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# The Local Structure of Trans-Sasakian Manifolds (*).

J. C. MARRERO

**Summary.** – *In this paper, we completely characterize the local structure of trans-Sasakian manifolds of dimension* $\geq 5$ *by giving suitable examples.*

## 0. – Introduction.

An almost Hermitian manifold $V$ is called locally conformal Kähler (l.c.K.) if its metric is conformally related to a Kähler metric in some neighbourhood of every point of $V$. Such manifolds have been studied by various authors (see, for instance, [L], [GH], [V1], [V2] and [V3]).

In [O], J. A. OUBIÑA has studied a new class of almost contact metric structure, called trans-Sasakian, which is, in some sense, an analogue of a locally conformal Kähler structure on an almost Hermitian manifold (see definition in § 1).

On the other hand, in [ChG] the authors have introduced two subclasses of trans-Sasakian structures, the $\mathcal{C}_5$ and $\mathcal{C}_6$-structures, which contain the Kenmotsu and Sasakian structures, respectively.

In this paper, we completely characterize the local nature of the trans-Sasakian structures on connected differentiable manifolds of dimension $\geq 5$. In section 1, we recall some results on almost contact metric manifolds. In section 2 and 3, we characterize the local nature of $\mathcal{C}_5$ and $\mathcal{C}_6$ structures, respectively (see Theorems 2.1 and 3.1), by using the techniques of [Ol]. In section 4 we prove that the trans-Sasakian structures are of class $\mathcal{C}_5$ or $\mathcal{C}_6$ (see Theorem 4.1). Finally, we obtain some examples of 3-dimensional trans-Sasakian structures which are neither of class $\mathcal{C}_5$ nor $\mathcal{C}_6$.

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Indirizzo dell'A.: Departamento de Matemática Fundamental, Universidad de la Laguna, La Laguna-Tenerife, Canary Island, Spain.
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## 1. – Preliminaries.

Let $V$ be a $C^\infty$ almost Hermitian manifold with metric $g$ and almost complex structure $J$. Denote by $\mathfrak{X}(V)$ the Lie algebra of $C^\infty$ vector fields on $V$. The Kähler form $\Omega$ is given by $\Omega(X,Y)=g(X,JY)$; and the Lee form is the 1-form $\theta$ defined by $\theta(X)=1/(n-1)\ \delta\Omega(JX)$, where $\delta$ denotes the coderivate, $\dim V=2n$ and $X,Y\in\mathfrak{X}(V)$.

Recall that $V$ is said to be Kähler if $d\Omega=0$ and $N_J=0$ and locally conformal Kähler (l.c.K.) if $d\Omega=\theta\wedge\Omega$ and $N_J=0$, where $N_J$ denotes the Nijenhuis tensor of $J$.

On the other hand, let $M$ be a $C^\infty$ almost contact metric manifold with metric $g$ and almost contact structure $(\varphi,\xi,\eta)$. Then we have,

$$\varphi^2=-I+\eta\otimes\xi,\qquad \eta(\xi)=1,$$

$$g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\,\eta(Y),$$

for $X,Y\in\mathfrak{X}(M)$, where $I$ denotes the identity transformation. The fundamental 2-form $\Phi$ of the almost contact metric manifold $(M,\varphi,\xi,\eta,g)$ is defined by $\Phi(X,Y)=g(X,\varphi Y)$ for all $X,Y\in\mathfrak{X}(M)$.

An almost contact structure $(\varphi,\xi,\eta)$ is said to be normal if the almost complex structure $J$ on $M\times\boldsymbol{R}$ given by

$$J(X,a\,d/dt)=(\varphi X-a\xi,\eta(X)\,d/dt),\tag{1.1}$$

where $a$ is a $C^\infty$ function on $M\times\boldsymbol{R}$, is integrable, which is equivalent to the condition $N_\varphi+2d\eta\otimes\xi=0$, where $N_\varphi$ denotes the Nijenhuis torsion of $\varphi$ (see [SH1] and [SH2]).

Now, let $(\varphi,\xi,\eta,g)$ be an almost contact metric structure on $M$. We define an almost Hermitian structure $(J,h)$ on $M\times\boldsymbol{R}$, where the almost complex structure $J$ is given by (1.1) and $h$ is the Riemannian metric following:

$$h((X,a\,d/dt),(Y,b\,d/dt))=g(X,Y)+ab\,.$$

An almost contact metric structure $(\varphi,\xi,\eta,g)$ is said to be trans-Sasakian (see [O]) if the almost Hermitian structure $(J,h)$ on $M\times\boldsymbol{R}$ is l.c.K.

In [O], the author proves that $(\varphi,\xi,\eta,g)$ is a trans-Sasakian structure if and only if it is normal and

$$d\Phi=2\alpha\eta\wedge\Phi,\tag{1.2}$$

$$d\eta=\beta\Phi,\tag{1.3}$$

where $\alpha=\operatorname{div}\xi/(2n)$ and $\beta=\delta\Phi(\xi)/(2n)$.

An almost contact metric structure $(\varphi,\xi,\eta,g)$ is said to be:

$\mathcal{C}_5$ if it is trans-Sasakian with $\beta=0$; Kenmotsu if it is $\mathcal{C}_5$ with $\alpha=1$; $\mathcal{C}_6$ if it is trans-Sasakian with $\alpha=0$; Sasakian if it is $\mathcal{C}_6$ with $\beta=1$; cosymplectic if it is trans-Sasakian with $\alpha=\beta=0$ (see [B2], [ChG] and [K]).

We say that the almost contact structure $(\varphi, \xi, \eta)$ has rank $r$ if and only if the 1-form $\eta$ has rank $r$. Consequently, $(\varphi, \xi, \eta)$ has rank $r = 2s$ if $(d\eta)^s \neq 0$ and $\eta \wedge (d\eta)^s = 0$, and has rank $r = 2s + 1$ if $\eta \wedge (d\eta)^s \neq 0$ and $(d\eta)^{s+1} = 0$.

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold and $\sigma$ a positive differentiable function on $M$. We put,

$$g' = \sigma g + (1 - \sigma)\eta \otimes \eta .$$

Then, $(\varphi, \xi, \eta, g')$ is also an almost contact metric structure on $M$. Moreover, if $(\varphi, \xi, \eta, g)$ is trans-Sasakian and we denote by $\Phi'$ the fundamental 2-form of the almost contact metric structure $(\varphi, \xi, \eta, g')$, we have:

$$d\Phi' = (d(\ln \sigma) + (\operatorname{div} \xi/n)\eta) \wedge \Phi' , \tag{1.4}$$

$$d\eta = (\beta/\sigma)\Phi' . \tag{1.5}$$

AGREEMENT. – Through the rest of this paper, $M$ always denotes a $(2n+1)$-dimensional $(n \geq 2)$ connected manifold unless stated otherwise.

## 2. – $\mathcal{C}_5$-structures.

In this section, we describe the local structure of manifolds of class $\mathcal{C}_5$. Before, we examine the following example:

EXAMPLE 1. – Let $M$ be the product manifold $L \times V$, where $L$ is the circle $S^1$ or an open interval $(a', b')$, $-\infty \leq a' < b' \leq \infty$, and $(V, J, G)$ is a $2n$-dimensional Kählerian manifold. Let $E$ be a nowhere vanishing vector field on $L$, $E^*$ its dual field of 1-forms and $\sigma$ a positive function on $L$. Put

$$\begin{cases} \varphi(aE, X) = (0, JX), \quad \xi = (E, 0), \quad \eta = (E^*, 0), \\ g((aE, X), (bE, Y)) = \sigma G(X, Y) + ab, \end{cases} \tag{2.1}$$

where $a$ and $b$ are differentiable functions on $M$, and $X, Y \in \mathfrak{X}(V)$. Then it is not difficult to check that $(\varphi, \xi, \eta, g)$ is an almost contact metric structure on $M$ of class $\mathcal{C}_5$.

We remark that in the above example the 1-form $(\operatorname{div} \xi)\eta = d(\ln \sigma^n)$ is closed. We generalize this result for trans-Sasakian manifolds.

Let $(\varphi, \xi, \eta, g)$ be a trans-Sasakian structure on $M$, then

LEMMA 2.1. – *The 1-form* $(\operatorname{div} \xi)\eta$ *is closed.*

PROOF. – From the definition of trans-Sasakian structure (see (1.2) and (1.3)), we obtain

$$d((\operatorname{div} \xi)\eta) \wedge \Phi = 0 , \tag{2.2}$$

$$d((\mathrm{div}\ \xi)\eta)\wedge\eta = (\beta\ \mathrm{div}\,\xi)\,\eta\wedge\Phi\,. \tag{2.3}$$

Let $p$ be a point of $M$. We shall prove that $d((\mathrm{div}\ \xi)\eta)_p = 0$. If $\beta(p) = 0$, we deduce the result from the relations (2.2) and (2.3), since $\dim M \geqslant 5$.

Now, we suppose that $\beta(p) \neq 0$. Let $U$ be a neighbourhood of $p$ such that $\beta \neq 0$ on $U$. We can suppose that $\beta > 0$ on $U$. Taking in (1.3) the exterior differential and using (1.2), one gets

$$(d\beta + \beta(\mathrm{div}\ \xi/n)\eta)\wedge\Phi = 0\,,$$

and since rang $\Phi \geqslant 4$ we obtain $d\beta + \beta(\mathrm{div}\,\xi/n)\eta = 0$, i.e., $d(\ln\beta) = -(\mathrm{div}\,\xi/n)\ \eta$ which also proves that $d((\mathrm{div}\ \xi)\eta)_p = 0$. ∎

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold of class $\mathcal{C}_5$. Next, we prove the following structure theorem, which generalizes a similar result obtained by KENMOTSU [K] for Kenmotsu manifolds.

THEOREM 2.1. – *The manifold $M$ is locally the product $(a', b')\times V$, where $(a', b')$ is an open interval and $V$ is a $2n$-dimensional Kählerian manifold, on which the structure $(\varphi, \xi, \eta, g)$ is given as in* (2.1).

PROOF. – Fix a point $p\in M$. From Lemma 2.1, there exists a neighbourhood $U'$ of $p$ on which $d(\ln\sigma) = (\mathrm{div}\ \xi/n)\eta$, for a certain positive function $\sigma$. Put, $g' = (1/\sigma)\,g + (1-1/\sigma)\,\eta\otimes\eta$. From the definition of structure of class $\mathcal{C}_5$ and using (1.4) and (1.5), $(\varphi, \xi, \eta, g')$ is cosymplectic on $U'$. Therefore the point $p$ has a neighbourhood $U = (a', b')\times V \subseteq U'$ such that $(\varphi, \xi, \eta, g')$ is given on $U$ by

$$\varphi(aE, X) = (0, JX)\,, \qquad \xi = (E, 0)\,, \qquad \eta = (E^*, 0)\,,$$

$$g'((aE, X), (bE, Y)) = G(X, Y) + ab\,,$$

where $(J, G)$ is a Kählerian structure on $V$, $E$ is a nowhere vanishing vector field on $(a', b')$ and $E^*$ its dual (see, for instance, [B1]). Finally, since $g = \sigma g' + (1-\sigma)\,\eta\otimes\eta$, we see that the structure $(\varphi, \xi, \eta, g)$ on $U$ is given as in (2.1). ∎

REMARK. – In [K], K. KENMOTSU has proved that a Kenmotsu manifold is not compact. However, taking in the example 1, $L = S^1$, $V$ a compact Kähler manifold, $\eta$ the length element of the circle $S^1$ and $\sigma$ a positive function (not constant) on $S^1$, we obtain an almost contact metric structure of class $\mathcal{C}_5$, which is not Kenmotsu, on the compact manifold $M = S^1\times V$.

Finally, we suppose that $(\varphi, \xi, \eta, g)$ is an almost contact metric structure of class $\mathcal{C}_5$ on a simply connected manifold $M$. From Lemma 2.1, we have $(\mathrm{div}\,\xi/n)\ \eta = d(\ln\sigma)$ for a certain positive function $\sigma$ on $M$. Put $g' = (1/\sigma)\,g + (1-1/\sigma)\,\eta\otimes\eta$. Then, $(\varphi, \xi, \eta, g')$ is a cosymplectic structure on $M$. Consequently, from Proposition 2.3 of [FM], we deduce

PROPOSITION 2.1. – *A compact simply connected manifold can not admit a structure of class $\mathcal{C}_5$.*

## 3. – $\mathcal{C}_6$-structures.

An almost contact metric structure $(\varphi, \xi, \eta, g)$ on $M$ is said to be $\gamma$-Sasakian ($\gamma \in \boldsymbol{R}$, $\gamma \neq 0$) if it is normal and $d\eta = \gamma\Phi$, where $\Phi$ is the fundamental 2-form (see [JV]). If $(\varphi, \xi, \eta, g)$ is Sasakian then it is 1-Sasakian, and if it is $\gamma$-Sasakian or cosymplectic then it is of class $\mathcal{C}_6$. Next, we prove the converse.

LEMMA 3.1. – *If $(\varphi, \xi, \eta, g)$ is an almost contact metric structure on $M$ of class $\mathcal{C}_6$, then it is $\gamma$-Sasakian or cosymplectic.*

PROOF. – Taking in (1.3) the exterior differential, we get that $d\beta \wedge \Phi = 0$ and, since $M$ is connected and dim $M \geq 5$, we obtain $\beta = \gamma =$ constant. Thus, if $\gamma \neq 0$, the structure $(\varphi, \xi, \eta, g)$ is $\gamma$-Sasakian and if $\gamma = 0$, it is cosymplectic. ■

Therefore, a not cosymplectic $\mathcal{C}_6$ manifold is essentially a Sasakian manifold. In fact, if the structure $(\varphi, \xi, \eta, g)$ is $\gamma$-Sasakian then the structure $(\varphi, (1/\gamma)\xi, \gamma\eta, \gamma^2 g)$ is Sasakian.

Now, let $M$ be the product manifold $L \times V$, where $L$ is the circle $S^1$ or an open interval $(a', b')$, $-\infty \leq a' < b' \leq \infty$, and $(V, J, G)$ is an almost Hermitian manifold of dimension $2n$. Let $E$ be a nowhere vanishing vector field on $L$, $E^*$ its dual field of 1-forms and $\omega$ a 1-form on $V$.

Put,

$$
(3.1) \qquad \begin{cases} \varphi(aE, X) = (-\omega(JX)E, JX), \qquad \xi = (E, 0), \qquad \eta = (E^*, \omega), \\ g((aE, X), (bE, Y)) = G(X, Y) + ab + \omega(X)\omega(Y) + \omega(X)b + \omega(Y)a, \end{cases}
$$

where $a$, $b$ are differentiable functions on $M$ and $X, Y \in \mathfrak{X}(V)$. By straightforward verification we can see that $(\varphi, \xi, \eta, g)$ is an almost contact metric structure on $M$. Moreover, if we denote by $N_J$ and $N_\varphi$ the Nijenhuis tensors of $J$ and $\varphi$, respectively, and by $\Omega$ the Kähler form of $(V, J, G)$, then it is not difficult to check the following:

PROPOSITION 3.1.

$$
(3.2) \qquad N_\varphi((aE, X), (bE, Y)) + 2d\eta((aE, X), (bE, Y))(E, 0) =
$$

$$
= ((-\omega(N_J(X, Y)) - 2d\omega(JX, JY) + 2d\omega(X, Y))E, N_J(X, Y)),
$$

$$
(3.3) \qquad \Phi((aE, X), (bE, Y)) = \Omega(X, Y),
$$

*for $X, Y \in \mathfrak{X}(V)$ and $a, b$ differentiable functions on $M$.*

Next, we describe the local structure of manifolds of class $\mathcal{C}_6$. Previously, we examine the following example.

EXAMPLE 2. – Let $M$, $V$, $L$, $J$, $E$ and $E^*$ be as in example 1 and $\omega$ a 1-form on $V$, such that $d\omega = \beta\Omega$ where $\beta$ is constant and $\Omega$ the Kähler form of $(V, J, G)$. We define $\varphi, \xi, \eta$ and $g$ as in (3.1). Then, from (3.2) and (3.3) we deduce

a) If $\beta = 0$, $(\varphi, \xi, \eta, g)$ is a cosymplectic structure.

b) If $\beta \neq 0$, $(\varphi, \xi, \eta, g)$ is a $\beta$-Sasakian structure.

Now, we prove that the converse holds locally. Let $(\varphi, \xi, \eta, g)$ be an almost contact metric structure of class $\mathcal{C}_6$ on $M$, then

THEOREM 3.1. – *The manifold $M$ is locally the product $(a', b') \times V$, where $(a', b')$ is an open interval and $V$ is a $2n$-dimensional Kählerian manifold, on which the structure $(\varphi, \xi, \eta, g)$ is given as in Example* 2.

PROOF. – Fix a point $p \in M$. Let $U$ be a coordinate neighbourhood of $p$, with coordinates $(x^0, x^1, \dots, x^{2n})$ such that $U = (-a, a) \times V$, $x^0$ is the coordinate on $(-a, a)$, $(x^1, \dots, x^{2n})$ are the coordinates on $V$ and $\xi = \partial/\partial x^0$ on $U$. Let $g_{ij}$, $\eta_i$, $\varphi_j^i$ be the components of $g$, $\eta$ and $\varphi$ in the coordinates $(x^0, x^1, \dots, x^{2n})$. From Lemma 3.1 we obtain the relations

$$\mathcal{L}_\xi g = \mathcal{L}_\xi \eta = \mathcal{L}_\xi \varphi = 0$$

where $\mathcal{L}_\xi$ denotes the Lie derivate with respect to $\xi$. Using the above relations we deduce that the components $g_{ij}$, $\eta_i$, $\varphi_j^i$ are independent of the coordinate $x^0$. Therefore they can be used to a description of an almost Hermitian structure on $V$. Thus, define

$$J(\partial/\partial x^j) = \sum_{i=1}^{2n} \varphi_j^i (\partial/\partial x^i) \qquad j = 1, \dots, 2n,$$

$$G(\partial/\partial x^i, \partial/\partial x^j) = g_{ij} - \eta_i \eta_j, \qquad i, j = 1, \dots, 2n.$$

It is clear that the pair $(J, G)$ is an almost Hermitian structure on $V$. Moreover, if we put

$$\omega(\partial/\partial x^i) = \eta_i \qquad (i = 1, \dots, 2n), \qquad E = \partial/\partial x^0, \qquad E^* = dx^0$$

then the almost contact metric structure $(\varphi, \xi, \eta, g)$ on $U$ and the almost Hermitian structure $(J, G)$ on $V$ are related by (3.1). Consequently, from relations (3.2) and (3.3), we deduce that $(J, G)$ is a Kähler structure on $V$. Finally, from the definition of structure of class $\mathcal{C}_6$, we obtain $d\omega = \beta\Omega$ with $\beta$ constant, where $\Omega$ is the Kähler form of $(V, J, G)$. ∎

## 4. – Trans-Sasakian manifolds.

First, we study the rank of a trans-Sasakian structure

PROPOSITION 4.1. – *Let $(M, \varphi, \xi, \eta, g)$ be a trans-Sasakian manifold and r the rank of $(\varphi, \xi, \eta)$. Then r cannot be even. Moreover, if $r = 2s + 1$, then $s = 0$ or $s = n$ and we have*

*a)* $(\varphi, \xi, \eta, g)$ *is of class* $\mathcal{C}_5$ *if and only if* $s = 0$.

*b)* $(\varphi, \xi, \eta, g)$ *is of class* $\mathcal{C}_6$ *not cosymplectic if and only if* $s = n$.

PROOF. – If $r = 2s$, from (1.3), we deduce that $\beta \neq 0$ at every point. On the other hand, since $\eta \wedge (d\eta)^s = 0$ and $\eta \wedge \Phi^n \neq 0$ we obtain $\beta = 0$, which is a contradiction.

The assertion *a)* is evident.

Now, we suppose that $r = 2s + 1$, $s \neq 0$. Then, it is clear that $\beta \neq 0$ at every point and thus $r = 2n + 1$. From Lemma 2.1, the 1-form $\alpha\eta = -(\mathrm{div}\,\xi/2n)\ \eta$ is closed. Therefore, by using 1.3, we obtain

$$d\alpha \wedge \eta + \alpha\beta\Phi = 0, \tag{4.1}$$

and since $\eta \wedge \Phi \neq 0$, we deduce $\alpha\beta = 0$, *i.e.*, $\alpha = 0$. Consequently, the structure $(\varphi, \xi, \eta, g)$ is of class $\mathcal{C}_6$ not cosymplectic.

Conversely, if $r = 2s + 1$ and $(\varphi, \xi, \eta, g)$ is of class $\mathcal{C}_6$ and it is not cosymplectic then $\beta \neq 0$ and thus $s = n$. ■

Next, we prove that a trans-Sasakian structure is of class $\mathcal{C}_5$ or $\mathcal{C}_6$.

THEOREM 4.1. – *If $(\varphi, \xi, \eta, g)$ is a trans-Sasakian structure, it is of class $\mathcal{C}_5$ or $\mathcal{C}_6$.*

PROOF. – Denote by $A$ the following set

$$A = \{x \in M / \beta(x) = 0\},$$

where $d\eta = \beta\Phi$.

Let $x_0$ be a point of $A$. From Lemma 2.1, $2\alpha\eta = (\mathrm{div}\,\xi/n)\ \eta$ is a closed 1-form. Then, there exists an open neighbourhood $U$ of $x_0$ on which $d(\ln\sigma) = 2\alpha\eta$, for a certain positive function $\sigma$. Put,

$$g' = (1/\sigma)g + (1 - 1/\sigma)\eta \otimes \eta. \tag{4.2}$$

From relations (1.4), (1.5) and by using the Lemma 3.1 we obtain that the almost contact metric structure $(\varphi, \xi, \eta, g')$ is of class $\mathcal{C}_6$ and $\beta/\sigma = c$ ($c$ constant). Thus, since $\beta(x_0) = 0$, $c = 0$ and therefore $U \subseteq A$.

Consequently, $A$ is an open subset of $M$. On the other hand, it is clear that $A$ is closed in $M$. Therefore, from the connectedness of $M$, we deduce that $A = M$ or $A = \phi$. If $A = M$, $(\varphi, \xi, \eta, g)$ is of class $\mathcal{C}_5$ (in this case $M$ may also be cosymplectic) and if $A = \phi$ the rank of the structure $(\varphi, \xi, \eta)$ is $2n+1$ and hence, using the Proposition 4.1, $(\varphi, \xi, \eta, g)$ is of class $\mathcal{C}_6$. ■

The Theorem 4.1 is not true for dim $M = 3$. In fact, if $(M, \varphi, \xi, \eta, g)$ is a 3-dimensional Sasakian manifold, and

$$g' = \sigma g + (1 - \sigma)\,\eta \otimes \eta\,,$$

where $\sigma$ is a positive function on $M$, then

PROPOSITION 4.2. – *$(\varphi, \xi, \eta, g')$ is a trans-Sasakian structure on $M$. Moreover, if $\xi(\sigma) \neq 0$, then $(\varphi, \xi, \eta, g')$ is neither of class $\mathcal{C}_5$ nor $\mathcal{C}_6$.*

PROOF. – It is clear that $(\varphi, \xi, \eta, g')$ is a normal structure on $M$. Moreover, if $\Phi'$ is the fundamental 2-form of the structure $(\varphi, \xi, \eta, g')$, we deduce, from (1.4) and (1.5), that

$$d\Phi' = d(\ln \sigma) \wedge \Phi'\,, \tag{4.3}$$

$$d\eta' = (1/\sigma)\,\Phi'\,. \tag{4.4}$$

Now take a $\varphi$-bassis $\{E_0, E_1, E_2\}$ for the structure $(\varphi, \xi, \eta, g')$ and its dual bassis of 1-forms $\{E_0^* = \eta, E_1^*, E_2^*\}$. Then, $\Phi' = 2E_2^* \wedge E_1^*$ and $d(\ln \sigma) = \xi(\ln \sigma)\,\eta + E_1(\ln \sigma)\,E_1^* + E_2(\ln \sigma)\,E_2^*$. Thus,

$$d\Phi' = \xi(\ln \sigma)\,\eta \wedge \Phi' \tag{4.5}$$

and $(\varphi, \xi, \eta, g')$ is a trans-Sasakian structure. Moreover, by using (4.4), we deduce that $(\varphi, \xi, \eta, g')$ is not of class $\mathcal{C}_5$.

On the other hand, from (4.4) and (4.5), $(\varphi, \xi, \eta, g')$ is of class $\mathcal{C}_6$ if and only if $\xi(\ln \sigma) = 0$, *i.e.*, $\xi(\sigma) = 0$.

This ends the proof of the proposition. ■

Next, by using the Proposition 4.2, we give an example of trans-Sasakian structure which is neither of class $\mathcal{C}_5$ nor $\mathcal{C}_6$.

Let $H(1,1)$ be the group of matrices of real numbers of the form

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix},$$

where $x$, $y$, $z \in \boldsymbol{R}$. $H(1,1)$ is a connected simply connected nilpotent Lie group of dimension 3, which is called Heisenberg group (for an extensive study of Heisenberg groups see [H], [CFL]).

A bassis for the left invariant 1-forms on $H(1,1)$ is given by

$$\alpha = dx\,, \qquad \beta = dy\,, \qquad \gamma = dz - x\,dy$$

and its dual bassis of left invariant vector fields on $H(1,1)$ is given by

$$X = \partial/\partial x\,, \qquad Y = \partial/\partial y + x\partial/\partial z\,, \qquad Z = \partial/\partial z\,.$$

Define an almost contact metric structure $(\varphi, \xi, \eta, g)$ on $H(1,1)$ by

$$\varphi X = Y\,, \qquad \varphi Y = -X\,, \qquad \xi = Z\,, \qquad \eta = \gamma\,,$$

$$g = 1/2\,(\alpha \otimes \alpha + \beta \otimes \beta + \gamma \otimes \gamma)\,.$$

Then, $(\varphi, \xi, \eta, g)$ is a Sasakian structure on $H(1,1)$.

Now, put

$$g' = e^z g + (1 - e^z)\,\eta \otimes \eta\,.$$

Then, by using the Proposition 4.2 (in this case $\sigma = e^z$), $(\varphi, \xi, \eta, g')$ is a trans-Sasakian structure on $H(1,1)$ which is neither of class $\mathcal{C}_5$ nor $\mathcal{C}_6$.

Finally, since the unit sphere $S^3$ carries an induced Sasakian structure as orientable hypersurface of $\boldsymbol{R}^4$ (see, for instance, [B2]), we also can obtain, by using the Proposition 4.2, a trans-Sasakian structure on $S^3$, which is neither of class $\mathcal{C}_5$ nor $\mathcal{C}_6$.

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