof of Proposition 5.3, we see that Lemmas 5.1 and 4.3 together imply, for any $i \in \mathbb{N}$ and small enough r depending on i, the inequality

$$(6.3)\quad k\log(1+\beta) = \log\frac{1}{p(r)} \le \Big(1+\frac{1}{i}\Big)\Big(\frac{K_I}{m}\Big)^{\frac{1}{n-1}}\log\frac{1}{r} = \Big(1+\frac{1}{i}\Big)\Big(\frac{K_I}{m}\Big)^{\frac{1}{n-1}}\sum_{j=0}^{k-1}\log\frac{p^{-1}((1+\beta)^{-j})}{p^{-1}((1+\beta)^{-j-1})}.$$

We shall use the notation

$$t_j := \frac{p^{-1}((1+\beta)^{-j})}{p^{-1}((1+\beta)^{-j-1})}.$$

Hence,

$$(6.4)\qquad \log(1+\beta) \le \Big(1+\frac{1}{i}\Big)\Big(\frac{K_I}{m}\Big)^{\frac{1}{n-1}}\log t_j \le \Big(1+\frac{1}{i}\Big)\Big(\frac{K_I}{2}\Big)^{\frac{1}{n-1}}\log t_j$$

holds for at least one $A_j = ((1+\beta)^{-j-1}, (1+\beta)^{-j})$. We fix such a j. By Proposition 5.3,

$$(6.5)\qquad \mu(\{s : \delta_s > \epsilon^{\frac{1}{4}}\}) \le 4\epsilon^{\frac{1}{4}}\log\frac{1}{p(r)} = 4k\epsilon^{\frac{1}{4}}\log(1+\beta) \le \frac{1}{i}\log(1+\beta)$$

by our choice of ϵ. By scaling we may assume that $A_j = ((1+\beta)^{-1}, 1)$. By (6.5) there exist s_1, s_2 such that

$$(6.6)\qquad (1+\beta)^{-1} < p(s_1) \le (1+\beta)^{\frac{1}{i}-1} < (1+\beta)^{\frac{-1}{i}} < p(s_2) < 1$$

and

$$\delta(B'_{s_1}), \delta(B'_{s_2}) \le \epsilon^{\frac{1}{4}}.$$

If we assume that $\epsilon < C_3^4$, where C_3 is as in (4.6), then Proposition 4.4 implies that

$$(6.7)\qquad \alpha(B'_{s_1}), \alpha(B'_{s_1}) \le \frac{1+A}{1-A}, \quad \text{where} \quad A = 2(n+1)^{\frac{1}{n}}C^*(n,K)C_2(\epsilon^{\frac{1}{4}}, n)^{\frac{1}{n^2}}$$

and C_2 is as in (4.3). We assume that ϵ is so small that

$$(6.8)\qquad \frac{1+A}{1-A} \le 1 + \frac{1}{i}.$$

Consider the family Λ_2 of all spheres S_t, $t \in (s_2, p^{-1}(1))$ (here it is assumed that the function p notices the scaling done before (6.6)), and denote $\Lambda'_2 = \{S'_t : S_t \in \Lambda_2\}$. By Theorem 3.4, Lemma 5.1, (6.6) and (6.1), we have

$$(6.9)\qquad \omega_{n-1}^{\frac{-1}{n-1}}\log\frac{p^{-1}(1)}{s_2} = M_S(\Lambda_2) \le \frac{K_O^{\frac{1}{n-1}}}{m}M_S^m(\Lambda'_2)$$
$$\le \Big(\frac{K_O m}{\omega_{n-1}}\Big)^{\frac{1}{n-1}}\log\frac{1}{p(s_2)} \le \Big(\frac{K_O m}{\omega_{n-1}}\Big)^{\frac{1}{n-1}}\frac{1}{i}\log(1+\beta)$$
$$\le \Big(\frac{10}{\omega_{n-1}}\Big)^{\frac{1}{n-1}}\frac{1}{i}\log(1+\beta),$$

where in the last inequality the fact $m \leq 10$ was used. A similar argument for s_1 yields

$$\log \frac{s_1}{p^{-1}((1+\beta)^{-1})} < \frac{10}{i} \log(1+\beta). \tag{6.10}$$

By (6.7) and (6.8) there exist some $y_1, y_2 \in B_1$ so that

$$B\Big(y_1, p(s_1)\Big(1+\frac{1}{i}\Big)^{-1}\Big) \subset B'_{s_1}, \quad B'_{s_2} \subset B\Big(y_2, p(s_2)\Big(1+\frac{1}{i}\Big)\Big). \tag{6.11}$$

Now we want to apply Lemma 6.1 to show that the points y_1 and y_2 are close to one another. We use a mapping T that is a composition of scaling and translation, so that $T(y_2) = 0$ and $T(B(y_2, p(s_2)(1+\frac{1}{i}))) = B_1$. Then $T(y_1) = x_1$ and $T(B(y_1, p(s_1)(1+\frac{1}{i})^{-1})) = B(x_1, 1-u)$. By (6.6), we have

$$\frac{1}{1-u} = \frac{p(s_2)(1+\frac{1}{i})^2}{p(s_1)} \leq (1+\beta)\Big(1+\frac{1}{i}\Big)^2. \tag{6.12}$$

We assume $i \geq 10000$, so that

$$\log\Big((1+\beta)\Big(1+\frac{1}{i}\Big)^2\Big) \leq \Big(1+\frac{1}{\sqrt{i}}\Big)\log(1+\beta). \tag{6.13}$$

By combining (6.4), the definition of t_j, (6.9) and (6.10), we have

$$\begin{aligned} \log(1+\beta) &\leq \Big(1+\frac{1}{i}\Big)\Big(\frac{K_I}{m}\Big)^{\frac{1}{n-1}} \log t_j \\ &= \Big(1+\frac{1}{i}\Big)\Big(\frac{K_I}{m}\Big)^{\frac{1}{n-1}} \Big(\log\frac{s_2}{s_1} + \log\frac{s_1}{p^{-1}((1+\beta)^{-1})} + \log\frac{p^{-1}(1)}{s_2}\Big) \\ &\leq \Big(1+\frac{1}{i}\Big)\Big(\frac{K_I}{m}\Big)^{\frac{1}{n-1}} \Big(\log\frac{s_2}{s_1} + \frac{20}{i}\log(1+\beta)\Big), \end{aligned}$$

and thus

$$\log(1+\beta) \leq \Big(1+\frac{40}{i}\Big) m^{\frac{-1}{n-1}} \log\frac{s_2}{s_1} \tag{6.14}$$

when $i \geq 100000$ and $K_I \leq 1+\frac{1}{i}$. If we denote the family of all spheres S_t, $t \in (s_1, s_2)$ by Λ_3, and if $\Lambda'_3 = \{S'_t : S_t \in \Lambda_3\}$, then Theorem 3.4, Lemma 6.1 and (6.6) give

$$\begin{aligned} \omega_{n-1}^{\frac{-1}{n-1}} \log\frac{s_2}{s_1} &= M_S(\Lambda_3) \leq \frac{K_O^{\frac{1}{n-1}}}{m} M_S^m(\Lambda'_3) \\ &\leq \Big(\frac{K_O m}{\omega_{n-1}}\Big)^{\frac{1}{n-1}} \log\frac{1}{\tanh\frac{\rho(L,u)}{2}}, \end{aligned} \tag{6.15}$$

where $\rho(L,u)$ is as in Lemma 6.1 and u as in (6.12). By combining (6.12), (6.13), (6.14) and (6.15), we have

$$\log\frac{1}{1-u} \le \Big(1+\frac{1}{\sqrt{i}}\Big)^3 \log\frac{1}{\tanh\frac{\rho(L,u)}{2}} \tag{6.16}$$

when $K \le 1+\frac{1}{i}$. Solving (6.16) with respect to L yields

$$L \le \frac{\sqrt{u^2(1+T)^2-(2-u)^2(1-T)^2}}{2u\sqrt{T}},$$

where $T=(1-u)^{\frac{1}{\nu}}$ and $\nu=(1+\frac{1}{\sqrt{i}})^3$. Hence, for each $I\in\mathbb{N}$ we can choose i to be large enough, so that

$$|x_1| \le I^{-1}u. \tag{6.17}$$

In conclusion, we have

$$B(0,1-(1+I^{-1})u) \subset B(x_1,1-u). \tag{6.18}$$

By combining (6.6), (6.11) and (6.18), we conclude that, after scaling and translating the domain Ω again, we have found an annulus $B_1\setminus B_{1-(1+I^{-1})u}$, where

$$u \le 1-(1+\beta)^{-1}\Big(1+\frac{1}{i}\Big)^{-2}, \tag{6.19}$$

so that

$$S'_t \subset B_1\setminus B_{1-(1+I^{-1})u} \quad \text{for all } t\in(s_1,s_2) \tag{6.20}$$

and (6.14) holds. Note that up to this point, the assumption $n\ge 3$ has not been used. In what follows, this assumption will be employed in the study of certain path families. We next describe these path families.

First, for each $t\in(s_1,s_2)$, S'_t must contain a branch point, since otherwise the restriction of f to S'_t would be a homeomorphism onto S_t, see [17], pages 69–70 (here the assumption $n\ge 3$ gets used). Take, for $t_b=\frac{s_1+s_2}{2}$, a branch point $b\in S'_{t_b}$. Then for each $t\in(s_1,s_2)$ there exists a point $a_t\in S'_t$ so that when a_t and b are connected by a line segment, 0 lies between a_t and b in this segment.

Next we denote

$$B_b = B\Big(f(b),\frac{s_2-s_1}{2}\Big).$$

Furthermore, the b-component of $f^{-1}(B_b)$ is denoted by B'_b. Then $\mu(f,B'_b)=m$ (recall that we assumed that for all branch points x of f, $i(x,f)\ge m$) and B'_b is a normal neighborhood of b. By Lemma 4.3, we have

$$L^*\Big(f(b),f,\frac{s_2-s_1}{2}\Big) \le C^*(n,K)\,l^*\Big(f(b),f,\frac{s_2-s_1}{2}\Big).$$

Since $B'_b \in B_1 \setminus B_{1-(1+I^{-1})u}$,

$$l^*\Big(f(b), f, \frac{s_2 - s_1}{2}\Big) \le \frac{(1+I^{-1})u}{2}.$$

Hence

$$L^*\Big(f(b), f, \frac{s_2 - s_1}{2}\Big) \le C^*(n,K)\frac{(1+I^{-1})u}{2} \le \frac{2}{3}C^*(n,K)u. \tag{6.21}$$

We consider the family Γ_b of all paths joining B_b and

$$A_b := \{f(y) : y = a_t \text{ for some } t \in (s_1, s_2)\}$$

in the annulus $B_{s_2} \setminus B_{s_1}$. Note that, by (6.21) and since $\mu(f, B'_b) = m$, $B_b \cap A_b = \emptyset$. Since both B_b and A_b intersect S_t for each $t \in (s_1, s_2)$, [21], Theorem 10.9 gives

$$M(\Gamma_b) \ge c_n \log \frac{s_2}{s_1}, \tag{6.22}$$

where

$$c_n = 2^{-1}\omega_{n-2}p_n^{1-n} \tag{6.23}$$

and

$$p_n = \int_0^\infty t^{\frac{2-n}{n-1}}(1+t^2)^{\frac{1}{1-n}}\, dt.$$

We need a sharp estimate for $M\Gamma'_b$, where

$$\Gamma'_b = \{\gamma' : \gamma' \text{ is a lift of some } \gamma \in \Gamma_b \text{ starting at some } a_t\}.$$

At this point it is important to notice that the fact $\mu(f, B'_b) = m$ implies that each $\gamma' \in \Gamma'_b$ terminates at B'_b. Also, each $\gamma \in \Gamma_b$ has a lift $\gamma' \in \Gamma'_b$, so that $M(\Gamma_b) = M(f\Gamma'_b)$.

By the proof of [21] Theorem 10.9,

$$\int_\gamma \rho\, ds \ge 1 \tag{6.24}$$

for all paths γ joining 0 and e_n in the sphere $S(\frac{1}{2}e_n, \frac{1}{2})$, where ρ is defined so that

$$\rho(g(x)) = p_n^{-1}|x|^{\frac{2-n}{n-1}}(1+|x|^2)^{\frac{n-2}{n-1}},$$

and g is the stereographic projection $g : S(\frac{e_n}{2}, \frac{1}{2}) \to \overline{\mathbb{R}}^{n-1}$;

$$g(x) = e_n + \frac{x - e_n}{|x - e_n|^2}.$$

Also, there is equality in (6.24) when γ is a geodesic. We set

$$\rho' : B_1 \setminus B_{1-(1+I^{-1})u} \to [0,\infty]; \quad \rho'(x) = \frac{1}{2(1-(1+I^{-1})u)}(\rho \circ g')(y)$$

for each $x \in [0, \frac{y}{|y|}] \cap (B_1 \setminus B_{1-(1+I^{-1})u})$, where g' is the restriction of a Möbius transformation that maps $S_{1-(1+I^{-1})u}$ onto $S(\frac{1}{2}e_n, \frac{1}{2})$ so that $g'(a) = 0$, where

$$\{a\} = S_{1-(1+I^{-1})u} \cap \left[0, \frac{a_t}{|a_t|}\right] \quad \text{for some } t \in (s_1, s_2).$$

Note that $a_t \in [0, \frac{a}{|a|}]$ for all $t \in (s_1, s_2)$. We choose a positive constant t_I so that

$$p_n^{-1} \int_{t_I}^{\infty} t^{\frac{2-n}{n-1}} (1+t^2)^{\frac{1}{1-n}} \, dt \le 1 - \Big(\frac{1}{1+\frac{1}{I}}\Big)^{\frac{1}{n}}. \tag{6.25}$$

Then we may further choose a constant t'_I so that

$$|(g \circ g')(x)| \ge t_I \tag{6.26}$$

for all $x \in B(-a, t'_I) \cap S_{1-(1+I^{-1})u}$. By estimate (6.21) we conclude that

$$\int_{\gamma'} \rho' \, ds \ge \Big(\frac{1}{1+\frac{1}{I}}\Big)^{\frac{1}{n}}$$

for all $\gamma' \in \Gamma'_b$, when u is small enough so that

$$C^*(n,K)u \le t'_I.$$

Hence

$$\begin{aligned} M(\Gamma'_b) &\le \Big(1+\frac{1}{I}\Big) \int_{B_1 \setminus B_{1-(1+I^{-1})u}} \rho'(x)^n \, dx \\ &= \Big(1+\frac{1}{I}\Big) \int_{1-(1+I^{-1})u}^{1} \int_{S_r} \rho'(y)^n \, dy \, dr \\ &\le c_n \Big(1+\frac{1}{I}\Big) \frac{(1+I^{-1})u}{(1-(1+I^{-1})u)^{n-1}}, \end{aligned} \tag{6.27}$$

where c_n is as in (6.23). By using (6.19), we see that with our choices of i and I, given at the end of the proof, we have

$$\begin{aligned} \Big(1+\frac{1}{I}\Big)^2 \frac{u}{(1-(1+I^{-1})u)^{n-1}} &\le 1.1^{\frac{1}{n-1}} u \le 1.2^{\frac{1}{n-1}} \log(1+u) \\ &\le 1.2^{\frac{1}{n-1}} \log\Big(1+1-(1+\beta)^{-1}\Big(1+\frac{1}{i}\Big)^{-2}\Big) \le 1.3^{\frac{1}{n-1}} \log(1+\beta). \end{aligned} \tag{6.28}$$

We combine estimates (6.27) and (6.28) with the Poletsky inequality in order to have

$$c_n \log \frac{s_2}{s_1} \le M(\Gamma_b) \le K_I M(\Gamma_b') \le 1.3^{\frac{1}{n-1}} K_I c_n \log(1+\beta). \tag{6.29}$$

By further combining (6.14) with (6.29), we finally arrive at a contradiction:

$$\begin{aligned} \log(1+\beta) &\le \Big(1+\frac{1}{\sqrt{i}}\Big) m^{\frac{-1}{n-1}} \log \frac{s_2}{s_1} \le 1.4^{\frac{1}{n-1}} K_I \Big(1+\frac{1}{\sqrt{i}}\Big) 2^{\frac{-1}{n-1}} \log(1+\beta) \\ &\le 0.7^{\frac{1}{n-1}} \log(1+\beta). \end{aligned}$$

We will finish the proof by giving an account on how the different constants should be chosen. First, if we initially assume that $K \le 2$, say, then some constants, originally depending on n and K, will only depend on n. For example, then $C^*(n,K) \le C^*(n,2)$. In the statement of the theorem it is assumed that

$$K \le (1+\epsilon)^{\frac{1}{3}}. \tag{6.30}$$

Now ϵ depends on n, β, i and r, and has to satisfy the following requirements:

$\epsilon \le C_3^4$, where C_3 is as in (4.6),

$\epsilon \le C(n,i)$ so that (6.8) is satisfied,

$\epsilon \le (2ki)^{-4}$, where k is as in (6.2).

Next, r is a constant depending on n, β and i, so that (6.3) is satisfied, and r can be estimated as in the proof of Proposition 5.3.

The constant i depends on n, β and I, and has to satisfy

$$\frac{1}{i} \le (1+\beta)^{\frac{-1}{2}} (2-(1+\beta)^{\nu})^{\frac{-1}{2}} - 1, \quad \nu = 1.001^{\frac{1}{n-1}},$$

and inequality (6.17).

Furthermore, the constant β depends on n and t_I', and should satisfy

$$\begin{aligned} 1-(1+\beta)^{-3} &\le 1.001^{\frac{1}{n-1}} \log(2-(1+\beta)^{-3}), \\ \beta &\le \Big(1-\frac{t_I'}{C^*(n,2)}\Big)^{\frac{-1}{3}} - 1. \end{aligned}$$

Finally, I is a dimensional constant so that

$$I^{-1} \le 1.001^{\frac{1}{8(n-1)^4}} - 1,$$

and t_I, t_I' are dimensional constants so that (6.25) and (6.26) hold true. The proof is complete.

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