THE LOCAL STRUCTURE OF TWISTED COVARIANCE ALGEBRAS

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1. Introduction

The fundamental problem in investigating the unitary representation theory of a separable locally compact group G is to determine its space $G^\wedge$ of (equivalence classes of) irreducible representations. It is known that when G is not type I, $G^\wedge$, with the Mackey Borel structure, is not standard, or even countably separated. This is generally interpreted to mean that the irreducible representations of such a group are not classifiable, and so the problem becomes to find a substitute for $G^\wedge$, simple enough to afford some hope that it can be described completely, yet complicated enough to reflect a significant part of the representation theory of G. Two promising candidates have been proposed, both defined using the group C^*-algebra $C^*(G)$ (which has the same representation theory as G): the space Prim G of primitive ideals of $C^*(G)$, which was shown by Effros [19] to be a standard Borel space in the Borel structure generated by the hull–kernel topology; and the space G_{nor} of quasi-equivalence classes of normal representations (traceable factor representations) of $C^*(G)$, shown by Halpern [35] to be standard in the Mackey Borel structure. (The results of [19] and [35] are actually valid for arbitrary separable C^*-algebras, not just those arising from groups.)

In the case that G is type I, both of these spaces may be naturally identified with $G^\wedge$. Striking evidence that they are natural objects of study may be found in the beautiful result [49] of Pukanszky, that for connected G they are "the same" in the sense that the map which associates to any element of G_{nor} the kernel of its members is a bijection of G_{nor} onto Prim $C^*(G)$. (It is easily shown that this bijection is in fact a Borel isomorphism.)

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This paper grew out of an attempt to extend Pukanszky's result to a wider class of groups, and more generally, to develop the basic theory of Prim G and G_{nor} for arbitrary groups. In pursuing these investigations it became apparent that it is convenient to work in the more general context of what we call "twisted covariant systems". Such a system consists of a triple $(G, A, \mathcal{T})$, in which G is a locally compact group, A a C^*-algebra on which G acts continuously by automorphisms, and $\mathcal{T}$ a homomorphism of a closed normal subgroup $N_{\mathcal{T}}$ of G into the unitary group of the multiplier algebra $\mathcal{M}(A)$ of A, such that $\mathcal{T}$ satisfies certain conditions with respect to the G-action on A. (See Section 1 for a precise definition.) To each such system can be associated a C^*-algebra $C^*(G, A, \mathcal{T})$ (the "twisted covariance algebra" of the system) whose representation theory coincides with the "covariant" representation theory of $(G, A, \mathcal{T})$. The usefulness of twisted covariance algebras in the study of group C^*-algebras stems from the following facts: given a closed normal subgroup K of a locally compact group G, there is a natural twisted covariant system $(G, C^*(K), \mathcal{T}^K)$ whose C^*-algebra is isomorphic to $C^*(G)$. Using this system it is frequently possible (and useful) to break up $C^*(G)$ into more manageable pieces, which are no longer group C^*-algebras, but which are twisted covariance algebras. Finally, the C^*-algebras associated to the projective representation theory of G can be naturally described as twisted covariance algebras (and in so doing, one avoids entirely the necessity of dealing with cocycles).

With the above facts and the further special case of covariance algebras as motivation, we have attempted in this paper to describe some of the basic local structure theory (with emphasis on the primitive ideal and trace structure) of twisted covariance algebras. This necessitated the development of a theory of induced representations of such algebras which generalizes that of Rieffel [51], [52] in the group case. This theory, involving Rieffel's concept of (strong) Morita equivalence, is much better suited to investigating the local structure of the C^*-algebras of non-type I groups than is the classical Mackey theory, especially now that Morita equivalence is beginning to be better understood ([30], [59]).

The outline of the paper is as follows (we let $(G, A, \mathcal{T})$ denote a fixed twisted covariant system): Section 1 contains the definition and some basic properties of twisted covariance algebras. (These algebras, under the name "produits croisés restreints", have also been studied in [12]; they can be regarded as special cases of the enveloping C^*-algebras of cross-sectional algebras of Banach *-algebraic bundles [24], and are related to the generalized L^1 algebras of [40], [41], [42]. There is little overlap in this paper with the methods or results of any of those references. Many of our results can be proved in the more general context of [24], but with greater technical complications.)

In Section 2 we develop a theory of induced representations of twisted covariance algebras, including an imprimitivity theorem. The essential techniques are those of [51], with some modifications and simplifications.

Section 3 contains results on continuity of the induction and restriction processes, generalizing results first obtained by Fell [22] for the case of separable locally compact groups. Our methods are quite different from Fell's, and rely on the special form of the imprimitivity theorem obtained in Section 2. Section 3 also contains a number of miscellaneous results, concerning among other things the implications for the inducing process of amenability of the groups involved.

In Section 4 we develop a "Mackey machine" for twisted covariance algebras which emphasizes structure theory. The first part, involving the reduction to stability groups (for the action of G on Prim A), generalizes Rieffel's theory in [52]; we have, however, substantially simplified his proof. The second part, involving the case of a G-stable primitive ideal P of A, is new in the form we give it, and states roughly that (at least in nice situations) the part of $C^*(G, A, \mathcal{T})$ which "lives over" P decomposes as a tensor product of simpler algebras.

Section 5 contains our results on Prim $C^*(G, A, \mathcal{T})$. These center around a generalized form of a conjecture of Effros and Hahn, which roughly speaking states that every primitive ideal of $C^*(G, A, \mathcal{T})$ should be "induced" from the algebra of a stability subgroup. Our results contain as special cases results of Gootman [27] and Zeller-Meier [57] concerning this conjecture. The methods used appear to be new, and simpler than those used in earlier attacks on the conjecture.

Section 6 concerns traces on $C^*(G, A, \mathcal{T})$. We begin with a result on inducing traces from A to $C^*(G, A, \mathcal{T})$, which generalizes a theorem essentially obtained by Pukanszky in [49, Section 2] for the group case. (A somewhat weaker result than ours was proved by Dang Ngoc in [12].) We then formulate an analogue of the Effros–Hahn conjecture for traces, and prove it in the special cases of "discrete" and "regular" systems. The result in the discrete case is obtained by filling in a slight gap in work of Zeller-Meier [57], but the result in the regular case (which corresponds to the case of a regularly embedded normal subgroup) appears to be new, and depends on a result concerning normal representations of Morita equivalent algebras.

In the seventh and final section we consider "abelian systems", i.e. those for which $G/N_{\mathcal{T}}$ (but not necessarily A) is abelian. We show first that the imprimitivity theorem for such systems may be regarded as a generalization of a weak form of the Takai Duality Theorem [54]. Using this, and the second part of the Mackey machine of Section 4, we are able to give new and considerably simplified proofs of results of Kleppner [38] and Baggett and Kleppner [3] concerning projective representations of abelian groups; we also answer a question raised in [38]. Our final theorem, which may be regarded as the culmination of the paper, concerns the trace and primitive ideal structure of systems which arise as "restrictions" of regular systems. This theorem has as a corollary the result that if G and H are separable locally compact groups, and π a continuous injective homomorphism of H into G whose image contains a type I regularly embedded normal subgroup K of G such that G/K is abelian, then every point of Prim H is locally closed and the natural map of H_{nor} into Prim H is a bijection. When combined with results of Pukanszky and Dixmier on algebraic groups this leads to a considerably simplified proof of the theorem of Pukanszky mentioned earlier, as well as of the result of Moore and Rosenberg [46] that points in the primitive ideal spaces of connected groups are always locally closed. In the proof of our theorem we have been greatly influenced by Pukanszky's arguments in [49].

Our methods throughout the paper are heavily algebraic in nature. We require practically no knowledge of group representations, developing all the necessary tools from scratch, but on the other hand we assume familiarity with the basic theory of C^*-algebras as contained in the first five chapters of [15]. Some acquaintance with Rieffel's theory of imprimitivity bimodules and strong Morita equivalence of C^*-algebras, as contained in [51, Sections 2 and 4–6], and [52, Section 3], would also be helpful.

We use $\mathbf{R}$, $\mathbf{C}$, and $\mathbf{T}$ to denote the reals, complexes, and unit circle in $\mathbf{C}$, respectively. Ends of proofs are denoted by "///".

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1. Twisted covariance algebras

We recall some basic facts (cf. [18], [55], [8]) about covariance algebras. A *covariant system* (G, A) consists of a locally compact group G, a C^*-algebra A, and a (left) action $(s, a)\mapsto {}^s a$ of G by *-automorphisms of A, which is strongly continuous in the sense (usual for representations of topological groups on Banach spaces) that for every a in A the map $s\mapsto {}^s a$ of G into A is continuous. A (*covariant*) *representation* L of (G, A) on a Banach space $\mathcal{B}=\mathcal{B}_L$ consists of a uniformly bounded strongly continuous representation V_L of G together with a norm decreasing non-degenerate representation M_L of A such that $V_L(s)\,M_L(a)\,V_L(s^{-1})=M_L({}^s a)$ (s and r are to denote arbitrary elements of G, a and b elements of A throughout the paper; we will frequently omit quantifiers involving these letters to avoid tedious repetition). By non-degeneracy of M_L we mean that $M_L(A)\,\mathcal{B}$ (the closed linear span of $\{M_L(a)\xi\colon a\in A,\ \xi\in\mathcal{B}\}$) is equal to $\mathcal{B}$; all algebra representations are assumed nondegenerate, unless otherwise stated. When $\mathcal{B}$ is a Hilbert space we further require that V_L be unitary and M_L be *-preserving.

To each covariant system we associate the Banach *-algebra $L^1(G, A)$ of all Bochner integrable A-valued measureable functions on G with respect to a fixed symmetric Haar measure ds (that is, $ds = \triangle(s)^{-\frac{1}{2}} d\lambda(s)$, where $d\lambda$ is a left Haar measure on G, and $\triangle = \triangle_G$ is the modular function of G), with multiplication and involution defined by

$$(f * g)(r) = \int f(s)\,{}^s(g(s^{-1}r))\,d\lambda(s); \; f^*(r) = {}^r(f(r^{-1})^*).$$

(Usually $L^1(G, A)$ is taken with respect to $d\lambda$, necessitating the introduction of $\triangle$ into the formula for the involution; the algebra defined above is isometrically *-isomorphic to the latter one via the map $f \mapsto \triangle^{-\frac{1}{2}} f$. The modified definition above results in some notational simplifications, which is our reason for introducing it.)

To each covariant representation L of (G, A) there corresponds a representation (also denoted by L) of $L^1(G, A)$, called the "integrated form" representation and defined by the formula

$$L(f)\,\xi = \int M_L(f(s))\, V_L(s)\,\xi\, ds$$

for each ξ in $\mathcal{B}$ and f in $L^1(G, A)$. In particular, the representation of $L^1(G, A)$ on itself by left multiplication is the integrated form of the covariant representation given by

$$(V(s)f)(r) = \triangle(s)^{\frac{1}{2}}\,{}^s(f(s^{-1}r))$$

and

$$(M(a)f)(r) = af(r).$$

These actions of G and A are easily seen to extend to actions on the enveloping C^*-algebra $C^*(G, A)$ of $L^1(G, A)$, and so induce homomorphisms R_G and R_A of G and A into the multiplier algebra [1] $\mathcal{M}(C^*(G, A))$; we will occasionally identify elements of G and A with their images in $\mathcal{M}(C^*(G, A))$, thus writing sa for $R_G(s)\,R_A(a)$, etc. (The usefulness of multiplier algebras in the theory of covariance algebras was first pointed out in [8], to which we refer the reader for further details.) If H is a closed subgroup of G and we set $R_H = R_G|_H$, then R_H and R_A define a covariant representation of (H, A) on $C^*(G, A)$, the integrated form of which gives a *-homomorphism of $L^1(H, A)$ into $\mathcal{M}(C^*(G, A))$; since the latter is a C^*-algebra this homomorphism "factors through $C^*(H, A)$. Similarly the integrated form of R_H gives a *-homomorphism (also denoted R_H) of $C^*(H)$ into $\mathcal{M}(C^*(G, A))$. We note the useful fact that the set $\{fa\,(= R_G(f)\,R_A(a))\,|\, f \in C^*(G), a \in A\}$ of products in $\mathcal{M}(C^*(G, A))$ is contained in $C^*(G, A)$, and generates it as a C^*-algebra—this follows from the easily proved fact that elements of $L^1(G, A)$ of the form $s \mapsto f(s)a$, where $f \in L^1(G, ds)$ and $a \in A$, have dense span in $C^*(G, A)$.

Any non-degenerate *-representation of $L^1(G, A)$ extends uniquely to $C^*(G, A)$, and hence to $\mathcal{M}(C^*(G, A))$, and so defines (by composition with R_G and R_A) a covariant *-representation of (G, A); viewed as a map from *-representations of $L^1(G, A)$ to covariant *-representations this process is inverse to the "integrated form" construction.

$C^*(G, A)$ is often referred to as the *covariance algebra*, or *crossed product algebra* of the system (G, A). We will at times use the same letter to denote a covariant *-representation of (G, A) and the uniquely associated *-representation of $C^*(G, A)$. In particular (i.e. the special case $A = \mathbf{C}$) the same letter will be used for representations of G and $C^*(G)$. Similarly we sometimes use the same letter for a *-representation of a C^*-algebra and the canonical extension (cf. [37]) to its multiplier algebra.

We now introduce a refinement of the concept of covariant system. A *twisted covariant system* $(G, A, \mathcal{T})$ consists of a covariant system (G, A) together with a continuous homomorphism $\mathcal{T}$ (called the *twisting map* of the system) of a closed normal subgroup $N_{\mathcal{T}}$ of G into the group of unitaries of $\mathcal{M}(A)$, equipped with the strict topology [7], such that $\mathcal{T}(n)\, a\, \mathcal{T}(n^{-1}) = {}^n a$ and $\mathcal{T}(sns^{-1}) = {}^s\mathcal{T}(n)$ (we extend here the automorphism defined by s to all of $\mathcal{M}(A)$) for all $n \in N$, $a \in A$, and $s \in G$. The *twisted covariance algebra* (or *twisted crossed product algebra*) $C^*(G, A, \mathcal{T})$ is defined to be the quotient of $C^*(G, A)$ by the unique minimal (closed two-sided) ideal $I_{\mathcal{T}}$ of $C^*(G, A)$ having the property that for any *-representation L of $C^*(G, A)$ with $\ker L \supseteq I_{\mathcal{T}}$, the following holds:

$$V_L(n) = M_L(\mathcal{T}(n)) \quad \text{for all } n \in N_{\mathcal{T}}. \tag{1}$$

To see that $I_{\mathcal{T}}$ is well defined, observe that a *-representation L "preserves $\mathcal{T}$", in the sense that (1) holds, iff the kernel of its extension to $\mathcal{M}(C^*(G, A))$ contains the ideal I of $\mathcal{M}(C^*(G, A))$ generated by $\{n a - \mathcal{T}(n) a \mid n \in N, a \in A\}$ and that by nondegeneracy of L this holds iff $\ker L$ contains the ideal $C^*(G, A) \cdot I = C^*(G, A) \cap I$ of $C^*(G, A)$. Thus we may define $I_{\mathcal{T}} = C^*(G, A) \cap I$.

**-Representations* of $(G, A, \mathcal{T})$ are defined to be those covariant *-representations of (G, A) which preserve $\mathcal{T}$. They are thus in natural 1–1 correspondence with *-representations of $C^*(G, A, \mathcal{T})$.

Throughout the remainder of the paper $(G, A, \mathcal{T})$ will denote a fixed twisted covariant system.

Whenever π is a continuous homomorphism of a locally compact group H into G we can form an associated twisted covariant system $(H, A, \mathcal{T}_H)$ (the "pull-back" of $(G, A, \mathcal{T})$

along π) by defining ${}^t a = {}^{\pi(t)}a$ for all t in H and a in A and $\mathcal{J}_H = \mathcal{J} \circ \pi|_{\pi^{-1}(N_\mathcal{J})}$ (so that $N_{\mathcal{J}_H} = \pi^{-1}(N_\mathcal{J})$). Note that when H is a closed subgroup of G containing $N_\mathcal{J}$ and π is the inclusion map, $\mathcal{J}_H$ is just $\mathcal{J}$.

In [12] Dang Ngoc discusses twisted covariance algebras under the name "produits croisés restreints". He gives a somewhat different construction of them which we now show is equivalent to ours. Let $C_c(G, A, \mathcal{J})$ denote the set of continuous A-valued functions f on G, having supports whose images in $G/N_\mathcal{J}$ are relatively compact, and such that

$$f(ns) = f(s)\,\mathcal{J}(n)^{-1} \quad \text{for all } n \in N_\mathcal{J},\ s \in G.$$

Fix left Haar measures on $N = N_\mathcal{J}$ and G/N, normalized so that

$$\int_G f(s)\,d\lambda(s) = \int_{G/N}\int_N f(sn)\,d\lambda(n)\,d\lambda(\dot{s}) \quad \text{for all } f \in C_c(G), \text{ (here } \dot{s} = sN),$$

and let dn, $d\dot{s}$ be the associated symmetric measures $\triangle_N^{-1/2}(n)\,d\lambda(n)$, $\triangle_{G/N}^{-1/2}(\dot{s})\,d\lambda(\dot{s})$. We give $C_c(G, A, \mathcal{J})$ the structure of a normed *-algebra by

$$f * g(s) = \int_{G/N} f(r)\,{}^r g(r^{-1}s)\,d\lambda(\dot{r});\ f^*(s) = {}^s f(s^{-1})^*$$

$$\|f\|(s) = \int_{G/N} \|f(r)\|\,d\dot{r}.$$

Let $\triangle_{G,N}(s) = d({}^{s^{-1}}n)/dn$ (so that $\triangle_G(s) = \triangle_{G/N}(\dot{s})\,\triangle_{G,N}(s)$), where ${}^{s^{-1}}n = s^{-1}ns$.

There is a norm-decreasing *-homomorphism π of $C_c(G, A)$ (the sub-algebra of $L^1(G, A)$ consisting of continuous functions of compact support) into $C_c(G, A, \mathcal{J})$, defined by $\pi(f)(s) = \int f(sn)\,\mathcal{J}({}^s n)\,dn$. The proof of [50, 10.9] (with slight modifications) shows that there is an isometric cross section for π. It follows that π extends to a *-homomorphism (also denoted π) of $L^1(G, A)$ onto the completion $L^1(G, A, \mathcal{J})$ of $C_c(G, A, \mathcal{J})$, and that $L^1(G, A, \mathcal{J})$ carries the quotient norm from $L^1(G, A)$. In particular $L^1(G, A, \mathcal{J})$ has a bounded approximate identity, and so it has an enveloping C^*-algebra B which is a quotient algebra of $C^*(G, A)$. Dang Ngoc defines the produit croisé restreint of $(G, A, \mathcal{J})$ to be this enveloping algebra B.

Observe that there are left actions of G and A on $C_c(G, A, \mathcal{J})$ given by

$$(rf)(s) = \triangle_{G/N}^{1/2}(\dot{r})\,{}^r f(r^{-1}s);\ (af)(s) = af(s),$$

the integrated form of which gives the natural left action of $C_c(G, A)$ on $C_c(G, A, \mathcal{J})$ arising from the homomorphism π. For $n \in N$, we check that $(nf)(s) = \mathcal{J}(n)f(s)$. It follows easily that for any *-representation L' of B, $L' \circ \pi$ preserves $\mathcal{J}$. On the other hand given a covariant *-representation L of (G, A) which preserves $\mathcal{J}$, we easily check that

$$L'(f)\,\xi = \int_{G/N} M_L(f(s))\,V_L(s)\,\xi\,d\dot{s}.$$

defines a norm decreasing *-representation L' of $C_c(G, A, \mathcal{J})$ satisfying $L(g) = (L' \circ \pi)(g)$ for all $g \in C_c(G, A)$. It follows that a *-representation of (G, A) preserves $\mathcal{J}$ iff its integrated form factors through π, showing that B is naturally isomorphic to $C^*(G, A, \mathcal{J})$ and thus that our definition agrees with Dang Ngoc's.

Among the most interesting twisted covariant systems are those arising from group extensions. We treat these systems as special cases of "iterated twisted covariant systems". (The latter are also discussed in Section 2 of [12], where a different proof of the proposition below is given.)

Let K be a closed normal subgroup of G containing $N_{\mathcal{J}}$. We may define an action of G by *-automorphisms of $C_c(K, A, \mathcal{J})$ by

$$({}^s f)(t) = \triangle_{G/N,\,K/N}(s)\,{}^s(f(s^{-1}ts)).$$

This action is isometric and strongly continuous with respect to the L^1 norm, and so by the universality property of the enveloping C^*-algebra it extends uniquely to a (strongly continuous) action on $C^*(K, A, \mathcal{J})$. Observe that the natural left actions defined earlier of G and A on $C_c(G, A, \mathcal{J})$ give rise to a covariant pair $(R^{\mathcal{J}}_G, R^{\mathcal{J}}_A)$ of homomorphisms of G and A into $\mathcal{M}(C^*(G, A, \mathcal{J}))$, which preserves $\mathcal{J}$ in the sense that, if $\tilde{R}^{\mathcal{J}}_A$ denotes the natural extension of $R^{\mathcal{J}}_A$ to a homomorphism of $\mathcal{M}(A)$ into $\mathcal{M}(C^*(G, A, \mathcal{J}))$, we have $R^{\mathcal{J}}_G(n) = \tilde{R}^{\mathcal{J}}_A(\mathcal{J}(n))$ for all $n \in N_{\mathcal{J}}$. ($\tilde{R}^{\mathcal{J}}_A$ exists because A acts non-degenerately on $C^*(G, A, \mathcal{J})$.) Similarly, we have homomorphisms $R^{\mathcal{J}}_K$ and $R^{\mathcal{J}}_{A,K}$ of K and A into $\mathcal{M}(C^*(K, A, \mathcal{J}))$. Then it is easily verified that $\mathcal{J}^K = R^{\mathcal{J}}_K$ is a twisting map for the system $(C, C^*(K, A, \mathcal{J}))$, so we can form the "iterated twisted covariance algebra" $B^{(G,K)} = C^*(G, C^*(K, A, \mathcal{J}), \mathcal{J}^K)$. We proceed now to show that this algebra is naturally isomorphic to $C^*(G, A, \mathcal{J})$:

Let R^1 and R^2 denote the natural homomorphisms of G and $C^*(K, A, \mathcal{J})$ into $\mathcal{M}(B^{(G,K)})$, and $\tilde{R}^2$ the extension of R^2 to $\mathcal{M}(C^*(K, A, \mathcal{J})$. Let $R^3 = \tilde{R}^2 \circ R^{\mathcal{J}}_{A,K}$. Then (R^1, R^3) is easily seen to be covariant and $\mathcal{J}$-preserving, and thus, via the integrated form construction, defines a *-homomorphism R of $C^*(G, A, \mathcal{J})$ into $\mathcal{M}(B^{(G,K)})$. Now $C^*(G, A, \mathcal{J})$ is generated by $R^{\mathcal{J}}_G(C^*(G))\,R^{\mathcal{J}}_A(A)$ (this fact, observed previously for covariance algebras, holds for twisted covariance algebras since they are quotients of covariance algebras), and $B^{(G,K)}$ by $R^1(C^*(G))((\tilde{R}^2 \circ R^{\mathcal{J}}_K(C^*(K)))(R^3(A)))$; but since $\tilde{R}_2 \circ R^{\mathcal{J}}_K = R^1|_K$, and since $R^1|_K(C^*(K))$ acts as multipliers on $R^1(C^*(G))$ (via the action of $C^*(K)$ by multipliers of $C^*(G)$), we have

$$R(C^*(G, A, \mathcal{J})) = R^1(C^*(G))\,R^3(A) = B^{(G,K)}.$$

On the other hand, we have a natural map R^4 of $C^*(K, A, \mathcal{T})$ into $\mathcal{M}(C^*(G, A, \mathcal{T}))$, and it is easily verified that the pair $(R_G^{\mathcal{T}}, R^4)$ defines a covariant homomorphism of $(G, C^*(K, A, \mathcal{T}), \mathcal{T}^K)$ into $\mathcal{M}(C^*(G, A, \mathcal{T}))$; let R' be its integrated form, $\tilde{R}'$ the extension to $\mathcal{M}(B^{(G,K)})$. Then $\tilde{R}' \circ R^1 = R_G^{\mathcal{T}}$, and $\tilde{R}' \circ R^3 = R_A^{\mathcal{T}}$. It follows easily that $\tilde{R}' \circ R$ is the identity map on $C^*(G, A, \mathcal{T})$, so that (since R has image $B^{(G,K)}$):

PROPOSITION 1. *With the above notation, R is an isomorphism of $C^*(G, A, \mathcal{T})$ onto the iterated algebra $B^{(G,K)} = C^*(G, C^*(K, A, \mathcal{T}), \mathcal{T}^K)$.* ///

Applying this to the case $A = \mathbf{C}$ with $\mathcal{T}$ trivial, we get

COROLLARY. *For each closed normal subgroup K of G there is a natural twisted covariant system $(C, C^*(K), \mathcal{T}^K)$, (with $N_{\mathcal{T}^K} = K$) such that $C^*(G, C^*(K), \mathcal{T}^K)$ is isomorphic to $C^*(G)$.* ///

2. Induced representations

We proceed now to develop a theory of induced representations for twisted covariance algebras which imitates and extends that of Rieffel [51] for the group case. As in [51] we find it convenient to work with "pre-C^*-algebras", in particular the dense subalgebra $C_c(G, A)$ of $C^*(G, A)$. In addition to its norm topology this algebra carries the "inductive limit topology": a net (f_α) in $C_c(G, A)$ converges to f in this topology iff it tends to f uniformly (for the norm on A), and for some α_0 and compact $K \subseteq G$ all the f_α with $\alpha \geqslant \alpha_0$ are supported in K. One sees easily that the restriction of the norm topology of $L^1(G, A)$ to $C_c(G, A)$ is weaker than the inductive limit topology, so that in particular the restriction of any *-representation of $L^1(G, A)$ to $C_c(G, A)$ is continuous for the inductive limit topology.

Let H be a closed subgroup of G, and define the "imprimitivity algebra" $E = E_H^G$ to be the covariance algebra of the system $(G, A \otimes C_\infty(G/H))$ ($C_\infty(G/H)$ denotes the algebra of continuous complex-valued functions on G/H vanishing at infinity, and $\otimes$ the minimal, or spatial, tensor product—see [53, p. 59]) where $A \otimes C_\infty(G/H)$ is given the diagonal action of G, defined on elementary tensors by ${}^s(a \otimes f) = {}^s a \otimes {}^s f$; here ${}^s f$ is defined by ${}^s f(C) = f(s^{-1}C)$ for $f \in C_\infty(G/H)$, $C \in G/H$. Using the canonical isomorphism of $A \otimes C_\infty(G/H)$ with $C_\infty(G/H, A)$ [53, pp. 59–60] we may alternatively define this action of G via ${}^s\psi(C) = {}^s(\psi(s^{-1}C))$ for all ψ in $C_\infty(G/H, A)$.

As in [51], we will construct an "imprimitivity bimodule" X between E and $B = C^*(H, A)$, which will then be used to define induced representations of $C^*(G, A)$.

To begin with we work with the subalgebras $E_0 = C_c(G, C_c(G/H, A))$ of E and $B_0 = C_c(H, A)$ of B; we regard elements of E_0 as continuous A-valued functions on $G \times G$ which are constant, in the second variable, on cosets of H. Then $X_0 = C_c(G, A)$ can be made into an $E_0 - B_0$ bimodule as follows: Let the algebra $C(G/H, A)$ of bounded continuous functions on G which are constant on H-cosets act to the left of X_0 by multiplication:

$$(\psi x)(r) = \psi(r)x(r) \quad (\psi \in C(G/H, A))$$

and let G act to the left on X_0 by

$$(sx)(r) = \triangle_G^{1/2}(s)\,{}^s(x(s^{-1}r)).$$

Here as in the rest of the paper x, y, and z denote elements of X_0 (or of its completion X, once that is defined). Then E_0 acts on X_0 via the integrated form representation

$$(fx)(r) = \int_G f(s, r)\,{}^s(x(s^{-1}r))\,d\lambda(s) \quad (f \in E_0).$$

B_0 acts to the right by

$$(xg)(r) = \int_H x(rt^{-1})^{rt^{-1}}g(t)\,d\varrho(t) \quad (g \in B_0),$$

the integrated form (with respect to right Haar measure ϱ on H) of the right A-action defined by

$$(xa)(r) = x(r)({}^r a)$$

and the right H-action defined by $xt(r) = x(rt^{-1})\,\triangle_H^{-1/2}(t)$ (t will denote elements of H). It is easily verified by use of the Fubini theorem that these formulae define commuting actions of E_0 and B_0. Next we introduce E_0 and B_0-valued inner products on X_0 as follows:

$$\langle x, y\rangle_{E_0}(s, r) = \int_H x(rt^{-1})\,{}^s(x(s^{-1}rt^{-1})^*)\,d\varrho(t)$$

$$\langle x, y\rangle_{B_0}(t) = \int_G {}^s(x(s^{-1})^*)\,{}^s(y(s^{-1}t))\,d\lambda(s).$$

It may readily be verified that $\langle x, y\rangle_{E_0}$ is constant under right translation of the second variable by elements of H, and that the resulting function on $G \times G/H$ is continuous of compact support. Similarly $\langle x, y\rangle_{B_0}$ is verified to lie in B_0. The following formulae follow from routine computations:

$$\langle x, yg\rangle_{B_0} = \langle x, y\rangle_{B_0}g, \ \langle fx, y\rangle_{E_0} = f\langle x, y\rangle_{E_0},$$

$$\langle fx, y\rangle_{B_0} = \langle x, f^*y\rangle_{B_0}, \ \langle x, yg\rangle_{E_0} = \langle xg^*, y\rangle_{E_0},$$

for all $g \in B_0$ and $f \in E_0$; and

$$\langle x, y\rangle_{E_0}^* = \langle y, x\rangle_{E_0};\ \langle x, y\rangle_{B_0}^* = \langle y, x\rangle_{B_0};\ x\langle y, z\rangle_{B_0} = \langle x, y\rangle_{E_0}z.$$

To complete the proof that X_0 with the above inner products is an E_0-B_0 imprimitivity bimodule, we need the following lemma. Recall that an approximate identity for an action of a topological algebra $\mathcal{A}$ on a topological module $\mathfrak{M}$ is a net (f_α) in $\mathcal{A}$ such that for each m in $\mathfrak{M}$, $(f_\alpha m)$ converges to m.

LEMMA 2. (i) *There exists a net (f_α) in E_0 which is an approximate identity with respect to the inductive limit topologies for the left and right actions of E_0 on itself and the left action on X_0, and which has the property that each f_α is a finite sum of elements of the form $\langle x, x\rangle_{E_0}$.*

(ii) *There exists (g_β) in B_0 with similar properties.*

Proof. (i) Let $(a_j)_{j\in\mathcal{J}}$ be an approximate identity for A consisting of positive elements of norm $\leqslant 1$. To each quadruple $\alpha=(U, K, j, \varepsilon)$, where U is a neighborhood of the identity in G, K is a compact subset of G/H, $j\in\mathcal{J}$, and $\varepsilon>0$, we associate an $f_\alpha\in E_0$ as follows:

Let C be a compact subset of G whose projection into G/H contains K. As in the proof of [51, 7.11] we choose a "truncated Bruhat approximate cross section" h in $C_c(G)$ such that $\int_H h(st^{-1})\,d\varrho(t)=1$ for all s in C. Let D be the support of h. Choose V to be a symmetric neighborhood of the identity of G such that $V^2\subseteq U$ and $\|{}^s a_j^{1/2}-a_j^{1/2}\|<\varepsilon$ for all s in V^2. Multiplying h pointwise by the elements of a suitable partition of unity, we obtain non-zero functions h_i, $i=1, \dots, n$ each having support in Vs_i for some $s_i\in G$, and such that $\sum_{i=1}^n \int_H h_i(st^{-1})\,d\varrho(t)=1$ for all s in C. Now let $\gamma_i=1/\int_G h_i(s^{-1})\,d\lambda(s)$, and define x_i in X_0 by $x_i(r)=(h_i(r)a_j^{1/2})\gamma_i^{1/2}$ for each $i=1, \dots, n$. Then $\langle x_i, x_i\rangle_{E_0}(s, r)=\int_H x_i(rt^{-1})\,{}^s(x_i(s^{-1}rt^{-1})^*)\,d\varrho(t)=\gamma_i\int_H h_i(rt^{-1})a_j^{1/2}\cdot h_i(s^{-1}rt^{-1})\,{}^s a_j^{1/2}\,d\varrho(t)$. In particular

$$\langle x_i, x_i\rangle_{E_0}(s, r)=0 \text{ for } s\notin V^2, \tag{2}$$

and for all r in C and all s we have

$$a_i=\sum_{i=1}^n\int_H h_i(rt^{-1})\,a_j\,d\varrho(t)=\sum_{i=1}^n\int_H h_i(rt^{-1})\int_G \gamma_i h_i(s^{-1}rt^{-1})\,a_j\,d\lambda(s)\,d\varrho(t),$$

so that

$$\begin{aligned}\left\|\sum_{i=1}^n\int_G\langle x_i, x_i\rangle_{E_0}(s, r)\,d\lambda(s)-a_j\right\| &= \left\|\sum_{i=1}^n\int_H\int_G h_i(rt^{-1})\,(a_j^{1/2}\,{}^s a_j^{1/2}-a_j)\,\gamma_i h_i(s^{-1}rt^{-1})\,d\lambda(s)\,d\varrho(t)\right\|\\ &\leqslant \sum_{i=1}^n\int_H\int_G h_i(rt^{-1})\,\varepsilon\gamma_i h_i(s^{-1}rt^{-1})\,d\lambda(s)\,d\varrho(t)\\ &= \varepsilon\sum_{i=1}^n\int_H h_j(rt^{-1})\,d\varrho(t)=\varepsilon.\end{aligned}$$

The inequality follows by using (2) and the fact that $\|a_j\|\leqslant 1$. Define $f_\alpha=\sum_{i=1}^n\langle x_i, x_i\rangle_{E_0}$, and direct the set of α's by $(U, K, j, \varepsilon)\leqslant(U', K', j', \varepsilon')$ iff $U'\subseteq U$, $K'\supseteq K$, $j'\geqslant j$, and $\varepsilon'\leqslant\varepsilon$.

Given $f \in E_0$ and $\varepsilon > 0$, put $D = p_1$ (supp f) and $K = p_2$ (supp f) where p_1 and p_2 denote the projections of $G \times G/H$ onto its first and second factors respectively. A routine compactness argument shows that we may choose j in $\mathfrak{J}$ and a neighborhood U of the identity of G such that $\|a_j f(s_0, r) - f(s_0, r)\| < \varepsilon$ and $\|{}^s f(s^{-1}s_0, s^{-1}r) - f(s_0, r)\| < \varepsilon$ for all s_0, r in G and s in U. A straightforward computation using the facts proved above shows that, with $\alpha = (U, K, j, \varepsilon)$, $f_\alpha * f$ has support contained in $UD \times UK$. Using

$$\int_G \|f_\alpha(s, r)\| \, d\lambda(s) \leqslant \sum_{i=1}^n \int_H h_i(rt^{-1}) \gamma_i \int_G h_i(s^{-1}rt^{-1}) \, d\lambda(s) \, d\varrho(t) = 1$$

and the facts above, we see that, for all s_0 and r,

$$\begin{aligned}
&\|f_\alpha * f(s_0, r) - f(s_0, r)\| \\
&\quad = \left\| \int_G f_\alpha(s, r)\, {}^s f(s^{-1}s_0, s^{-1}r) \, d\lambda(s) - f(s_0, r) \right\| \\
&\quad \leqslant \int_G \|f_\alpha(s, r)\| \cdot \|{}^s f(s^{-1}s_0, s^{-1}r) - f(s_0, r)\| \, d\lambda(s) + \left\| \int_G f_\alpha(s, r) \, d\lambda(s) \cdot f(s_0, r) - f(s_0, r) \right\| \\
&\quad \leqslant \varepsilon + \left\| \int_G f_\alpha(s, r) \, d\lambda(s) - a_j \right\| \cdot \|f(s_0, r)\| + \|a_j f(s_0, r) - f(s_0, r)\| \\
&\quad \leqslant 2\varepsilon + \varepsilon \|f(s_0, r)\|,
\end{aligned}$$

which shows that the f_α are a left approximate identity; continuity of the *-operation for the inductive limit topology then implies that they are a right approximate identity. A similar computation shows that they are an approximate identity for the action on X_0.

(ii) has a similar, but easier proof; we may in fact take the g_β to be of the form $\langle x, x \rangle_{B_0}$, with the x having supports that shrink to the identity. ///

From this lemma it follows that the span of the range of $\langle\ ,\ \rangle_{E_0}$ contains an approximate identity for the inductive limit topology, hence since it is an ideal in E_0 it is dense in this topology and so a fortiori in the C^* topology. Similarly the range of $\langle\ ,\ \rangle_{B_0}$ has dense span in B_0.

To verify positivity of the inner product, consider any $x \in X_0$. From the lemma we can find x_α tending to x in the inductive limit topology, such that each x_α is of the form $f_\alpha x = \sum_{i=1}^n \langle x_i, x_i \rangle_{E_0} x$. Then

$$\langle x, x_\alpha \rangle_{B_0} = \sum_{i=1}^n \langle x, \langle x_i, x_i \rangle_{E_0} x \rangle_{B_0} = \sum_{i=1}^n \langle x, x_i \langle x_i, x \rangle_{B_0} \rangle_{B_0} = \sum_{i=1}^n \langle x, x_i \rangle_{B_0} \langle x, x_i \rangle_{B_0}^*$$

is a positive element of B_0. It is readily verified that $(x, x_\alpha \rangle_{B_0}$ tends to $\langle x, x \rangle_B$ in the inductive limit, and so in the C^*-topology; thus $\langle x, x \rangle_{B_0}$ is positive. A similar argument shows that $\langle x, x \rangle_{E_0}$ is positive.

To complete the proof that X_0 is an E_0-B_0 imprimitivity bimodule we need only verify the norm conditions [51, p. 235]: $\langle ex, ex\rangle_{B_0} \leqslant \|e\|^2\langle x, x\rangle_{B_0}$, and $\langle xb, xb\rangle_{E_0} \leqslant \|b\|^2\langle x, x\rangle_{B_0}$ for each $e \in E_0$, $b \in B_0$. To do this, first observe that the previously defined left action of $C(G/H, A)$ on X_0 (by multiplication) satisfies the norm condition; this we see from the computation

$$\|\psi\|_\infty^2\langle x, x\rangle_{B_0} - \langle\psi x, \psi x\rangle_{B_0} = \langle(\|\psi\|_\infty^2 - \psi^*\psi)x, x\rangle_{B_0} = \langle\psi_0 x, \psi_0 x\rangle_{B_0} \geqslant 0 \tag{3}$$

where ψ_0 denotes the element $(\|\psi\|_\infty^2 - \psi^*\psi)^{1/2}$ of $C(G/H, \mathcal{M}(A))$ and $\| \|_\infty$ the norm on $C(G/H, A)$. Given any f (regarded now as a function on G) in E_0 and x in X_0, the function $s \mapsto f(s)(sx)$ is continuous for the inductive limit topology, and so a fortiori for the topology on X_0 defined by the inner product $p(\langle\ ,\ \rangle_{B_0})$ where p is any state on B_0. Denoting by $\| \|_p$ the norm from that inner product, we compute easily using the above formula (3) that $\|f(s)(sx)\|_p^2 \leqslant \|f(s)\|_\infty^2\|x\|_p^2$, and so $\|fx\|_p = \|\int f(s)(sx)\,ds\|_p \leqslant (\int\|f(s)\|_\infty ds)\|x\|_p$.

Since this is true for all states p of B_0, it follows that f acts as a *bounded operator* (see [51, Def. 2.3]) on the pre-B_0-Hilbert space X_0, and that the homomorphism so obtained of E_0 into the pre-C^*-algebra $\mathcal{L}(X_0)$ of such operators (see [51, pp. 194–199]) is norm decreasing for the L^1 norm on E_0. By the universality property of the enveloping C^*-algebra of $L^1(G, C_\infty(G/H, A))$ we see that this homomorphism is also norm-decreasing for the C^* norm; from this the norm condition

$$\langle fx, fx\rangle_{B_0} \leqslant \|f\|_\infty^2\langle x, x\rangle_{B_0}$$

follows immediately. The argument for the other norm condition is similar (but easier).

We have completed the proof of the following "imprimitivity theorem":

PROPOSITION 3. *X_0 is an E_0-B_0 imprimitivity bimodule.* ///

COROLLARY (of the proof). *Any *-representation of $C_c(G, A)$ which is continuous for the inductive limit topology is continuous for the C^* norm topology.*

Proof. Take $H=G$; then $E_0 \cong C_c(G, A) \cong B_0$. In the proof above we can replace the C^* norms on E_0 and B_0 by any stronger norms which induce topologies weaker than the inductive limit topologies, and still get an imprimitivity bimodule. Taking any such norm on E_0, but the original C^*-norm on B_0, we deduce that the homomorphism of E_0 into $\mathcal{L}(X_0)$ is still an isometry for the new norm (since this always holds for an imprimitivity bimodule); that is, the new norm coincides with the usual one. It follows that given any *-representation of E_0 continuous for the inductive limit topology, the sup of the C^* semi-norm on E_0

induced by this representation and the usual C^* norm coincides with the latter; i.e. the representation is continuous for the usual C^* norm. ///

We now apply the argument following [52, Prop. 3.1] to see that X_0 may be completed in the semi-norm $\|x\| = \|\langle x, x\rangle_{B_0}\|^{1/2}$ (after factoring out the elements of norm 0) to obtain an $E-B$ imprimitivity bimodule X. It is easy to show that the actions of $G, H, A, C(G/H, A)$ etc. on X_0 which were considered above extend to continuous actions on X. Given a (closed 2-sided) ideal J of B, we denote by X_J the closed $E-B$ sub-module XJ of X, and by J^E the ideal of E which correspond to it (see [52, Thm. 3.2]). Then the quotient $X^J = X/X_J$ carries the natural structure of an $E/J^E-B/J$ imprimitivity bimodule. We will frequently identify elements of X_0 with their images in X^J when no confusion is likely to result.

Given a *-representation L of B we may induce it up to a *-representation of E via the tensor product construction described in [51, Thm. 5.1]; this then extends to a *-representation of the multiplier algebra $\mathcal{M}(E)$, and since we have a natural homomorphism of $C^*(G, A)$ into $\mathcal{M}(E)$ (the "integrated form" of the homomorphisms of G and A into $\mathcal{M}(E)$), we get a *-representation $\operatorname{Ind} L = \operatorname{Ind}_H^G(L)$ of $C^*(G, A)$. It is easy to see that the restriction of this representation to $C_c(G, A)$ is the same as that obtained from the natural left action of $C_c(G, A)$ on $X_0 \otimes \mathcal{H}_L$ defined by $f(x\otimes\xi) = fx\otimes\xi$ (for f in $C_c(G, A)$), where the action of $C_c(G, A)$ on X_0 is taken to be the integrated form of the left action of G on X_0 given earlier and left multiplication by A.

To extend our theory of induced representations to twisted covariance algebras we need to know that the induction process described above "preserves twisting". This will follow from the next lemma, which is adapted from Lemma 2.4 of [52]. We assume for the rest of the paper that H is a closed subgroup of G containing $N_\mathcal{T}$, and continue to denote by $E = E_H^G$ the imprimitivity algebra constructed earlier.

Lemma 4. *Let $t \in N_\mathcal{T}$, and $a, a' \in A$. If $x \in X_0$ is of the form $x(s) = \psi(s)b$, for some $\psi \in C_c(G)$ and $b \in A$, then*

(i) $x(ta - a') \in$ *(the closed linear span in X of $\{b({}^r t\, {}^r a - {}^r a')X \mid r \in \operatorname{supp} x\}$).*

(ii) $(ta - a')x \in$ *the closed linear span of*

$$\{X({}^r t\, {}^r a - {}^r a')\, {}^r b \mid r^{-1} \in \operatorname{supp} x\}$$

(Here ${}^r t = rtr^{-1}$ and $(ta - a')$ is regarded as an operator on the left and right of X by means of the H- and A-actions considered earlier.) Similar statements hold for an arbitrary x in X, if "supp x" is replaced by "G" in the above formulae and b ranges over all of A.

Proof. Assume first that x is of the form ψb, and choose $\varepsilon > 0$. Then if y is any element of X_0, we compute (letting $*$ denote convolution in the algebra $C_c(G, A)$, and using the fact that $\triangle_G(t) = \triangle_{N_\mathcal{J}}(t) = \triangle_H(t)$)

$$x * (ta - a')y = x(ta - a') * y.$$

Multiplying x pointwise by the elements of a suitable partition of unity of G, we may write it as a finite sum of elements of the form x_i, $x_i(s) = \psi_i(s)b$, where each ψ_i has support contained in an open subset U_i of G with the property that $\|{}^{r_1}c - {}^{r_2}c\| < \varepsilon$ for all $r_1, r_2 \in U_i$, and c denotes $ta - a'$, regarded as an operator acting to the left on the Banach space X. Then

$$\begin{aligned}(x_i * (ta - a')y)(r) &= \int \psi_i(s)\, b\, {}^s((ta - a')\, y(s^{-1}r))\, d\lambda(s)\\ &= \int b({}^s t\, {}^s a - {}^s a')\, \psi_i(s)\, {}^s y(s^{-1}r)\, d\lambda(s)\\ &= z_i(r) + \int b({}^s t\, {}^s a - {}^s a' - {}^{s_i} t\, {}^{s_i} a - {}^{s_i} a')\, \psi_i(s)\, {}^s y(s^{-1}r)\, d\lambda(s)\end{aligned}$$

where

$$z_i(r) = b({}^{s_i} t\, {}^{s_i} a - {}^{s_i} a') \int \psi_i(s)\, {}^s y(s^{-1}r)\, d\lambda(s).$$

Thus

$$\begin{aligned}(x(ta - a) * y)(r) &= \sum_{i=1}^n (x_i(ta - a') * y)(r)\\ &= \sum_{i=1}^n z_i(r) + \int \sum_{i=1}^n b({}^s t\, {}^s a - {}^s a' - {}^{s_i} t_i\, {}^{s_i} a - {}^{s_i} a')\, \psi_i(s)\, {}^s y(s^{-1}r)\, d\lambda(s)\end{aligned}$$

The norm of the second term is $\leqslant$

$$\begin{aligned}&\sup_{1 \leqslant i \leqslant n,\, s \in U_i} \|b({}^s t\, {}^s a - {}^s a' - {}^{s_i} t\, {}^{s_i} a - {}^{s_i} a')\| \cdot \int |\psi(s)| \cdot \|{}^s y(s^{-1}r)\|\, d\lambda(s)\\ &\leqslant \|b\| \cdot \varepsilon \cdot \int |\psi(s)|\, \|y(s^{-1}r)\|\, d\lambda(s).\end{aligned}$$

Since ε is arbitrary and the z_i are in the closed span of

$$\{b({}^r t\, {}^r a - {}^r a')X \mid r \in \operatorname{supp} x\},$$

so is $x(ta - a') * y$. Now if we choose (y_α) to be a right approximate identity for $C_c(G, A)$ for the inductive limit topology, then $x(ta - a') * y_\alpha$ tends to $x(ta - a')$ in norm, so that $x(ta - a')$ also lies in the indicated subspace.

The argument for (ii) is similar. Since finite linear combinations of elements x of this form are dense in X these results extend immediately to arbitrary x in X. ///

The following corollary permits us to define induced representations of twisted covariant systems.

COROLLARY 5. *Let I be a closed ideal of $C^*(H, A)$. Then*

(i) *The natural covariant representation of (G, A) on the left of X^I preserves $\mathcal{J}$ iff $I \supseteq I_{\mathcal{J}}$.*

(ii) *If L is a *-representation of $C^*(H, A)$, then $\operatorname{Ind}_H^G L$ preserves $\mathcal{J}$ iff L does.*

(iii) *$E/(I_{\mathcal{J}})^E$, the imprimitivity algebra for the $B/I_{\mathcal{J}}$-rigged space $X^{\mathcal{J}} = X^{I_{\mathcal{J}}}$, is naturally isomorphic to $C^*(G, C_\infty(G/H) \otimes A, \mathcal{J}^\wedge)$, where $N_{\mathcal{J}^\wedge} = N_{\mathcal{J}}$ and $\mathcal{J}^\wedge(n)(f \otimes a) = f \otimes \mathcal{J}(n)a$ for all $f \in C_\infty(G/H)$, $n \in N_{\mathcal{J}}$, and $a \in A$.*

Proof. (i) Let $n \in N_{\mathcal{J}}$ and $a \in A$, and put $a' = \mathcal{J}(n)a$. If $I \supseteq I_{\mathcal{J}}$ then, since ${}^s a' = \mathcal{J}({}^s n)\,{}^s a$, we have $X({}^s n\,{}^s a - {}^s a') \subseteq X_I$ for all s in G; by the lemma this implies that $(na - a')X \subseteq X_I$, so that the covariant representation of (G, A) on X^I preserves $\mathcal{J}$. The other implication is proved similarly.

(ii) follows from (i), since whether or not a *-representation preserves $\mathcal{J}$ depends only on its kernel in the covariance algebra. (Note that the kernel of $\operatorname{Ind}_H^G L$ consists of those elements f of $C^*(G, A)$ such that $fX \subseteq X_{\ker L}$).

(iii) follows from (i) also, together with the observation that, since EX is dense in X, the kernel of the action of $C^*(G, A)$ on $X^{\mathcal{J}}$ consists of those $f \in C^*(G, A)$ for which $(fe)X \subseteq X_{I_{\mathcal{J}}}$ for all $e \leqslant E$ (recall $C^*(G, A)$ acts as multipliers on E). ///

The above corollary allows us to induce *-representations of $C^*(H, A, \mathcal{J})$ up to *-representations of $C^*(G, A, \mathcal{J})$ by means of the bimodule $X^{\mathcal{J}}$. We note that $X^{\mathcal{J}}$ may be viewed (as a $C^*(H, A, \mathcal{J})$-rigged space) to be the (Hausdorff) completion of X_0 with respect to the $C_0 = C_c(H, A, \mathcal{J})$-valued inner product $\pi_H(\langle \cdot, \cdot \rangle_{B_0})$, where $B_0 = C_c(H, A)$ and π_H is the canonical homomorphism of B_0 onto C_0. However it is easily checked that $\pi_H(\langle x, y \rangle_{B_0})(t) = (\pi_G(x)^* * \pi_G(y))(t)$, so that we may define $X^{\mathcal{J}}$ to be the completion of $X_0^{\mathcal{J}} = C_c(G, A, \mathcal{J})$ with respect to the inner product defined by $\langle x, y \rangle_{C_0}(t) = (x^* * y)(t)$. The natural left action of $C_c(G, A, \mathcal{J})$ on $X^{\mathcal{J}}$ then restricts on $X_0^{\mathcal{J}}$ to ordinary convolution. This description of the induced bimodule will be convenient later. It can be used, as in the proof of [51, 5.12] to relate our definition of induced representations to the more "classical" ones, such as are given in [55] and [12].

We will use $E^{\mathcal{J}}$ to denote the imprimitivity algebra $C^*(G, C_\infty(G/H) \otimes A, \mathcal{J}^\wedge)$ in the following. $X^{\mathcal{J}}$ will always denote the bimodule for inducing from $(H, A, \mathcal{J})$ to $(G, A, \mathcal{J})$; when it is necessary to emphasize the groups involved, we write ${}_G(X^{\mathcal{J}})_H$. The following "Imprimitivity theorem" now follows easily from Rieffel's imprimitivity theorem for C^*-algebras [51, 6.29]:

THEOREM 6. *Let L be a *-representation of $(G, A, \mathcal{J})$. Then L is induced from $(H, A, \mathcal{J})$ iff there exists a *-representation M of $C_\infty(G/H)$ on $\mathcal{H}_L$ whose image commutes with $M_L(A)$, and such that (with (M, M_L) denoting the resulting *-representation of $C_\infty(G/H)\otimes A$) the pair $(V_L, (M, M_L))$ defines a covariant representation of $(G, C_\infty(G/H)\otimes A, \mathcal{J}^\wedge)$.* ///

PROPOSITION 7. *Let K be a closed normal subgroup of G, and suppose $H \supseteq K$. Then (with the notation of Proposition 1) $X^{\mathcal{J}}$ is naturally isomorphic to the bimodule $Y^{\mathcal{J}^K}$ for inducing representations of $B^{(H,K)} = C^*(H, A, \mathcal{J})$ up to $B^{(G,K)} \cong C^*(G, A, \mathcal{J})$. (Thus the latter two isomorphisms "respect the inducing process".)*

Proof. The homomorphism R of Proposition 1, when restricted to $C_c(G, A, \mathcal{J}) = X_0^{\mathcal{J}}$ is easily seen to implement the desired isomorphism. ///

PROPOSITION 8. ("Induction in Stages.") *Let $H \supseteq K$ be closed subgroups of G containing $N_{\mathcal{J}}$, and L a *-representation of $(K, A, \mathcal{J})$. Then* $\mathrm{Ind}_H^G (\mathrm{Ind}_K^H L)$ *is unitarily equivalent to* $\mathrm{Ind}_K^G L$.

Proof. Let ${}_G(X_0)_H$, ${}_H(X_0)_K$, ${}_G(X_0)_K$ denote the bimodules for the respective inducing processes, and define a bilinear map

$$ {}_G(X_0)_H \times {}_H(X_0)_K \to {}_G(X_0)_K $$

by $(x, y) \mapsto xy$, where on the right hand side of the arrow we regard y as an element of $C_c(H, A)$ acting on ${}_G(X_0)_H$. It is easily verified that this map is $B_0 = C_c(H, A)$ balanced and thus defines a linear map T of

$$ {}_G(X_0)_H \otimes_{B_0} {}_H(X_0)_K $$

into ${}_G(X_0)_K$ which preserves the left-$C_c(G, A)$ and right-$C_c(K, A)$ actions. It also preserves $C_0 = C_c(K, A)$-valued inner products when we define $\langle\ ,\ \rangle_{C_0}$ on the tensor product by $\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle_{C_0} = \langle\langle x_2, x_1\rangle_{B_0} y_1, y_2\rangle_{C_0}$. From Lemma 2 we may deduce that the range of T is dense in the inductive limit topology, and hence in the norm topology. Thus the completions of the tensor product bimodule and of ${}_G(X_0)_K$ are isomorphic. The proposition now follows from [51, Thm. 5.9]. ///

3. Induced ideals

We turn now to some results on continuity properties of the induction and restriction processes which generalize those of Fell [22] for the group case. It will be convenient for this purpose to view these processes as giving maps between spaces of ideals (rather than of representations, as in [22]) of the C^*-algebras involved. We give the space $\mathfrak{J}(D)$ of (closed 2-sided) ideals of a C^*-algebra D the topology having as a sub-base for its open sets the family $\{Q_I\}_{I\in\mathfrak{J}(D)}$, where

$$Q_I = \{J \in \mathfrak{J}(D) \,|\, J \not\supseteq I\}.$$

This topology restricts on the space Prim D of primitive ideals to the usual (Jacobson) hull–kernel topology. It is essentially the same as Fell's "inner hull-kernel topology" [22], which may be defined as the topology induced on the set of unitary equivalence classes of *-representations of D (on some "large" Hilbert space) via the map $L \mapsto \ker L$ of this set into $\mathfrak{J}(D)$.

Note that the canonical bijection of ideal spaces $\mathfrak{J}(D) \simeq \mathfrak{J}(F)$ induced by an $F-D$ imprimitivity bimodule (cf. [52, Section 3]) is a homeomorphism, since it is a lattice isomorphism. We write I^F for the ideal of F corresponding to $I \in \mathfrak{J}(D)$.

We also remark that the action of G on $\mathfrak{J}(A)$ defined by

$${}^sI = \{{}^sa \,|\, a \in I\}$$

is jointly continuous—this follows from trivial modifications of Glimm's proof [26, Lemma 1.3] that the action on Prim A is jointly continuous.

In the following H will continue to denote a closed subgroup of G containing $N_{\mathcal{T}}$. For L a *-representation of $C^*(G, A, \mathcal{T})$, $\mathrm{Res}_H^G L$ (the "restriction" of L to $C^*(H, A, \mathcal{T})$) denotes the *-representation of $C^*(H, A, \mathcal{T})$ defined by the covariant representation $(V_L|_H, M_L)$ of (H, A). The continuity properties we require are given by the following result:

PROPOSITION 9. (i) *Let D, F be C^*-algebras, and P a *-homomorphism of F into a C^*-algebra D' containing D as an ideal. Then*

$$P_*\colon \mathfrak{J}(F) \to \mathfrak{J}(D)$$
$$J \mapsto \text{the ideal generated by } \{P_f d \,|\, f \in J,\, d \in D\}$$

preserves arbitrary unions (the "union" of a collection of ideals being the ideal they generate), and

$$P^*\colon \mathfrak{J}(D) \to \mathfrak{J}(F),\ I \mapsto \{f \in F \,|\, P_f \cdot D \subseteq I\}$$

is continuous and preserves arbitrary intersections. Furthermore P_* and P^* are order preserving, and form a "Galois correspondence" in the sense that the following relations hold:

$$P^*P_*P^* = P^*;\ P_*P^*P_* = P_*$$
$$P^*P_*(J) \supseteq J,\ P_*P^*(I) \subseteq I \quad \text{for all } I \in \mathfrak{J}(D),\ J \in \mathfrak{J}(F).$$

(ii) *Let* $F=C^*(H, A, \mathcal{T})$, $D=C^*(G, A, \mathcal{T})$, *and let* P *be the canonical homomorphism of* F *into* $D'=\mathfrak{M}(C^*(G, A, \mathcal{T}))$. *Define*

$$\mathrm{Res}_H^G=P^*\colon \mathcal{I}(C^*(G, A, \mathcal{T}))\to \mathcal{I}(C^*(H, A, \mathcal{T})), \quad \textit{and}$$
$$\mathrm{Ex}_H^G=P_*\colon \mathcal{I}(C^*(H, A, \mathcal{T}))\to \mathcal{I}(C^*(G, A, \mathcal{T})).$$

($\mathrm{Ex}_H^G\, I$ *will be called the "extension" of* I.) *Then* $\mathrm{Res}_H^G(\ker L)=\ker \mathrm{Res}_H^G L$ *for any* $*$-*representation* L *of* $C^*(G, A, \mathcal{T})$.

(iii) *Let* $F=C^*(G, A, \mathcal{T})$, $D=$ *the imprimitivity algebra* $E^{\mathcal{T}}$ *of* $X^{\mathcal{T}}$, *and let* P *be the canonical homomorphism of* F *into* $D'=\mathfrak{L}(X^{\mathcal{T}})$. *Define*

$$\mathrm{Ind}_H^G\colon \mathcal{I}(C^*(H, A, \mathcal{T}))\to \mathcal{I}(C^*(G, A, \mathcal{T})),\ I\mapsto P^*(I^{E^{\mathcal{T}}})$$

and

$$\mathrm{Sub}_H^G\colon \mathcal{I}(C^*(G, A, \mathcal{T}))\to \mathcal{I}(C^*(H, A, \mathcal{T})),\ J\mapsto (\textit{the unique}$$
$$I\in \mathcal{I}(C^*(H, A, \mathcal{T}))\quad \textit{such that } I^{E^{\mathcal{T}}}=P_*(J)).$$

Then $\mathrm{Ind}_H^G(\ker L)=\ker(\mathrm{Ind}_H^G L)$ *for each* $*$-*representation* L *of* $C^*(H, A, \mathcal{T})$.

Proof. (i) The fact that $P^*(I)\not\leq F$ (i.e. $P^*(I)\in\mathcal{I}(F)$) when $I\not\leq D$ is a consequence of the fact that $I\not\leq D'$. Given $J\in\mathcal{I}(F)$, we have $(P^*)^{-1}(Q_J)=\{K\in\mathcal{I}(D)\,|\,P^*(K)\nsupseteq J\}=\{K\in\mathcal{I}(D)\,|\,P(J)\cdot D\nsubseteq K\}=Q_{J'}$, where J' is the ideal of D generated by $P(J)\cdot D$; this proves continuity of P^*. The other statements of part (i) are easy.

(ii) is an easy consequence of the nondegeneracy of L, while (iii) follows from the facts that the representation of $E^{\mathcal{T}}$ induced from L has kernel $(\ker L)^{E_{\mathcal{T}}}$ [52, Prop. 3.7] and is nondegenerate. ///

In the case that $H=N_{\mathcal{T}}$ we can obtain more precise results on the induction, restriction, and extension processes. First we observe that the map $f\mapsto f(e)$ gives an isometric $*$-isomorphism of $C_c(N_{\mathcal{T}}, A, \mathcal{T})$ onto A. Hence $C_c(N_{\mathcal{T}}, A, \mathcal{T})=L^1(N_{\mathcal{T}}, A, \mathcal{T})=C^*(N_{\mathcal{T}}, A, \mathcal{T})$ and so we can identify $C^*(N_{\mathcal{T}}, A, \mathcal{T})$ with A. We will use this identification to view $\mathrm{Ind}_{N_{\mathcal{T}}}^G$, $\mathrm{Res}_{N_{\mathcal{T}}}^G$, and $\mathrm{Ex}_{N_{\mathcal{T}}}^G$ as giving maps between $\mathcal{I}(A)$ and $\mathcal{I}(C^*(G, A, \mathcal{T}))$.

Recall (from the discussion preceding Proposition 1) that when K is a closed normal subgroup of G containing $N_{\mathcal{T}}$, there is a natural action of G on $C^*(K, A, \mathcal{T})$. This action induces an action on the collection of $*$-representations of $C^*(K, A, \mathcal{T})$ via $({}^sL)(b)=L({}^{s^{-1}}b)$, for all $b\in C^*(K, A, \mathcal{T})$ and any $*$-representation L of $C^*(K, A, \mathcal{T})$.

LEMMA 10. *Let $K_1 \supseteq K_2$ be closed normal subgroups of G containing $N_{\mathcal{J}}$, and let $s \in G$.*

(i) *For any *-representation L of $C^*(K_2, A, \mathcal{J})$, $\mathrm{Ind}_{K_2}^{K_1}({}^sL)$ is unitarily equivalent to ${}^s(\mathrm{Ind}_{K_2}^{K_1}(L))$.*

(ii) *For any $I \lhd C^*(K_2, A, \mathcal{J})$, $\mathrm{Ind}_{K_2}^{K_1}({}^sI) = {}^s(\mathrm{Ind}_{K_2}^{K_1}(I))$.*

Proof. (i) We write $\mathcal{H}$ for $\mathcal{H}_L$, and $\mathcal{H}_s$ for the same Hilbert space when regarded as a $B_0 = C_c(K_2, A)$ module via ${}^sL$. The underlying Hilbert space of $\mathrm{Ind}_{K_2}^{K_1} L$ may be defined as the completion of $X_0 \otimes_{\mathbf{C}} \mathcal{H}$ (where $X_0 = {}_{K_1}(X_0)_{K_2}$) endowed with the pre-inner product $\langle x_1 \otimes \xi_1, x_2 \otimes \xi_2 \rangle = \langle \langle x_2, x_1 \rangle_{B_0} \xi_1, \xi_2 \rangle_{\mathcal{H}}$. (This is equivalent to Rieffel's definition [52, p. 222], which uses $X_0 \otimes_{B_0} \mathcal{H}$ instead of $X_0 \otimes_{\mathbf{C}} \mathcal{H}$, because elements in $X_0 \otimes \mathcal{H}$ of the form $xb \otimes \xi - x \otimes b\xi$ (for $b \in B_0$) have length 0.) Then if we define ${}^sx(t) = \triangle_{G, K_1}(s)\, {}^s(x(s^{-1}ts))$ for x in X_0, it may readily be checked that $U: X_0 \otimes \mathcal{H} \to X_0 \otimes \mathcal{H}_s$, $x \otimes \xi \mapsto {}^sx \otimes \xi_s$ (ξ_s denotes the vector ξ when viewed as an element of $\mathcal{H}_s$) preserves pre-inner products, and that it intertwines the action of $C_c(K_1, A)$ on $X_0 \otimes \xi$ defined by

$$f(x \otimes \xi) = ({}^{s^{-1}}f)x \otimes \xi$$

with that on $X_0 \otimes \mathcal{H}_s$ given by

$$f(x \otimes \xi_s) = fx \otimes \xi_s.$$

It follows that U extends to a unitary intertwining operator for ${}^s(\mathrm{Ind}_{K_2}^{K_1} L)$ and $\mathrm{Ind}_{K_2}^{K_1} {}^sL$.

(ii) is immediate from (i) in virtue of the relation $\ker {}^sL = {}^s(\ker L)$. ///

Definition. The *G-hull* ${}^GI$ of $I \lhd A$ is the G-invariant ideal generated by I. The *G-kernel* ${}_GI = \bigcap_{s \in G} {}^sI$ of I is the largest G-invariant ideal contained in I. ///

PROPOSITION 11. *Let* $\mathrm{Res} = \mathrm{Res}_{N_{\mathcal{J}}}^G$, *and similarly for* Ex, Ind *and* Sub.

(i) Res I *and* Sub I *are G-invariant ideals of A, and* $\mathrm{Ex}\,(\mathrm{Res}\, I) \subseteq I \subseteq \mathrm{Ind}\,(\mathrm{Sub}\, I)$, *for all $I \lhd C^*(G, A, \mathcal{J})$. The smallest ideal of $C^*(G, A, \mathcal{J})$ having the same restriction as I is* $\mathrm{Ex}\,(\mathrm{Res}\, I)$.

(ii) $\mathrm{Ind}\, I = \mathrm{Ind}\,({}_GI)$, $\mathrm{Ex}\, I = \mathrm{Ex}\, {}^GI$, $\mathrm{Res}\,\mathrm{Ex}\, I = {}^GI$ *and* $\mathrm{Res}\,\mathrm{Ind}\, I = {}_GI$, *for all $I \lhd A$. When restricted to the set ${}^G\mathfrak{I}(A)$ of G-invariant ideals of A,* Ind *and* Ex *are $1-1$, and* Ind *is a homeomorphism onto its image; furthermore* $\mathrm{Ex}\, I \subseteq \mathrm{Ind}\, I$ *for all $I \in {}^G\mathfrak{I}(A)$.*

Proof. (i) That Res I is G-invariant follows from the easily proved fact that the restriction map is G-equivariant and the fact that I is G-invariant (since it is also an ideal in $\mathcal{M}(C^*(G, A, \mathcal{J}))$). By Proposition 9 (i) and (iii), Sub I is the minimal ideal J of A such that $\mathrm{Ind}\, J \supseteq I$; by Lemma 10, if $\mathrm{Ind}\, J \supseteq I$, so does $\mathrm{Ind}\, {}^sJ$ for any s in G, so Sub I must be G-invariant. The remaining statements of (i) follow immediately from Proposition 9.

(ii) Lemma 4 implies that when I is a G-invariant ideal of A we have $IX = XI$ (where $X = {}_G(X)_{N_{\mathcal{T}}}$); since Res Ind I is G-invariant and is the largest $I' \triangleleft A$ such that $I'X \subseteq XI$, and since $XI' \neq XI$ if $I' \neq I$ (we use [52, Thm. 3.2]), we must have $I = \text{Res Ind } I$. In particular the restriction of Ind to ${}^G\mathfrak{J}(A)$ is a homeomorphism onto its image, since Res and Ind are both continuous. The facts that Res and Ind are intersection preserving and that Ind ${}^sI = \text{Ind } I$ then imply that Res Ind $I = {}_G I$ for all $I \in \mathfrak{J}(A)$.

Now again assume $I \in {}^G\mathfrak{J}(A)$, and let $J = \text{Ind } I$. Then by Proposition 9 and the preceding, Res Ex I = Res Ex Res J = Res $J = I$. Since Res and Ex are order preserving and Res Ex $I \supseteq I$ for arbitrary $I \in \mathfrak{J}(A)$, we see that in general Res Ex $I = {}^G I$, and so also Ex I = Ex ${}^G I$. Finally, to prove that Ex $I \subseteq$ Ind I when $I \in {}^G\mathfrak{J}(A)$ we apply the relation Ex Res $J \subseteq J$ (for $J \in \mathfrak{J}(C^*(G, A, \mathcal{T}))$ to $J = \text{Ind } I$. ///

For any $I \triangleleft A$ there are natural homomorphisms of $\mathcal{M}(A)$ into $\mathcal{M}(I)$ and $\mathcal{M}(A/I)$. It follows easily that when I is G-invariant there are natural twisting maps $\mathcal{T}_I$ and $\mathcal{T}^I$ for the systems (G, I) and $(G, A/I)$ respectively. The following result relates the corresponding twisted covariance algebras to $C^*(G, A, \mathcal{T})$.

PROPOSITION 12. *Let I be a G-invariant ideal of A.*

(i) *The inclusion map of $C_c(G, I, \mathcal{T}_I)$ into $C_c(G, A, \mathcal{T})$ extends to a *-isomorphism $\alpha_I = \alpha_{I,G}$ of $C^*(G, I, \mathcal{T}_I)$ onto $\mathrm{Ex}^G_{N_{\mathcal{T}}} I$.*

(ii) *The map α_c^I of $C_c(G, A, \mathcal{T})$ into $C_c(G, A/I, \mathcal{T}^I)$ defined by $\alpha_c^I(f)(r) = f(r) + I$ extends to a *-homomorphism $\alpha^I = \alpha^{I,G}$ of $C^*(G, A, \mathcal{T})$ onto $C^*(G, A/I, \mathcal{T}^I)$ with kernel $\mathrm{Ex}^G_{N_{\mathcal{T}}} I$.*

(iii) *These homomorphisms respect the inducing process in the following sense:*

The bimodule $Y^{\mathcal{T}}$ for the $(H, I, \mathcal{T}_I) - (G, I, \mathcal{T}_I)$ inducing process is isomorphic, as a right $C^*(H, I, \mathcal{T}_I)$-rigged (see [51, Def. 2.8]), left $C^*(G, I, \mathcal{T}_I)$ module, to $(X^{\mathcal{T}})_J$, where $J = \mathrm{Ex}^H_{N_{\mathcal{T}}} I$ is identified with $C^*(H, I, \mathcal{T}_I)$ via $\alpha^{-1}_{I,H}$, and $\mathrm{Ex}^G_{N_{\mathcal{T}}}$ is identified with $C^*(G, I, \mathcal{T}_I)$ via $\alpha^{-1}_{I,G}$.

Similarly, the bimodule $Z^{\mathcal{T}}$ for the $(H, A/I, \mathcal{T}^I) - (G, A/I, \mathcal{T}^I)$ inducing process is isomorphic to $(X^{\mathcal{T}})^J$.

Proof. Since $C^*(G, A, \mathcal{T})$ is generated as a C^*-algebra by $AC_c(G)$ (where we identify A and $C_c(G)$ with their images in $\mathcal{M}(C^*(G, A, \mathcal{T}))$), $\mathrm{Ex}^G_{N_{\mathcal{T}}} I$ is generated as an ideal of $C^*(G, A, \mathcal{T})$ by $IAC_c(G) = IC_c(G)$. However products of elements in $IC_c(G)$ by elements of A or $C_c(G)$ give elements in the C^*-algebra generated by $IC_c(G)$, so $\mathrm{Ex}^G_{N_{\mathcal{T}}} I$ is in fact generated as a C^*-algebra by $IC_c(G)$. Trivial modifications of the proof of [29, Lemma 1] (which is the special case where A is abelian and $\mathcal{T}$ is trivial) now yield (i) and (ii).

(iii) It is readily verified that the inclusion map of $Y_0^{\mathcal{J}} = C_c(G, I, \mathcal{J}_I)$ into $X_0^{\mathcal{J}}$ preserves the bimodule structure and $C_c(H, I, \mathcal{J}_I)$-valued inner product, and so extends uniquely to an isometric embedding of $Y^{\mathcal{J}}$ into $X^{\mathcal{J}}$. The image of $Y^{\mathcal{J}}$ contains $(X_0^{\mathcal{J}})I$, and hence also $X_0^{\mathcal{J}}(C_c(H, A, \mathcal{J})I)$, as a dense subspace: but the closure of the latter space is just $X^{\mathcal{J}}J = (X^{\mathcal{J}})_J$.

$(X^{\mathcal{J}})^J$ may be defined as the Hausdorff completion of $X_0^{\mathcal{J}}$ with respect to the norm defined by the $C^*(H, A/I, \mathcal{J}^I)$-valued inner product defined by $\langle x, y\rangle = \alpha^{I,H}(\langle x, y\rangle_B)$ (where $B = C^*(H, A, \mathcal{J})$). Arguments similar to those above now show that the map $\alpha_c^{I,G}$: $X_0^{\mathcal{J}} \to Z_0^{\mathcal{J}}$ extends to the required isomorphism of $(X^{\mathcal{J}})^J$ with $Z^{\mathcal{J}}$. ///

The above proposition is typically used to "break up" twisted covariance algebras into more manageable pieces; this technique will be of great use in the following sections.

In the remainder of this section we discuss a situation in which the inducing process is better behaved than in general.

Let μ be a quasi-invariant measure for the action of G on G/H, and let V be the *-representation of $C^*(G)$ on $L^2(G/H, \mu)$ associated to the "quasi-regular" representation of G (the latter is defined as left translation of functions, modified by Radon–Nikodym derivatives so as to give a unitary representation). The coset space G/H is said to be *amenable* if the kernel of the trivial (one-dimensional) representation of $C^*(G)$ contains the kernel of V. A number of equivalent conditions, involving the existence of G-invariant means for various spaces of functions on G/H, are given in [21] and [32]; when H is normal, the condition reduces to amenability in the usual sense (of [31]) for the group G/H.

Our interest in the concept arises from the following proposition, the first part of which generalizes [54, Prop. 2.2] and [12, Thm. VIII-2]. The method of proof used in those references is quite different from the one used here, which goes back to an idea of Fell [22].

PROPOSITION 13. *Assume G/H is amenable.*

(i) $\mathrm{Ind}_H^G(0) = (0)$. (*In other words, the canonical homomorphism of $C^*(G, A, \mathcal{J})$ into $\mathcal{M}(C^*(G, C_\infty(G/H, A)), \mathcal{J}^\wedge))$ is faithful.*)

(ii) *Let also $H = N_{\mathcal{J}}^*$: then for any G-invariant ideal I of A, $\mathrm{Ind}_H^G I = \mathrm{Ex}_H^G I$.*

Proof. (i) Let L_1 be any faithful *-representation of $C^*(G, A, \mathcal{J})$. We define a representation L_2 of $(G, A \otimes C_\infty(G/H))$ on the Hilbert space tensor product $\mathcal{H}_{L_1} \otimes L^2(G/H, \mu)$ by taking V_{L_2} to be the (inner) tensor product of V_{L_1} with the quasi-regular representation V, and M_{L_2} to be the spatial tensor product representation $M_{L_1} \otimes M$ of $A \otimes C_\infty(G/H)$ (where M denotes the representation of $C_\infty(G/H)$ by multiplication of functions on $L^2(G/H, \mu)$). It is easily checked that L_2 preserves $\mathcal{J}$ and so defines a representation of $C^*(G, A \otimes C_\infty(G/H), \mathcal{J}^\wedge)$.

To prove (i), it suffices to show that the restriction, L, to $C^*(G, A, \mathcal{J})$ of the extension of L_2 to $\mathcal{M}(C^*(G, A \otimes C_\infty(G/H), \mathcal{J}^\wedge))$ is faithful. Let $D = L_1(C^*(G, A, \mathcal{J})) \otimes V(C^*(G))$; then $\mathcal{M}(D)$ may be identified with an algebra of operators on $\mathcal{H}_{L_1} \otimes L^2(G/H, \mu)$. Noting that $V_L = V_{L_1} \otimes V$, and that M_L is given by $a \mapsto M_{L_1}(a) \otimes 1$, we see that $V_L(G)$ and $M_L(A)$ lie in $L_1(\mathcal{M}(C^*(G, A, \mathcal{J}))) \otimes V(\mathcal{M}(C^*(G)))$ and hence also in the (generally larger) algebra $\mathcal{M}(D)$. Thus also $L(C^*(G, A, \mathcal{J}) \subseteq \mathcal{M}(D)$, the corresponding action of $C^*(G, A, \mathcal{J})$ on D being the integrated form of the G and A actions. We may thus regard L as a homomorphism of $C^*(G, A, \mathcal{J})$ into $\mathcal{M}(D)$.

Let V_0 denote the direct sum of V with the trivial representation on a one-dimensional space $\mathcal{H}_1$. By our hypothesis that G/H is amenable, V_0 factors through $V(C^*(G))$, and so can be regarded as a representation of $V(C^*(G))$. Then the representation of $C^*(G, A, \mathcal{J})$ obtained by composing L with the extension to $\mathcal{M}(D)$ of $I \otimes V_0$ (I denotes the identity representations of $L_1(C^*(G, A, \mathcal{J}))$) contains as a subrepresentation (on the subspace $\mathcal{H}_{L_1} \otimes \mathcal{H}_1$) a representation equivalent to L_1. As L_1 is faithful by assumption, so is L, and (i) is proved.

(ii) Let $I \in {}^G\mathfrak{J}(A)$. By (i), applied to the system $(G, A/I, \mathcal{J}^I)$, the (0)-ideal of A/I induces to the (0)-ideal of $C^*(G, A/I, \mathcal{J}^I)$; from Proposition 13 (ii) and (iii) it now follows easily that $\operatorname{Ind}_{N_{\mathcal{J}}}^G I = \ker \alpha^{I,G} = \operatorname{Ex}_{N_{\mathcal{J}}}^G I$. ///

We conclude this section by applying the preceding result to show that nuclearity (in the sense of [39]) of twisted covariance algebras is preserved under "amenable extensions" of the groups. The perception that amenability is related to nuclearity appears to have originated with Guichardet, and in fact the germ of the following proof can be seen in his observation that the C^*-algebras of amenable groups are nuclear (see [34]).

PROPOSITION 14. *Suppose G/H is amenable and $C^*(H, A, \mathcal{J})$ is nuclear. Then $C^*(G, A, \mathcal{J})$ is nuclear.*

Proof. Let B be an arbitrary C^*-algebra; we must show that the maximal tensor product (see [39]) $C^*(G, A, \mathcal{J}) \otimes_{\max} B$ is the same as $C^*(G, A, \mathcal{J}) \otimes B$. We construct a system $(G, A \otimes_{\max} B, \mathcal{J}')$ by letting G act on $A \otimes_{\max} B$ via the inner tensor product of the action on A with the trivial action on B, and defining $\mathcal{J}' = \mathcal{J} \otimes 1$ (with $N_{\mathcal{J}'} = N_{\mathcal{J}}$). Then *-representations L of $(G, A \otimes_{\max} B, \mathcal{J}')$ correspond to triples (V, M_A, M_B) such that $L' = (V, M_A)$ is a representation of $(G, A, \mathcal{J})$, and M_B is a representation of B whose image commutes with $V(G)$ and $M_A(A)$ (and hence with $L'(C^*(G, A, \mathcal{T})))$. As $L(C^*(G, A \otimes_{\max} B, \mathcal{T}')))$ is generated by $V(C^*(G)) \cdot M_A(A) M_B(B)$ and so by $L'(C^*(G, A, \mathcal{T}))) M_B(B)$, it follows easily that $C^*(G, A \otimes_{\max} B, \mathcal{T}')$ is naturally isomorphic to $C^*(G, A, \mathcal{T}) \otimes_{\max} B$.

Similarly $C^*(H, A \otimes_{\max} B, \mathcal{T}')$ is isomorphic to $C^*(H, A, \mathcal{T}) \otimes_{\max} B$. Choose faithful representations L_1 of $C^*(H, A, \mathcal{T})$, and M_1 of B; our assumption that $C^*(H, A, \mathcal{T})$ is nuclear then implies that $L_2 = L_1 \otimes M_1$ is a faithful representation of $C^*(H, A, \mathcal{T}) \otimes_{\max} B$. We regard L_2 as a faithful representation of $C^*(H, A \otimes_{\max} B, \mathcal{T}')$, and form $L = \operatorname{Ind}_H^G L_2$. An inspection of $Y_0^{\mathcal{T}}$, the bimodule for the $(H, A \otimes_{\max} B, \mathcal{T}') - (G, A \otimes_{\max} B, \mathcal{T}')$ induction process, shows that it contains a dense subspace of the form $X_0^{\mathcal{T}} \otimes_{\mathbf{C}} B$ (where $X_0^{\mathcal{T}}$ is the bimodule for the $(H, A, \mathcal{T}) - (G, A, \mathcal{T})$ induction process); it is then easy to see that the representation L (which has as underlying space the completion of $Y_0^{\mathcal{T}} \otimes \mathcal{H}_{L_2}$) decomposes (when regarded as a representation of $C^*(G, A, \mathcal{T}) \otimes_{\max} B$) as the spatial tensor product $\operatorname{Ind}_H^G L_1 \otimes M_1$. By Proposition 13 (i), L and $\operatorname{Ind}_H^G L_1$ are both faithful, so $C^*(G, A, \mathcal{T}) \otimes_{\max} B$ coincides with $C^*(G, A, \mathcal{T}) \otimes B$ as desired.

When H is normal we can give a different proof, using the recent characterization [10] of nuclear algebras as those C^*-algebras for which the commutant of every *-representation is an injective von Neumann algebra. Namely, given any *-representation L of $C^*(G, A, \mathcal{T})$, its commutant $L(C^*(G, A, \mathcal{T}))'$ may be identified with the fixed point subalgebra of $(\operatorname{Res}_H^G L)'$, for the natural action of G on $(\operatorname{Res}_H^G L)'$ induced by conjugation by unitaries in $V_L(G)$. Since $V_L(H)$ commutes with $(\operatorname{Res}_H^G L)'$ this action drops to an action of G/H, and as G/H is amenable and $(\operatorname{Res}_H^G L)'$ is injective (by our assumption that $C^*(H, A, \mathcal{T})$ is nuclear), this fixed point algebra is injective (see [11, Section 6]). (This argument apparently does not generalize to non-normal H, and since in any case [10] depends on the very deep results of [11] our earlier direct proof seems preferable.) ///

4. The "Mackey machine"

In this section we develop a version of Mackey's [44] normal subgroup analysis (also called the "orbit method") for twisted covariance algebras. The first theorem below generalizes to our context a result of Rieffel [52] which gives a more precise formulation, involving Morita equivalence, of one part of the orbit method. (Actually, our result is more general than Rieffel's even in the group case, in that it incorporates an idea due to Moore [2, Chapter 2] for extending the Mackey analysis.) In proving it we make use of a well-known result of Dixmier [13] which states that there is a natural action T of $C(\operatorname{Prim} A)$ on A, uniquely determined by the condition $T_f a - f(J) a \in J$ for all $a \in A$, $f \in C(\operatorname{Prim} A)$, $J \in \operatorname{Prim} A$, and that this action identifies $C(\operatorname{Prim} A)$ with the center of $\mathcal{M}(A)$.

LEMMA 15. *Let W be a locally compact Hausdorff space, D a C^*-algebra, and P_1, P_2 two *-homomorphisms of $C_\infty(W)$ into D whose images commute. Assume* (i) *the image of P_i does not annihilate any non-zero element of D, for $i=1, 2$; and* (ii) *for any two distinct points $w_1, w_2 \in W$ there exist $f_1, f_2 \in C_\infty(W)$ such that $f_i(w_i) \neq 0$ for $i=1, 2$, and $P_1(f_1)P_2(f_2)=0$. Then $P_1=P_2$.*

Proof. Since the images of P_1 and P_2 commute we may assume that D is commutative, so let $D=C_\infty(W_0)$ for W_0 a locally compact Hausdorff space. The hypothesis (i) implies that the "dual" maps P_i^*: $W_0 \to W$ are well-defined. If $P_1 \neq P_2$ then $P_1^* \neq P_2^*$, so that we may choose $w \in W_0$ such that $w_1=P_1^*(w)$ is distinct from $w_2=P_2^*(w)$; choosing f_1, f_2 as in (ii), we have

$$0 = (P_1(f_1)P_2(f_2))(w) = f_1(P_1^*(w))f_2(P_2^*(w)) \neq 0,$$

a contradiction. Thus $P_1=P_2$. ///

The following lemma (which generalizes [51, Example 2.14]) is perhaps of some independent interest.

LEMMA 16. *Let E and B be C^*-algebras, Y an $E-B$ imprimitivity bimodule. Then the natural injection of E into $\mathfrak{L}(Y)$ extends to an isomorphism of $\mathfrak{M}(E)$ onto $\mathfrak{L}(Y)$.*

Proof. Regard E as a right E-rigged space via the inner product $\langle e_1, e_2\rangle_E=e_1^* e_2$. The calculation of [51, Example 2.14] shows that the left action of $\mathfrak{M}(E)$ on E is by bounded operators with respect to this inner product. It follows ([51, Thm. 5.9]) that $\mathfrak{M}(E)$ acts naturally by bounded operators (with respect to the B-valued inner product) on the completion Y^1 of the tensor product bimodule $E \otimes_E Y$. But one easily verifies that the map $e \otimes y \mapsto ey$ extends to an isomorphism (as B-rigged spaces) of Y^1 onto Y, thus giving a *-homomorphism α of $\mathfrak{M}(E)$ into $\mathfrak{L}(Y)$; α extends the natural injection of E into $\mathfrak{L}(Y)$, so (as no nonzero element of $\mathfrak{M}(E)$ annihilates E) it is faithful. Now since E is an ideal of $\mathfrak{L}(Y)$ with trivial annihilator, α has an inverse, so it is surjective. ///

THEOREM 17. *Let π: $\operatorname{Prim} A \to G/H$ be a continuous G-equivariant map. Let I be the H-invariant ideal* $\ker(\pi^{-1}(\{eH\}))$ *of A, and $J=\mathrm{Ex}_{N_\mathcal{T}}^H I$. Then the natural homomorphism of $C^*(G, A, \mathcal{T})$ into $\mathfrak{L}((X^\mathcal{T})^J)$ is an isomorphism onto the imprimitivity algebra of $(X^\mathcal{T})^J$. In particular $C^*(G, A, \mathcal{T})$ is Morita equivalent to $C^*(H, A, \mathcal{T})/J$, and so* (*by Proposition 12* (ii)) *to $C^*(H, A/I, \mathcal{T}^I)$.*

Proof. Let D_0 denote the imprimitivity algebra for $(X^\mathcal{T})^J$; thus D_0 is isomorphic to $E^\mathcal{T}/J^{E^\mathcal{T}}$ (in the notation of Section 2). By Lemma 16 we may identify $D=\mathfrak{M}(D_0)$ with

$\mathcal{L}((X^{\mathcal{J}})^J)$. The natural left actions of $C_\infty(G/H)$ and A on $(X^{\mathcal{J}})^J$ give *-homomorphisms P^1 and Q, respectively, of those algebras into D. Since the action of A on D_0 induced by Q is non-degenerate, Q extends to a homomorphism $\tilde{Q}$ of $\mathcal{M}(A)$ into D; composing $\tilde{Q}$ with π^* (the homomorphism of $C_\infty(G/H)$ into $\mathcal{M}(A)$ induced by π) we get another homomorphism P^2 of $C_\infty(G/H)$ into D.

We wish to show that $P^1 = P^2$. As the images of P^1 and Q commute, so do the images of P^1 and P^2. Furthermore the condition (i) of Lemma 15 follows easily from non-degeneracy of the $C_\infty(G/H)$ and A actions on $(X^{\mathcal{J}})^J$, so we need only check condition (ii).

It is sufficient to show that for any $f_1, f_2 \in C_\infty(G/H)$ of disjoint support, $P^1_{f_1}P^2_{f_2} = 0$, or equivalently (by non-degeneracy of the A action) that $P^1_{f_1}P^2_{f_2}Q(A) = 0$. Thus let $a \in A$ be arbitrary, and let $d = T_{\pi^*(f_2)}a$, so that $P^2_{f_2}Q(a) = Q(d)$. Regard f_1 and f_2 as functions on G which are constant on H-cosets; the condition that they have disjoint support then implies that $d \in {}^sI$, and hence that ${}^{s^{-1}}d \in I$, for all $s \in \operatorname{supp} f_1$. Let $x \in X_0$ be of the form $x(s) = \psi(s)b$, for some $\psi \in C_c(G)$ and $b \in A$. By Lemma 4, $d(f_1x)$ is in the closed span of $\{X^s d^s b$: $s^{-1} \in \operatorname{supp} f_1\}$, and hence in XI. Thus if $\tilde{x}$ denotes the canonical image of x in $(X^{\mathcal{J}})^J$, we have $d(f_1\tilde{x}) \in (X^{\mathcal{J}})^J I = ((X^{\mathcal{J}})^J C^*(H, A, \mathcal{J}))I \subseteq (X^{\mathcal{J}})^J J = 0$. Since elements of the form $\tilde{x}$ have dense span in $(X^{\mathcal{J}})^J$,

$$P^2_{f_2}Q(a)\,P^1_{f_1}(X^{\mathcal{J}})^J = Q(d)\,P^1_{f_1}(X^{\mathcal{J}})^J = (0),$$

as was to be proved. Thus $P^1 = P^2$.

Now D_0 is generated by $C^*(G)AC_\infty(G/H)$ (where we identify elements of $C^*(G)$, etc., with the corresponding operators on $(X^{\mathcal{J}})^J$), and so (since, by equality of P^1 and P^2, $C_\infty(G/H)$ "multiplies" A) by $C^*(G)A$. Thus the natural homomorphism of $C^*(G, A, \mathcal{J})$ into $\mathcal{L}((X^{\mathcal{J}})^J)$ has image equal to the imprimitivity algebra, and it remains only to show that this homomorphism is faithful.

To do this, we begin by observing that there is a natural homomorphism R' of $C_\infty(G, H) \otimes A$ into $\mathcal{M}(C^*(G, A, \mathcal{J}))$, defined by $R'(f \otimes a) = R^{\mathcal{J}}_A(T_{\pi^*(f)}a)$. It is then easily verified that the pair (R_G, R') is covariant and $\mathcal{J}^\wedge$ preserving, and that its integrated form gives a *-homomorphism R of $E^{\mathcal{J}} = C^*(G, C_\infty(G/H \otimes A), \mathcal{J}^\wedge)$ onto $C^*(G, A, \mathcal{J})$. Let J' denote the kernel of R; then $J' = (J_1)^{E_{\mathcal{J}}}$ for a unique ideal J_1 of $C^*(H, A, \mathcal{J})$, and the natural homomorphism R_1 of $E^{\mathcal{J}}$ onto the imprimitivity algebra E_1 of $(X^{\mathcal{J}})^{J_1}$ has kernel J'. One easily verifies that $R_1 = Q_1 \circ R$, where Q_1 denotes the natural homomorphism of $C^*(G, A, \mathcal{J})$ into $\mathcal{L}((X^{\mathcal{J}})^{J_1})$. This implies that Q_1 is faithful, since otherwise (as R is surjective) R_1 would have kernel larger than J'. It follows that for any $J_2 \subseteq J_1$ the natural homomorphism Q_2 of $C^*(G, A, \mathcal{J})$ into $\mathcal{L}((X^{\mathcal{J}})^{J_2})$ is faithful, since Q_1 is the composition of Q_2 with the natural homomorphism of $\mathcal{L}((X^{\mathcal{J}})^{J_2})$ into $\mathcal{L}((X^{\mathcal{J}})^{J_1})$.

Thus it suffices to show $J \subseteq J_1$. This is equivalent to $(X^{\mathcal{J}})^{J_1} J = 0$, and hence to $(X^{\mathcal{J}})^{J_1} I = 0$. Thus assume $(X^{\mathcal{J}})^{J_1} I \neq 0$, and choose $d \in I$ and $x \in (X^{\mathcal{J}})^{J_1}$ such that $\|xd\| = 4$ and $\|x\| = 1$. There exists an open symmetric neighborhood U of the identity of G such that $\|{}^s d - d\| \leqslant 1$ for all $s \in U$. Since ${}^s d \in {}^s I$, the norm of the image of d in $A/{}^s I$ is $\leqslant 1$ for $s \in U$. Now choose $f \in C(G/H)$ of norm 1 such that f is identically 1 outside $e(U)$ (where e denotes the projection of G onto G/H) and which vanishes on a neighborhood V of eH.

Identify $C(G/H)$ with its image in $\mathcal{M}(A)$ under π^*, and let $d' = T_f d$. I claim that $\|d' - d\| \leqslant 2$: By [15, 2.7.1] it is enough to verify that for any $P \in \operatorname{Prim} A$, $\|\beta_P(d' - d)\| \leqslant 2$, where β_P denotes the projection of A onto A/P. For P in the complement of $\pi^{-1}(e(U))$ we have $d' - d \in P$ (by definition of T), so $\|\beta_P(d' - d)\| = 0$. Thus we may assume $P \in \pi^{-1}(e(U))$; choose $s \in U$ such that $\pi(P) = e(s)$. By equivariance of π, $\pi({}^{s^{-1}}P) = eH$, so $P \supseteq {}^s I$. Thus

$$\|\beta_P(d' - d)\| \leqslant \|\beta_{({}^s I)}(d' - d)\| \leqslant \|T_f\| \, \|\beta_{({}^s I)} d\| + \|\beta_{({}^s I)} d\| \leqslant 2.$$

The claim follows, and we deduce that $\|xd'\| \geqslant 2$.

Now d' is in $\ker(\pi^{-1}(V))$; thus if W is a symmetric neighborhood of the identity of G such that $e(W^3) \subseteq V$, we must have ${}^s(d') \in \bigcap_{r \in W^2} {}^r I$ for all $s \in W$. Using Lemma 2 (ii) (or rather the more precise form thereof stated in the proof) we may find $y \in X_0$ of the form $y(s) = \psi(s) b$ (for some $\psi \in C_c(G)$, $b \in A$), such that $\|x\langle y, y\rangle_{B_0} - x\| \leqslant 1/\|d'\|$ (where $B_0 = C_c(H, A)$) and such that $\operatorname{supp} y \subseteq W$. Choose $g \in C_\infty(G/H)$ vanishing outside $e(W^2)$ and identically one on $e(W)$. Then (working now in X_0, and using Lemma 4) we have

$$yd' = gyd' \in g[A\, {}^s d' X : s \in W] \subseteq [g(\bigcap_{r \in W^2} {}^r I)\, X]$$

where the square brackets denote closed linear span. We observe next that because the natural homomorphism R_1 of $E^{\mathcal{J}}$ into $\mathcal{L}((X^{\mathcal{J}})^{J_1})$ factors through R, the natural homomorphism of $C_\infty(G/H)$ into $\mathcal{L}((X^{\mathcal{J}})^{J_1})$ arising from the left action (by multiplication) of $C_\infty(G/H)$ on $(X^{\mathcal{J}})^{J_1}$ coincides with the homomorphism obtained by viewing $C_\infty(G/H)$ as a subalgebra of $\mathcal{M}(A)$. Thus the image of $[g(\bigcap_{r \in W^2} {}^r I) X]$ in $(X^{\mathcal{J}})^{J_1}$ is $[T_g(\bigcap_{r \in W^2} {}^r I)(X^{\mathcal{J}})^{J_1}]$, which is 0 since g vanishes outside $e(W^2)$.

Let y_1 denote the image of y in $(X^{\mathcal{J}})^{J_1}$; the above shows $y_1 d' = 0$. Let $B_1 = C^*(H, A, \mathcal{J})/J_1$. Then $\langle y_1, y_1\rangle_{B_1}$ is just the image of $\langle y, y\rangle_{B_0}$ in B_1, so $x\langle y_1, y_1\rangle_{B_1} = x\langle y, y\rangle_{B_0}$. Thus $1 \geqslant \|x\langle y_1, y_1\rangle_{B_1} d' - xd'\| = \|\langle x, y_1\rangle_{E_1} y_1 d' - xd'\| = \|xd'\|$, a contradiction. Hence we must have $J_1 \supseteq J$, as was to be shown. ///

The hypotheses on $(G, A, \mathcal{J})$ in the preceding theorem are apparently quite special, but in practice it is frequently possible, by means of Proposition 12, to break up twisted covariance algebras into pieces to which the theorem applies. (More precise comments on this process appear in the following section.)

Note that when the map π of the theorem is a homeomorphism, the ideal I is maximal, so that A/I is in particular primitive. Thus in this case the theorem "reduces" (modulo Morita equivalence) the study of $C^*(G, A, \mathcal{J})$ to the study of a system in which the algebra acted upon is primitive. The second half of the "Mackey machine", to which we now turn, concerns such systems.

Let $\mathcal{U} = \mathcal{U}(\mathcal{M}(A))$ denote the unitary group of $\mathcal{M}(A)$, equipped with the strict topology, and $\mathcal{D}\mathcal{U}$ the quotient group of $\mathcal{U}$ by its center. With the quotient topology $\mathcal{D}\mathcal{U}$ is a topological group. The process of associating to any $u \in \mathcal{U}$ the corresponding "generalized inner" automorphism $a \mapsto uau^*$ of A induces a continuous injection of $\mathcal{D}\mathcal{U}$ into Aut A (endowed with the topology of pointwise norm convergence on A.) The system $(G, A, \mathcal{J})$ is said to be *generalized inner* if the action of G on A is induced by a continuous homomorphism of G into $\mathcal{D}\mathcal{U}$; this is stronger than requiring merely that each element of G act via a generalized inner automorphism, since in general the topology on $\mathcal{D}\mathcal{U}$ is stronger than that on Aut A.

Let $(G, A, \mathcal{J})$ be a generalized inner system, and α the corresponding homomorphism of G into $\mathcal{D}\mathcal{U}$. Assume also that A is primitive; then the center of $\mathcal{M}(A)$ consists of scalar multiples of the identity, so that the natural homomorphism β of $\mathcal{U}$ onto $\mathcal{D}\mathcal{U}$ has kernel isomorphic to $\mathbf{T}$. Let G'' denote the fiber product of $\mathcal{U}$ and G with respect to the homomorphisms α and β; thus

$$G'' = \{(s, u) \in G \times U \colon \alpha(s) = \beta(u)\},$$

with the relative product topology and group structure. There is a natural short exact sequence of topological groups

$$(1) \longrightarrow \mathbf{T} \xrightarrow{\gamma_{\mathbf{T}}} G'' \xrightarrow{\alpha''} G \longrightarrow (1)$$

defined by $\gamma_{\mathbf{T}}(t) = (1_G, t1_{\mathcal{M}(A)})$ and $\alpha''(s, u) = s$. In particular G'' is locally compact. The map $\gamma_N \colon n \mapsto (n, \mathcal{J}(n))$ is a continuous isomorphism of $N = N_{\mathcal{J}}$ onto a closed normal subgroup of G''. The subgroups $\gamma_{\mathbf{T}}(\mathbf{T})$ and $\gamma_N(N)$ intersect in the identity element of G'', so $N'' = \gamma_{\mathbf{T}}(\mathbf{T})\gamma_N(N)$ may be naturally identified with $\mathbf{T} \times N$; we let $P_{\mathbf{T}}$ and P_N denote the corresponding projections of N'' onto its factors.

We now describe two twisted covariant systems associated with G''. First, $\mathcal{J}_1 \colon n'' \mapsto P_{\mathbf{T}}(n'')^-$ (the bar denotes complex conjugate) defines a twisting map of N'' into $\mathbf{T}$, giving rise to a system $(G'', \mathbf{C}, \mathcal{J}_1)$. Observe that ker $R_{G''}^{\mathcal{J}_1} \supseteq \gamma_N(N)$; thus if we define $G' = G''/\gamma_N(N)$, and $\mathcal{J}' \colon N''/\gamma_N(N) \to \mathbf{T}$, $n''\gamma_N(N) \mapsto P_{\mathbf{T}}(n'')^-$, there is a natural covariant homomorphism of

$(G', \mathbf{C}, \mathcal{J}')$ into $\mathcal{M}(C^*(G'', \mathbf{C}, \mathcal{J}_1))$, and it is easy to see (using the fact that $C^*(G'', \mathbf{C}, \mathcal{J}_1) = R^{\mathcal{J}_1}_{G''}(C^*(G''))\, R^{\mathcal{J}_1}_{\mathbf{C}}(\mathbf{C}) = R^{\mathcal{J}_1}_{G''}(C^*(G''))$) that the integrated form of this homomorphism gives an isomorphism of $C^*(G', \mathbf{C}, \mathcal{J}')$ onto $C^*(G'', \mathbf{C}, \mathcal{J}_1)$. We refer to $(G', \mathbf{C}, \mathcal{J}')$ as the *Mackey system* for $(G, A, \mathcal{J})$. Note that $G'/N_{\mathcal{J}'} \cong G''/N'' \cong G/N_{\mathcal{J}}$.

We obtain another system, $(G'', A, \mathcal{J}_2)$, by letting G'' act on A via the composition of α'' with the given homomorphism of G into $\operatorname{Aut} A$, and taking $\mathcal{J}_2 = \mathcal{J} \circ P_N$. An argument similar to that in the preceding paragraph shows that $C^*(G'', A, \mathcal{J}_2)$ is naturally isomorphic to $C^*(G, A, \mathcal{J})$.

We now proceed to construct an isomorphism between $A \otimes_{\max} C^*(G'', \mathbf{C}, \mathcal{J}_1)$ and $C^*(G'', A, \mathcal{J}_2)$. First define a map R' of G'' into $\mathcal{M}(C^*(G'', A, \mathcal{J}_2))$, continuous with respect to the strict topology on the latter, by

$$R'(s, u) = R^{\mathcal{J}_2}_A(u^{-1})\, R^{\mathcal{J}_2}_{G''}(s, u).$$

Using the fact that u and s induce the same automorphism of A we easily verify that the image of R' commutes with $R^{\mathcal{J}_2}_A(\mathcal{M}(A))$. Thus

$$\begin{aligned} R'(s_1 s_2, u_1 u_2) &= R^{\mathcal{J}_2}_A(u_2^{-1})\, R^{\mathcal{J}_2}_A(u_1^{-1})\, R^{\mathcal{J}_2}_{G''}(s_1, u_1)\, R^{\mathcal{J}_2}_{G''}(s_2, u_2) \\ &= R^{\mathcal{J}_2}_A(u_1^{-1})\, R^{\mathcal{J}_2}_{G''}(s_1, u_1)\, R^{\mathcal{J}_2}_A(u_2^{-1})\, R^{\mathcal{J}_2}_{G''}(s_2, u_2) = R'(s_1, u_1)\, R'(s_2, u_2), \end{aligned}$$

so R' is a homomorphism.

Another simple calculation shows that (R', S), where S denotes the canonical homomorphism of $\mathbf{C}$ onto scalar multiples of the identity of $\mathcal{M}(C^*(G'', \mathbf{C}, \mathcal{J}_1))$, is a $\mathcal{J}_1$-preserving covariant homomorphism; its integrated form R'' thus defines a *-homomorphism of $C^*(G'', \mathbf{C}, \mathcal{J}_1)$ into $\mathcal{M}(C^*(G'', A, \mathcal{J}_2))$. As the image of R'' commutes with $R^{\mathcal{J}_2}_A(A)$ we see that $(R^{\mathcal{J}_2}_A, R'')$ defines a homomorphism R of $A \otimes_{\max} C^*(G'', \mathbf{C}, \mathcal{J}_1)$ into $\mathcal{M}(C^*(G'', A, \mathcal{J}_2))$.

We wish to show that the image $[R^{\mathcal{J}_2}_A(A)\, R''(C^*(G'', \mathbf{C}, \mathcal{J}_1))]$ of R is precisely $C^*(G'', A, \mathcal{J}_2)$. Since $R''(C^*(G'', \mathbf{C}, \mathcal{J}_1)) = R'(C^*(G''))$, it is enough to show that $R^{\mathcal{J}_2}_A(A)\, R'(L^1(G''))$ spans a dense subspace of $C^*(G'', A, \mathcal{J}_2)$. Thus choose $a \in A$, $f \in C_c(G'')$. Given $\varepsilon > 0$, we may partition $\operatorname{supp} f$ into finitely many Borel sets $(B_i)_{i=1}^n$, with representatives $(s_i, u_i)_{i=1}^n$, such that for all $(s, u) \in B_i$ we have

$$\| a R^{\mathcal{J}_2}_A(u)^{-1} - a R^{\mathcal{J}_2}_A(u_i)^{-1} \| < \varepsilon / (\boldsymbol{n} \sup_{s'' \in G''} f(s''))$$

where $\boldsymbol{n}$ denotes the $d(s, u)$-measure of $\operatorname{supp} f$. Let f_i denote the (pointwise) product of f with the characteristic function of B_i; then $f_i \in L^1(G'')$. Define T to be the element

$$R^{\mathcal{J}_2}_A(a)\, R'(f) - \sum_{i=1}^n R^{\mathcal{J}_2}_A(a u_i^{-1})\, R^{\mathcal{J}_2}_{G''}(f_i)$$

of $\mathcal{M}(C^*(G'', A, \mathcal{T}_2))$. A straightforward calculation shows that for any $g \in C^*(G'', A, \mathcal{T}_2)$ of norm $\leqslant 1$, $\|Tg\| \leqslant \varepsilon$. Thus T itself is of norm $\leqslant \varepsilon$, so that, as ε was arbitrary,

$$R_A^{\mathcal{T}_2}(A)\, R'(C_c(G)) \subseteq [R_A^{\mathcal{T}_2}(A)\, R_{G''}^{\mathcal{T}_2}(C^*(G))] \subseteq C^*(G'', A, \mathcal{T}_2).$$

Similar reasoning shows that for any $a \in A$, $h \in C_c(G'')$, the element $R_A^{\mathcal{T}_2}(a)\, R_{G''}^{\mathcal{T}_2}(h)$ can be approximated arbitrarily closely by elements in the span of $R_A^{\mathcal{T}_2}(A)\, R'(L^1(G''))$. It follows that R has image equal to $C^*(G'', A, \mathcal{T}_2)$.

To see that R is injective, we define homomorphisms R_1 and R_2 of A and G'', respectively, into $\mathcal{M}(A) \otimes_{\max} \mathcal{M}(C^*(G'', \mathbf{C}, \mathcal{T}_1)) \subseteq \mathcal{M}(A \otimes_{\max} C^*(G'', \mathbf{C}, \mathcal{T}_1))$, by $R_1(a) = a \otimes 1$, $R_2(s, u) = u \otimes R_{G''}^{\mathcal{T}_1}(s, u)$. Then (R_1, R_2) is covariant and $\mathcal{T}_2$-preserving, hence its integrated form gives a *-homomorphism R_3 of $C^*(G'', A, \mathcal{T}_2)$ into $C = \mathcal{M}(A \otimes_{\max} C^*(G'', \mathbf{C}, \mathcal{T}_2))$. Let R_0 denote the canonical extension of R_3 to $\mathcal{M}(C^*(G'', A, \mathcal{T}_2))$. We identify $\mathcal{M}(C^*(G'', \mathbf{C}, \mathcal{T}_1))$ with the subalgebra $1 \otimes \mathcal{M}(C^*(G'', \mathbf{C}, \mathcal{T}_1))$ of C; then it is easily verified that $R_0 \circ R'$ equals $R_{G''}^{\mathcal{T}_1}$, hence its integrated form $R_0 \circ R''$ is the canonical embedding of $C^*(G'', \mathbf{C}, \mathcal{T}_1)$ into C. Furthermore $R_0 \circ R_A^{\mathcal{T}_2}$ is the canonical embedding of A into C. It follows that R_0 is a right inverse to R. Thus R is injective, and so an isomorphism onto $C^*(G'', A, \mathcal{T}_2)$.

In virtue of the isomorphisms indicated earlier of $C^*(G'', A, \mathcal{T}_2)$ with $C^*(G, A, \mathcal{T})$ and $C^*(G'', \mathbf{C}, \mathcal{T}_1)$ with $C^*(G', \mathbf{C}, \mathcal{T}')$, we have proved

THEOREM 18. *Let $(G, A, \mathcal{T})$ be generalized inner, with A primitive, and let $(G', \mathbf{C}, \mathcal{T}')$ be the associated Mackey system. Then there is a natural isomorphism of $A \otimes_{\max} C^*(G', \mathbf{C}, \mathcal{T}')$ onto $C^*(G, A, \mathcal{T})$.* ///

Systems $(G', \mathbf{C}, \mathcal{T}')$ (such as the Mackey systems) for which $\mathcal{T}'$ is an isomorphism of $N_{\mathcal{T}'}$ onto $\mathbf{T}$ are said to be reduced. (Note that for such a system $N_{\mathcal{T}'}$ is always central in G'.) We will take up the study of reduced systems such that $G'/N_{\mathcal{T}'}$ is abelian in section 7.

Remark. Systems $(G, A, \mathcal{T})$ for which A is isomorphic to the algebra $\mathcal{K}(\mathcal{H})$ of compact operators on a Hilbert space $\mathcal{H}$ are always generalized inner. This essentially well-known result follows from two observations:

1. $\mathcal{M}(\mathcal{K}(\mathcal{H}))$, with the strict topology, is naturally isomorphic to $\mathcal{L}(\mathcal{H})$ with the strong topology (cf. [1]), so $\mathcal{P}\mathcal{U}(\mathcal{K}(\mathcal{H}))$ is isomorphic to the projective unitary group $\mathcal{P}(\mathcal{H})$ with the strong topology;

2. Every automorphism of $\mathcal{K}(\mathcal{H})$ is induced by a unitary of $\mathcal{H}$, and the topology of pointwise norm convergence on Aut $\mathcal{K}(\mathcal{H})$ coincides with the strong topology on $\mathcal{P}(\mathcal{H})$ (this follows from the argument of [52, Lemma 8.4]).

I originally proved Theorem 18 in the special case $A = \mathcal{K}(\mathcal{H})$, using arguments suggested by the proof of [17, Prop. 6]. The proof of Theorem 18 which is given above is based in part on a suggestion by Marc Rieffel for simplifying the original proof. We note that [17, Prop. 6] follows easily from Theorem 18, since Prim $(\mathcal{K}(\mathcal{H})\otimes C^*(G', \mathbf{C}, \mathcal{T}))$ is naturally homeomorphic to Prim $C^*(G', \mathbf{C}, \mathcal{T}')$. ///

5. Primitive ideals and the Effros-Hahn conjecture

In this section we apply the theory of induced representations developed in the preceding sections to the problem of determining the structure of twisted covariance algebras. The first step in investigating the structure of any C^*-algebra is to describe its primitive ideal space; we are far from being able to do this for general twisted covariance algebras, but at any rate Mackey's "orbit method" suggests a general method of attack. The idea is to break up the problem into two parts by analysing the action of G on Prim A: first, show that, at least in reasonably nice situations, all primitive ideals of $C^*(G, A, \mathcal{T})$ are induced from primitive ideals of $C^*(H, A, \mathcal{T})$, where H is the stability group G_P of some $P\in\operatorname{Prim} A$; second, determine the primitive ideals of $C^*(H, A, \mathcal{T})$. We proceed now to give a more precise formulation of a conjecture (first raised by Effros and Hahn [20, 7.4] in the special case in which A is abelian and $\mathcal{T}$ is trivial) concerning how the first step in this process should work.

Definitions. The G-quasi-orbit $Q_P=Q_P^G$ of $P\in\operatorname{Prim} A$ is the set of all primitive ideals of A whose G-kernels are the same as that of P—or equivalently, whose G-orbit closures in Prim A coincide with that of P. The common G-kernel of the elements in a quasi-orbit Q is denoted by ker Q and is called the *G-primitive ideal* corresponding to Q. A primitive ideal J of $C^*(G, A, \mathcal{T})$ is said to *live on* the quasi-orbit Q if $\operatorname{Res}_{N\mathcal{T}}^{G} J=\ker Q$; if it exists, this Q is denoted by $Q^G(J)$. One says that $(G, A, \mathcal{T})$ is *quasi-regular* if every primitive ideal of $C^*(G, A, \mathcal{T})$ lives on a quasi-orbit. ///

We endow the collection $\mathbf{Q}$ of all G-quasi-orbits in Prim A with the quotient topology arising from the surjection π_Q: $P\mapsto Q_P$ of Prim A onto $\mathbf{Q}$. The following lemma shows that this topology coincides with the hull–kernel topology on the set of G-primitive ideals when we identify quasi-orbits with their corresponding G-kernels.

LEMMA. *π_Q is open, and the map $Q\mapsto\ker(Q)$ is a homeomorphism onto its image in ${}^G\mathcal{J}(A)$.*

Proof. We show that $P\mapsto {}_GP$ is a continuous open map of Prim A onto its image in ${}^G\mathcal{J}(A)$; from this both parts of the lemma follow immediately (since the image of Prim A in ${}^G\mathcal{J}(A)$ will have the quotient topology).

Continuity follows from continuity of Res and Ind and the fact that ${}_GP = \operatorname{Res}^G_{N_{\mathcal{J}}} \operatorname{Ind}^G_{N_{\mathcal{J}}} P$ (Proposition 11); it can also be proved directly. To verify openness choose a basic open set $O = Q_I \cap \operatorname{Prim} A$ of Prim A, and let $J = {}^G I$. For any $P \in \operatorname{Prim} A$, if $P \in O$ then ${}_GP$ does not contain J, so ${}_GP \in Q_J$. On the other hand if ${}_GP \in Q_J$, then there must exist s in G for which ${}^sP \nsupseteq I$, as otherwise we would have ${}_GP = \bigcap_{s \in G} {}^sP \supseteq {}^G I$. Since ${}_G({}^sP) = {}_GP$ it follows that the image of O in ${}^G\mathfrak{J}$ is equal to the intersection of Q_J with the image of Prim A. Hence $P \mapsto {}_GP$ is open. ///

We proceed now to prove some (essentially known) sufficient conditions on Q for $(G, A, \mathcal{J})$ to be quasi-regular. A topological space is *totally Baire* if each locally closed subset of it is Baire (with the relative topology). By [15, 3.4.13, 3.2.1, and 3.2.2], Prim A is totally Baire; since the image of a totally Baire space under an open continuous map is totally Baire, the preceding lemma implies that Q is totally Baire. A topological space is *irreducible* if it is not the union of two proper closed subsets, or equivalently if every non-empty open set is dense. An *almost Hausdorff space* is one in which every closed subset contains a dense relatively open Hausdorff subset.

LEMMA. *Let T be a totally Baire space which is either second countable or almost Hausdorff; then a non-empty closed subset F of T is irreducible if and only if it is the closure of a single point.*

Proof. The subset F satisfies the same hypotheses as T, so we may assume that T is irreducible and non-empty. Then any non-empty open subset is dense, so if T has a countable base for its topology it must have, by the Baire property, a point which is contained in every non-empty open subset and is thus dense. If on the other hand T is almost Hausdorff, then a dense open Hausdorff subset must consist of a single point, since otherwise it would contain two disjoint non-empty open subsets.

For the "if" direction, observe that *any* topological space containing a dense point is irreducible. ///

COROLLARY 19. *If Q is second countable or almost Hausdorff then $(G, A, \mathcal{J})$ is quasi-regular.*

Proof. There is a natural bijection between ${}^G\mathfrak{J}(A)$ and the collection of closed subsets of Q, which to each $I \in {}^G\mathfrak{J}(A)$ associates the image in Q of the hull of I (in Prim A). We show that, for an arbitrary $P \in \operatorname{Prim} C^*(G, A, \mathcal{J})$, the closed subset C of Q corresponding to $\operatorname{Res}^G_{N_{\mathcal{J}}} P$ is irreducible.

Thus assume that $C = C_1 \cup C_2$, where C_1 and C_2 are proper closed subsets of C. Then the ideals I_1 and I_2 of A corresponding to C_1 and C_2 satisfy $I_1 \cap I_2 = I$, with I the ideal corresponding to C. Since $C^*(G, I_1, \mathcal{J}_{I_1}) C^*(G, I_2, \mathcal{J}_{I_2}) \subseteq C^*(G, I, \mathcal{J}_I)$, $\operatorname{Ex} I_1 \cap \operatorname{Ex} I_2 \subseteq \operatorname{Ex} I = \operatorname{Ex} \operatorname{Res} P \subseteq P$. Since P is prime we have (relabeling I_1 and I_2 if necessary) $\operatorname{Ex} I_1 \subseteq P$, so $I_1 = \operatorname{Res} \operatorname{Ex} I_1 \subseteq \operatorname{Res} P = I$, contradicting our assumption that C_1 is properly contained in C. It follows that C is irreducible.

By the preceding lemma there exists $Q \in \mathbf{Q}$ with closure equal to C; the ideal corresponding to C is then just $\ker Q$. Thus P lives on the quasi-orbit Q. ///

Remark. Dixmier has asked whether every prime ideal of a C^*-algebra is primitive. This is equivalent to asking whether every irreducible closed subset of the primitive ideal space of a C^*-algebra is a point closure. One may ask more generally whether quasi-orbit spaces always have this property—if so, every $(G, A, \mathcal{J})$ is quasi-regular. I know of no counterexamples. There are totally Baire spaces which do not have the property: for example, any uncountable set with the topology consisting of complements of finite subsets, together with the empty set; this space is irreducible but has no dense point. ///

We now introduce some stronger regularity properties which $\operatorname{Prim}(G, A, \mathcal{J})$ ($=\operatorname{Prim} C^*(G, A, \mathcal{J})$) may satisfy. Recall that by [5, Lemma 1], the stabilizer, G_P, in G of any $P \in \operatorname{Prim} A$ is closed; since $N_{\mathcal{J}}$ acts by unitaries of $\mathcal{M}(A)$, G_P always contains $N_{\mathcal{J}}$.

Definitions. A quasi-orbit Q is *EH-regular* if, for every $J \in \operatorname{Prim}(G, A, \mathcal{J})$ which lives on Q, there exists $P \in Q$ and $I \in \operatorname{Prim}(G_P, A, \mathcal{J})$ such that $\operatorname{Res}^{G_P}_{N_{\mathcal{J}}} I = P$ and $\operatorname{Ind}^G_{G_P} I = J$. Q is *regular* if it is locally closed, and if there exists $P \in Q$ such that the map $sG_P \mapsto {}^sP$ is a homeomorphism of G/G_P onto Q. The system $(G, A, \mathcal{J})$ is *EH-regular* (resp. *regular*) if it is quasi-regular, and each quasi-orbit is EH-regular (resp. regular). ///

The *generalized Effros–Hahn conjecture* is then that every system $(G, A, \mathcal{J})$ for which $G/N_{\mathcal{J}}$ is amenable is EH-regular. In some situations (one of which is given below) EH-regularity holds even when $G/N_{\mathcal{J}}$ is not amenable, but it is unlikely that the conjecture would hold in general if this hypothesis were dropped. The remainder of this section is devoted to showing that certain systems are EH-regular.

In the proof of the following proposition we make use of the well-known fact (cf. [15, 2.11.5]) that for any ideals I, J of a C^*-algebra D with $I \supseteq J$, $\operatorname{Prim}(I/J)$ is naturally homeomorphic to the locally closed subset S of $\operatorname{Prim} D$ consisting of primitive ideals which contain J but not I; the map $S \to \operatorname{Prim} I/J$ is given by $P \mapsto P \cap I/J$, while its inverse associates to $P'/J \in \operatorname{Prim}(I/J)$ the unique ideal of D maximal with respect to the property that its intersection with I be P'.

PROPOSITION 20. *A regular quasi-orbit is EH-regular.*

Proof. Let Q be a regular quasi-orbit, J a primitive ideal of $(G, A, \mathcal{T})$ which lives on Q, and $P \in Q$. As Q is locally closed and G-invariant there are G-invariant ideals I_1 and I_2 of A with hulls in Prim A equal to $Q^- \setminus Q$ and Q^-, respectively. Then $I_2 = \ker Q = \mathrm{Res}^G_{N_\mathcal{T}} J$. Let $J_i = \mathrm{Ex}^G_{N_\mathcal{T}} I_i$, $i=1, 2$. By Proposition 11, $J \supseteq J_2$ but $J \not\supseteq J_1$, so in particular $(J \cap J_1)/J_2$ is a primitive ideal of J_1/J_2. By Proposition 12, J_1/J_2 is naturally isomorphic to $C^*(G, I_1/I_2, (\mathcal{T}_{I_1})^{I_2})$. Let P' denote the primitive ideal $(P \cap I_1)/I_2$ of I_1/I_2. Our hypothesis of regularity implies that $sG_P \mapsto {}^sP'$ is homeomorphism of G/G_P onto Prim $I_1/I_2 \cong Q$, so (taking π to be its inverse) we may apply Theorem 17 to deduce that the inducing bimodule gives a Morita equivalence of $C^*(G, I_1/I_2, (\mathcal{T}_{I_1})^{I_2})$ with $C^*(G_P, I_1/(P \cap I_1), (\mathcal{T}_{I_1})^{P \cap I_1})$. The latter algebra is (again, by Proposition 12) isomorphic to the subquotient J'_1/J'_2 of $C^*(G_P, A, \mathcal{T})$, where $J'_1 = \mathrm{Ex}^G_{N_\mathcal{T}} I_1$ and $J'_2 = \mathrm{Ex}^G_{N_\mathcal{T}} (P \cap I_1)$. In virtue of the Morita equivalence there is by [52, 3.8] a unique primitive ideal J'_0/J'_2 of J'_1/J'_2 which induces up to $(J \cap J_1)/J_2$. Let L_0 be an irreducible *-representation of J'_1/J'_2 with kernel J'_0/J'_2; then L_0 extends canonically to an irreducible representation of $C^*(G_P, A, \mathcal{T})/J'_2$, which in turn lifts to an irreducible representation L of $C^*(G_P, A, \mathcal{T})$. It is then easy to see, using [52, 3.6 and 3.7], that the kernel of $\mathrm{Ind}^G_{G_P} L$ is the unique primitive ideal of $C^*(G, A, \mathcal{T})$ whose intersection with J_1 is $J \cap J_1$; that is, $\ker \mathrm{Ind}^G_{G_P} L = J$. On the other hand $\mathrm{Res}^{G_P}_{N_\mathcal{T}} L_0$ has kernel $(P \cap I_1)/(P \cap I_1)$ (since $I_1/P \cap I_1)$ is simple), so $\mathrm{Res}^{G_P}_{N_\mathcal{T}} L$, as the canonical extension of $\mathrm{Res}^G_{N_\mathcal{T}} L_0$ to A, has kernel equal to the unique ideal of A maximal with respect to the property that its intersection with I_1 be $P \cap I_1$; i.e. $\ker \mathrm{Res}^{G_P}_{N_\mathcal{T}} L = P$. Thus, taking $J' = \ker L$, we have $\mathrm{Ind}^G_{G_P} J' = J$, $\mathrm{Res}^{G_P}_{N_\mathcal{T}} J' = P$, which proves EH-regularity. ///

LEMMA 21. *Let H be open in G. Then the natural isomorphism i_H of $C_c(H, A, \mathcal{T})$ onto the subalgebra of $C_c(G, A, \mathcal{T})$ consisting of functions which vanish off H extends to an isomorphism of $C^*(H, A, \mathcal{T})$ onto a C^*-subalgebra of $C^*(G, A, \mathcal{T})$.*

Proof. Let Y_0 denote the subspace of $X_0^\mathcal{T}$ consisting of functions which vanish off H; then Y_0 is a submodule for the right and left actions of $C_c(H, A, \mathcal{T})$ on $X_0^\mathcal{T}$, and it is easily verified that Y_0 is isomorphic (as a right $C_c(H, A, \mathcal{T})$-rigged space) to the bimodule for inducing representations of $C_c(H, A, \mathcal{T})$ up to itself. From the proof of the corollary to Proposition 3 it follows that the norm induced on $C_c(H, A, \mathcal{T})$ from its left action on Y_0 is just the ordinary C^*-norm, so the same is true of the action on $X_0^\mathcal{T}$; thus the natural homomorphism of $C_c(H, A, \mathcal{T})$ into $\mathfrak{L}(X_0^\mathcal{T})$ is an isometry. Since this homomorphism "factors through" i_H, and since i_H, being norm decreasing for the L^1 norms, is also norm decreasing for the C^*-norms (by universality), we see that i_H is itself an isometry. Thus it extends to an embedding of $C^*(H, A, \mathcal{T})$ into $C^*(G, A, \mathcal{T})$, as claimed. ///

We will use this lemma to identify $C^*(H, A, \mathcal{T})$ with a subalgebra of $C^*(G, A, \mathcal{T})$, when H is open. In particular when $N_{\mathcal{T}}$ is open we identify $A \cong C^*(N_{\mathcal{T}}, A, \mathcal{T})$ with a subalgebra of $C^*(G, A, \mathcal{T})$.

LEMMA 22. *Suppose $G/N_{\mathcal{T}}$ is compact, and let $P \in \operatorname{Prim} A$.*

(i) *$sG_P \mapsto {}^sP$ is a homeomorphism of G/G_P onto $Q = Q(P)$.*

(ii) *If $\{P\}$ is locally closed in* Prim A, *then Q is regular.*

Proof. (i) That $sG_P \mapsto {}^sP$ is a homeomorphism of G/G_P onto the orbit ${}^G\{P\}$ of P follows from [24, Lemma 17.2], so we need to show only that ${}^G\{P\} = Q$. Observe first that ${}^G(\{P\}^-) = \bigcup_{s \in G} {}^s(\{P\}^-)$ is closed in Prim A—this is a consequence of the fact that for any topological transformation group, the union of the translates of a given closed subset of the space by the elements of a given compact subset of the group is always closed. It follows that the closure of ${}^G\{P\}$ is precisely ${}^G(\{P\}^-)$. Suppose there exists $P' \in Q$ not lying in the orbit of P; then $P' \in {}^s\{P\}^- = \{{}^sP\}^-$ for some $s \in G$. As Prim A is T_0, $\{P'\}^-$ is a subset of $\{{}^sP\}^-$ not containing ${}^sP$. Since ${}^G\{P\}$, being homeomorphic to G/G_P, is T_1, ${}^s\{P\}^- \cap {}^G\{P\} = \{{}^sP\}$, so $\{P'\}^-$, and hence also ${}^G\{P'\}^-$ does not intersect ${}^G\{P\}$. It follows that P and P' have distinct orbit closures and so lie in distinct quasi-orbits. Thus $Q = {}^G\{P\}$, as was to be shown.

To prove (ii) we observe that if $\{P\}^- \setminus \{P\}$ is closed in $\{P\}^-$, then ${}^G(\{P\}^-) \setminus {}^G\{P\} = {}^G(\{P\}^- \setminus \{P\})$ is closed in ${}^G\{P\}^-$, i.e. ${}^G\{P\}$ is locally closed. ///

We say $(G, A, \mathcal{T})$ is *separable* if both G and A are; this implies that $C^*(G, A)$ and hence also $C^*(G, A, \mathcal{T})$ are separable C^*-algebras. Note that separable systems are always quasi-regular, by Corollary 19.

PROPOSITION 23. *Suppose $(G, A, \mathcal{T})$ is separable, and let Q be a quasi-orbit all points of which have the same stabilizer K. (K is then equal to its conjugates in G, hence normal.) Suppose that G/K is amenable, and that either*

(i) *G/K has a compact open subgroup, and every point of* Prim $(K, A, \mathcal{T})$ *which lives on (a K-quasi-orbit contained in) Q is locally closed; or* (ii) *G/K is discrete.*

Then Q is EH-regular.

Proof. Let J be a primitive ideal of $(G, A, \mathcal{T})$ living on Q. Since the iterated system $(G, C^*(K, A, \mathcal{T}), \mathcal{T}^K)$ is separable, J also lives on a G-quasi-orbit Q' in Prim $C^*(K, A, \mathcal{T})$. I claim that it suffices to show that Q' is EH-regular: Let $J' \in Q'$, so that $\operatorname{Res}_K^G J = \bigcap_{s \in G} {}^sJ'$. By quasi-regularity of $(K, A, \mathcal{T})$ we may choose $P \in \operatorname{Prim} A$ such that $\bigcap_{t \in K} {}^tP = \operatorname{Res}_{N_{\mathcal{T}}}^K J'$. Then $\bigcap_{s \in G} {}^sP = \bigcap_{s \in G} {}^s(\operatorname{Res}_{N_{\mathcal{T}}}^K J') = \operatorname{Res}_{N_{\mathcal{T}}}^K (\bigcap_{s \in G} {}^sJ') = \operatorname{Res}_{N_{\mathcal{T}}}^G J$. (We are using the facts that

$\mathrm{Res}^K_{N_{\mathcal{J}}}$ is intersection preserving (Proposition 9) and G-equivariant.) Thus $P \in Q$, so K stabilizes P, and $P = \mathrm{Res}^K_{N_{\mathcal{J}}} J'$. As ${}^sP = \mathrm{Res}^K_{N_{\mathcal{J}}} {}^sJ'$, the stabilizer in G of J' is precisely K. Since J' is an arbitrary element of Q' we see (using the fact that all elements of a quasi-orbit induce to the same ideal of $C^*(G, A, \mathcal{J})$, by Proposition 11) that EH-regularity of Q' will imply $\mathrm{Ind}^G_K J' = J$, yielding EH-regularity of Q.

Thus, passing to the iterated system $(G, C^*(K, A, \mathcal{J}), \mathcal{J})$, we may replace A by $C^*(K, A, \mathcal{J})$ and Q by Q'. Let H be an open subgroup of G, containing K, and such that H/K is compact (when K is itself open, we take $H = K$). Let L_1 be an irreducible *-representation of $C^*(G, A, \mathcal{J})$ with kernel J. By [19], we may decompose $\mathrm{Res}^G_H L_1$ as a direct integral $\int L^P d\mu(P)$ of homogeneous representations over Prim $(H, A, \mathcal{J})$, with $\ker L^P = P$ for all $P \in \mathrm{Prim}\,(H, A, \mathcal{J})$. Let $\mathcal{O} = \{P \in \mathrm{Prim}\,(H, A, \mathcal{J}) \mid P \supseteq \mathrm{Res}^G_H J\}$; the complement of $\mathcal{O}$ then has μ-measure 0. Assume that no point of $\mathcal{O}$ lives on Q, or in other words that for all $P \in \mathcal{O}$ the G-kernel of $\mathrm{Res}^H_K P$ properly contains $\ker Q$. Then since Q is second countable, we can find a sequence $\{I_i\}_{i=1}^{\infty}$ of G-invariant ideals, each properly containing $\ker Q$, and such that for each $P \in \mathcal{O}$, $\mathrm{Res}^H_K P \supseteq I_i$ for some i. We may thus find an n such that $\{P \in \mathcal{O} \mid \mathrm{Res}^H_K P \supseteq I_n\}$ (which is closed in Prim $(H, A, \mathcal{J})$, by continuity of Res^H_K) has positive μ-measure. But then the subspace of $\mathcal{H}_{L_1}$ annihilated by $(\mathrm{Res}^G_K L_1)(I_n)$ is non-zero; since this subspace is G- and A-invariant, it must be all of $\mathcal{H}_{L_1}$ by irreducibility of L_1. Thus $\ker(\mathrm{Res}^G_K L_1) \supseteq I_n \supsetneqq \ker Q$, contradicting the fact that J lives on Q. Therefore we may choose $J' \in \mathrm{Prim}\,(H, A, \mathcal{J})$ containing $\mathrm{Res}^G_H J = J \cap C^*(H, A, \mathcal{J})$ and living on Q.

Now let L_2 be an irreducible *-representation of $C^*(H, A, \mathcal{J})$ with kernel J'. Applying [15, 2.10.2] to the subalgebra $C^*(H, A, \mathcal{J})/(J \cap C^*(H, A, \mathcal{J}))$ of $C^*(G, A, \mathcal{J})/J$, we may find an irreducible *-representation L of $C^*(G, A, \mathcal{J})$ on a Hilbert space $\mathcal{H}$ containing $\mathcal{H}_2 = \mathcal{H}_{L_2}$, such that $\ker L \supseteq J$ and $\mathrm{Res}^G_H L$ contains L_2 as a subrepresentation. Covariance of (V_L, M_L) implies that $V_L(s)\mathcal{H}_2$ is A-invariant for each $s \in G$, and that the corresponding subrepresentation of M_L is unitarily equivalent to ${}^sM_{L_2}$.

I claim that for $s_1, s_2 \in G$, ${}^{s_1}M_{L_2}$ and ${}^{s_2}M_{L_2}$ are disjoint representations unless s_1 and s_2 are in the same element of G/H: When $H = K$, this is clear, since then $M_{L_2} = L_2$ is irreducible, and K is the stabilizer in G of $J' = \ker L_2$. When H properly contains K, then by our hypotheses and Lemma 22, J' lives on a locally closed H-orbit Q_0 contained in Q. Let $I = \ker(Q_0^- \setminus Q_0)$. Then I is an H-invariant ideal of A, so the subspace of $\mathcal{H}_2$ annihilated by $M_{L_2}(I)$ is $C^*(H, A, \mathcal{J})$-invariant and is thus either (0) or all of $\mathcal{H}_2$; but if it were $\mathcal{H}_2$, then $\ker M_{L_2}$ would contain I, contradicting the fact that $\ker M_{L_2} = \mathrm{Res}^H_K J' = \ker Q_0$. It follows that the Glimm projection valued measure [26] on Prim A associated to M_{L_2} is concentrated on Q_0. But then the measure associated to ${}^{s_i}M_{L_2}$ is concentrated on ${}^{s_i}Q_0$, for $i = 1, 2$, which (since Q_0 is an H orbit, and K is the stabilizer of points in Q) are disjoint Borel subsets of Prim A

unless s_1 and s_2 lie in the same H-coset. The claim follows, and so $V_L(s_1)\mathcal{H}_2$ is orthogonal to $V_L(s_2)\mathcal{H}_2$ when $s_1H \neq s_2H$.

Therefore, as L is irreducible, $\mathcal{H}$ decomposes as a Hilbert space direct sum $\oplus_{\alpha\in G/H} V_L(s_\alpha)\mathcal{H}_2$, where $s_\alpha \in \alpha$ for each $\alpha \in G/H$. We can thus define a *-representation M' of $C_\infty(G/H, A)$ on $\mathcal{H}$ by letting $M'(f)\xi = M_L(f(\alpha))\xi$ for $\xi \in V_L(s_\alpha)\mathcal{H}_2$ and $f \in C_\infty(G/H, A)$. One checks easily that (V_L, M') defines a *-representation of the imprimitivity algebra $C^*(G, C_\infty(G/H, A), \mathcal{T}^\wedge)$, hence L is induced from some representation of $C^*(H, A, \mathcal{T})$. Now from the observation made in the proof of Lemma 21, that (in the notation of that proof) the submodule Y_0 is just the module for the $C_c(H, A, \mathcal{T}) - C_c(H, A, \mathcal{T})$ induction process, we see that for any representation L' of $C^*(H, A, \mathcal{T})$ the restriction of $\mathrm{Ind}_H^G L'$ to the range space of the characteristic function $\chi_{\{H\}}$ (regarded as an element of $C_\infty(G/H)$) is equivalent to L'. Thus in particular our representation L is induced from L_2, so $\mathrm{Ind}_H^G J' = \ker L \supseteq J$. But by Lemma 22 and Proposition 20, Q_0 is EH-regular, so $J' = \mathrm{Ind}_K^H P$ for some $P \in Q_0 \subseteq Q$.

Thus $\mathrm{Ind}_K^G P \supseteq J$. On the other hand from Propositions 11 and 13 we have $\mathrm{Ind}_K^G P = \mathrm{Ind}_K^G ({}_GP) = \mathrm{Ex}_K^G\, {}_GP = \mathrm{Ex}_K^G \mathrm{Res}_K^G J \subseteq J$. So $\mathrm{Ind}_K^G P = J$, and it follows that Q is EH-regular. ///

Remarks. Gootman, extending earlier work of Effros and Hahn, has obtained in [27, Cor., p. 102] and [28] the above result under quite restrictive additional hypotheses; namely, that A is commutative, $\mathcal{T}$ is trivial, K is central in G, and the connected component of G is compact and abelian. (He has also however obtained in [27] another positive result on the Effros–Hahn conjecture not subsumed by our proposition.) These hypotheses imply in particular that points in Prim $(K, A, \mathcal{T})$ which live on Q are in fact kernels of one-dimensional representations, and so are actually closed. The methods of [27] are quite different (and more complicated) than ours, and do not appear to extend to the more general situation of twisted covariance algebras.

It should be noted that when A has Hausdorff primitive ideal space the hypothesis that all points in Q have the same stabilizer is equivalent to the weaker hypothesis that their stabilizers all be normal; this follows easily from the fact that the set of points stabilized by a normal subgroup of G is closed and G-invariant (provided Prim A is Hausdorff). Thus in particular when $G/N_{\mathcal{T}}$ and A are abelian this hypothesis is automatic.

Finally, we note that by making use of Fell's Mackey machine for Banach *-algebraic bundles [24] it is possible to prove Proposition 23 under the hypothesis (i), without assuming that points in Prim $(K, A, \mathcal{T})$ which live on Q be locally closed. We plan to discuss this and other partial results on the Effros–Hahn conjecture in a future paper. ///

We say $(G, A, \mathcal{T})$ is *essentially free* if the stabilizer of each point in Prim A is $N_{\mathcal{T}}$.

THEOREM 24. *Let $(G, A, \mathcal{T})$ be essentially free and separable. If also either*

(i) *$(G, A, \mathcal{T})$ is regular; or*

(ii) *$G/N_{\mathcal{T}}$ is amenable and has a compact open subgroup, and points in* Prim A *are locally closed; or*

(iii) *$G/N_{\mathcal{T}}$ is discrete and amenable, then $Q \mapsto \operatorname{Ind}_{N_{\mathcal{T}}}^{G} (\ker Q)$ is a homeomorphism of $\mathbf{Q}$ onto* Prim $(G, A, \mathcal{T})$, *and $I \mapsto \operatorname{Ind}_{N_{\mathcal{T}}}^{G} I$ is a homeomorphism of ${}^{G}\mathcal{J}(A)$ onto $\mathcal{J}(C^*(G, A, \mathcal{T}))$ (with inverse $\operatorname{Res}_{N_{\mathcal{T}}}^{G}$).*

Proof. By Propositions 20 and 23 and Corollary 19, $(G, A, \mathcal{T})$ is EH-regular (under any of the hypotheses (i), (ii), (iii)). Thus $\operatorname{Ind}_{N_{\mathcal{T}}}^{G} (\mathcal{J}(A))$ contains Prim $(G, A, \mathcal{T})$, hence (since, by Proposition 9, Ind is intersection preserving, and all ideals are intersections of primitive ones) all of $\mathcal{J}(C^*(G, A, \mathcal{T}))$. From Propositions 9 and 11 we thus see that $\operatorname{Ind}_{N_{\mathcal{T}}}^{G}$ is a homeomorphism of ${}^{G}\mathcal{J}(A)$ onto $\mathcal{J}(C^*(G, A, \mathcal{T}))$, with inverse $\operatorname{Res}_{N_{\mathcal{T}}}^{G}$.

We now show that $\operatorname{Ind}_{N_{\mathcal{T}}}^{G} (\ker Q)$ is primitive, for every $Q \in \mathbf{Q}$. Observe first that for $I_1, ..., I_n \in {}^{G}\mathcal{J}(A)$ properly containing $\ker Q$, the intersection $I_1 \cap ... \cap I_n$ properly contains $\ker Q$: otherwise, given $P \in Q$ we must have $I_j \subseteq P$ for some j (since P is prime), hence $I_j \subseteq {}_{G}P = \ker Q$, a contradiction. Thus $\ker Q$ is "irreducible" in ${}^{G}\mathcal{J}(A)$, hence as $\operatorname{Ind}_{N_{\mathcal{T}}}^{G}$ preserves intersections, $\operatorname{Ind}_{N_{\mathcal{T}}}^{G} (\ker Q)$ is irreducible in $\mathcal{J}(C^*(G, A, \mathcal{T}))$, and so by [15, 3.9.1] is primitive, as claimed. On the other hand by EH-regularity all primitive ideals of $C^*(G, A, \mathcal{T})$ are of the form $\operatorname{Ind}_{N_{\mathcal{T}}}^{G} (\ker Q)$ for some Q. Thus the restriction of $\operatorname{Ind}_{N_{\mathcal{T}}}^{G}$ to the set of kernels of quasi-orbits is a homeomorphism onto Prim $(G, A, \mathcal{T})$. ///

The above result, under the hypothesis that G be discrete, was essentially proved by Zeller–Meier [57, 5.15], using different methods. We remark that with somewhat more work one can prove Theorem 24 with the hypothesis of separability weakened to quasi-regularity.

6. Traces

In this section we investigate (semi-finite, lower semi-continuous) traces on twisted covariance algebras. The reader may consult [15, Chap. 6] for the basic properties of traces used in the following.

For any trace ν on a C^*-algebra D, we let m_ν denote the "ideal of definition" of ν, $\tilde{\nu}$ the canonical extension of ν to a positive linear functional on m_ν, and L_ν the traced *-representation of D associated to ν. We will make frequent use of the fact that, for any $c \in m_\nu$, the functional $d \mapsto \tilde{\nu}(cd)$ is continuous on D. (This is most easily proved as follows: We can assume c is positive, since any element of m_ν is a linear combination of positive elements of m_ν. Then $\tilde{\nu}(c\cdot)=\tilde{\nu}(c^{\frac{1}{2}}\cdot c^{\frac{1}{2}})$, which, as a positive functional on D, must be continuous.)

Part (i) of the following result was proved, using quite different methods, by Dang-Ngoc [12, Section 6] under the additional hypothesis that $(G, A, \mathcal{T})$ be separable.

PROPOSITION 25. (i) *Let ν be a trace on A such that $\nu({}^s a)=\triangle^{-1}_{G/N_\mathcal{T}}(sN_\mathcal{T})\nu(a)$ for all $a\in A^+$ (where A^+ denotes the set of positive elements of A). Then there is a canonical trace $\operatorname{Ind}\nu=\operatorname{Ind}^G_{N_\mathcal{T}}\nu$ on $C^*(G, A, \mathcal{T})$, with $\operatorname{Ind}\nu(f^*\ast f)=\nu((f^*\ast f)(e))$ for suitable $f\in C_c(G, A, \mathcal{T})$, and such that $L_{\operatorname{Ind}\nu}=\operatorname{Ind}^G_{N_\mathcal{T}}L_\nu$.*

(ii) *Let ν_1 and ν_2 be traces on A satisfying the condition of* (i). *Then* $\operatorname{Ind}\nu_1=\operatorname{Ind}\nu_2$ *iff* $\nu_1=\nu_2$.

Proof. (i) Assume first that ν is "densely defined", in the sense that m_ν is dense in A. Then m_ν contains the Pedersen ideal [47] $\varkappa(A)$ of A. Let C denote the linear span in $C_c(G, A, \mathcal{T})$ of

$$C'=\{f\in C_c(G, A, \mathcal{T})\mid \exists\, a, b\in\varkappa(A)\quad\text{such that } af=f=fb\}.$$

(Here af and fb denote products in the multiplier algebra of $C^*(G, A, \mathcal{T})$, of which we regard $C_c(G, A, \mathcal{T})$ as a subalgebra.) It is easy to see that C is a *-subalgebra of $C_c(G, A, \mathcal{T})$. Furthermore C is dense in $C_c(G, A, \mathcal{T})$, in the inductive limit topology: Given $a\in A^+$ and $\varepsilon>0$, we may choose $\psi, \psi_1\in C(0,\infty])$ with values in the non negative reals, such that ψ vanishes on $[0,\varepsilon]$, ψ_1 vanishes on $[0,\varepsilon/2]$, $\psi_1\psi=\psi$, and $\|\psi(a)-a\|<\varepsilon$; then $\psi_1(a)\in\varkappa(A)$ and $\psi_1(a)\psi(a)=\psi(a)=\psi(a)\psi_1(a)$. Choosing a from among the elements of an approximate identity for A, and taking ε arbitrarily small, we may approximate any $f\in C_c(G, A, \mathcal{T})$ as closely as desired by elements of the form $\psi(a)f\psi(a)$, which then clearly lie in C.

Let $X_0^\mathcal{T}={}_G(X_0^\mathcal{T})_{N_\mathcal{T}}$. Via the canonical identification of A with $C_c(N_\mathcal{T}, A, \mathcal{T})$, the $C_c(N_\mathcal{T}, A, \mathcal{T})$-valued inner product on $X_0^\mathcal{T}$ becomes an A-valued inner product, given by $\langle x, y\rangle_A=(x^*\ast y)(e)$. We regard C as a subspace of $X_0^\mathcal{T}$; then for $x, y\in C'$ we have, for some $b\in\varkappa(A)$, $\langle x, y\rangle_A=\langle x, yb\rangle_A=\langle x, y\rangle_A b\in\varkappa(A)$, so $\langle C, C\rangle_A\subseteq\varkappa(A)$. We may thus define a sesquilinear form β on C by $\beta(x, y)=\tilde{\nu}(\langle y, x\rangle_A)$.

We wish to show that β extends to a maximal bitrace on $C^*(G, A, \mathcal{T})$ by verifying that it satisfies the properties (i)–(v) of [15, 17.2.1]. Properties (i), (iii), and (iv) are immediate consequences of the properties of an imprimitivity bimodule. To verify (ii), let $x, y\in C'$, and let $a, b\in\varkappa(A)$ be such that $yb=y$ and $ax=x$. Then (with $\dot{r}$ denoting $rN_\mathcal{T}$)

$$\beta(x, y)=\tilde{\nu}\Big(b^*\int_{G/N_\mathcal{T}}y^*(r)\,{}^r x(r^{-1})\,d\lambda(\dot{r})\Big)$$

$$=\int_{G/N}\tilde{\nu}(y^*(r)\,{}^r x(r^{-1}))\,d\lambda(\dot{r})\quad\text{(we use the fact that } \tilde{\nu}(b^*\cdot)\text{)}$$

is continuous to bring it inside the integral)

$$= \int_{G/N_{\mathcal{J}}} \tilde{\nu}({}^{r}({}^{r^{-1}}y^*(r)\,x(r^{-1})))\,d\lambda(\dot{r})$$

$$= \int_{G/N_{\mathcal{J}}} \Delta_{G/N}^{-1}(\dot{r})\,\tilde{\nu}(ax(r^{-1})\,{}^{r^{-1}}y^*(r))\,d\lambda(\dot{r})$$

$$= \tilde{\nu}\left(a\int_{G/N_{\mathcal{J}}} x(r)\,{}^{r}y^*(r^{-1})\,d\lambda(\dot{r})\right) = \beta(y^*, x^*),$$

from which (ii) follows. To verify (v) it is enough to show that for each $x \in C'$, there is a net y_α in C such that $\beta(x-xy_\alpha, x-xy_\alpha) \underset{\alpha}{\longrightarrow} 0$. Choose $b \in \varkappa(A)$ such that $xb = x$, and let $y_\alpha = by'_\alpha b$, where (y'_α) is an approximate identity for the inductive limit topology on $C_c(G, A, \mathcal{J})$. Then $\langle x-xy'_\alpha, x-xy'_\alpha\rangle_A$ tends to 0 in A (in the norm topology). But $\langle x-xy_\alpha, x-xy_\alpha\rangle_A = b^*\langle x-xy'_\alpha, x-xy'_\alpha\rangle_A b$. Since the functional $\tilde{\nu}(b^*\cdot b)$ is continuous on A, we see that $\beta(x-xy_\alpha, x-xy_\alpha) = \tilde{\nu}(b^*\langle x-xy'_\alpha, x-xy'_\alpha\rangle_A b)$ tends to 0, as desired.

By [15, 17.2.1] there is a unique maximal bitrace on $C^*(G, A, \mathcal{J})$ extending β; let $\operatorname{Ind}\nu$ denote the trace on $C^*(G, A, \mathcal{J})$ corresponding to it. We must show that $\operatorname{Ind}_{N_{\mathcal{J}}}^{G} L_\nu = L_{\operatorname{Ind}\nu}$. Let $A' \subseteq \varkappa(A)$ be the linear span of

$$\{a \in A \mid \exists\, b, c \in \varkappa(A) \quad \text{such that } ab = a = ca\}.$$

By [15, 6.3.6] A' endowed with the inner product $\langle a_1, a_2\rangle = \tilde{\nu}(a_1 a_2^*)$ is dense in the Hilbert space $\mathcal{H}_\nu = \mathcal{H}_{L_\nu}$ of L_ν, so (as C is dense in $X^{\mathcal{J}}$) $C \otimes_{\mathbf{C}} A'$ is dense in $\mathcal{H}_{\operatorname{Ind} L_\nu}$. Let C_β denote C endowed with the inner product β; then $T\colon C\otimes_{\mathbf{C}} A' \to C_\beta$, $x \otimes a \mapsto xa$ is easily seen to have dense image, and to preserve inner products. Furthermore T intertwines the left C-actions on the two spaces. Since C is dense in $C^*(G, A, \mathcal{J})$, and since by [15, 6.3.6] C_β is dense in $\mathcal{H}_{\operatorname{Ind}\nu}$, it follows that T extends to a unitary intertwining operator for $L_{\operatorname{Ind}\nu}$ and $\operatorname{Ind}_{N_{\mathcal{J}}}^{G} L_\nu$.

When ν is not densely defined, observe that in any case m_ν is G-invariant. We thus may apply the preceding to the system $(G, J, \mathcal{J}_J)$, where $J = (m_\nu)^-$ has trace $\nu_J = \nu|_{J^+}$, to get a trace $\operatorname{Ind}\nu_J$ on $C^*(G, J, \mathcal{J}_J) \cong \operatorname{Ex}_{N_{\mathcal{J}}}^{G} J$. Then $\operatorname{Ind}\nu_J$ extends canonically to a trace $\operatorname{Ind}\nu$ on $C^*(G, A, \mathcal{J})$ such that $L_{\operatorname{Ind}\nu_J} = (L_{\operatorname{Ind}\nu})\,|\,C^*(G, J, \mathcal{J}^J)$; by [52, 3.6] and the fact that L_ν is the canonical extension to A of L_{ν_J}, the canonical extension of $\operatorname{Ind}_{N_{\mathcal{J}}}^{G} L_{\nu_J}$ to $C^*(G, A, \mathcal{J})$ is equivalent to $\operatorname{Ind}_{N_{\mathcal{J}}}^{G} L_\nu$, so that $\operatorname{Ind}_{N_{\mathcal{J}}}^{G} L_\nu$ is equivalent to $L_{\operatorname{Ind}\nu}$ as desired.

(ii) Let ν_1 and ν_2 be distinct traces on A, satisfying the condition in (i). We must show $\operatorname{Ind}\nu_1 \neq \operatorname{Ind}\nu_2$. Suppose first that both ν_1 and ν_2 are densely defined. Then it is easy to show, using the properties of imprimitivity bimodules, that (in the notation of (i)) $\langle C, C\rangle_A$ is a dense *-subalgebra of A, which by [15, 6.5.3] implies that the restrictions of $\tilde{\nu}_1$ and $\tilde{\nu}_2$

to $\langle C, C\rangle_A$ must still be distinct. It follows that the bitraces on C constructed from ν_1 and ν_2 are distinct, so $\operatorname{Ind}\nu_1$ and $\operatorname{Ind}\nu_2$ are.

Now drop the assumption that ν_1 and ν_2 are densely defined. The above argument shows more generally that if the restrictions of ν_1 and ν_2 to $J=(m_{\nu_1})^- \cap (m_{\nu_2})^-$ are distinct, then $\operatorname{Ind}_{\nu_1}$ and $\operatorname{Ind}_{\nu_2}$ differ on $\operatorname{Ex}_{N_{\mathcal{J}}}^G J$. Thus we may assume $\nu_1|_J=\nu_2|_J$. Since ν_1 and ν_2 are distinct, at least one of them—say ν_1—is *not* the canonical extension to A of $\nu_1|_J$. This implies that $L_{\nu_1}(J)\mathcal{H}$ is a proper subspace of $\mathcal{H}$, where $\mathcal{H}$ denotes the underlying Hilbert space of L_{ν_1}. But since $L_{\nu_1}(m_{\nu_1})$ acts non-degenerately on $\mathcal{H}$, we have $L_{\nu_1}(m_{\nu_2}^-)\mathcal{H}= L_{\nu_1}(m_{\nu_2}^-)L_{\nu_1}(m_{\nu_1}^-)\mathcal{H}\subseteq L_{\nu_1}(J)\mathcal{H}$. It follows that there is a non-zero subrepresentation of L_{ν_1} which kills $m_{\nu_2}^-$. This subrepresentation induces to a non-zero subrepresentation of $\operatorname{Ind}_{N_{\mathcal{J}}}^G L_{\nu_1}$ which kills $\operatorname{Ex}_{N_{\mathcal{J}}}^G(m_{\nu_2}^-)$, but then since $\operatorname{Ex}_{N_{\mathcal{J}}}^G(m_{\nu_2}^-)$ acts non-degenerately on the space of $\operatorname{Ind}\nu_2$, and hence of $\operatorname{Ind}_{N_{\mathcal{J}}}^G L_{\nu_2}$, we see that $\operatorname{Ind}_{N_{\mathcal{J}}}^G L_{\nu_1}\neq\operatorname{Ind}_{N_{\mathcal{J}}}^G L_{\nu_2}$, implying $\operatorname{Ind}\nu_1\neq \operatorname{Ind}\nu_2$. ///

The following special case (of (i)) was essentially proved by Pukanszky in [49, Section 2]. (Cf. the remarks in [58, 2.6].)

COROLLARY. *Let N be a closed normal subgroup of G such that G/N is unimodular, and ν a G-invariant trace on $C^*(N)$. Then there is a canonical trace* $\operatorname{Ind}\nu$ *on $C^*(G)$, with the property that $L_{\operatorname{Ind}\nu}$ is unitarily equivalent to* $\operatorname{Ind}_N^G L_\nu$. ///

In order to classify the the traces on twisted covariance algebras one would like to have a "Mackey machine" for relating traces on $C^*(G, A, \mathcal{J})$ to those on $C^*(H, A, \mathcal{J})$, where H is a stability subgroup for the action of G on Prim A. In particular, it would be interesting to know whether the following analogue of the Effros–Hahn conjecture (in the case of essentially free action) holds:

Conjecture. Suppose that $(G, A, \mathcal{J})$ is essentially free, and that $G/N_{\mathcal{J}}$ is amenable. Then every trace μ on $C^*(G, A, \mathcal{J})$ is induced from A, in the sense that there is a trace ν on A such that $\nu({}^s a)=\triangle_{G/N}^{-1}(sN_{\mathcal{J}})\nu(a)$ for all $a\in A^+$ and s in G, and $\mu=\operatorname{Ind}\nu$. ///

Extending ideas of Guichardet [33], Zeller–Meier showed [57, Section 9] that this conjecture holds for the case $G/N_{\mathcal{J}}$ discrete, provided one restricts attention to those μ satisfying a certain technical condition (namely, that the image in $L_\mu(C^*(G, A, \mathcal{J}))$ of those elements in $C_c(G, A, \mathcal{J})$ of finite trace be dense in the weak closure of $L_\mu(C^*(G, A, \mathcal{J}))$). In the result below we show that it is unnecessary to assume this technical condition.

PROPOSITION 26. *Suppose $(G, A, \mathcal{J})$ is essentially free and separable, and that $G/N_{\mathcal{J}}$ is discrete. Let μ be a trace on $C^*(G, A, \mathcal{J})$. If either*

(i) *μ is densely defined; or*

(ii) *$G/N_{\mathcal{J}}$ is amenable,*

then μ is induced from A. In particular, when $G/N_{\mathcal{J}}$ is amenable there is a $1-1$ correspondence (given by Ind*) between G-invariant traces on A and traces on $C^*(G, A, \mathcal{J})$.*

Proof. Assume first that μ is densely defined. It is easy to see from Pedersen's construction in [47] that the Pedersen ideal of a C^*-algebra contains the Pedersen ideal of any C^*-subalgebra; applied to the C^*-subalgebra A of $C^*(G, A, \mathcal{J})$, this shows that the restriction of μ to A^+ is a densely defined trace ν on A. We proceed to show, using arguments of [33] and [57], that $\mu = \operatorname{Ind} \nu$.

It is sufficient to show that the maximal bitraces corresponding to μ and $\operatorname{Ind}\nu$ are the same, and thus by [15, 6.5.3] that the restrictions of these bitraces to the subalgebra C are the same (with C defined as in the proof of the preceding proposition). Choose a family $(s_i)_{i\in G/N_{\mathcal{J}}}$ of representatives for the $N_{\mathcal{J}}$ cosets. By discreteness of $G/N_{\mathcal{J}}$ every element of C can be written as a finite sum of elements of the form $s_i a_i$, where a_i lies in the dense subalgebra A' of A constructed in the proof of the preceding proposition. It is thus sufficient to show that for elements $s_i a_i$, $s_j a_j$ of this form we have

$$\tilde{\mu}((s_j a_j)^*(s_i a_i)) = \tilde{\nu}(\langle s_j a_j, s_i a_i\rangle_A). \tag{4}$$

But it is easily computed that

$$\langle s_j a_j, s_i a_i\rangle_A = \begin{cases} 0 & s_i \neq s_j \\ a_j^* a_i & s_i = s_j \end{cases}.$$

When $s_i = s_j$ we have $\tilde{\mu}((s_j a_j)^*(s_i a_i)) = \tilde{\mu}(a_j^* a_i) = \tilde{\nu}(a_j^* a_i)$ (by definition of ν), so that in this case (4) does hold. We must therefore show that when $s_i \neq s_j$ we have $\mu(a_j^* s_j^{-1} s_i a_i) = 0$. Let $s = s_j^{-1} s_i$. Since Prim A is second countable, there is a countable family $(W_k')_{k=1}^{\infty}$ of Borel subsets of Prim A such that for any two distinct elements P, Q of Prim A there exists k with $P \in W_k'$, $Q \notin W_k'$. Define $W_0 = \varnothing$, and, inductively for $k = 1, \ldots,$ $W_k = W_k' \setminus ({}^{s^{-1}}(W_k') \cup W_{k-1})$; then ${}^sW_k \cap W_k = \varnothing$, and since by our assumptions s does not lie in $N_{\mathcal{J}}$, so that s does not fix any point of Prim A, we see that the W^k form a partition of Prim A.

Consider now the representation L_μ of $C^*(G, A, \mathcal{J})$. The functional $\tilde{\mu}$ "extends" to a functional $\tilde{\mu}_w$ on a weakly dense ideal $\mathcal{M}_\mu$ of the weak closure $\mathcal{N}$ of $L_\mu(C^*(G, A, \mathcal{J}))$, such that $\mathcal{M}_\mu \supseteq L_\mu(m_\mu)$ and $\tilde{\mu}_w(L_\mu(f)) = \tilde{\mu}(f)$ for all $f \in m_\mu$, and such that for any $T \in \mathcal{M}_\mu$ the functional $\tilde{\mu}_w(T\cdot)$ is normal on $\mathcal{N}$. Let M denote the Glimm projection valued measure [26] on Prim A associated with M_{L_μ}; then $\sum_{k=1}^{n} M(W_k) \xrightarrow[n]{} I$ in the weak topology, so that

$$\tilde{\mu}_w\left(\sum_{k=1}^{n} M(W_k)\, L_\mu(a_j^* s a_i)\right) \xrightarrow[n]{} \tilde{\mu}_w(L_\mu(a_j^* s a_i)) = \tilde{\mu}(a_j^* s a_i).$$

But

$$\begin{aligned}\tilde{\mu}_w(M(W_k)L_\mu(a_j^* sa_i)) &= \tilde{\mu}_w(M(W_k)L_\mu(a_j^* sa_i)M(W_k))\\ &= \tilde{\mu}_w(M(W_k)M_{L_\mu}(a_j^*)M_{L_\mu}({}^s a_i)V_{L_\mu}(s)M(W_k))\\ &= \tilde{\mu}_w(M_{L_\mu}(a_j^*)M_{L_\mu}({}^s a_i)M(W_k)M({}^s W_k)L_{V_\mu}(s))\\ &= \tilde{\mu}_w(0) \quad (\text{since } M(W_k)M({}^s W_k)=0)=0.\end{aligned}$$

Hence $\tilde{\mu}(a_j^* sa_i)=0$, so that the two bitraces on C do coincide as claimed and Ind $\nu=\mu$.

If μ is not densely defined but $G/N_{\mathcal{J}}$ is amenable, let $J=(m_\mu)^-$. By Theorem 24 and Proposition 13, $J=\mathrm{Ex}_{N_{\mathcal{J}}}^G I$ for some G-invariant ideal I of A. Thus $\mu_J=\mu|_{J^+}$ is a densely defined trace on $C^*(G, I, \mathcal{J}_I)$, and the preceding shows that $\mu_J=\mathrm{Ind}\, v_I$ for some trace ν_I on I. Then if ν denotes the canonical extension of ν_I to A, we see easily (since μ is the canonical extension of μ_J to $C^*(G, A, \mathcal{J})$) that $\mu=\mathrm{Ind}\,\nu$.

The statement that in the case $G/N_{\mathcal{J}}$ amenable there is a bijection between G-invariant traces on A and traces on $C^*(G, A, \mathcal{J})$ follows from combining the above with the preceding proposition. ///

The other case in which we can obtain positive results concerning our conjecture is that of factor traces, or characters, on C^*-algebras of regular systems.

We begin with a lemma which, though surely known, does not seem to be accessible in the literature. A *normal* representation of a C^*-algebra is a traceable factor representation.

LEMMA 27. *Let L be a (non-zero) *-representation of a C^*-algebra D, such that the weak closure of $L(D)$ is a semi-finite factor $\mathcal{N}$. Then L is normal iff $L(D)$ has non-zero intersection with the (unique) minimal norm-closed ideal of $\mathcal{N}$.*

Proof. L is normal iff $L(D)$ has non-trivial intersection with the ideal $\mathcal{M}$ of traceable operators of $\mathcal{N}$. But $\mathcal{M}$, being a hereditary ideal of $\mathcal{N}$, certainly contains the Pedersen ideal $\varkappa$ of $\mathcal{M}^-$ (the *norm* closure of $\mathcal{M}$), and since $\varkappa$ contains the Pedersen ideal of any sub-C^*-algebra of $\mathcal{M}^-$, $L(D)$ will intersect $\varkappa$ non-trivially iff it intersects $\mathcal{M}^-$ non-trivially.

Hence it suffices to show that $\mathcal{M}^-$ is the minimal norm closed ideal of $\mathcal{N}$. Observe first that for any finite projection e in $\mathcal{N}$, $e\mathcal{N}e$ is a finite factor, hence in particular [16, Cor. 3, p. 257] is a simple C^*-algebra. By [51, 6.7] $e\mathcal{N}e$ is Morita equivalent to $\mathcal{N}e\mathcal{N}$, the norm closed ideal generated by e; so the latter ideal is also simple and hence minimal. But there can be at most one minimal norm closed ideal in a factor, for if there were two such the supports of their weak closures would be orthogonal central projections.

Thus the minimal norm closed ideal is unique and contains all finite projections. Since every element of $\mathcal{M}^-$ can be approximated in norm by finite linear combinations of finite projections, this minimal ideal is $\mathcal{M}^-$. ///

The following result is of some independent interest—see [30].

PROPOSITION 28. *Let E and B be C^*-algebras, Y an $E-B$ imprimitivity bimodule, L a *-representation of B. Then L is normal iff the representation* $\operatorname{Ind} L$ *of E induced via Y is normal.*

Proof. Suppose L is normal, and let $\mathcal{N}_1$ and $\mathcal{N}_2$ denote the weak closures of $L(B)$ and $(\operatorname{Ind} L)(E)$, respectively. Because Ind gives an equivalence between the categories of *-representations of B and E, there is a natural isomorphism between the commutants $\mathcal{N}_1'$ and $\mathcal{N}_2'$, arising from the action of $\mathcal{N}_1'$ on $Y \otimes_{\mathbf{C}} \mathcal{H}_L$ (regarded, by abuse of notation, as a subspace of $\mathcal{H}_{\operatorname{Ind} L}$) defined by $\mathcal{U}(y \otimes \xi) = y \otimes \mathcal{U}\xi$, for $\mathcal{U} \in \mathcal{N}_1'$. Thus we may regard both $\mathcal{H}_L$ and $\mathcal{H}_{\operatorname{Ind} L}$ as modules over the W^*-algebra $\mathcal{N} = \mathcal{N}_1'$. Let Y' denote the collection of all bounded linear maps from $\mathcal{H}_L$ to $\mathcal{H}_{\operatorname{Ind} L}$ which intertwine the $\mathcal{N}$-actions. Then Y' is stable under left and right multiplication by elements of $\mathcal{N}_2$ and $\mathcal{N}_1$, and is thus an $\mathcal{N}_2 - \mathcal{N}_1$ bimodule. Furthermore there are natural $\mathcal{N}_1$- and $\mathcal{N}_2$-valued inner products on Y', defined respectively by $\langle T_1, T_2\rangle_1 = T_1^* T_2$, $\langle T_1, T_2\rangle_2 = T_1 T_2^*$. Let I_i denote the norm closure of $\langle Y', Y'\rangle_i$, $i=1, 2$. Then I_i is a norm closed ideal of $\mathcal{N}_i$, and it is easily verified that with above inner products Y' becomes an $I_2 - I_1$ imprimitivity bimodule.

Now since L is normal, the ideal I of B consisting of elements whose images lie in the minimal (norm closed) ideal of $\mathcal{N}_1$ must properly contain $\ker L$. Using [52, 3.3] we may find $y \in Y$ such that $\langle y, y\rangle_B \in I \setminus \ker L$. It is easily verified that the operator T_y: $\mathcal{H}_L \to \mathcal{H}_{\operatorname{Ind} L}$, $\xi \mapsto y \otimes \xi$ lies in Y', and that $\langle T_y, T_y\rangle_1 = L(\langle y, y\rangle_B)$. Thus $\langle T_y, T_y\rangle_1$ is a non-zero element of the minimal norm closed ideal of I_1, hence by [52, 3.2] $\langle T_y, T_y\rangle_2$ is a non-zero element of the minimal norm-closed ideal of I_2. But it is readily shown that $\langle T_y, T_y\rangle_2 = (\operatorname{Ind} L)(\langle y, y\rangle_E)$, so by the lemma $\operatorname{Ind} L$ is normal.

Conversely, since L is unitarily equivalent to the representation induced from $\operatorname{Ind} L$ via the dual bimodule $\tilde{Y}$ (see [51, Section 6]) we see that normality of $\operatorname{Ind} L$ implies normality of L. ///

PROPOSITION 29. *Let Q be a regular quasi-orbit in A, and let $P \in Q$. There is a natural bijection, given by* $\operatorname{Ind}_{G_P}^G$, *between the (unitary equivalence classes of) normal representations of $(G_P, A, \mathcal{J})$ whose kernels live on the G_P quasi-orbit $\{P\}$, and the normal representations of $(G, A, \mathcal{J})$ whose kernels live on Q.*

Proof. As in the proof of Proposition 20, we use Proposition 12 to reduce to the case in which Q is all of Prim A. The result now follows from the preceding proposition together with Theorem 17. ///

Each normal representation of A determines a character, or factor trace, on A, unique up to a scalar multiple. Thus the above result implies (at least in the presence of a separability assumption, so as to ensure that kernels of factor representations are primitive) that when $(G, A, \mathcal{T})$ is essentially free and regular, every factor trace ν on $C^*(G, A, \mathcal{T})$ arises from some factor trace μ on A. In general μ is not (relatively) G-invariant, so ν is not induced from μ in the sense of Proposition 25; however it can be shown that it is possible to "smooth out" μ under an appropriate averaging process so as to obtain a relatively invariant trace μ' for which $\nu = \operatorname{Ind} \mu'$. Thus our conjecture may be verified in this case as well. We do not give the details here since the above result is more convenient for applications, factor traces being in general easier to work with than the smoothed out invariant ones.

7. Abelian systems

We assume throughout this section that $(G, A, \mathcal{T})$ is *abelian*, meaning that $G/N_{\mathcal{T}}$ (but not necessarily A) is abelian. $\tilde{G}$ will denote the abelian group $G/N_{\mathcal{T}}$. Such systems have a number of interesting properties, which we proceed now to investigate.

Observe that there is a natural action by *-automorphisms of $\tilde{G}^\wedge$ on $C_c(G, A, \mathcal{T})$, given by

$$({}^p f)(s) = \langle p, s\rangle^- f(s) \quad \text{for } p \in \tilde{G}^\wedge.$$

(Here the bar denotes complex conjugate, and $\langle p, s\rangle = p(sN_{\mathcal{T}})$.) It is easily verified that this action is isometric and strongly continuous with respect to the L^1-norm, and so by universality extends to a (strongly continuous) action on $C^*(G, A, \mathcal{T})$.

We wish to relate the crossed product algebra $C^*(\tilde{G}^\wedge, C^*(G, A, \mathcal{T}))$ to the imprimitivity algebra $E^{\mathcal{T}} = C^*(G, C_\infty(G/N_{\mathcal{T}}) \otimes A, \mathcal{T}^\wedge)$. Observe first that there are natural homomorphisms $R^1_{\tilde{G}^\wedge}$, R^1_G, and R^1_A of $\tilde{G}^\wedge$, G, and A, respectively, into $\mathcal{M}(D)$, $D = C^*(\tilde{G}^\wedge, C^*(G, A, \mathcal{T}))$. (The latter two arise from composing R_G and R_A with the natural homomorphism of $\mathcal{M}(C^*(G, A, \mathcal{T}))$ into $\mathcal{M}(D)$, which exists because the action (by multipliers) of $C^*(G, A, \mathcal{T})$ on $C^*(\tilde{G}^\wedge, C^*(G, A, \mathcal{T}))$ is essential.) Similarly, there are natural homomorphisms $R^2_{\tilde{G}^\wedge}$, R^2_G, and R^2_A into $\mathcal{M}(E^{\mathcal{T}})$, where $R^2_{\tilde{G}^\wedge}$ is defined by composing the homomorphism of $\tilde{G}^\wedge$ into $C(G/N_{\mathcal{T}}) \cong \mathcal{M}(C_\infty(G/N_{\mathcal{T}}))$ given by $p \mapsto f_p$, $f_p(sN_{\mathcal{T}}) = p(s^{-1}N_{\mathcal{T}})$ (the integrated form of which is just the Fourier transform isomorphism of $C^*(\tilde{G}^\wedge)$ onto $C_\infty(G/N_{\mathcal{T}})$) with the natural homomorphism of $\mathcal{M}(C_\infty(G/N_{\mathcal{T}}))$ into $\mathcal{M}(E^{\mathcal{T}})$. If we let R^2 denote the integrated form of (R^2_G, R^2_A), then it is easily seen that the pair $(R^2_{\tilde{G}^\wedge}, R^2)$ is a covariant homomorphism of $(\tilde{G}^\wedge, C^*(G, A, \mathcal{J}))$ into $\mathcal{M}(E^{\mathcal{J}})$. Let R be its integrated form. Using the facts that D is generated by $C^*(\tilde{G}^\wedge)\, C^*(G, A, \mathcal{J})$ (interpreted as a set of products in $\mathcal{M}(D)$), and that $E^{\mathcal{J}}$ is generated by $C^*(G, A, \mathcal{J})\, C_\infty(G/N_{\mathcal{J}})$, we see easily that the image of R is exactly $E^{\mathcal{J}}$.

On the other hand, if we let R^1 denote the homomorphism of $C_\infty(G/N_{\mathcal{J}}) \otimes A$ into $\mathcal{M}(D)$ obtained from the pair $(R^1_{\tilde{G}^\wedge}, R^1_A)$—where we regard (the integrated form of) $R^1_{\tilde{G}^\wedge}$ as a homomorphism of $C_\infty(G/N_{\mathcal{J}})$ into $\mathcal{M}(D)$ by means of the inverse Fourier transform—then the pair (R^1_G, R^1) is easily seen to define a covariant homomorphism of $(G, C_\infty(G/H) \otimes A, \mathcal{J}^\wedge)$ into $\mathcal{M}(D)$; furthermore one may readily verify that its integrated form inverts R. Thus:

PROPOSITION 30. *With the above notation, R is an isomorphism of $C^*(\tilde{G}^\wedge, C^*(G, A, \mathcal{J}))$ onto $E^{\mathcal{J}}$.* ///

In conjunction with the imprimitivity theorem, this yields

COROLLARY 31. *$C^*(\tilde{G}^\wedge, C^*(G, A, \mathcal{J}))$ is Morita equivalent to A.* ///

Remark. For the special case of trivial $\mathcal{J}$ (so that $G = \tilde{G}$), the covariance algebra $C^*(G^\wedge, C^*(G, A))$ is precisely the dual crossed product algebra considered by Takai in [54, Section 3]. His duality theorem states that it is isomorphic to $A \otimes \mathcal{K}(L^2(G))$, which implies in particular that it is Morita equivalent to A. Thus the above Corollary 31 may be regarded as a weak form of the duality theorem, valid for non-trivial $\mathcal{J}$. In fact I have recently obtained, in the case of separable, not necessarily abelian, $(G, A, \mathcal{J})$, a substantial generalization of Takai's theorem, which states that (for arbitrary closed H containing $N_{\mathcal{J}}$) the algebra $C^*(G, C_\infty(G/H) \otimes A, \mathcal{J}^\wedge)$ is isomorphic to $C^*(H, A, \mathcal{J}) \otimes \mathcal{K}(L^2(G/H))$. Details and applications, as well as an analogous result for von Neumann algebras which extends Takesaki's duality theorem, will appear in a subsequent paper. ///

We turn now to a consideration of *reduced* abelian systems $(G, \mathbf{C}, \mathcal{J})$, i.e. those for which $\mathcal{J}$ is an isomorphism of $N_{\mathcal{J}}$ onto $\mathbf{T}$. Note that as $N_{\mathcal{J}}$ is then central and $\tilde{G}$ is unimodular, G is itself unimodular, so that we can drop the modular functions from all our formulae. We begin by observing that for any $r, s \in G$, the commutator $srs^{-1}r^{-1}$ is in $N_{\mathcal{J}}$; centrality of $N_{\mathcal{J}}$ implies that this commutator depends only on the cosets $\dot{r}$ $(= rN_{\mathcal{J}})$ and $\dot{s}$, and we easily verify that the resulting pairing

$$\beta(\dot{r}, \dot{s}) = \mathcal{J}(srs^{-1}r^{-1})$$

is a homomorphism in each variable when the other is held fixed. Hence there is a well-defined continuous homomorphism $h_{\mathcal{J}}$ of $\tilde{G}$ into $\tilde{G}^\wedge$, given by $h_{\mathcal{J}}(\dot{r})(\dot{s}) = \beta(\dot{r}, \dot{s})$. We readily verify that the dual map $h_{\mathcal{T}}^{\wedge}$, regarded as a homomorphism of $\tilde{G}$ into $\tilde{G}^{\wedge}$ via the canonical identification of $\tilde{G}$ with $\tilde{G}^{\wedge\wedge}$, is given by $h_{\mathcal{T}}^{\wedge}(\dot{r})(\dot{s}) = \beta(\dot{s}, \dot{r}) = \beta(\dot{r}, \dot{s}))^{-1}$, so that $h_{\mathcal{T}}^{\wedge}(\dot{r}) = (h_{\mathcal{T}}(\dot{r}))^{-1}$.

The homomorphism $h_{\mathcal{T}}$ allows us to relate the action of $\tilde{G}^{\wedge}$ on $C^*(G, \mathbf{C}, \mathcal{T})$ with the generalized inner action of G which arises from the homomorphism R_G into the unitary group of $\mathcal{M}(C^*(G, \mathbf{C}, \mathcal{T}))$: For each $f \in C_c(G, \mathbf{C}, \mathcal{T})$ and $s \in G$, we have (using the fact that G acts trivially on $\mathbf{C}$)

$$^s f(r) = {}^s(f(s^{-1}rs)) = f((s^{-1}rsr^{-1})r) = f(r)\,\mathcal{T}(rs^{-1}r^{-1}s) = h_{\mathcal{T}}(\dot{s}^{-1})(\dot{r})f(r) = {}^{h_{\mathcal{T}}(\dot{s})}f(r).$$

By density of $C_c(G, \mathbf{C}, \mathcal{T})$ in $C^*(G, \mathbf{C}, \mathcal{T})$ we see that the actions of s and $h_{\mathcal{T}}(\dot{s})$ on $C^*(G, \mathbf{C}, \mathcal{T})$ coincide.

The reduced abelian system $(G, \mathbf{C}, \mathcal{T})$ is said to be *totally skew* if the center of G is exactly $N_{\mathcal{T}}$. This is equivalent to the condition that $h_{\mathcal{T}}$ be $1-1$, which in turn (since $h_{\mathcal{T}}^{\wedge}(\dot{r}) = (h_{\mathcal{T}}(\dot{r}))^{-1}$) is equivalent to $1-1$-ness of $h_{\mathcal{T}}^{\wedge}$, and thus (by the Pontrjagin theory) to the condition that $h_{\mathcal{T}}$ have dense range.

The following basic fact about totally skew systems is due to Kleppner [38]. Our proof is different, and relies on the observation that, by Corollary 31, $C^*(\tilde{G}^{\wedge}, C^*(G, \mathbf{C}, \mathcal{T}))$ is Morita equivalent to $\mathbf{C}$, and so is isomorphic to the algebra of compact operators on some Hilbert space. We say that a C^*-algebra *has unique trace* if it has a non-zero trace which is unique to within a scalar multiple.

PROPOSITION 32. *Let $(G, \mathbf{C}, \mathcal{T})$ be totally skew. Then $C^*(G, \mathbf{C}, \mathcal{T})$ is simple and has a unique trace.*

Proof. From the preceding remarks, elements in the dense subgroup $h_{\mathcal{T}}(\tilde{G})$ of $\tilde{G}^{\wedge}$ act on $C^*(G, \mathbf{C}, \mathcal{T})$ as conjugation by unitaries in $\mathcal{M}(C^*(G, \mathbf{C}, \mathcal{T}))$, hence in particular they act trivially on the ideal space $\mathcal{I}(C^*(G, \mathbf{C}, \mathcal{T}))$. But as the action of $\tilde{G}^{\wedge}$ on $\mathcal{I}(C^*(G, \mathbf{C}, \mathcal{T}))$ is jointly continuous, and $\mathcal{I}(C^*(G, \mathbf{C}, \mathcal{T}))$ is T_0, stabilizers of points must be closed by [5, Lemma 1], so that $\tilde{G}^{\wedge}$ fixes every point in $\mathcal{I}(C^*(G, \mathbf{C}, \mathcal{T}))$, and $^{\tilde{G}^{\wedge}}\mathcal{I}(C^*(G, \mathbf{C}, \mathcal{T}))$ coincides with $\mathcal{I}(C^*(G, \mathbf{C}, \mathcal{T}))$. If there were two distinct non-zero ideals I and J of $C^*(G, \mathbf{C}, \mathcal{T})$, then by Proposition 11 $\mathrm{Ex}_{N_{\mathcal{T}}}^{G}\, I$ and $\mathrm{Ex}_{N_{\mathcal{T}}}^{G}\, J$ would be distinct non-zero ideals of $C^*(\tilde{G}, C^*(G, \mathbf{C}, \mathcal{T}))$, contradicting the fact that the algebra of compact operators on any Hilbert space is simple. It follows that $C^*(G, \mathbf{C}, \mathcal{T})$ is simple.

Uniqueness of the trace follows from an argument which is formally quite similar. We show first (repeating the calculation of [38, Lemma 1.1]) that any trace ν on $C^*(G, \mathbf{C}, \mathcal{T})$ is $\tilde{G}^{\wedge}$-invariant: Let p be an arbitrary element of $\tilde{G}^{\wedge}$, and let (p_α) be a net in $h_{\mathcal{T}}(\tilde{G})$

converging to p^{-1}. Since the p_α act by unitaries in $\mathcal{M}(C^*(G, A, \mathcal{J}))$ we have $\nu({}^{p_\alpha}f) = \nu(f)$ for all $f \in C^*(G, \mathbf{C}, \mathcal{J})^+$. By lower semi-continuity of ν, and the fact that ${}^{p_\alpha}f \xrightarrow[\alpha]{} {}^{p^{-1}}f$, we have $\nu({}^{p^{-1}}f) \leqslant \nu(f)$. Applying this to $f = {}^{p}g$ for an arbitrary $g \in C^*(G, A, \mathcal{J})^+$ we get $\nu(g) \leqslant \nu({}^{p}g)$; but since ${}^{p_\alpha^{-1}}g \to {}^{p}g$ we have also $\nu({}^{p}g) \leqslant \nu(g)$. Thus $\tilde{G}^\wedge$ leaves ν invariant.

By Proposition 25 we may induce up the canonical trace on $\mathbf{C}$ to get a non-zero trace ν on $C^*(G, \mathbf{C}, \mathcal{J})$. Suppose there is another trace ν_1 on $C^*(G, \mathbf{C}, \mathcal{J})$, not a scalar multiple of ν. As ν and ν_1 are $\tilde{G}^\wedge$-invariant by the preceding paragraph, they induce by Proposition 25 to traces on $C^*(\tilde{G}^\wedge, C^*(G, \mathbf{C}, \mathcal{J}))$ which are not scalar multiples of each other; but this contradicts the fact that the compact operators have a unique trace.

Alternatively, one may argue that since any trace μ is $\tilde{G}^\wedge$ invariant, there is a natural action V of $\tilde{G}^\wedge$ on the underlying Hilbert space of L_μ. It is easily checked that (V, L_μ) is a covariant representation of $(\tilde{G}^\wedge, C^*(G, \mathbf{C}, \mathcal{J}))$, which implies (since $C^*(\tilde{G}^\wedge, C^*(G, \mathbf{C}, \mathcal{J}))$ is naturally isomorphic to the imprimitivity algebra for $\mathrm{Ind}_{N_\mathcal{J}}^G$) that L_μ is induced from $\mathbf{C}$; the quasi-equivalence class of L_μ is thus well-defined. This implies that L_μ is a factor representation, since any decomposition of L_μ into a direct sum of disjoint subrepresentations would give non-quasiequivalent traceable representations. But two traces giving the same factor representation (to within quasiequivalence) must be scalar multiples of each other. (This alternative argument avoids assuming the fact that $C^*(\tilde{G}^\wedge, C^*(G, \mathbf{C}, \mathcal{J}))$ is isomorphic to the algebra of compact operators.) ///

Although we will not need the following proposition (which is the main result of [3]) we prove it here as an illustration of the usefulness of Theorem 18. The proof in [3] is quite complicated and relies heavily on structure theoretic arguments.

PROPOSITION 33. ([3, Thm. 3.2 and 3.3].) *Let $(G, \mathbf{C}, \mathcal{J})$ be totally skew. The following are equivalent:*

(i) *$C^*(G, \mathbf{C}, \mathcal{J})$ is a type I C^*-algebra.*

(ii) *$C^*(G, \mathbf{C}, \mathcal{J})$ is isomorphic to $\mathcal{K}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$.*

(iii) *$h_\mathcal{J}$ is a homeomorphism onto $\tilde{G}^\wedge$.*

Proof. The equivalence of (i) and (ii) follows from the simplicity of $C^*(G, \mathbf{C}, \mathcal{J})$ (Proposition 32) and the Glimm–Sakai theorem [53, 4.6.4].

(ii) $\Rightarrow$ (iii): It is sufficient to prove that $h_\mathcal{J}$ is open onto its image in $\tilde{G}^\wedge$. By the remark following Theorem 18 the action of $\tilde{G}^\wedge$ on $C^*(G, \mathbf{C}, \mathcal{J}) \simeq \mathcal{K}(\mathcal{H})$ is generalized inner, and so induces a continuous homomorphism R of $\tilde{G}^\wedge$ into the projective unitary group $\mathcal{P}$ of $\mathcal{M}(C^*(G, \mathbf{C}, \mathcal{J}))$. The map $R \circ h_\mathcal{J}$ is just the map R' of $\tilde{G}$ into $\mathcal{P}$ arising from the (generalized inner) action of G on $C^*(G, \mathbf{C}, \mathcal{T})$; we show that R' is open onto its image in $\mathcal{D}$, which will imply that $h_{\mathcal{T}}$ is open onto its image in $\tilde{G}^\wedge$ as desired.

Thus let $\tilde{O}$ be any neighborhood of the identity of $\tilde{G}$, $\tilde{O}_1$ a symmetric open neighborhood of the identity with $\tilde{O}_1^2 \subseteq \tilde{O}$, and let O, O_1 be their inverse images in G. Let μ be the trace on $C^*(G, \mathbf{C}, \mathcal{T})$ induced from the canonical trace on $\mathbf{C}$. We may choose $f \in C_c(G, \mathbf{C}, \mathcal{T})$ such that $(\operatorname{supp} f)^2 \subseteq O_1$, and such that $\mu(f * f^*) = 1$. (Note that the space C' (constructed in the proof of Proposition 25) is all of $C_c(G, \mathbf{C}, \mathcal{T})$.) For any $s \notin O$ the support of $sf * f^*$ does not contain the identity, so $\mu(sf * f^*) = 0$ (by definition of μ). Therefore, regarding f as an element of the underlying Hilbert space of L_μ, we see that $s \mapsto |\langle V_{L_\mu}(s) f, f \rangle|$ vanishes off O and is 1 at $s = e$. Since the function $\dot{U} \mapsto |\langle Uf, f \rangle|$ is continuous on $\mathcal{P}\mathcal{U}(\mathcal{H})$, it follows easily that the natural homomorphism of $\tilde{G}$ into $\mathcal{P}\mathcal{U}(\mathcal{H}_{L_\mu})$ is open onto its image. As this homomorphism factors continuously through the map R' of $\tilde{G}$ into $\mathcal{D}$, the latter map is open onto its image.

(iii) $\Rightarrow$ (i): If $h_{\mathcal{T}}$ is a homeomorphism, then $R' \circ h_{\mathcal{T}}^{-1}$ (with R' defined as above) is continuous, so that the action of $\tilde{G}^\wedge$ on $C^*(G, \mathbf{C}, \mathcal{T})$ is generalized inner. Thus as $C^*(G, \mathbf{C}, \mathcal{T})$ is simple Theorem 18 applies to the system $(\tilde{G}^\wedge, C^*(G, \mathbf{C}, \mathcal{T}))$ and we deduce that there is a reduced system $(G', \mathbf{C}, \mathcal{T}')$ such that $C^*(\tilde{G}^\wedge, C^*(G, \mathbf{C}, \mathcal{T})) = C^*(G, \mathbf{C}, \mathcal{T}) \otimes_{\max} C^*(G', \mathbf{C}, \mathcal{T}')$. If $C^*(G, \mathbf{C}, \mathcal{T})$ had a non-type I representation L, we could construct a non-type I representation of $C^*(G, \mathbf{C}, \mathcal{T}) \otimes_{\max} C^*(G', \mathbf{C}, \mathcal{T}')$ by taking the tensor product of L with an irreducible representation of $C^*(G', \mathbf{C}, \mathcal{T}')$; but this would contradict the fact that $C^*(\tilde{G}^\wedge, C^*(G, \mathbf{C}, \mathcal{T}))$ is isomorphic to the compact operators on some Hilbert space. ///

We conclude our discussion of reduced systems with a result on non-totally skew systems. Except for the statement about the topology of Prim $C^*(G, \mathbf{C}, \mathcal{T})$ (which answers a question raised in [38]), this result is essentially contained in [3] and [38], but we give a complete proof anyway.

PROPOSITION 34. *Let $(G, \mathbf{C}, \mathcal{T})$ be a reduced abelian system, and let Z denote the center of G.*

(i) *There is a totally skew system $(G', \mathbf{C}, \mathcal{T}')$ such that $G'/N_{\mathcal{T}'} \cong G/Z$, and such that for any $P \in \operatorname{Prim} C^*(G, \mathbf{C}, \mathcal{T})$, $C^*(G, \mathbf{C}, \mathcal{T})/P$ is isomorphic to $C^*(G', \mathbf{C}, \mathcal{T}')$.*

(ii) *For any $P \in \operatorname{Prim} C^*(G, \mathbf{C}, \mathcal{T})$, the stabilizer in $\tilde{G}^\wedge$ of P is equal to the annihilator $Z^\perp$ of $Z/N_{\mathcal{T}}$, and the map $tZ^\perp \mapsto {}^tP$ is a homeomorphism of $\tilde{G}^\wedge/Z^\perp$ onto* Prim $C^*(G, \mathbf{C}, \mathcal{T})$.

Proof. Since Z is abelian, $C^*(Z, \mathbf{C}, \mathcal{T})$ is commutative; its dual $C^*(Z, \mathbf{C}, \mathcal{T})^\wedge \cong \operatorname{Prim} C^*(Z, \mathbf{C}, \mathcal{T})$ may be naturally identified (via the map $L \mapsto V_L$) with the subset of $Z^\wedge$ consisting of those characters whose restrictions to $N_{\mathcal{T}}$ equal $\mathcal{T}$.

We consider the iterated system $(G, C^*(Z, \mathbf{C}, \mathcal{T}), \mathcal{T}^Z)$. By Proposition 1 its C^*-algebra is isomorphic to $C^*(G, \mathbf{C}, \mathcal{T})$. Let $I \in \operatorname{Prim} C^*(Z, \mathbf{C}, \mathcal{T})$, say corresponding to the character χ of Z, and let $J = \operatorname{Ex}_Z^G I$. Since Z is central I is fixed by the action of G, so that by Proposition 12, $C^*(G, C^*(Z, \mathbf{C}, \mathcal{T}), \mathcal{T}^Z)/J$ is naturally isomorphic to $B = C^*(G, C^*(Z, \mathbf{C}, \mathcal{T})/I, (\mathcal{T}^Z)^I)$. But $C^*(Z, \mathbf{C}, \mathcal{T})/I$ may be naturally identified with $\mathbf{C}$ in such a way that $(\mathcal{T}^Z)^I$ becomes the character χ on Z; thus B is isomorphic to $C^*(G, \mathbf{C}, \chi)$.

Let K denote the kernel of χ, let $G' = G/K$, and define $\mathcal{T}'$ on Z/K by $\mathcal{T}'(sK) = \chi(s)$ for all $s \in Z$. Elements of $C_c(G, \mathbf{C}, \chi)$ are constant on cosets of K, so that (with suitable choice of Haar measure on G') there is a natural isometric *-isomorphism of $C_c(G, \mathbf{C}, \chi)$ onto $C_c(G', \mathbf{C}, \mathcal{T}')$. Hence $C^*(G, \mathbf{C}, \chi)$ is naturally isomorphic to $C^*(G', \mathbf{C}, \mathcal{T}')$. As $\mathcal{T}'$ is faithful, and also surjective (since $\chi|_{N_\mathcal{T}} = \mathcal{T}$), we see that $(G', \mathbf{C}, \mathcal{T}')$ is reduced. Let $rK \in G' \setminus (Z/K)$. As $r \notin Z$, there exists s with $rsr^{-1}s^{-1} \neq e_G$; then $\mathcal{T}'(rKsK(rK)^{-1}(sK)^{-1}) = \mathcal{T}(rsr^{-1}s^{-1}) \neq 1$, so in particular rK does not commute with sK and thus is not central. Therefore the center of G' is precisely Z/K, so $(G', \mathbf{C}, \mathcal{T}')$ is totally skew.

The preceding together with Proposition 32 shows that $C^*(G, \mathbf{C}, \mathcal{T})/\operatorname{Ex}_Z^G I$ is a simple C^*-algebra, i.e. that $\operatorname{Ex}_Z^G I$ is maximal, for any $I \in \operatorname{Prim} C^*(Z, \mathbf{C}, \mathcal{T})$. Let P be an arbitrary primitive ideal of $C^*(G, \mathbf{C}, \mathcal{T})$. As G acts trivially on $C^*(Z, \mathbf{C}, \mathcal{T})$, the quasi-orbit space is just $\operatorname{Prim} C^*(Z, \mathbf{C}, \mathcal{T})$, which is Hausdorff; by Corollary 19, P lives on a quasi-orbit, so in particular $\operatorname{Res}_Z^G P$ is primitive. (One can also see this directly by using the centrality of Z.) Since $\operatorname{Ex}_Z^G \operatorname{Res}_Z^G P \subseteq P$ (Proposition 11) we see by the above that $P = \operatorname{Ex}_Z^G \operatorname{Res}_Z^G P$; and as P is thus maximal, $P = \operatorname{Ind}_Z^G \operatorname{Res}_Z^G P$ by Proposition 11. By continuity of Ind and Res the map $P \mapsto \operatorname{Res}_Z^G P$ is a homeomorphism of $\operatorname{Prim} C^*(G, \mathbf{C}, \mathcal{T})$ onto $\operatorname{Prim} C^*(Z, \mathbf{C}, \mathcal{T})$.

There is a natural action of $\tilde{G}^\wedge$ on $C^*(Z, \mathbf{C}, \mathcal{T})$, obtained by composing the natural homomorphism of $\tilde{G}^\wedge$ onto $(Z/N_\mathcal{T})^\wedge$ with the natural action of $(Z/N_\mathcal{T})^\wedge$ on $C^*(Z, \mathbf{C}, \mathcal{T})$. One sees easily that Res_Z^G: $\operatorname{Prim} C^*(G, \mathbf{C}, \mathcal{T}) \to \operatorname{Prim} C^*(Z, \mathbf{C}, \mathcal{T})$ is equivariant for the associated actions of $\tilde{G}^\wedge$ on these two spaces. Now $\operatorname{Prim} C^*(Z, \mathbf{C}, \mathcal{T})$ may be identified with the coset of $(Z/N_\mathcal{T})^\wedge$ in $Z^\wedge$ consisting of those characters which restrict to $\mathcal{T}$ on $N_\mathcal{T}$, and it is easy to see that the action of $(Z/N_\mathcal{T})^\wedge$ on $\operatorname{Prim} C^*(Z, \mathbf{C}, \mathcal{T})$ just corresponds to the action by translation on that coset. Part (ii) of the proposition now follows easily from the above equivariance of the action.

Since the action of $\tilde{G}^\wedge$ on $\operatorname{Prim} C^*(G, \mathbf{C}, \mathcal{T})$ is in particular transitive, all primitive quotients of $C^*(G, \mathbf{C}, \mathcal{T})$ are isomorphic, and (i) now follows from the facts proved earlier. ///

Our final result is motivated by the beautiful theorem of Pukanszky [49] that for a connected Lie group G, every primitive ideal of $C^*(G)$ is the kernel of a unique quasi-equivalence class of normal representations. Roughly speaking, we prove that the same result holds for the C^*-algebras of those abelian systems which arise as the "restrictions" of regular systems, from which we then deduce Pukanszky's result (for a more general class of locally compact groups) as a corollary. The crucial fact about such systems is that "locally" they can be broken up as a series of iterated systems, each of which is either regular, or discrete (in the sense that the domain of the twisting map is open); we are then in a position to apply the results of Sections 5 and 6. This idea (for the case $G/N_{\mathcal{T}}$ a connected Lie group) can already be found, in considerably disguised form, in Pukanszky's discussion of the primitive ideals of $C^*(G)$ in [49, Section 1]. By making more systematic use of it than he, and by using it to investigate the trace structure, as well as the primitive ideal structure, we are able to considerably simplify, unify, and generalize his proof. (The "Mackey machine" results of Section 4, and Kleppner's result on totally skew systems (Proposition 32), also play crucial roles in our argument.)

The lemma on which this decomposition into regular and discrete systems depends is the following:

LEMMA 35. *Let G and H be locally compact abelian groups, and π a continuous injective homomorphism of H into G. Then there are closed subgroups $(1_H)=H_0\subseteq H_1\subseteq H_2\ldots\subseteq H_n=H$ of H such that, with π_i denoting the natural homomorphism of H_i/H_{i-1} into $\pi(H_i)^-/\pi(H_{i-1})^-$, either*

(i) *π_i is a homeomorphism, or*

(ii) *π_i is injective and H_i/H_{i-1} is discrete, for each $i=1, \ldots, n$.*

(Note that as π_i is injective in either case, $\pi^{-1}(\pi(H_i)^-)=H_i$ for all i.)

Proof. We argue by induction on the dimension $m=m(H)$ of the vector part of H. When $m=0$, H has a compact open subgroup which we take as H_1, and the series $(1)\subseteq H_1\subseteq H$ suffices. Hence assume $m>0$, and that the lemma holds for groups having vector part of dimension less than m. Let H' be a closed subgroup of H isomorphic to $\mathbf{R}$, let $G'=\pi(H')^-$, and let $H''=\pi^{-1}(G')$. Since $m(H/H'')<m$ we can construct a series of the desired form for the natural injection of H/H'' into G/G', so we may as well assume that $H=H''$ and that $G=G'$. When $\pi'=\pi|_{H'}$ is a homeomorphism the result is obvious, so we assume it is not. Then G, being a "solenoid group", is compact. Thus $G^\wedge$ is discrete, and since $(\pi')^\wedge$ is injective we may choose an infinite cyclic subgroup D of $G^\wedge$. The dual map to $(\pi')^\wedge|_D$ is a surjective map of H' onto $D^\wedge\cong\mathbf{T}$ which factors through G; let H_1' denote its kernel. Then $\pi(H_1')^-$ does not contain $\pi(H')$ since it is contained in the kernel of the natural surjection of G onto $D^\wedge$. Thus if we let $H'''=\pi^{-1}(\pi(H_1')^-)$, then $m(H''')$ and $m(H/H''')$ are

both smaller than m. We now obtain the desired composition series for H by piecing together those for H''' and H/H''' (which exist by the induction hypothesis). ///

We say that a C^*-algebra B is *locally simply traced* if for each $P \in \operatorname{Prim} B$ there exists $P' \triangleleft B$, containing P, such that P'/P is simple and has a unique trace. As P is prime, the condition that P'/P be simple implies that P' is uniquely determined, and that $\{P\}$ is locally closed in Prim B. Furthermore P'/P determines the local representation theory of B at P, in the sense that the factor representations of B with kernel P are in natural $1-1$ correspondence with the factor representations of P'/P. In particular for a separable locally simply traced algebra the map $L \mapsto \ker L$ induces a bijection between the quasi-equivalence classes of normal representations and Prim B. (The separability hypothesis is needed to insure that the kernel of any normal representation is primitive.)

Given a system $(G, A, \mathcal{T})$ with A locally simply traced, and given $P \in \operatorname{Prim} A$, there is a natural action of G_P on the simple algebra P'/P. We say that $(G, A, \mathcal{T})$ is *locally generalized inner* if this action is generalized inner, and *locally trace preserving* if it leaves invariant the trace on P'/P, for all $P \in \operatorname{Prim} A$. Note that locally generalized inner systems are always locally trace preserving, and that (by the remark following Theorem 18) systems in which A is type I are always locally generalized inner.

LEMMA 36. *Let A be locally simply traced, and let $(G, A, \mathcal{T})$ be abelian and locally trace preserving. Let $P \in \operatorname{Prim} A$ and suppose that $sG_P \mapsto {}^sP$ is a homeomorphism of G/G_P onto* Prim A. *Then there is a non-zero densely defined G-invariant trace on A.*

Proof. As Prim A is Hausdorff, P is maximal; let π denote the canonical homomorphism of A onto A/P, and ν a non-zero trace on A/P. The hypothesis that $(G, A, \mathcal{T})$ is locally trace preserving allows us to associate to each $a \in A^+$ a function f_a: $G/G_P \to [0, \infty]$, defined by

$$f_a(\dot{s}) = \nu(\pi({}^s a)).$$

Lower semi-continuity of ν implies that f_a is lower semi-continuous, so that we can define a map μ: $A^+ \to [0, \infty]$ by

$$\mu(a) = \int_{G/G_P} f_a(\dot{s})\, d\lambda(\dot{s}).$$

Fatou's lemma together with the lower semi-continuity of ν imply that μ is lower semi-continuous, and one easily checks that μ satisfies the other criteria for a trace, and that it is G-invariant.

Since A/P is simple, ν is densely defined and faithful (in the sense that no non-zero positive element has zero trace). For any positive element a of the Pedersen ideal $\varkappa(A)$, $\pi({}^s a)$ is in $\varkappa(A/P)$ for all $s \in G$, so that f_a is everywhere finite. Let O be an open subset of G/G_P with compact closure. A simple compactness argument shows that we may find $a \in \varkappa(A)^+$ such that $f_a(\dot{s}) \geqslant 1$ for all $\dot{s} \in O$. By [15, B 18, p. 356] there exists a point of continuity $\dot{s}_0$ for f_a in O, so that in particular we can find an open subset O_0 on which f_a is bounded. Let g be a continuous, non-zero, non-negative function on G/G_P with support contained in O_0. Then we may regard g as a function on Prim A via the canonical identification of G/G_P with Prim A. Applying the multiplier corresponding to g to the element a, we obtain a positive element b of A with the property that f_b is bounded, of compact support, and $\geqslant g$. Then $\mu(b)$ is finite and non-zero. Thus μ is non-zero; furthermore its ideal of definition m_μ is G-invariant and non-zero, and so (since G acts transitively on Prim A) is dense in A. Thus μ is densely defined, and so has all the properties claimed in the lemma. ///

THEOREM 37. *Assume A is locally simply traced, and that $(G, A, \mathcal{T})$ is separable, regular, abelian, and locally generalized inner. Let H be a separable locally compact group, and π an injective continuous homomorphism of H into G whose image contains $N_\mathcal{T}$. Then $(H, A, \mathcal{T}_H)$ (the "pull-back" of $(G, A, \mathcal{T})$ along π) is EH-regular, and $C^*(H, A, \mathcal{T}_H)$ is locally simply traced.*

Proof. Let $G' = G \times (H/\pi^{-1}(N_\mathcal{T}))$ act on A by ${}^{(s,t)}a = {}^s a$, and define $\pi' \colon H \to G'$ by $\pi'(t) = (\pi(t), t\pi^{-1}(N_\mathcal{T}))$. The image of π' is easily seen to be closed, so by the open mapping theorem π' is a homeomorphism onto its image. Furthermore $(G', A, \mathcal{T})$ (we identify $N_\mathcal{T}$ with $N_\mathcal{T} \times \{1\}$) satisfies the same hypotheses as $(G, A, \mathcal{T})$, and the pull-back of $(G', A, \mathcal{T})$ along π' is equal to $(H, A, \mathcal{T}_H)$. Thus, replacing G by G' and H by $\pi'(H)$, we may assume that H is actually a closed subgroup of G containing $N_\mathcal{T}$.

Let $P \in \operatorname{Prim} C^*(H, A, \mathcal{T})$. As A is separable, $\operatorname{Res}^H_{N_\mathcal{T}} P$ lives on an H-quasi-orbit, which in turn is contained in a unique G-quasi-orbit Q. Since $(G, A, \mathcal{T})$ is regular Q is locally closed and is the G-orbit of some $I \in \operatorname{Prim} A$. Let $I_1 = \ker(Q^-)$, and $I_2 = \ker(Q^- \setminus Q)$. Then I_1 and I_2 are G-invariant. If we let $J_i = \operatorname{Ex}^H_{N_\mathcal{T}} I_i$, $i = 1, 2$, then by Proposition 11, $P \supseteq J_1$ but $P \not\supseteq J_2$; thus P is contained in the locally closed subset (Hull $J_1 \setminus$ Hull J_2) of Prim $C^*(H, A, \mathcal{T})$. In particular $\{P\}$ is locally closed in Prim $C^*(H, A, \mathcal{T})$ if and only if the corresponding primitive ideal $(P \cap J_2)/J_1$ of J_2/J_1 is locally closed in Prim J_2/J_1. Furthermore the normal representations of $C^*(H, A, \mathcal{T})$ with kernel P are in $1-1$ correspondence with the normal representations of J_2/J_1 with kernel $(P \cap J_2)/J_1$. Since, by Proposition 12, $J_2/J_1 \simeq C^*(H, I_2/I_1, (\mathcal{T}_{I_2})^{I_1})$, without loss of generality we may replace A by I_2/I_1, and thus assume that G acts transitively on Prim A. Then as $G/N_\mathcal{T}$ is abelian all points in

Prim A have the same stabilizer K_G in G; let $K=K_G\cap H$ denote the common stabilizer in H.

We consider the system $(K, A, \mathcal{J})$, and show that its C^*-algebra is locally simply traced: The K-quasi-orbits in Prim A consist of single points. Thus for any $P\in$ Prim $C^*(K, A, \mathcal{J})$, $\operatorname{Res}^K_{N_\mathcal{J}} P$ is a primitive ideal I_P of A. As Prim A is Hausdorff, I_P is maximal and so $A_P=A/I_P$ is simple, and hence (as A is locally simply traced) has a unique trace. Since, by Proposition 11, $P\supseteq \operatorname{Ex}^K_{N_\mathcal{J}} I_P$, it is sufficient to show that the quotient algebra $C^*(K, A, \mathcal{J})/\operatorname{Ex}^K_{N_\mathcal{J}} I_P$, which we may identify with $C^*(K, A_P, \mathcal{J}^{I_P})$ by means of Proposition 12, is locally simply traced. By Theorem 18 there is a reduced abelian system $(K', \mathbf{C}, \mathcal{J}')$ such that $C^*(K, A_P, \mathcal{J}^{I_P})\cong A_P\otimes_{\max} C^*(K', \mathbf{C}, \mathcal{J}')$. Since $C^*(K', \mathbf{C}, \mathcal{J}')$ is nuclear by Proposition 14, and since A_P is simple, with unique trace, results of Blackadar [4, Thm 3.3 and 3.8] imply that Prim $(K, A_P, \mathcal{J}^{I_P})$ is naturally homeomorphic to Prim $(K', \mathbf{C}, \mathcal{J}')$ and that factor traces of $C^*(K, A_P, \mathcal{J}^{I_P})$ with a given primitive ideal J as kernel (of the corresponding normal representation) are in $1-1$ correspondence with factor traces of $C^*(K', \mathbf{C}, \mathcal{J}')$ having the primitive ideal corresponding to J as kernel. (When A is type I, so that A_P is isomorphic to the algebra of compact operators on some Hilbert space, it is easy to prove these facts directly without invoking [4]). Thus as $C^*(K', \mathbf{C}, \mathcal{J}')$ is locally simply traced by Propositions 34 and 32, the same is true of $C^*(K, A_P, \mathcal{J}^{I_P})$ and hence (as remarked earlier) of $C^*(K, A, \mathcal{J})$.

Let I be a primitive ideal of A. The argument of the preceding paragraph shows that there is a reduced system $(K', \mathbf{C}, \mathcal{J}')$ and a natural homomorphism β_I of $C^*(K, A, \mathcal{J})$ onto $B=A/I\otimes_{\max} C^*(K', \mathbf{C}, \mathcal{J}')$, with kernel $\operatorname{Ex}^K_{N_\mathcal{J}} I$. If we give B the action of $(K/N_\mathcal{J})\hat{}$ arising from the natural action of $(K/N_\mathcal{J})\hat{}\cong(K'/N_\mathcal{J})\hat{}$ on the right hand factor, then it is straightforward to show that β_I is equivariant for the $(K/N_\mathcal{J})\hat{}$ actions. Thus the homeomorphism induced by β_I of Prim B onto the hull of $\operatorname{Ex}^K_{N_\mathcal{J}} I$ is $(K/N_\mathcal{J})\hat{}$-equivariant. But by the preceding paragraph, Prim B is naturally homeomorphic to Prim $(K', \mathbf{C}, \mathcal{J}')$, and it then follows easily from Proposition 34 that the $(K/N_\mathcal{J})\hat{}$ action on Prim B is transitive. On the other hand, using Proposition 11 and the fact that I is maximal we see that the hull of $\operatorname{Ex}^G_{N_\mathcal{J}}$ consists precisely of the primitive ideals of $(K, A, \mathcal{J})$ which restrict to I. Now it is readily computed that the natural action of G on $C^*(K, A, \mathcal{J})$ commutes with that of $(K/N_\mathcal{J})\hat{}$, so G permutes the $(K/N_\mathcal{J})\hat{}$ orbits on Prim $C^*(K, A, \mathcal{J})$; the fact that G acts transitively on Prim A, together with G-equivariance of $\operatorname{Res}^K_{N_\mathcal{J}}$ and the fact observed in the preceding paragraph that every $P\in$ Prim $(K, A, \mathcal{J})$ restricts to some maximal ideal of A, now imply that $G''=G\times(K/N_\mathcal{J})\hat{}$ acts transitively on Prim $(K, A, \mathcal{J})$. This idea of throwing in the action of $(K/N_\mathcal{J})\hat{}$ to produce a transitive action is due to Pukanszky; it is of great importance for the following.

Let $B = C^*(K, A, \mathcal{T})$, and let K'' denote the common stabilizer in G'' of points in Prim B. Fixing $I \in \text{Prim } B$, we see from the above, together with standard Baire category arguments (using separability of G'') that we may identify G''/K'' with Prim B via the map $s''K'' \mapsto {}^{s''}I$. Regarding now H as a subgroup of G'' by identifying it with $H \times \{1\}$, we see easily from the H-equivariance of $\text{Res}^K_{N_\mathcal{T}}$ that $H \cap K'' = K$. In particular the system $(H, B, \mathcal{T}^K)$ is essentially free. Let π denote the natural injection of H/K into G''/K'', and choose a series $H_0 = K \subseteq H_1 \subseteq \dots \subseteq H_n = H$ of closed subgroups of H such that the H_i/K have the properties of Lemma 35 with respect to π.

By Proposition 1, $C^*(H, B, \mathcal{T}^K)$ is naturally isomorphic to $C^*(H, A, \mathcal{T})$. We prove, by induction on n, that there is a unique factor trace at each primitive ideal of $C^*(H, B, \mathcal{T}^K)$, and that Ind^H_K gives a surjective map of Prim B onto Prim $(H, B, \mathcal{T}^K)$. The latter fact implies EH-regularity of $(H, A, \mathcal{T})$; by Propositions 9 and 11 it also implies that Prim $(H, B, \mathcal{T}^K)$ is homeomorphic to the space Q_H of H-quasi-orbits of B. Since it is easily seen that, via the above homeomorphism of Prim B with G''/K'', Q_H is identified with the coset space $G''/(HK'')^-$, this will show in particular that every primitive ideal of $(H, B, \mathcal{T}^K)$ is maximal, implying that $C^*(H, B, \mathcal{T}^K)$ is locally simply traced and completing the proof of the theorem.

The case $n = 0$ (i.e. $H = K$) was proved above, so we assume $n > 0$ and that the result holds for series of length $< n$. Let $C = C^*(H_{n-1}, B, \mathcal{T}^K)$. By the induction hypothesis, C is locally simply traced and Prim C is homeomorphic, via the map (by restriction) taking each primitive ideal to its H_{n-1}-quasi-orbit in B, to $G''/(H_{n-1}K'')^-$. Furthermore H-equivariance of $\text{Res}^{H_{n-1}}_K$ implies that the action of H on Prim C corresponds to left translation by (the image of) H in $G''/(H_{n-1}K'')^-$. By our choice of the H_i, $(H_{n-1}K'')^- \cap H = H_{n-1}$, so that the stabilizer in H of any point in Prim C is just H_{n-1}. Thus the iterated system $(H, C, \mathcal{T}^{H_{n-1}})$ (whose C^*-algebra is isomorphic to that of $(H, A, \mathcal{T})$, by Proposition 1) is essentially free; furthermore it is either regular, or discrete (in the sense that H/H_{n-1} is discrete), according to the two possibilities of Lemma 35. In either case we easily deduce from Theorem 24 that $\text{Ind}^H_{H_{n-1}}$ takes Prim C onto Prim $(H, B, \mathcal{T}^K)$, and thus by "Induction in Stages" that Ind^H_K takes Prim B onto Prim $(H, B, \mathcal{T}^K)$.

It remains only to prove that there is a unique factor trace at each primitive ideal of $C^*(H, C, \mathcal{T}^{H_{n-1}})$. In the case that $(H, C, \mathcal{T}^{H_{n-1}})$ is regular, this follows from Proposition 29 and the induction hypothesis, so we may assume that H/H_{n-1} is discrete. By Lemma 36 there is a G-invariant trace on A, so we may construct the induced trace ν on $C \cong C^*(H_{n-1}, A, \mathcal{T})$. Since H_{n-1} is a normal subgroup of G'' whose action on B is the restriction of that of G'', there is a natural action of G'' on $C \cong C^*(H_{n-1}, B, \mathcal{T}^K)$, and it can be readily verified that this action preserves the bitrace corresponding to ν (which was constructed in the proof of Proposition 25)—this is most easily done by checking the G and $(K/N_{\mathcal{J}})^{\wedge}$ actions separately. It follows that ν is G''-invariant.

Now let $\mathcal{N}$ denote the space of quasi-equivalence classes of normal representations of C. Since by [35] and [19] $\mathcal{N}$ and Prim C are both standard Borel spaces, and the natural map $\mathcal{N}\to$Prim C taking any normal representation to its kernel is Borel, the fact that C is locally simply traced implies that $\mathcal{N}$ and Prim C are isomorphic as Borel spaces. Thus by [15, 8.8.2] there exist a measure μ on Prim C, and traces ν_J lifted from the simple quotients C/J for each $J \in \text{Prim } C$, such that

$$\nu(c) = \int_{\text{Prim } C} \nu_J(c)\, d\mu(J)$$

for all $c \in C^+$. Since K'' stabilizes each point in Prim C, the uniqueness part of [15, 8.8.2] is easily seen to imply that, for any $t \in K''$, ν_J is t-invariant for almost all J, and hence for at least one J; but transitivity of the G'' action on Prim C (which follows from G'' equivariance of $\text{Res}_K^{H_{n-1}}$) then implies that all ν_J are t-invariant. Now since *any* trace on C decomposes as a direct integral of scalar multiples of the ν_J (because C is locally simply traced), it must be K'' invariant. Thus any H-invariant trace ν' is also HK''-invariant, but since the stabilizer of any trace is closed (cf. the argument of Kleppner that was given in the proof of Proposition 32) it follows that ν' is also $(HK'')^-$ invariant. Thus if ν' lives on an H-quasi-orbit—which is just an $(HK'')^-$ orbit—the measure μ' of its direct integral decomposition must be $(HK'')^-$ quasi-invariant, and so we may assume that it is Haar measure on $(HK'')^-/(H_{n-1}K'')^-$; but then the corresponding ν'_J are all translates of each other (i.e. $\nu'_{tJ} = \nu'_J$ for all $t \in (HK'')^-$). It follows easily that the H-invariant traces on C living on H-quasi-orbits are unique (up to a scalar), hence as by Proposition 26 every factor trace of $C^*(H, C, \mathcal{J}^{H_{n-1}})$ is induced from such a trace, they too are unique, provided they exist. However we can mimic the argument used above in constructing ν to produce a G''-invariant trace on $C^*(H, A, \mathcal{J}) \cong C^*(H, C, \mathcal{J}^{H_{n-1}})$, and then by direct integral theory there exists a factor trace at some primitive ideal of $C^*(H, C, \mathcal{J}^{H_{n-1}})$. Factor traces at all primitive ideals are then obtained by translation, under the transitive G'' action on $\text{Prim}(H, C, \mathcal{J}^{H_{n-1}})$. ///

COROLLARY 38. *Let G and H be separable locally compact groups, π: $H \to G$ a continuous injective homomorphism. Suppose $\pi(H)$ contains a type I regularly embedded normal subgroup K of G such that G/K is abelian. Then $C^*(H)$ is locally simply traced.*

Proof. This follows easily from the above theorem, once we use the corollary to Proposition 1 to construct systems $(G, C^*(K), \mathcal{J}^K)$, $(H, C^*(K), \mathcal{J}^K)$ whose C^*-algebras are isomorphic to those of G and H. (Note that since $C^*(K)$ is type I the system $(G, C^*(K), \mathcal{T}^K)$ is automatically locally generalized inner.) ///

COROLLARY 39. ([49, Thm. 1]; [46, Cor. 3].) *Let H be a connected locally compact group. Then $C^*(H)$ is locally simply traced.*

Proof. A result of Moore [45, Prop. 2.2], combined with the fact that if H' is a quotient group of H, then $C^*(H')$ is a quotient algebra of $C^*(H)$, allow us to reduce to the case that H is a simply connected Lie group. We now repeat an argument of Pukanszky [49, pp. 84–86]: Represent the Lie algebra $\mathbf{h}$ of H faithfully on a finite dimensional vector space, let $\mathbf{g}$ be its algebraic hull on that space, and let $\mathbf{k} = [\mathbf{g}, \mathbf{g}]$. By [9, Thm. 15, p. 177 and Thm. 13, p. 173] $\mathbf{k}$ is algebraic and contained in $\mathbf{h}$. Furthermore, if G denotes the simply connected Lie group corresponding to $\mathbf{g}$, then the subgroups $H' = \exp \mathbf{h}$ and $K = \exp \mathbf{k}$ corresponding to $\mathbf{h}$ and $\mathbf{k}$ are closed and simply connected, so in particular H' is isomorphic to H. By [48, Thm., p. 379] K is regularly embedded in G, and by [14, Prop. 2.1, p. 425] it is type I, so Corollary 38 applies. ///

We conclude with several remarks concerning topics related to the above, which will be pursued in subsequent papers.

1. Theorem 37 can be generalized considerably. In particular one can replace the hypothesis that $G/N_{\mathcal{T}}$ is abelian with the hypothesis that it is amenable, provided that one assumes that stabilizers of points in Prim A are abelian (mod $N_{\mathcal{T}}$) and normal in G, and that H is closed and normal in G. It appears that yet more general results, with weakened hypotheses on the stabilizers, should hold.

2. It would be of great interest to classify the simple subquotients (i.e. the simple algebras of the form P'/P, where P is a primitive ideal and P' the minimal ideal properly containing it) of the algebras $C^*(H, A, \mathcal{T})$ arising from systems which satisfy the hypotheses of Theorem 37, and for which in addition A is type I. By the theorem, these simple algebras always carry a unique trace. Furthermore it is easy to see from the proof of Theorem 37 that they can be described as the C^*-algebras of essentially free systems $(H', A', \mathcal{T}')$ for which Prim A' is homeomorphic to some locally compact abelian group G', and such that $H'/N_{\mathcal{T}'}$ can be regarded as a dense subgroup of G' in such a way that the action of $H'/N_{\mathcal{T}'}$ on Prim A' corresponds to the action by translation on G'. The "canonical examples" of such algebras are covariance algebras $C^*(H, C_\infty(G))$, where H and G are locally compact abelian groups, and the action of H on $C_\infty(G)$ comes from an injective homomorphism of H onto a dense subgroup of G. There are several interesting examples of these algebras: In [56] Takesaki shows that when H is taken to be a direct sum of a sequence of cyclic

groups, and G the corresponding direct product, one obtains Glimm's UHF algebras. I have been able to show that the simple uniquely traced algebras constructed in [6] are also of the form $C^*(H, C_\infty(G))$, where in this case one takes $G = \mathbf{T}$, and H a dense torsion subgroup of $\mathbf{T}$ (with the discrete topology). (The classification of these algebras given in [6] then turns out to correspond to the classification of the dense torsion subgroups of the circle.)

I have also been able to show that any simple subquotient of the C^*-algebra of a connected group is either finite dimensional (in which case of course the primitive ideal is maximal), or else *stable* in the sense that it is isomorphic to its tensor product with the algebra of compact operators on a separable infinite dimensional Hilbert space. (This "explains" the fact that connected groups never have II_1 factor representations, since stable algebras never do.) In view of [59], this shows that the problem of classifying these algebras reduces to the problem of classifying their Morita equivalence classes.

3. In the course of proving Theorem 37 we essentially obtained a description of Prim $(H, A, \mathcal{J})$, which in the special case of the system associated to a simply connected Lie group turns out to be somewhat simpler and more natural than that obtained by Pukanszky in [49, Section 1]. (It is fairly easy, using Proposition 34, to reconcile these two descriptions.)

4. By refining the arguments of Proposition 25, it is possible to obtain a character formula for the systems of Theorem 37 which generalizes that obtained by Pukanszky in [49, Thm. 2].

5. The deduction of Corollary 39 from Corollary 38 given above suggests that it should be possible, using recent results on algebraic p-adic groups, to derive an analogue of Corollary 39 for the C^*-algebras of (not necessarily algebraic) p-adic groups. I have been able to do this, but using a definition of p-adic groups which is probably unnecessarily restrictive.

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