THE LOCAL HOMOLOGY OF CUT LOCI IN RIEMANNIAN MANIFOLDS

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In 1935, Myers [11] exhibited a decomposition of the cut locus of a point p in a complete, two-dimensional, real analytic Riemannian manifold as a one-dimensional simplicial complex. He showed that for each cut point q there is a direct relation between the number of minimal geodesics connecting p to q—this number being called the order of the cut point q—and the position of q in the simplicial decomposition. Cut points of order one are extreme points of the simplicial complex; those of order two are interior to an edge; and those of order $k \geq 3$ are vertices where k edges meet. In short, the topology of a neighborhood of a point in the cut locus is determined by the order of the cut point.

Recently, Ozols [12] and Buchner [3] have shown that the cut locus of a point p in a real analytic Riemannian manifold admits a simplicial decomposition. Moreover, Ozols [12] describes the structure of the cut locus near a non-conjugate cut point q as a finite (depending on the order of q) intersection of hyperspaces and half-planes, while Buchner [4] completely classifies the local structure of *generic* cut loci in low dimensional manifolds. In general, however, the relation between the set, called the *link*, of minimal geodesics connecting p to a cut point q and the structure of the cut locus near q remains obscure.

This paper establishes, as a consequence of Poincaré duality, a duality between the Čech cohomology of the link and the local homology groups of cut loci in smooth Riemannian manifolds, thereby weakly generalizing the result of Myers on the order and local topology of cut loci in real analytic surfaces. Using standard arguments from algebraic topology, we show that certain local homology groups of cut loci are torsion free. Finally, we prove interconnections between the dimension of the cut locus and the vanishing of high dimensional local homology which lead up to a generalization of a theorem of Bishop [2] on the decomposition of cut loci.

1. **Duality.** Fixing notation, throughout this paper M denotes a complete n-dimensional smooth (C^∞) Riemannian manifold, and p an arbitrary but fixed point of M having the cut locus $C = C(p)$ in M. Given p, q always denotes a cut point in C. $S = S_q$ always denotes the

unit tangent sphere to M at q, and the *link*, $\Lambda = \Lambda(p, q)$, denotes the non-empty closed subset of S consisting of all unit tangent vectors at q that are tangent to minimal geodesic segments connecting p to q but directed from q to p. (See [1, p. 135].) X always denotes an element of Λ. Finally, $\Sigma = \Sigma(p, q)$ denotes the subset of M consisting of all points that lie on some minimal geodesic segment connecting p to q. In other words, Σ is the union of all such geodesic segments, and hence, Σ is homeomorphic to the suspension of Λ. Thus, for an abelian group G,

$$(1.1) \qquad \check{H}^{i+1}(\Sigma, p; G) \cong \check{H}^i(\Lambda, X; G)$$

by the well known relation between the Čech cohomology of a space and that of its suspension [6, p. 51].

The complement of the cut locus C in M is the largest open neighborhood of p for which normal coordinates about p can be used. Thus every point in the complement of C, except for p, lies on a unique minimal geodesic emitted from p. If $U \subset C$ is a relatively open subset of C having compact closure, then U is a strong deformation retract of the open set $\tilde{U}$ consisting of all the points of M, except for p, that lie on a minimal geodesic segment connecting p to some point in U, where the deformation is to push a point in $\tilde{U}$ into U along the minimal geodesic segment that it lies upon. Let U be a relatively open subset of C having compact closure, let q be a point in U, and let D be a closed geodesic ball centred at p such that $D \cap C = \varnothing$. Then the sets $K = (M - \tilde{U}) \cup D \cup \Sigma$ and $L = (M - \tilde{U}) \cup D$ are such that $L \subset K$ and the closure of $K - L$ is a compact contractible set. See Figure. Thus M is
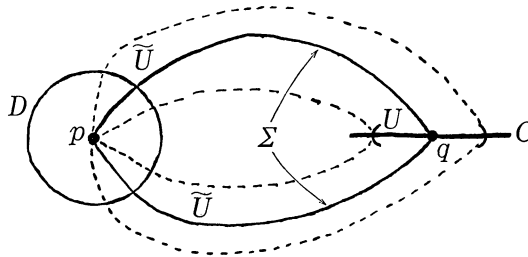
oriented along the closure of $K - L$, and therefore a version of Poincaré duality can be applied. (See [6, 7.10, p. 296].) Hence, by Poincaré duality, a deformation retraction, and excision:

$$
\begin{aligned}
(1.2) \quad \check{H}^{i+1}(K, L; G) &\cong H_{n-1-i}(M - L, M - K; G) \\
&= H_{n-1-i}(\tilde{U} - D, \tilde{U} - (D \cup \Sigma); G) \\
&\cong H_{n-1-i}(U, U - q; G) \cong H_{n-1-i}(C, C - q; G) .
\end{aligned}
$$

However, by excision and by collapsing D to a point (see [6, 6.20, p. 287].):

$$(1.3) \quad \check{H}^{i+1}(K, L; G) \cong \check{H}^{i+1}(D \cup \Sigma, D; G) \cong \check{H}^{i+1}((D \cup \Sigma)/D, \{D\}; G)$$
$$\cong \check{H}^{i+1}(\Sigma, p; G).$$

Therefore (1.1), (1.2), and (1.3) imply the following duality relation.

THEOREM 1.4. $\check{H}^i(\Lambda, X; G) \cong H_{n-1-i}(C, C - q; G)$.

REMARK. By a similar proof, Theorem 1.4 also holds for the cut locus of a properly embedded submanifold of M, the link now being defined as the set of unit tangent vectors to geodesics minimizing the distance from the submanifold to the given cut point. Recall that a submanifold is properly embedded if the embedding is a proper map.

2. **Local homology groups of** $C(p)$. The groups $H_*(C, C - q; G)$ are known as the local homology groups of C at q with coefficients in the abelian group G. Throughout this section, integer coefficients are assumed. Recall that the word *countable* allows both finite and infinite.

PROPOSITION 2.1. (1) $H_*(C, C - q)$ *is countably generated.* (2) $H_i(C, C - q) = 0$ *for* $i \geqq n$. (3) $H_i(C, C - q)$ *is torsion free for* $i = 0, 1, n - 2,$ *and* $n - 1$.

PROOF. By Theorem 1.4, by Poincaré duality in the tangent sphere S, and by the contractibility of $S - X$,

$$H_i(C, C - q) \cong \check{H}^{n-1-i}(\Lambda, X) \cong H_i(S - X, S - \Lambda) \stackrel{\partial_*}{\cong} \tilde{H}_{i-1}(S - \Lambda)$$

where the tilde indicates reduced homology. Since $S - \Lambda$ is a non-compact open subset of the $(n - 1)$-dimensional sphere S, (1) and (2) are true.

As for (3), each case is treated separately. $H_0(C, C - q)$ is isomorphic to the integers if q is a full path component of C and is the trivial group otherwise. $H_1(C, C - q) \cong \tilde{H}_0(S - \Lambda)$ is a free abelian group whose rank plus one equals the number of connected components of $S - \Lambda$. Now a standard argument (see [6, 3.5, p. 261 and exercise 3, p. 266]) proves that under certain conditions on an m-dimensional manifold N and closed subset F, $H_{m-1}(N, N - F)$ is torsion free. Thus $H_{n-2}(C, C - q) \cong H_{n-2}(S - X, S - \Lambda)$ has no torsion because $N = S - X$ is an oriented $(n - 1)$-dimensional manifold containing the closed subset $F = \Lambda - X$, while $H_{n-1}(C, C - q) \cong H_{n-1}(\tilde{U} - D, \tilde{U} - (D \cup \Sigma))$ (notation as in (1.2)) has no torsion because $N = \tilde{U} - D$ is an n-dimensional manifold oriented along the closed connected set $F = \Sigma - (D \cap \Sigma)$. q.e.d.

REMARK. According to Buchner [3], cut loci of compact real analytic Riemannian manifolds are finite simplicial complexes. Thus the local homology groups are finitely generated. On the other hand, Gluck and Singer's example of a nontriangulable cut locus has an infinitely generated local homology group at some point [7].

PROPOSITION 2.2. *The following are equivalent.*

(1) $H_0(C, C - q) \neq 0$.

(2) $\Lambda(p, q) = S_q$.

(3) $C(p) = \{q\}$.

(4) $H_i(C, C - q)$ *is the group of integers for* $i = 0$, *and is the trivial group otherwise.*

If any one, and hence all, of the above hold, then M is homeomorphic to a sphere.

PROOF. As in Proposition 2.1, $H_0(C, C - q) \cong H_0(S - X, S - \Lambda)$. Since $S - X$ is connected, the second group is non-trivial if and only if $S = \Lambda$. This shows (1) implies (2). The remaining implications are trivial. The last last statement follows from [1, p. 142].   q.e.d.

If M is of dimension less than or equal to four, then $H_*(C, C - q)$ is countably generated and torsion free.

There is the possibility of torsion when $n \geq 5$. Gluck and Singer's work on the deformation of geodesic fields allow the construction of such an example. Let q be the north pole and p be the south pole of the unit sphere S^n. Consider the north polar cap consisting of all points whose distance from q is at most $\pi/4$. The boundary of the polar cap is an $(n - 1)$-dimensional sphere which is called the arctic sphere. If Y is a "nice" subset of the arctic sphere, the geodesic cone C over Y with vertex at q, i.e., the union of all minimal geodesic segments connecting Y to q, is a good candidate for a cut locus. For by drawing off a family of geodesics starting in C which head in a southerly direction, while simultaneously drawing up the geodesics from the south pole, if the conditions of [7] are satisfied, then the metric can be altered near the equator so that the two families of geodesics match up, thereby making C the cut locus of p. Observe $H_i(C, C - q) \cong \tilde{H}_{i-1}(Y)$, since C is the cone on Y with vertex at q.

Now, the Veronese surface V is an embedding of the real projective plane in S^4 ([5, p. 88]). If $-$ denotes the antipodal map of S^4, V and $-V$ are disjoint sets. Thus the set $Y = V \cup (-V)$ is the disjoint union of two real projective planes. Considering S^4 as the arctic sphere of S^5, the above construction gives an example of a cut locus whose local

homology has torsion.

3. Cut points and conjugate points. The conjugate points of p are the singularities of the exponential map $\exp: T_p(M) \to M$, that is, the points in the tangent space $T_p(M)$ where the differential of the exponential map $\exp_*$ fails to be surjective. The multiplicity of a conjugate point is the dimension of the kernel of $\exp_*$ at that point. Warner [13] showed that the locus of regular conjugate points—every regular conjugate point has a neighborhood in $T_p(M)$ that meets each ray in at most one conjugate point—is a smooth hypersurface of $T_p(M)$ that is dense in the full conjugate locus. The set of conjugate points nearest the origin is called the first conjugate locus. Conjugate points arise in the study of the cut locus since for every cut point q, either there are at least two minimal goedesics connecting p to q, or q is the first conjugate point to p along the unique minimal geodesic between the two points [10].

PROPOSITION 3.1. *If q is a cut point which is not conjugate along any minimal qeodesic from p to q, then $\Lambda(p, q)$ is a finite set containing at least two points. Hence $\check{H}^0(\Lambda, X) \cong H_{n-1}(C, C - q)$ is a non-trivial finitely generated free abelian group and $\check{H}^i(\Lambda, X) \cong H_{n-1-i}(C, C - q)$ is trivial for $i \neq 0$.*

PROOF. The minimal geodesics from p to q are isolated. Thus Λ is finite and contains at least two points. q.e.d.

Thus if $H_{n-1}(C, C - q) \cong 0$, then the cut point q is conjugate to p along some minimal geodesic connecting p to q.

PROPOSITION 3.2. *The set of cut points which are conjugate along some minimal geodesic has dimension less than or equal to $n - 2$.*

PROOF. By [14, Lemma 1.1], the image under exp of the union of the set of conjugate points of multiplicity one for which the kernel of $\exp_*$ is tangent to the conjugate locus with the set of conjugate points having multiplicity at least two has dimension at most $n - 2$. Hence, it suffices to show that the kernel of $\exp_*$ is tangent to the conjugate locus at every conjugate point that is also a cut point. Argue by contradiction. Let Y be a multiplicity one conjugate point in $T_p(M)$ for which the kernel of $\exp_*$ is not tangent to the conjugate locus. In a neighborhood of Y in the conjugate locus, the kernel of $\exp_*$ decomposes uniquely into a non-zero radial component and a component tangent to the conjugate locus. Take an integral curve σ of the tangential component that passes through Y. Let Y_0 be a point on σ which is closer

to the origin than Y. If we let γ and γ_0 be the geodesics determined by Y and Y_0, respectively, then the curve γ_0 plus $\exp(\sigma)$, which is not a geodesic, has the same length, by construction, as γ. Hence the cut point along γ occurs before $\exp(Y)$.   q.e.d.

In particular, the set of q in C with $H_{n-1}(C, C - q) = 0$ has dimension at most $n - 2$. This can be refined.

PROPOSITION 3.3. *Let $U \subset C$ be a relatively open subset of C. Then $\dim(U) \leqq k$ if and only if $H_i(C, C - q) = 0$ for all q in U and all $i > k$.*

PROOF. Suppose $H_i(C, C - q) = 0$ for all $q \in U$ and $i > k$. Since $\dim(C) \leqq n - 1$ and $H_i(C, C - q) = 0$ for all $q \in C$ and $i \geqq n$, it suffices to consider $k < n - 1$.

Let $V = \exp^{-1}(U) \cap K$ where K is the tangent cut locus of p. Then V is relatively open in K. Hence V is an $n - 1$ dimensional topological manifold, and V is contained in the first conjugate locus, since $k < n - 1$. Let μ be the least multiplicity of a conjugate point in V. Then the set of conjugate points in V with multiplicity μ are regular conjugate points, the reason being that every neighborhood of a singular (non-regular) conjugate contains a conjugate point of strictly lower multiplicity. (See Property (R3) of a regular exponential map in [13].) Thus the exponential map restricted to the set of points in V of multiplicity μ is a submersion into M. (This is even true when $\mu = 1$ since the proof of Proposition 3.2 shows that the kernel of $\exp_*$ is tangent to the conjugate locus.) Hence, by cutting down this set if necessary, its image under $\exp$ is a smooth d-dimensional manifold N contained in U where $d = n - 1 - \mu$. Furthermore, since the multiplicity of every conjugate point in V is at least μ, and the multiplicity of every singular conjugate point is at least $\mu + 1$, the argument in [14, p. 202] shows $\dim(U) \leqq d$. Thus to show $\dim(U) \leqq k$ it suffices to show $k \geqq d$, which we can do, by assumption on the local homology groups, by showing $H_d(C, C - q) \neq 0$ for some $q \in U$.

If q is a point in the manifold N, then $H_d(N, N - q) \neq 0$. Now one can reason that the inclusion homomorphism $H_d(N, N - q) \rightarrow H_d(U, U - q) \approx H_d(C, C - q)$ is a monomorphism, thereby proving that $H_d(C, C - q) \neq 0$. The way to do this is as follows.

Let W be an open set in M such that $U = W \cap C$. Let B be an open n-dimensional ball about q in M whose closure is contained in W and whose intersection with N is an open d-dimensional ball in N. The map f taking $U - B$ to the south pole $\{*\}$ of the sphere S^d and taking the open cell $B \cap N$ homeomorphically onto $S^d - \{*\}$ is a continuous map of

the relatively closed subset $N \cup (U - B)$ of U into S^d. Since $\dim(U) \leq d$, f extends to a map F of U into S^d ([9, p. 83]). Thus the diagram

$$H_d(N, N - B) \overset{f_*}{\underset{\sim}{\Longrightarrow}} H_d(S^d, *)$$
$$\underset{\iota_*}{\searrow} \qquad \underset{F_*}{\nearrow}$$
$$H_d(U, U - B)$$

shows the inclusion homomorphism ι_* is a monomorphism. Since the family of all such B forms a base for the neighborhoods of q, passing to a direct limit (or the argument in [6, Lemma 2.2, p. 252], if one prefers) implies that $H_d(N, N - q) \to H_d(U, U - q)$ is a monomorphism.

The converse is easier. $H_i(C, C - q) \cong H_i(U, U - q)$ is the direct limit of the groups $H_i(U, U - B)$ for open neighborhoods B of q. If $\dim(U) \leq k$, then the latter groups vanish for $i > k$. Thus $H_i(C, C - q)$ vanishes for $i > k$.   q.e.d.

In general the vanishing of high dimensional local homology does not imply that a finite dimensional space has low dimension. Knaster and Kuratowski [9, pp. 22–25] have an example of a totally disconnected subset of the plane which is one-dimensional. Being totally disconnected, each connected component of this set is a point, and thus the local homology groups vanish in all dimensions except zero. Proposition 3.3 has several consequences. For example take $U = C$.

COROLLARY 3.4. $\dim(C) \leq k$ *if and only if* $H_i(C, C - q) = 0$ *for all* q *in* C *and all* $i > k$.

The following is a new necessary and sufficient condition for the coincidence of the tangent cut locus and the first conjugate locus. (See [8] for similar conditions.)

COROLLARY 3.5. *The following are equivalent*:
(1) *The tangent cut locus and the first conjugate locus at* p *coincide.*
(2) *Every* q *in* C *is conjugate to* p *along some minimal geodesic.*
(3) $H_{n-1}(C, C - q) = 0$ *for all* q *in* C.

PROOF. (1) implies (2) is trivial. By Proposition 3.2, (2) implies that $\dim(C) \leq n - 2$. Thus (3) follows by Corollary 3.4. If (1) does not hold, then $\dim(C) = n-1$. Thus (3) implies (1), again by Corollary 3.4.   q.e.d.

Looking at local homology suggests the following naïve decomposition of the cut locus. For each $j = 0, \cdots, n$, let

$$C_j = \{ q \in C \colon H_i(C, C - q) = 0 \quad \text{for all} \quad i \geq j \} \, .$$

Thus $C_0 \subset C_1 \subset \cdots \subset C_n = C$ and, by Proposition 3.3, $\dim(\overset{\circ}{C}_j) \leqq j - 1$, where $\overset{\circ}{C}_j$ is the relative interior of C_j in C.

Define a cut point q to be a *singular cut point* if there is exactly one minimial geodesic connecting p to q, and to be an *ordinary cut point* otherwise. If q is singular, then $\check{H}(\Lambda, X) = 0$ and thus, by duality, $H_*(C, C - q) = 0$. Hence the singular cut points are contained in C_0. But $\overset{\circ}{C}_0$ is empty, since $\dim(\overset{\circ}{C}_0) \leqq -1$. Thus the set of singular cut points has empty interior. Consequently, we have the following:

THEOREM 3.5 (Bishop [2]). *The ordinary cut points are dense in the cut locus.*

REMARK. Similar results hold for the cut locus and focal locus of a properly embedded submanifold in a complete Riemannian manifold.

REFERENCES

[1] A. BESSE, Manifolds all of whose geodesics are closed, Springer-Verlag, Berlin, Heidelberg, and New York, 1979.
[2] R. BISHOP, Decomposition of cut loci, Proc. Amer. Math. Soc. 65 (1976), 133-136.
[3] M. BUCHNER, Simplicial structure of the real analytic cut locus, Proc. Amer. Math. Soc. 66 (1977), 118-121.
[4] M. BUCHNER, The structure of the cut locus in dimension less than or equal to six, Compositio Math. 37 (1978), 103-119.
[5] B.-Y. CHEN, Geometry of submanifolds, Marcel Decker, New York, 1973.
[6] A. DOLD, Lectures on algebraic topology, Springer-Verlag, Berlin, Heidelberg, and New York, 1972.
[7] H. GLUCK AND D. SINGER, Scattering of geodesic fields, I, Ann. of Math. 108 (1978), 347-372.
[8] J. HEBDA, Conjugate and cut loci and the Cartan-Ambrose-Hicks Theorem, Indiana U. Math. J. 31 (1982), 17-25.
[9] W. HUREWICZ AND H. WALLMAN, Dimension Theory, Princeton University Press, Princeton, 1941.
[10] S. KOBAYASHI, On conjugate and cut loci, Studies in Global Geometry and Analysis (S. S. Chern, ed.), MAA Studies in Mathematics (1967), 96-122.
[11] S. B. MYERS, Connections between differential geometry and topology I and II, Duke Math. J. 1 (1935), 376-391, ibid. 2 (1936), 95-102.
[12] V. OZOLS, Cut loci in Riemannian manifolds, Tôhoku Math. J. 26 (1974), 219-227.
[13] F. W. WARNER, The conjugate locus of a Riemannian manifold, Amer. J. Math. 87 (1965), 575-604.
[14] F. W. WARNER, Conjugate loci of constant order, Ann. of Math. 86 (1967), 192-212.

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