

## THE LOCAL THETA CORRESPONDENCE OF IRREDUCIBLE TYPE 2 DUAL REDUCTIVE PAIRS

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(Received May 9, 1994, revised April 6, 1995)

**Abstract.** The explicit Howe duality correspondence is partially solved in the case of irreducible type 2 dual reductive pairs defined over a non-Archimedean local field.

**Introduction.** Let  $(GL_n, GL_m)$  be an irreducible type 2 dual reductive pair defined over a non-Archimedean local field  $F$ . The Weil representation  $\omega_{n,m}$  of  $GL_n(F) \times GL_m(F)$  on the Schwartz-Bruhat space  $\mathcal{S}(M_{n,m}(F))$  is given by

$$\omega_{n,m}(h, g)f(x) = |\det h|^{-m/2} |\det g|^{n/2} f(h^{-1}xg) \quad (h \in GL_n(F), g \in GL_m(F)).$$

Then a problem on the (explicit) Howe correspondence for  $(GL_n, GL_m)$  is stated as follows. For a given irreducible admissible representation  $\sigma$  of  $GL_n(F)$ , determine an irreducible admissible representation  $\sigma'$  of  $GL_m(F)$  such that  $\text{Hom}_{GL_n(F) \times GL_m(F)}(\omega_{n,m}, \sigma \otimes \sigma') \neq 0$ . The purpose of this paper is to study this problem in the case where  $m = n + 1$  and  $\sigma$  is generic.

Our starting point is a global theta series lifting of a cusp form on the adèle group  $GL_n(\mathcal{A})$ . For a cusp form  $\varphi$  on  $GL_n(\mathcal{A})$  and a Schwartz-Bruhat function  $f \in \mathcal{S}(M_{n,m}(\mathcal{A}))$ , we define a theta series lifting  $\varphi_f^s$ , where  $s$  is a complex parameter with  $\text{Re}(s) \ll 0$ . This  $\varphi_f^s$  is an automorphic form on  $GL_m(\mathcal{A})$ . In Section 1, we calculate a Whittaker function  $W_{\varphi_f^s}$  of  $\varphi_f^s$  and prove that  $W_{\varphi_f^s}$  is identically zero if  $m \neq n, n + 1$ . In the case where  $m = n$  or  $m = n + 1$ , the function  $W_{\varphi_f^s}$  is represented by a convolution of the Whittaker function  $W_\varphi$  of  $\varphi$  and a certain function  $\Phi_m(f)$  related to  $f$ . More precisely, we have a formula of the form

$$W_{\varphi_f^s}(g) = \int_{U_n(\mathcal{A}) \backslash GL_n(\mathcal{A})} W_\varphi(h) |\det h|_A^s \Phi_m(\omega_{n,m}(g)f)(h) dh, \quad (m = n, n + 1).$$

On the basis of this formula, we can define a local theta series lifting of a local Whittaker function. This is the reason why we study the Howe correspondence in the case where  $m = n + 1$  and  $\sigma$  is generic. The case  $m = n$  will be investigated in another paper [17].

We state the results of this paper. Let  $\sigma$  be an irreducible generic representation of  $GL_n(F)$ . By using a local analogue of the formula mentioned above, one can construct

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Partly supported by the Grants-in-Aid for Encouragement of Young Scientists, The Ministry of Education, Science and Culture, Japan.

1991 *Mathematics Subject Classification*. Primary 11F70; Secondary 11F27, 22E50.

a local theta series lifting  $\tilde{\sigma}$  of  $\sigma$ . This  $\tilde{\sigma}$  is an admissible representation of  $GL_{n+1}(F)$  realized in the space of Whittaker functions and satisfies the following:

(0.1)  $\text{Hom}_{GL_n(F) \times GL_{n+1}(F)}(\omega_{n,n+1}, \sigma^\vee \otimes \tilde{\sigma}) \neq 0$ , where  $\sigma^\vee$  denotes the contragredient representation of  $\sigma$ .

To describe the properties of  $\tilde{\sigma}$ , we denote by  $\sigma_1$  the normalized induced representation  $\text{Ind}_{Q_{n+1}(F)}^{GL_{n+1}(F)} \sigma \otimes 1$  of  $GL_{n+1}(F)$ , where  $Q_{n+1}(F)$  denotes the standard upper triangular parabolic subgroup of  $GL_{n+1}(F)$  with Levi factor  $GL_n(F) \times GL_1(F)$ . In this introduction, we assume  $\sigma_1$  to be irreducible for simplicity. (If  $\sigma_1$  is not irreducible, we must modify the definition of  $\sigma_1$  as will be mentioned in Section 2.) Then we show the following:

(0.2)  $\sigma_1$  is a unique irreducible subrepresentation of  $\tilde{\sigma}$ . Furthermore, the quotient representation  $\tilde{\sigma}/\sigma_1$  has no nonzero vectors fixed by the closed subgroup

$$\left\{ \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \middle| k \in GL_n(\mathcal{O}) \right\},$$

where  $GL_n(\mathcal{O})$  is the maximal compact subgroup of  $GL_n(F)$  consisting of integral matrices.

(0.3)  $\tilde{\sigma}$  is of Whittaker type in the sense of Jacquet, Piatetski-Shapiro and Shalika [8, (2.1)].

By (0.3), one can define the gamma factor  $\gamma(s, \tilde{\sigma} \times \tau, \psi)$  (cf. [8, (3.1)]) for each irreducible generic representation  $\tau$  of  $GL_m(F)$ . Then (0.2) implies that

$$\gamma(s, \tilde{\sigma} \times \tau, \psi) = \gamma(s, \sigma_1 \times \tau, \psi)$$

for all irreducible generic representations  $\tau$  of  $GL_m(F)$ . In light of these results, one can expect that  $\tilde{\sigma} = \sigma_1$  for any generic  $\sigma$ . If  $\sigma$  is a generic spherical representation, we really have  $\tilde{\sigma} = \sigma_1$ .

Prasad [14, (4.6.5)] stated a conjectural form of an irreducible admissible representation  $\sigma'$  of  $GL_{n+1}(F)$  corresponding to  $\sigma$  by the Howe duality. Since  $\sigma_1 \neq \sigma'$ , this conjectural form  $\sigma'$  is not consistent with the Howe correspondence if  $\sigma$  is a generic spherical representation.

NOTATION. For an associative ring  $R$  with the identity element, we denote by  $R^\times$  the group of all invertible elements of  $R$  and by  $M_{n,m}(R)$  the set of all  $n \times m$  matrices with entries in  $R$ . If  $n = m$ , we write  $M_n(R)$  for  $M_{n,n}(R)$ . For  $A \in M_{n,m}(R)$ ,  ${}^tA$  stands for its transpose. For  $A \in M_n(R)$ ,  $\det A$  stands for its determinant. The identity matrix in  $M_n(R)$  is denoted by  $1_n$ .

When a base field  $F$  is given, we set  $G_n = GL(n, F)$ . If  $m < n$ , we will regard  $G_m$  as a subgroup of  $G_n$  by the embedding

$$g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1_{n-m} \end{pmatrix}.$$

We define algebraic subgroups of  $G_n$  as

$$\begin{aligned} &B_n \text{ the set of upper triangular matrices,} \\ &U_n \text{ the unipotent radical of } B_n, \\ &T_n \text{ the set of diagonal matrices,} \\ &Z_n \text{ the center of } G_n, \\ &P_n = \left\{ \begin{pmatrix} g & u \\ 0 & 1 \end{pmatrix}, g \in G_{n-1}, u \in M_{n-1,1}(F) \right\}, \\ &Q_n = Z_n P_n. \end{aligned}$$

The Weyl group of  $G_n$  will be identified with the symmetric group  $S_n$  of degree  $n$ .

If  $G$  is a locally compact abelian group, then  $\mathcal{S}(G)$  denotes the space of Schwartz-Bruhat functions on  $G$ .

**1. The global theta lifting.** In this section, let  $k$  be a global field and  $\mathcal{A}$  the adèle ring of  $k$ . For a  $k$ -subgroup  $G$  of  $G_n = GL(n, k)$ ,  $G(\mathcal{A})$  denotes the corresponding adèle group. We fix a nontrivial additive character  $\psi$  of  $k \setminus \mathcal{A}$  and define the character  $\psi_n$  of  $U_n(\mathcal{A})$  by

$$\psi_n(u) = \psi(u_{12} + u_{23} + \cdots + u_{n-1n}) \quad (u = (u_{ij}) \in U_n(\mathcal{A})).$$

The Weil representation  $(\omega_{n,m}, \mathcal{S}(M_{n,m}(\mathcal{A})))$  of  $G_n(\mathcal{A}) \times G_m(\mathcal{A})$  is defined as follows: for  $f \in \mathcal{S}(M_{n,m}(\mathcal{A}))$ ,  $h \in G_n(\mathcal{A})$  and  $g \in G_m(\mathcal{A})$ ,

$$\omega_{n,m}(h, g)f(x) = |\det h|_{\mathcal{A}}^{-m/2} |\det g|_{\mathcal{A}}^{n/2} f(h^{-1}xg).$$

Let  $\mu$  be a character of  $Z_n \setminus Z_n(\mathcal{A})$ . For  $f \in \mathcal{S}(M_{n,m}(\mathcal{A}))$  and  $s \in \mathbb{C}$ , we define a modified theta series  $\theta(s, \mu, f)$  as

$$\theta(s, \mu, f) = \int_{Z_n \setminus Z_n(\mathcal{A})} \mu(z) |\det z|_{\mathcal{A}}^{s-m/2} \sum_{\substack{x \in M_{n,m}(k) \\ x \neq 0}} f(z^{-1}x) dz.$$

From [4, Lemmas 11.5 and 11.6], it follows that the integral on the right-hand side is absolutely convergent for  $\text{Re}(s) < -m/2$  and the function  $(h, g) \mapsto \theta(s, \mu, \omega_{n,m}(h, g)f)$  is slowly increasing on  $(G_n \setminus G_n(\mathcal{A})) \times (G_m \setminus G_m(\mathcal{A}))$ . Let  $\varphi$  be a cusp form on  $G_n(\mathcal{A})$  satisfying  $\varphi(zg) = \mu(z)\varphi(g)$  for any  $z \in Z_n(\mathcal{A})$ . Then we define a modified theta lifting  $\varphi_f^s$  of  $\varphi$  by

$$\varphi_f^s(g) = \int_{G_n \setminus G_n(\mathcal{A})} \varphi(h) |\det h|_{\mathcal{A}}^s \sum_{\substack{x \in M_{n,m}(k) \\ x \neq 0}} \omega_{n,m}(h, g)f(x) dh$$

$$= \int_{Z_n(\mathcal{A})G_n \setminus G_n(\mathcal{A})} \varphi(h) |\det h|_{\mathcal{A}}^s \theta(s, \mu, \omega_{n,m}(h, g) f) dh .$$

Since  $\varphi(h)$  is rapidly decreasing on  $Z_n(\mathcal{A})G_n \setminus G_n(\mathcal{A})$ , this integral is absolutely convergent for  $\text{Re}(s) < -m/2$ , and hence  $\varphi_f^s$  defines an automorphic form on  $G_m(\mathcal{A})$ . The purpose of this section is to calculate a Whittaker function of  $\varphi_f^s$ . Namely we compute the integral

$$W_{\varphi_f^s}(g) = \int_{U_m \setminus U_m(\mathcal{A})} \psi_m(u)^{-1} \varphi_f^s(ug) du .$$

We set

$$W_{\varphi}(h) = \int_{U_n \setminus U_n(\mathcal{A})} \psi_n(u)^{-1} \varphi(uh) du .$$

If  $m \geq n$ , we define the function  $\Phi_m(f)$  on  $G_n(\mathcal{A})$  by

$$\Phi_m(f)(h) = \int_{U_m(\mathcal{A})} \psi_m(u)^{-1} \omega_{n,m}(h, u) f(\varepsilon_{n,m}) du ,$$

where we put  $\varepsilon_{n,m} = (1_n, 0) \in M_{n,m}(k)$ .

**PROPOSITION 1.** *Let  $\varphi$  be a cusp form on  $G_n(\mathcal{A})$ ,  $f \in \mathcal{S}(M_{n,m}(\mathcal{A}))$  and  $s \in \mathbb{C}$  with  $\text{Re}(s) < -m/2$ .*

- (1) *If  $m \leq n - 1$  or  $n + 2 \leq m$ , then  $W_{\varphi_f^s}$  is identically zero.*
- (2) *If  $m = n$ , then*

$$W_{\varphi_f^s}(g) = \int_{U_n(\mathcal{A}) \setminus G_n(\mathcal{A})} W_{\varphi}(h) |\det h|_{\mathcal{A}}^s \Phi_n(\omega_{n,n}(g) f)(h) dh ,$$

*provided that the integral on the right-hand side is absolutely convergent.*

- (3) *If  $m = n + 1$ , then*

$$W_{\varphi_f^s}(g) = \int_{U_n(\mathcal{A}) \setminus G_n(\mathcal{A})} W_{\varphi}(h) |\det h|_{\mathcal{A}}^s \Phi_{n+1}(\omega_{n,n+1}(g) f)(h) dh .$$

*Here the integral on the right-hand side always converges absolutely.*

**PROOF.** For a matrix  $x \in M_{n,m}(k)$ , let  $x_j$  denote the  $j$ -th column vector of  $x$ . We write  $(x_1, x_2, \dots, x_m)$  for  $x$ . We define the subset  $Y_j$  of  $M_{n,m}(k)$  as

$$Y_0 = \{x \in M_{n,m}(k) - \{0\} \mid x_1 = 0\}$$

$$Y_j = \{x \in M_{n,m}(k) \mid \text{rank}(x_1, \dots, x_j) = \text{rank}(x_1, \dots, x_j, x_{j+1}) = j\} ,$$

for  $1 \leq j \leq \min(n, m - 1)$ . If  $m \leq n$ , we also set

$$Y_m = \{x \in M_{n,m}(k) \mid \text{rank}(x_1, \dots, x_m) = m\} .$$

Then  $M_{n,m}(k) - \{0\}$  is a disjoint union of  $Y_j$ ,  $0 \leq j \leq \min(n, m)$ , and each  $Y_j$  is left  $G_n$ - and right  $U_m$ -invariant. Let  $Y_j$  be a complete set of representatives for  $Y_j/U_m$ . If  $0 \leq j \leq \min(n, m-1)$ , then we can take  $Y_j$  so that each  $x \in Y_j$  has  $x_{j+1} = 0$ . In the following, for  $x \in M_{n,m}(k)$ ,  $Z(x, U_m)$  stands for the stabilizer of  $x$  in  $U_m$ . Then  $W_{\varphi_j^2}(g)$  equals

$$\begin{aligned} & \int_{G_n \setminus G_n(\mathcal{A})} \varphi(h) |\det h|_{\mathcal{A}}^s \int_{U_m \setminus U_m(\mathcal{A})} \psi_m(u)^{-1} \sum_{\substack{x \in M_{n,m}(k) \\ x \neq 0}} \omega_{n,m}(h, ug) f(x) dudh \\ &= \int_{G_n \setminus G_n(\mathcal{A})} \varphi(h) |\det h|_{\mathcal{A}}^s \int_{U_m \setminus U_m(\mathcal{A})} \psi_m(u)^{-1} \sum_{j=0}^{\min(n,m)} \sum_{x \in Y_j} \sum_{\gamma \in Z(x, U_m) \setminus U_m} \omega(h, ug) f(x\gamma) dudh \\ &= \sum_{j=0}^{\min(n,m)} I_j^m, \end{aligned}$$

where we set

$$\begin{aligned} I_j^m &= \int_{G_n \setminus G_n(\mathcal{A})} \varphi(h) |\det h|_{\mathcal{A}}^s \sum_{x \in Y_j} \left( \int_{Z(x, U_m) \setminus Z(x, U_m(\mathcal{A}))} \psi_m(u') du' \right) \\ &\quad \times \int_{Z(x, U_m(\mathcal{A})) \setminus U_m(\mathcal{A})} \psi_m(u)^{-1} \omega(h, ug) f(x) dudh. \end{aligned}$$

By our choice of representatives,  $\psi_m$  is nontrivial on  $Z(x, U_m(\mathcal{A}))$  if  $x \in Y_j$  and  $0 \leq j \leq \min(n, m-2)$ . This implies  $I_j^m = 0$  for  $0 \leq j \leq \min(n, m-2)$ . Therefore we have

$$W_{\varphi_j^2}(g) = \begin{cases} I_{m-1}^m + I_m^m & \text{if } m \leq n \\ I_{m-1}^m & \text{if } m = n + 1 \\ 0 & \text{if } m \geq n + 2. \end{cases}$$

We consider the case  $m \leq n$ . We regard  $U_m$  as a subgroup of  $G_n$ . Let  $M_n^m$  be the stabilizer of the matrix  ${}^t\varepsilon_{m,n} = {}^t(1_m, 0) \in M_{n,m}(k)$  in  $G_n$ , i.e.

$$M_n^m = \left\{ \left( \begin{array}{cc} 1_m & u \\ 0 & g \end{array} \right) \middle| g \in G_{n-m}, u \in M_{m,n-m}(k) \right\}.$$

Since  $Y_m = G_n {}^t\varepsilon_{m,n}$ ,  $I_m^m$  equals

$$\begin{aligned} & \int_{G_n \setminus G_n(\mathcal{A})} \varphi(h) |\det h|_{\mathcal{A}}^s \int_{U_m \setminus U_m(\mathcal{A})} \psi_m(u)^{-1} \sum_{\gamma \in M_n^m \setminus G_n} \omega_{n,m}(rh, ug) f({}^t\varepsilon_{m,n}) dudh \\ &= \int_{M_n^m \setminus G_n(\mathcal{A})} \varphi(h) |\det h|_{\mathcal{A}}^s \int_{U_m \setminus U_m(\mathcal{A})} \psi_m(u)^{-1} \omega_{n,m}(u^{-1}h, g) f({}^t\varepsilon_{m,n}) dudh \\ &= \int_{M_n^m(\mathcal{A}) \setminus G_n(\mathcal{A})} \left( \int_{M_n^m \setminus M_n^m(\mathcal{A})} \varphi(h_0 h) |\det h_0 h|_{\mathcal{A}}^s dh_0 \right) \end{aligned}$$

$$\times \int_{U_m \setminus U_m(\mathcal{A})} \psi_m(u)^{-1} \omega_{n,m}(u^{-1}h, g) f({}^t \varepsilon_{m,n}) dudh .$$

If  $m < n$ , the cuspidality condition of  $\varphi$  implies  $I_m^m = 0$ . If  $m = n$ , by formal computation, we have

$$\begin{aligned} I_m^m &= \int_{G_n(\mathcal{A})} \left( \int_{U_n \setminus U_n(\mathcal{A})} \psi_n(u)^{-1} \varphi(uh) du \right) |\det h|_{\mathcal{A}}^s \omega_{n,n}(h, g) f(1_n) dh \\ &= \int_{U_n(\mathcal{A}) \setminus G_n(\mathcal{A})} W_\varphi(h) |\det h|_{\mathcal{A}}^s \Phi_n(\omega_{n,n}(g)f)(h) dh . \end{aligned}$$

Next, let  $M_n^{m-1}$  denote the stabilizer  ${}^t \varepsilon_{m-1,n} = {}^t(1_{m-1}, 0) \in M_{n,m-1}(k)$  in  $G_n$ . Then we have

$$Y_{m-1} = \{ \gamma^{-1}({}^t \varepsilon_{m-1,n}, x_m) \mid \gamma \in M_n^{m-1} \setminus G_n, \text{rank}({}^t \varepsilon_{m-1,n}, x_m) = m-1 \} .$$

Therefore,  $I_{m-1}^m$  equals

$$\begin{aligned} & \int_{G_n \setminus G_n(\mathcal{A})} \varphi(h) |\det h|_{\mathcal{A}}^s \int_{U_m \setminus U_m(\mathcal{A})} \psi_m(u)^{-1} \\ & \quad + \sum_{\gamma \in M_n^{m-1} \setminus G_n} \sum_{\substack{x_m \in M_{n,1}(k) \\ \text{rank}({}^t \varepsilon_{m-1,n}, x_m) = m-1}} \omega_{n,m}(\gamma h, ug) f({}^t \varepsilon_{m-1,n}, x_m) dudh \\ &= \int_{M_n^{m-1}(\mathcal{A}) \setminus G_n(\mathcal{A})} \left( \int_{M_n^{m-1} \setminus M_n^{m-1}(\mathcal{A})} \varphi(h_0 h) |\det h_0 h|_{\mathcal{A}}^s dh_0 \right) \int_{U_m \setminus U_m(\mathcal{A})} \psi_m(u)^{-1} \\ & \quad \times \sum_{\substack{x_m \in M_{n,1}(k) \\ \text{rank}({}^t \varepsilon_{m-1,n}, x_m) = m-1}} \omega_{n,m}(h, ug) f({}^t \varepsilon_{m-1,n}, x_m) dudh . \end{aligned}$$

The cuspidality of  $\varphi$  implies  $I_{m-1}^m = 0$  for all  $m \leq n$ . This completes the proof of the statements (1) and (2).

We consider the case  $m = n + 1$ . By calculation similar to that above, we have

$$I_{m-1}^m = \int_{U_n(\mathcal{A}) \setminus G_n(\mathcal{A})} W_\varphi(h) |\det h|_{\mathcal{A}}^s \Phi_{n+1}(\omega_{n,n+1}(g)f)(h) dh .$$

We prove that the integral on the right-hand side converges absolutely. It is sufficient to show that the integral

$$\int_{T_n(\mathcal{A})} W_\varphi(t) |\det t|_{\mathcal{A}}^s \Phi_{n+1}(f)(t) \delta_n(t)^{-1} dt$$

converges absolutely, where  $\delta_n$  denotes the modular character of  $B_n(\mathcal{A})$ . By definition, for  $t = \text{diag}(a_1, \dots, a_n) \in T_n(\mathcal{A})$ ,

$$\begin{aligned} \Phi_{n+1}(f)(t) &= |\det t|^{-(n+1)/2} \int_{U_{n+1} \setminus U_{n+1}(\mathcal{A})} \varphi(u_{12} + \cdots + u_{nn+1})^{-1} \\ &\quad \times f \left( \begin{pmatrix} a_1^{-1} & a_1^{-1}u_{12} & \cdots & a_1^{-1}u_{1n} & a_1^{-1}u_{1n+1} \\ 0 & a_2^{-1} & \cdots & a_2^{-1}u_{2n} & a_2^{-1}u_{2n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_n^{-1} & a_n^{-1}u_{nn+1} \end{pmatrix} \right) du_{12} \cdots + du_{nn+1}. \end{aligned}$$

This integral is regarded as a partial Fourier transform of  $f$ . Hence there exists a function  $\Phi \in \mathcal{S}(\mathcal{A}^n \oplus \mathcal{A}^n)$  such that

$$|\Phi_{n+1}(f)(t)| \leq \Phi(t, t^{-1}).$$

Furthermore  $W_\varphi(t)$  is majorized by a gauge function  $\xi$  on  $G_n(\mathcal{A})$  (cf. [6, Proposition 2.3.6, Lemma 8.3.3 and (12.1)]), i.e.

$$|W_\varphi(t)| \leq \xi(t), \quad (t \in T(\mathcal{A})).$$

Then it is easy to see that the integral

$$\int_{T_n(\mathcal{A})} \xi(t) |\det t|_A^s \Phi(t, t^{-1}) \delta_n(t)^{-1} dt$$

is convergent. This completes the proof.

**2. The local theta lifting.** From now on, we fix a local non-Archimedean field  $F$ . In this section, we define a local theta lifting from the set of generic representations of  $G_n = GL(n, F)$  to the set of smooth representations of  $G_{n+1}$  by using an integral analogous to that in Proposition 1 (3).

First we define some notation and recall certain notions. Let  $\mathcal{O}$  denote the ring of integers of  $F$ ,  $\varpi$  a prime element of  $F$  and  $q$  the order of  $\mathcal{O}/\varpi\mathcal{O}$ . The absolute valuation of  $F$  is denoted by  $|\cdot|_F$ , which is normalized as  $|\varpi|_F = q^{-1}$ . We fix, once and for all, a nontrivial additive character  $\psi$  of  $F$  with the conductor  $\mathcal{O}$ . We denote by  $K_n$  the maximal compact subgroup  $GL(n, \mathcal{O})$  of  $G_n$  and by  $\mathcal{H}_n$  the convolution algebra consisting of all locally constant and compactly supported functions on  $G_n$ . The character  $\psi_n$  of  $U_n$  is defined to be

$$\psi_n(u) = \psi(u_{12} + u_{23} + \cdots + u_{n-1n})$$

for  $u = (u_{ij}) \in U_n$ .

Let  $\mathcal{W}(\psi_n)$  be the space of all locally constant functions  $W$  on  $G_n$  satisfying  $W(ug) = \psi_n(u)W(g)$  for any  $u \in U_n$  and  $g \in G_n$ , i.e.  $\mathcal{W}(\psi_n) = \text{Ind}_{U_n}^{G_n} \psi_n$ . Then  $g \in G_n$  acts on  $\mathcal{W}(\psi_n)$  by right translation:  $\rho(g)W(g_0) = W(g_0g)$ . An admissible representation  $\sigma$  of  $G_n$  is said to be of Whittaker type if  $\sigma$  is finitely generated and  $\dim \text{Hom}_{G_n}(\sigma, \mathcal{W}(\psi_n)) = 1$ . If  $\sigma$  is of Whittaker type, we denote by  $\mathcal{W}(\sigma, \psi_n)$  the image of the unique (up to constant)

nonzero  $G_n$ -morphism from  $\sigma$  to  $\mathcal{W}(\psi_n)$ . Note that the representation  $(\rho, \mathcal{W}(\sigma, \psi_n))$  need not be isomorphic to  $\sigma$ . If  $\sigma$  is of Whittaker type and irreducible, then  $\sigma$  is said to be generic.

A classification of irreducible generic representations of  $G_n$  is known by Bernstein and Zelevinsky [18]. In the following, for a given smooth representation  $\pi$  of  $G_n$  and a complex number  $z$ , we denote by  $\pi[z]$  the twist of  $\pi$  by  $|\cdot|^z$ , i.e.  $\pi[z](g) = |\det g|^z \pi(g)$ . Let  $Q$  be a standard upper triangular parabolic subgroup of  $G_n$  with Levi factor  $G_{n_1} \times G_{n_2} \times \cdots \times G_{n_k}$ ,  $n_1 + \cdots + n_k = n$ . Let  $\pi^i$ ,  $1 \leq i \leq k$ , be an irreducible tempered representation of  $G_{n_i}$  and  $r_1 > r_2 > \cdots > r_k$  real numbers. We set

$$(2.1) \quad \text{Ind}_Q^{G_n}(\pi^1[r_1] \otimes \pi^2[r_2] \otimes \cdots \otimes \pi^k[r_k]).$$

Bernstein and Zelevinsky proved that if  $\sigma$  is an irreducible generic representation of  $G_n$ , then  $\sigma$  must be equivalent to a representation of the form (2.1), where the parabolic subgroup  $Q$ , the tempered representations  $\pi^1, \dots, \pi^k$  and the real numbers  $r_1 > \cdots > r_k$  are uniquely determined by  $\sigma$  (see also [9]). We note that, by a theorem of Jacquet [11], the irreducible tempered representation  $\pi^i$  of  $G_{n_i}$  must be equivalent to a representation of the form

$$\text{Ind}_{R_i}^{G_{n_i}}(\pi^{i,1} \otimes \pi^{i,2} \otimes \cdots \otimes \pi^{i,p_i}),$$

where  $R_i$  denotes a standard upper triangular parabolic subgroup of  $G_{n_i}$  with Levi factor  $G_{n_{i1}} \times G_{n_{i2}} \times \cdots \times G_{n_{ip_i}}$ ,  $n_{i1} + \cdots + n_{ip_i} = n_i$  an  $\pi^{i,j}$  and irreducible square integrable representation of  $G_{n_{ij}}$  for each  $1 \leq j \leq p_i$ .

Let  $\sigma$  be an irreducible generic representation of the form (2.1). We define the representation  $\sigma_1$  of  $G_{n+1}$  as follows. Assume that  $r_1 > r_2 > \cdots > r_j \geq 0 > r_{j+1} > \cdots > r_k$ . Let  $Q'$  be a standard upper triangular parabolic subgroup of  $G_{n+1}$  with Levi factor  $G_{n_1} \times \cdots \times G_{n_j} \times G_1 \times G_{n_{j+1}} \times \cdots \times G_{n_k}$ . Then we set

$$(2.2) \quad \sigma_1 = \text{Ind}_{Q'}^{G_{n+1}}(\pi^1[r_1] \otimes \cdots \otimes \pi^j[r_j] \otimes 1 \otimes \pi^{j+1}[r_{j+1}] \otimes \cdots \otimes \pi^k[r_k]).$$

This  $\sigma_1$  has the following properties (cf. [9, Proposition 3.2] and [18, Theorem 4.2]).

LEMMA 1. (1)  $\sigma_1$  has a unique irreducible quotient representation.

(2) The representation  $(\rho, \mathcal{W}(\sigma_1, \psi_{n+1}))$  is isomorphic to  $\sigma_1$  itself even if  $\sigma_1$  is not irreducible.

(3)  $\sigma_1$  is reducible if and only if there exists at least one  $\pi^{i,j}[r_i]$  such that  $\pi^{i,j}[r_i] = \text{St}_{n_{ij}}[\pm(n_{ij} + 1)/2]$ , where  $\text{St}_{n_{ij}}$  denotes the Steinberg representation of  $G_{n_{ij}}$  (cf. [4, Theorem 7.11]).

It is known by [2, Theorem 2.9] that the induced representation  $\text{Ind}_{Q_{n+1}}^{G_{n+1}} \sigma \otimes 1$  given in the Introduction has the same composition factors as that of  $\sigma_1$ . However, if  $\text{Ind}_{Q_{n+1}}^{G_{n+1}} \sigma \otimes 1$  is not irreducible, it does not always satisfy the properties (1) and (2) above, and, furthermore, we cannot apply [8, Proposition (9.4)] to this representation. ([8, Proposition (9.4)] will be used in Section 3, (3.3) below.) This is the reason why



we define  $\sigma_1$  not by  $\text{Ind}_{\mathcal{O}_{n+1}^\times}^{G_{n+1}} \sigma \otimes 1$  but by (2.2).

The local Weil representation  $(\omega_{n,m}, \mathcal{S}(M_{n,m}(F)))$  of  $G_n \times G_m$  is defined as follows: for  $f \in \mathcal{S}(M_{n,m}(F))$ ,  $h \in G_n$  and  $g \in G_m$ ,

$$\omega_{n,m}(h, g)f(x) = |\det h|_F^{-m/2} |\det g|_F^{n/2} f(h^{-1}xg).$$

We write simply  $\omega$  for  $\omega_{n,n+1}$  and  $\omega_1$  for  $\omega_{n,n}$ . For  $f \in \mathcal{S}(M_{n,n+1}(F))$ , we define the function  $\Phi(f)$  on  $G_n$  by

$$\Phi(f)(h) = \int_{U_{n+1}} \psi_{n+1}(u)^{-1} \omega(h, u) f(\varepsilon_n) du,$$

where we put  $\varepsilon_n = (1_n, 0) \in M_{n,n+1}(F)$ . For each  $W \in \mathcal{W}(\psi_n)$ , we set

$$V_{(W,f)}(g) = \int_{U_n \setminus G_n} W(h) \Phi(\omega(g)f)(h) dh \quad (g \in G_{n+1}).$$

Since  $\Phi(f)$  has compact support in  $G_n$  modulo  $U_n$ , the integral on the right-hand side reduces to a finite sum. Furthermore, as a function in  $g \in G_{n+1}$ ,  $V_{(W,f)}$  is contained in  $\mathcal{W}(\psi_{n+1})$ . Therefore we have a correspondence

$$\mathcal{W}(\psi_n) \times \mathcal{S}(M_{n,n+1}(F)) \rightarrow \mathcal{W}(\psi_{n+1}),$$

which satisfies the relation

$$(2.3) \quad \rho(g)V_{(\rho(h)W,f)} = V_{(W,\omega(h^{-1},g)f)} \quad (h \in G_n, g \in G_{n+1}).$$

Let  $\sigma$  be an irreducible generic representation of  $G_n$ . We set

$$\mathcal{V}(\sigma, \psi_{n+1}) = \{V_{(W,f)} \mid W \in \mathcal{W}(\sigma, \psi_n), f \in \mathcal{S}(M_{n,n+1}(F))\}.$$

Then the theta lift  $\tilde{\sigma}$  of  $\sigma$  is defined to be a smooth subrepresentation  $(\rho, \mathcal{V}(\sigma, \psi_{n+1}))$  of  $\mathcal{W}(\psi_{n+1})$ . It is known by [13, Chapter 3, Section III, Corollary 3] that  $\tilde{\sigma}$  is of finite length. Thus, by [1, Theorem 4.1],  $\tilde{\sigma}$  is admissible and finitely generated.

Let  $\xi_n \in \mathcal{H}_n$  be the characteristic function of  $K_n$ . We define the action of  $\xi_n$  on  $\mathcal{W}(\psi_{n+1})$  by

$$\rho(\xi_n)W_1(g) = \int_{K_n} W_1\left(g \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}\right) dk \quad (W_1 \in \mathcal{W}(\psi_{n+1}), g \in G_{n+1}).$$

The main result of this paper is stated as follows.

**THEOREM 1.** *Let  $\sigma$  be an irreducible generic representation of  $G_n$ . Then the intersection  $\rho(\xi_n)\mathcal{V}(\sigma, \psi_{n+1}) \cap \rho(\xi_n)\mathcal{W}(\sigma_1, \psi_{n+1})$  contains a nonzero element. Furthermore, if  $\sigma_1$  is irreducible, we have*

$$\rho(\xi_n)\mathcal{V}(\sigma, \psi_{n+1}) = \rho(\xi_n)\mathcal{W}(\sigma_1, \psi_{n+1}).$$

This theorem will be proved in Section 5. We note that  $\rho(\xi_n)\mathcal{W}(\sigma_1, \psi_{n+1})$  has infinite dimension.

REMARK. When  $z$  is a complex number with  $\text{Re}(z) \ll 0$ , the representation  $\sigma[z]_1$  is just the irreducible representation  $\text{Ind}_{Q'}^{GL_{n+1}} 1 \otimes \sigma[z]$ . Here  $Q'$  denotes the standard upper triangular parabolic subgroup with Levi factor  $GL_1 \times GL_n$ . Then, Jacquet, Piatetski-Shapiro and Shalika [8, Proposition (6.1)] essentially proved that, for any irreducible generic representation  $\sigma$ ,

$$(2.4) \quad \mathcal{W}(\sigma[z]_1, \psi_{n+1}) \subset \mathcal{V}(\sigma[z], \psi_{n+1})$$

if  $\text{Re}(z) \ll 0$ . This fact is derived as follows. Let  $\mathcal{S}'(M_{n,n+1}(F))$  be the subset consisting of functions  $f \in \mathcal{S}(M_{n,n+1}(F))$  with  $\text{supp } f \subset \varepsilon_n G_{n+1}$ . The subspace  $\mathcal{S}'(M_{n,n+1}(F))$  is  $\omega(G_n \times G_{n+1})$ -invariant. For each  $f \in \mathcal{S}'(M_{n,n+1}(F))$  and  $W \in \mathcal{W}(\sigma, \psi_n)$ , the integral

$$(2.5) \quad \phi_{(W, f)}(g; m) = |\det m|_F^z \int_{G_n} \omega(h, g) f(\varepsilon_n W(mh)) |\det h|_F^z dh \quad (g \in G_{n+1}, m \in G_n)$$

is convergent, and as a function in  $g \in G_{n+1}$ ,  $\phi_{(W, f)}$  is an element of  $\text{Ind}_{Q''}^{\mathcal{G}_{n+1}}(\sigma[z] \otimes 1)$  (cf. loc. cit. p. 430), where  $Q''$  denotes the standard lower parabolic subgroup with Levi factor  $GL_n \times GL_1$ . We note that the representation  $\text{Ind}_{Q''}^{\mathcal{G}_{n+1}}(\sigma[z] \otimes 1)$  is isomorphic to  $\sigma[z]_1$ . Since  $\sigma[z]_1$  is irreducible, the correspondence  $(W, f) \mapsto \phi_{(W, f)}$  is a surjection from  $\mathcal{W}(\sigma, \psi_n) \times \mathcal{S}'(M_{n,n+1}(F))$  onto  $\text{Ind}_{Q''}^{\mathcal{G}_{n+1}}(\sigma[z] \otimes 1)$ . Furthermore, for  $\phi = \phi_{(W, f)}$ , the integral

$$W'_\phi(g) = \int_{M_{n,1}(F)} \phi_{(W, f)}\left(\begin{pmatrix} 1_n & x \\ 0 & 1 \end{pmatrix} g; 1_n\right) \psi_{n+1}\left(\begin{pmatrix} 1_n & x \\ 0 & 1 \end{pmatrix}\right)^{-1} dx$$

is absolutely convergent by the assumption  $\text{Re}(z) \ll 0$ . Then the space  $\{W'_\phi \mid \phi \in \text{Ind}_{Q''}^{\mathcal{G}_{n+1}}(\sigma[z] \otimes 1)\}$  gives a Whittaker model of  $\sigma[z]_1$ . By replacing  $\phi_{(W, f)}$  by its expression (2.5) and changing order of integrations, we obtain  $W'_\phi = V_{(W \otimes |\cdot|^z, f)}$ . Therefore we have (2.4).

**3. Some results of Jacquet, Piatetski-Shapiro and Shalika.** First, we recall class 1 Whittaker functions of  $G_n$  by Shintani [16]. For an  $n$ -tuple  $k = (k_1, \dots, k_n) \in \mathbf{Z}^n$  of rational integers, we denote by  $t_k$  the diagonal matrix in  $T_n$  whose  $i$ -th diagonal entry is  $\varpi^{k_i}$  for  $1 \leq i \leq n$ . We set

$$A_n = \{k \in \mathbf{Z}^n \mid k_1 \geq k_2 \geq \dots \geq k_n\}.$$

Let  $C[X_1, X_1^{-1}, X_2, X_2^{-1}, \dots, X_n, X_n^{-1}]$  be the Laurent polynomial ring in indeterminates  $X_1, \dots, X_n$  and  $A_n$  the subalgebra consisting of the elements in  $C[X_1, X_1^{-1}, X_2, X_2^{-1}, \dots, X_n, X_n^{-1}]$  which are invariant by permutations of indeterminates. We define the function  $W(\cdot; X_1, \dots, X_n; \psi_n^{-1})$  on  $G_n$  with values in  $A_n$  as follows: for  $u \in U_n$ ,  $t_k \in T_n$  and  $k \in A_n$ ,

$$\begin{aligned}
 &W(ut_k k; X_1, \dots, X_n; \psi_n^{-1}) \\
 &= \psi_n(u)^{-1} \times \begin{cases} \delta_n(t_k)^{1/2} \prod_{1 \leq i < j \leq n} (X_i - X_j)^{-1} \sum_{\tau \in S_n} \operatorname{sgn} \tau \prod_{i=1}^n X_{\tau(i)}^{k_i + n - i} & \text{if } k \in A_n \\ 0 & \text{if } k \notin A_n, \end{cases}
 \end{aligned}$$

where  $\delta_n$  denotes the modular character of  $B_n$ , i.e.  $\delta_n(t_k) = \prod_{i=1}^n |\varpi|^{(n+1-2i)k_i}$ . Then, for each  $n$ -tuple  $z = (z_1, \dots, z_n) \in (\mathbb{C}^\times)^n$  of complex numbers, the class 1 Whittaker function  $W_z$  on  $G_n$  is given by a specialization of  $W(\cdot; X_1, \dots, X_n; \psi_n^{-1})$  at  $(z_1, \dots, z_n)$ , i.e.  $W_z(h) = W(h; z_1, \dots, z_n; \psi_n^{-1})$ . We denote by  $W_z(\psi_n^{-1})$  the submodule of  $W(\psi_n^{-1})$  generated by  $W_z$ , i.e.  $W_z(\psi_n^{-1}) = \rho(\mathcal{H}_n)W_z$ .

For  $z \in (\mathbb{C}^\times)^n$ , we define an unramified character  $\chi_z$  of  $B_n$  by

$$\chi_z(t_k t' u) = z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}, \quad (k \in \mathbb{Z}^n, t' \in T_n \cap K_n, u \in U_n).$$

Let  $\operatorname{Ind}_{B_n}^{G_n} \chi_z$  be the representation of  $G_n$  induced from  $\chi_z$ . Then  $\operatorname{Ind}_{B_n}^{G_n} \chi_z$  is of Whittaker type (cf. [3]). We take a  $\tau \in S_n$  so that  $\tau(z) = (z_{\tau(1)}, z_{\tau(2)}, \dots, z_{\tau(n)})$  satisfies  $|z_{\tau(1)}| \leq |z_{\tau(2)}| \leq \dots \leq |z_{\tau(n)}|$ . It is known by the argument in the proof of [12, Theorem 2.2] that the space  $W(\operatorname{Ind}_{B_n}^{G_n} \chi_{\tau(z)}, \psi_n^{-1})$  coincides with  $W_z(\psi_n^{-1})$ . We denote the representation  $(\rho, W_z(\psi_n^{-1}))$  by  $\pi_z$ . Since  $\pi_z$  is isomorphic to a quotient representation of  $\operatorname{Ind}_{B_n}^{G_n} \chi_{\tau(z)}$ , it is also of Whittaker type.

Next, we recall results of Jacquet, Piatetski-Shapiro and Shalika, which will play an essential role in our proof of Theorem 1. We set

$$w_n = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & 1 & \vdots \\ 1 & \cdots & 0 \end{pmatrix} \in G_n, \quad \eta_n = {}^t(0, \dots, 0, 1) \in M_{n,1}(F).$$

If  $h \in G_n$ , we denote by  $h^t$  the inverse transpose of  $h$ , i.e.  $h^t = {}^t h^{-1}$ . Let  $\sigma$  be an admissible representation of  $G_n$  which is of Whittaker type. We define the representation  $\sigma^t$  of  $G_n$  by  $\sigma^t(h) = \sigma(h^t)$  for  $h \in G_n$ . For  $W \in W(\sigma, \psi_n)$ , we also define the function  $\tilde{W}$  on  $G_n$  by

$$\tilde{W}(h) = W(w_n h^t).$$

Then the set of  $\tilde{W}$  with  $W \in W(\sigma, \psi_n)$  coincides with the space  $W(\sigma^t, \psi_n^{-1})$ .

For the moment, we fix an irreducible generic representation  $\sigma$  of  $G_n$  and the representation  $\sigma_1$  of  $G_{n+1}$  defined in Section 2. Let  $\pi$  be another irreducible generic representation of  $G_n$ . We denote by  $\omega_\pi$  the central character of  $\pi$ . Furthermore, the local factor and the epsilon factor of  $\pi$  given by Godement and Jacquet [4, Theorem 3.3] is denoted by  $L(s, \pi)$  and  $\varepsilon(s, \pi, \psi)$ , respectively. For  $W \in W(\sigma, \psi_n)$ ,  $W' \in W(\pi, \psi_n^{-1})$  and  $\varphi \in \mathcal{S}(M_{n,1}(F))$ , we set

$$\Psi(s, W, W'; \varphi) = \int_{U_n \backslash G_n} W(h) W'(h) \varphi({}^t h \eta_n) |\det h|_F^s dh.$$

In a similar fashion, for  $W_1 \in \mathcal{W}(\sigma_1, \psi_{n+1})$ ,  $W' \in \mathcal{W}(\pi, \psi_n^{-1})$ , we set

$$\Psi(s, W_1, W') = \int_{U_n \setminus G_n} W_1 \left( \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \right) W'(h) |\det h|_F^{s-1/2} dh.$$

Then Jacquet, Piatetski-Shapiro and Shalika [8, Theorems (2.7), (3.1) and Proposition (9.4)] proved the following. Each of the integrals  $\Psi(s, W, W'; \varphi)$  and  $\Psi(s, W_1, W')$  is absolutely convergent for  $\operatorname{Re}(s) \gg 0$  and they are rational functions of  $q^{-s}$ . The integrals  $\Psi(s, W, W'; \varphi)$  span a fractional ideal  $\mathbf{C}[q^s, q^{-s}]L(s, \sigma \times \pi)$  of the ring  $\mathbf{C}[q^s, q^{-s}]$ . The factor  $L(s, \sigma \times \pi)$  has the form

$$(3.1) \quad L(s, \sigma \times \pi) = P_{\sigma \times \pi}(q^{-s})^{-1}, \quad P_{\sigma \times \pi} \in \mathbf{C}[X], \quad P_{\sigma \times \pi}(0) = 1.$$

Furthermore, there is a factor  $\varepsilon(s, \sigma \times \pi, \psi)$  of the form  $cq^{-ms}$  such that

$$(3.2) \quad \frac{\Psi(1-s, \tilde{W}, \tilde{W}'; \hat{\varphi})}{L(1-s, \sigma' \times \pi')} = \omega_\pi(-1)^{n-1} \varepsilon(s, \sigma \times \pi, \psi) \frac{\Psi(s, W, W'; \varphi)}{L(s, \sigma \times \pi)},$$

where  $\hat{\varphi}$  denotes the Fourier transform of  $\varphi$ , that is,

$$\hat{\varphi}(x) = \int_{M_{n,1}(F)} \psi(\langle xy \rangle) \varphi(y) dy.$$

Similarly, the integrals  $\Psi(s, W_1, W')$  span a fractional ideal  $\mathbf{C}[q^s, q^{-s}]L(s, \sigma_1 \times \pi)$ . Here the factor  $L(s, \sigma_1 \times \pi)$  has the form

$$(3.3) \quad L(s, \sigma_1 \times \pi) = L(s, \sigma \times \pi)L(s, \pi).$$

There is a functional equation

$$(3.4) \quad \frac{\Psi(1-s, \tilde{W}_1, \tilde{W}')}{L(1-s, \sigma'_1 \times \pi')} = \omega_\pi(-1)^n \varepsilon(s, \sigma \times \pi, \psi) \varepsilon(s, \pi, \psi) \frac{\Psi(s, W_1, W')}{L(s, \sigma_1 \times \pi)}.$$

We set

$$\gamma(s, \sigma_1 \times \pi, \psi) = \varepsilon(s, \sigma \times \pi, \psi) \varepsilon(s, \pi, \psi) \frac{L(1-s, \sigma'_1 \times \pi')}{L(s, \sigma_1 \times \pi)}.$$

**4. The gamma factor of  $\tilde{\sigma} \times \pi$ .** We fix two irreducible generic representations  $\sigma$  and  $\pi$  of  $G_n$ . The purpose of this section is to calculate the integral

$$\Psi(s, V, W') = \int_{U_n \setminus G_n} V \left( \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \right) W'(g) |\det g|_F^{s-1/2} dg$$

for  $V \in \mathcal{V}(\sigma, \psi_{n+1})$  and  $W' \in \mathcal{W}(\pi, \psi_n^{-1})$ . Since we do not yet know whether  $\tilde{\sigma}$  is of Whittaker type, we cannot apply the results of Jacquet, Piatetski-Shapiro and Shalika. However we can compute this integral directly in the same way as in [8, (6.3)]. In the

following, we explain briefly this method. See [8, (6.3)] for details.

We may assume  $V$  to be the form  $V = V_{(W, f_1 \otimes f_2)}$ , where  $W \in \mathcal{W}(\sigma, \psi_n)$ ,  $f_1 \in \mathcal{S}(M_n(F))$  and  $f_2 \in \mathcal{S}(M_{n,1}(F))$ . We set

$$\Phi_1(f_1)(h, g) = \int_{U_n} \psi_n(u)^{-1} \omega_1(h, g) f_1(u) du .$$

From easy calculation, it follows that

$$(4.1) \quad \Phi \left( \omega \left( \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \right) (f_1 \otimes f_2) \right) (h) = |\det h|_F^{1/2} \hat{f}_2({}^t h \eta_n) \Phi_1(f_1)(h, g) .$$

We regard  $\Psi(s, V, W')$  as a formal Laurent series in  $X = q^{-s}$ . Thus we write  $\Psi(s, V, W')$  as

$$(4.2) \quad \sum_{m=-\infty}^{\infty} \Psi_m(V, W') X^m .$$

This Laurent series has only finitely many nonzero negative terms. Each coefficient  $\Psi_m(V, W')$  is given by

$$\Psi_m(V, W') = \int_{U_n \setminus G_n^m} V \left( \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \right) W'(g) |\det g|_F^{-1/2} dg ,$$

where  $G_n^m$  denotes the set of  $g \in G_n$  with  $|\det g|_F = q^{-m}$ . By (4.1),  $\Psi_m(V, W')$  equals

$$\begin{aligned} & \int_{U_n \setminus G_n^m} \left\{ \int_{U_n \setminus G_n} W(h) \Phi_1(f_1)(h, g) \hat{f}_2({}^t h \eta_n) |\det h|_F^{1/2} dh \right\} W'(g) |\det g|_F^{-1/2} dg \\ &= \int_{U_n \setminus G_n^m} \left\{ \int_{G_n} W(h) f_1(h^{-1}g) \hat{f}_2({}^t h \eta_n) |\det h|_F^{1/2 - n/2} dh \right\} W'(g) |\det g|_F^{n/2 - 1/2} dg \\ &= \int_{U_n \setminus G_n^m} W'(g) \int_{G_n} W(gh^{-1}) f_1(h) \hat{f}_2(h {}^t g \eta_n) |\det h|_F^{n/2 - 1/2} dh dg . \end{aligned}$$

This double integral is absolutely convergent. By changing  $g$  to  $gh$ , we obtain

$$(4.3) \quad \int_{U_n \setminus G_n} W(g) \hat{f}_2({}^t g \eta_n) \int_{G_n} \chi_m(gh) W'(gh) f_1(h) |\det h|_F^{n/2 - 1/2} dh dg ,$$

where  $\chi_m$  denotes the characteristic function of  $G_n^m$ . We take an open compact subgroup  $\Omega$  of  $G_n$  such that  ${}^t \Omega = \Omega$  and  $f_1(\omega h) = f_1(h)$  for  $\omega \in \Omega$ . Let  $\mathcal{W}(\pi, \psi_n^{-1})^\Omega$  be the subspace of  $\mathcal{W}(\pi, \psi_n^{-1})$  consisting of all elements fixed by  $\Omega$  and  $\{W'_1, \dots, W'_p\}$  a basis of  $\mathcal{W}(\pi, \psi_n^{-1})^\Omega$ . Then there exist matrix coefficients  $\phi_1, \dots, \phi_p$  of  $\pi$  such that

$$(4.4) \quad \int_{\Omega} W'(g \omega h) d\omega = \sum_{1 \leq j \leq p} W'_j(g) \phi_j(h) .$$

Thus we have

$$\int_{G_n} \chi_m(gh)W'(gh)f_1(h)|\det h|_F^{n/2-1/2}dh = \sum_{j=1}^p W'_j(g) \int_{G_n} \chi_m(gh)\phi_j(h)f_1(h)|\det h|_F^{n/2-1/2}dh .$$

Furthermore, we have

$$(4.5) \quad \chi_m(gh) = \sum_{m'+m''=m} \chi_{m'}(g)\chi_{m''}(h) ,$$

where  $m'$  and  $m''$  are bounded from below. Consequently, by (4.2), (4.3), (4.4) and (4.5), we obtain

$$(4.6) \quad \Psi(s, V_{(W, f_1 \otimes f_2)}, W') = \sum_{j=1}^p \Psi(s, W, W'_j; \hat{f}_2) \int_{G_n} f_1(h)\phi_j(h)|\det h|_F^{s+n/2-1/2}dh .$$

Here we note that the integral

$$Z(f_1, s+n/2-1/2, \phi_j) = \int_{G_n} f_1(h)\phi_j(h)|\det h|_F^{s+n/2-1/2}dh$$

is a zeta-integral defined by Godement and Jacquet [4] and there is a functional equation

$$(4.7) \quad \frac{Z(\hat{f}_1, 1-s+n/2-1/2, \phi_j^!)}{L(1-s, \pi^!)} = \varepsilon(s, \pi, \psi) \frac{Z(f_1, s+n/2-1/2, \phi_j)}{L(s, \pi)} ,$$

where  $\phi_j^!$  is a matrix coefficient of  $\pi^!$  given by  $\phi_j^!(h) = \phi_j(h')$ . Similarly, by using [8, Proposition 6.2], we obtain

$$(4.8) \quad \Psi(1-s, \tilde{V}_{(W, f_1 \otimes f_2)}, \tilde{W}') = \sum_{j=1}^p \Psi(1-s, \tilde{W}, \tilde{W}'_j; f_2)\omega_\pi(-1)Z(\hat{f}_1, 1-s+n/2-1/2, \phi_j^!)$$

Therefore, by (4.6), (4.7), (4.8) and (3.2), we have

$$(4.9) \quad \frac{\Psi(1-s, \tilde{V}, \tilde{W}')}{L(1-s, \sigma^! \times \pi^!)L(1-s, \pi^!)} = \omega_\pi(-1)^n \varepsilon(s, \sigma \times \pi, \psi) \varepsilon(s, \pi, \psi) \frac{\Psi(s, V, W')}{L(s, \sigma \times \pi)L(s, \pi)}$$

as a formal Laurent series. However, (4.9) itself implies that both sides are polynomials in  $(X, X^{-1})$  (cf. [8, (4.4)]). Thus (4.9) may be regarded as an identity of analytic functions. As a consequence, we obtain the following result.

**PROPOSITION 2.** *For each  $V \in \mathcal{V}(\sigma, \psi_{n+1})$  and  $W' \in \mathcal{W}(\pi, \psi_n^{-1})$ , the integral  $\Psi(s, V, W')$  is absolutely convergent for  $\text{Re}(s) \gg 0$  and  $L(s, \sigma_1 \times \pi)^{-1}\Psi(s, V, W')$  is an element of  $\mathcal{C}[q^s, q^{-s}]$ . Moreover, there is an equation*

$$\Psi(1-s, \tilde{V}, \tilde{W}') = \omega_\pi(-1)^n \gamma(s, \sigma_1 \times \pi, \psi) \Psi(s, V, W') .$$

If  $\pi$  is a spherical representation  $\pi_z$ , then we can prove the assertion of Proposition

2 in another way. Namely, we can calculate the integral  $\Psi(s, V, W_z)$  more directly. Since the integral  $\Psi(s, V, W_z)$  will be used in Section 5, we explain this calculation in the rest of this section. We may assume again that  $V$  is of the form  $V_{(W, f_1 \otimes f_2)}$ . We set

$$J(s, f_1, W_z)(h) = \int_{U_n \setminus G_n} \Phi_1(f_1)(h, g) W_z(g) |\det g|_F^{s-1/2} dg.$$

By (4.1), we have formally

$$(4.10) \quad \Psi(s, V_{(W, f_1 \otimes f_2)}, W_z) = \int_{U_n \setminus G_n} W(h) J(s, f_1, W_z)(h) \hat{f}_2(th\eta_n) |\det h|_F^{1/2} dh.$$

Let  $f_1^0 \in \mathcal{S}(M_n(F))$  be the characteristic function of  $M_n(\mathcal{O})$ . Then  $\omega_1(1_n, \xi_n)$  is a projection from  $\mathcal{S}(M_n(F))$  to the subspace  $\mathcal{S}(M_n(F))^{\omega_1(1_n, K_n)}$  consisting of functions invariant by  $\omega_1(1_n, K_n)$ . By Howe [5, Theorem 10.2], the space  $\mathcal{S}(M_n(F))^{\omega_1(1_n, K_n)}$  coincides with the space  $\omega_1(\mathcal{H}_n, 1_n) f_1^0$ . Thus, corresponding to  $f_1$ , there exists  $\varphi_1 \in \mathcal{H}_n$  such that

$$(4.11) \quad \omega_1(1_n, \xi_n) f_1 = \omega_1(\varphi_1, 1_n) f_1^0.$$

Then we have

$$J(s, f_1, W_z) = J(s, f_1, \rho(\xi_n) W_z) = J(s, \omega_1(\varphi_1, 1_n) f_1^0, W_z) = \rho(\varphi_1) J(s, f_1^0, W_z).$$

We compute the integral  $J(s, f_1^0, W_z)(h)$ . The next lemma follows from simple calculation.

LEMMA 2. *Let  $k = (k_1, \dots, k_n) \in \mathbf{Z}^n$  and  $p = (p_1, \dots, p_n) \in \mathbf{Z}^n$ . If  $p_1 \geq k_1 \geq p_2 \geq k_2 \geq \dots \geq p_n \geq k_n$ , then we have*

$$\Phi_1(f_1^0)(t_k, t_p) = |\det t_k|_F^{-1/2} \delta_n(t_k)^{1/2} |\det t_p|_F^{1/2} \delta_n(t_p)^{1/2}.$$

Otherwise,  $\Phi_1(f_1^0)(t_k, t_p)$  is zero.

LEMMA 3. *The integral  $J(s, f_1^0, W_z)$  absolutely converges if  $\text{Re}(s)$  is sufficiently large, and we have*

$$J(s, f_1^0, W_z)(h) = \left( \prod_{i=1}^n (1 - q^{-s z_i}) \right)^{-1} |\det h|_F^{s-1/2} W_z(h).$$

PROOF. By Lemma 2 and an explicit formula for  $W_z$ ,  $J(s, f_1^0, W_z)(t_k)$  equals

$$\begin{aligned} & \sum_{p \in A_n} \Phi_1(f_1^0)(t_k, t_p) W_z(t_p) |\det t_p|_F^{s-1/2} \delta_n(t_p)^{-1} \\ &= |\det t_k|_F^{-1/2} \delta_n(t_k)^{1/2} \sum_{\substack{p \in A_n \\ k_{i-1} \geq p_i \geq k_i (1 \leq i \leq n)}} |\det t_p|_F^s W_z(t_p) \delta_n(t_p)^{-1/2} \\ &= |\det t_k|_F^{-1/2} \delta_n(t_k)^{1/2} \left( \prod_{1 \leq i < j \leq n} (z_i - z_j) \right)^{-1} \end{aligned}$$

$$\times \sum_{\tau \in S_n} \operatorname{sgn} \tau \prod_{i=1}^n z_{\tau(i)}^{n-i} \left\{ \sum_{k_{i-1} \geq p_i \geq k_i} (q^{-s z_{\tau(i)}})^{p_i} \right\}.$$

Here we put  $k_0 = +\infty$  for convenience. The sum of  $(q^{-s z_{\tau(i)}})^{p_i}$  over  $+\infty > p_1 \geq k_1$  absolutely converges if  $\operatorname{Re}(s) > \max_{1 \leq i \leq n} (\log_q |z_i|)$ . Then, by calculation of determinants, we obtain

$$\sum_{\tau \in S_n} \operatorname{sgn} \tau \prod_{i=1}^n z_{\tau(i)}^{n-i} \left\{ \sum_{k_{i-1} \geq p_i \geq k_i} (q^{-s z_{\tau(i)}})^{p_i} \right\} = \left( \prod_{i=1}^n (1 - q^{-s z_i}) \right)^{-1} \sum_{\tau \in S_n} \operatorname{sgn} \tau \prod_{i=1}^n z_{\tau(i)}^{n-i} (q^{-s z_{\tau(i)}})^{k_i}.$$

Thus implies the assertion.

For  $\varphi_1 \in \mathcal{H}_n$  satisfying (4.11), we define  $\varphi_1^s \in \mathcal{H}_n$  by

$$(4.12) \quad \varphi_1^s(h) = |\det h|_F^{s-1/2} \varphi_1(h).$$

By Lemma 3 and (4.10), we have a relation

$$(4.13) \quad \Psi(s, V_{(W, f_1 \otimes f_2)}, W_z) = \left( \prod_{i=1}^n (1 - q^{-s z_i}) \right)^{-1} \Psi(s, W, \rho(\varphi_1^s) W_z; \hat{f}_2).$$

Next, we compute the integral

$$\Psi(1-s, \tilde{V}_{(W, f_1 \otimes f_2)}, \tilde{W}_z) = \int_{U_n \setminus G_n} V_{(W, f_1 \otimes f_2)} \left( w_{n+1} \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \right) W_z(w_n g) |\det g|_F^{1/2-s} dg.$$

By changing  $g$  to  $w_n g' w_n$ , this integral equals

$$\begin{aligned} & \int_{U_n \setminus G_n} V_{(W, f_1 \otimes f_2)} \left( w_{n+1} \begin{pmatrix} w_n & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \right) W_z(g) |\det g|_F^{s-1/2} dg \\ &= \int_{U_n \setminus G_n} \left\{ \int_{U_n \setminus G_n} W(h) \Phi \left( \omega \left( \begin{pmatrix} 0 & 1 \\ g & 0 \end{pmatrix} \right) (f_1 \otimes f_2) \right) (h) dh \right\} W_z(g) |\det g|_F^{s-1/2} dg \\ &= \int_{U_n \setminus G_n} W(h) \left\{ \int_{U_n \setminus G_n} \Phi \left( \omega \left( \begin{pmatrix} 0 & 1 \\ g & 0 \end{pmatrix} \right) (f_1 \otimes f_2) \right) (h) W_z(g) |\det g|_F^{s-1/2} dg \right\} dh. \end{aligned}$$

For  $u \in U_{n+1}$ , we denote by  $u_1$  the  $n \times n$  matrix obtained by eliminating the first column vector and the  $(n+1)$ -st row vector from  $u$ . Then

$$\begin{aligned} & \Phi \left( \omega \left( \begin{pmatrix} 0 & 1 \\ g & 0 \end{pmatrix} \right) (f_1 \otimes f_2) \right) (h) \\ &= |\det h|_F^{-1/2} f_2 \left( h^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right) \int_{U_{n+1}} \psi_{n+1}(u)^{-1} \omega_1(h, g) f_1(u_1) du. \end{aligned}$$



We take  $\varphi_1^s \in \mathcal{H}_n$  to be the same as (4.12) for given  $f_1 \in \mathcal{S}(M_n(F))$ . It follows from calculation similar to that in the proof of Lemma 3 that

$$\int_{U_n \setminus G_n} \left\{ \int_{U_{n+1}} \psi_{n+1}(u)^{-1} \omega_1(h, g) f_1(u_1) du \right\} W_z(g) |\det g|_F^{s-1/2} dg \\ = \left( \prod_{i=1}^n \frac{-q^{1-s} z_i}{1 - q^{1-s} z_i} \right) |\det h|_F^{s-1/2} \rho(\varphi_1^s) W_z(h),$$

and hence

$$\Psi(1-s, \tilde{V}_{(W, f_1 \otimes f_2)}, \tilde{W}_z) \\ = \left( \prod_{i=1}^n \frac{-q^{1-s} z_i}{1 - q^{1-s} z_i} \right) \int_{U_n \setminus G_n} W(h) \rho(\varphi_1^s) W_z(h) f_2 \left( h^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right) |\det h|_F^{s-1} dh.$$

By the change of variable  $h \mapsto w_n h^t w_n$ , we obtain

$$(4.14) \quad \Psi(1-s, \tilde{V}_{(W, f_1 \otimes f_2)}, \tilde{W}_z) = \left( \prod_{i=1}^n \frac{-q^{1-s} z_i}{1 - q^{1-s} z_i} \right) \Psi(1-s, \tilde{W}, \widetilde{\rho(\varphi_1^s) W_z}; f_2).$$

Therefore, by (4.13), (4.14) and (3.2), we have

$$(4.15) \quad \frac{\Psi(1-s, \tilde{V}_{(W, f_1 \otimes f_2)}, \tilde{W}_z)}{L(1-s, \sigma_1^t \times \pi_z^t)} = \varepsilon(s, \sigma \times \pi_z, \psi) \frac{\Psi(s, V_{(W, f_1 \otimes f_2)}, W_z)}{L(s, \sigma_1 \times \pi_z)}.$$

**5. Proof of Theorem 1.** Let  $\sigma$  be an irreducible generic representation of  $G_n$ . We set  $U = V(\sigma, \psi_{n+1}) + W(\sigma_1, \psi_{n+1})$ . By (3.4) and Proposition 2, each  $V \in U$  satisfies the functional equations

$$(5.1) \quad \Psi(1-s, \tilde{V}, \tilde{W}') = \omega_\pi(-1)^n \gamma(s, \sigma_1 \times \pi, \psi) \Psi(s, V, W'),$$

for all irreducible generic representations  $\pi$  of  $G_n$  and  $W' \in \mathcal{W}(\pi, \psi_n^{-1})$ .

**LEMMA 4.** *Let  $R$  be the restriction map  $V \mapsto V|_{P_{n+1}}$  from  $U$  to  $\text{Ind}_{U_{n+1}}^{P_{n+1}} \psi_{n+1}$ . Then  $R$  is injective.*

**PROOF.** We denote by  $\tilde{U}$  the space of functions  $\tilde{V}$  with  $V \in U$  and define the  $P_{n+1}$ -morphism  $\tilde{R}: \tilde{U} \rightarrow \text{Ind}_{U_{n+1}}^{P_{n+1}} \psi_{n+1}^{-1}$  by the restriction  $\tilde{V} \mapsto \tilde{V}|_{P_{n+1}}$ . If  $V \in \text{Ker } R$ , then we have  $\Psi(s, V, W') = 0$ , and hence, by (5.1),  $\Psi(1-s, \tilde{V}, \tilde{W}') = 0$  for all irreducible generic representations  $\pi$  of  $G_n$  and  $W' \in \mathcal{W}(\pi, \psi_n^{-1})$ . Then, by [10, Lemma (3.2)] (cf. [7, Lemma (3.5)]), we have  $\tilde{V} \in \text{Ker } \tilde{R}$ . Similarly, if  $\tilde{V} \in \text{Ker } \tilde{R}$ , then  $V \in \text{Ker } R$ . Therefore,  $V \in \text{Ker } R$  is equivalent to  $\tilde{V} \in \text{Ker } \tilde{R}$ . Let  $V \in \text{Ker } R$  and  $p^t = {}^t p^{-1} \in P_{n+1}^t$ . Since  $\rho(p^t) \tilde{V} = \rho(p) \tilde{V}$  and  $\text{Ker } \tilde{R}$  is  $P_{n+1}$ -invariant, we have  $\rho(p^t) \tilde{V} \in \text{Ker } \tilde{R}$ , and hence  $\rho(p^t) V \in \text{Ker } R$ . As a result,  $\text{Ker } R$  is  $P_{n+1}^t$ -invariant. Since the action of the center  $Z_{n+1}$  on  $U$  is through the scalar

multiplication by the central character of  $\sigma$ ,  $\text{Ker } R$  is both  $\mathcal{Q}_{n+1}$ - and  $\mathcal{Q}_{n+1}^i$ -invariant. Consequently,  $\text{Ker } R$  is  $G_{n+1}$ -invariant. Thus we have  $V(g) = \rho(g)V(1_n) = 0$  for  $V \in \text{Ker } R$ .  $\blacksquare$

Next we consider the integral  $\Psi(s, V, W_z)$  for  $V \in \rho(\xi_n)U$ . By (3.4) and (4.15), there is an equation

$$(5.2) \quad \frac{\Psi(1-s, \tilde{V}, \tilde{W}_z)}{L(1-s, \sigma_1^1 \times \pi_z^1)} = \varepsilon(s, \sigma \times \pi_z, \psi) \frac{\Psi(s, V, W_z)}{L(s, \sigma_1 \times \pi_z)}$$

for each  $V \in \rho(\xi_n)U$  and  $z \in (\mathbf{C}^\times)^n$ . We replace the parameter  $(z_1, \dots, z_n)$  by indeterminates  $X_1, \dots, X_n$ . Namely, we consider the "integral"

$$\Psi(X, V, X_1, \dots, X_n) = \int_{U_n \setminus G_n} V \left( \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \right) \mathcal{W}(g, X_1, \dots, X_n; \psi_n^{-1}) |\det g|_{\mathbb{F}}^{s-1/2} dg,$$

where we put  $X = q^{-s}$ . This  $\Psi(X, V, X_1, \dots, X_n)$  is regarded as an element in the ring  $\Delta_n[[X, X^{-1}]]$  of formal Laurent series with coefficients in  $\Delta_n$ . Then (5.2) and the argument in the proof of [7, Theorem 4.1] implies that each  $V \in \rho(\xi_n)U$  satisfies the equation

$$(5.3) \quad \Psi(q^{-1}X^{-1}, \tilde{V}, X_1^{-1}, \dots, X_n^{-1}) \prod_{i=1}^n P_{\sigma_i}(q^{-1}X^{-1}X_i^{-1})(1 - q^{-1}X^{-1}X_i^{-1}) \\ = \prod_{i=1}^n \varepsilon_{\sigma_i}(XX_i, \psi) \Psi(X, V, X_1, \dots, X_n) \prod_{i=1}^n P_{\sigma_i}(XX_i)(1 - XX_i).$$

Here polynomials  $P_{\sigma_i}(X)$  and  $\varepsilon_{\sigma_i}(X, \psi)$  are given by

$$L(s, \sigma) = P_{\sigma}(q^{-s})^{-1}, \quad \varepsilon(s, \sigma, \psi) = \varepsilon_{\sigma}(q^{-s}, \psi).$$

From (5.3) and the fact that  $\Psi(X, V, X_1, \dots, X_n)$  has a finite number of nonzero terms with negative exponents in  $X$  (cf. [7, Section 3]), it follows that  $\Psi(X, V, X_1, \dots, X_n)$  is contained in the polynomial ring  $\Delta_n[X, X^{-1}]$  and there exists an element  $\Xi(V, X_1, \dots, X_n) \in \Delta_n$  such that

$$(5.4) \quad \Xi(V, XX_1, \dots, XX_n) = \Psi(X, V, X_1, \dots, X_n) \prod_{i=1}^n P_{\sigma_i}(XX_i)(1 - XX_i).$$

LEMMA 5. Let  $V \in \rho(\xi_n)U$ . If  $\Xi(V, X_1, \dots, X_n) = 0$ , then we have  $V = 0$ .

PROOF. If  $\Xi(V, X_1, \dots, X_n) = 0$ , then  $\Psi(X, V, X_1, \dots, X_n) = 0$ . By [7, Lemma (3.5)], we have  $R(V) = 0$ . Therefore, by Lemma 4, we have  $V = 0$ .  $\blacksquare$

PROOF OF THEOREM 1. We denote by  $I(\tilde{\sigma})$  (resp.  $I(\sigma_1)$ ) the subset of  $\Delta_n$  consisting of  $\Xi(V, X_1, \dots, X_n)$  with  $V \in \rho(\xi_n)V(\sigma, \psi_{n+1})$  (resp.  $V \in \rho(\xi_n)W(\sigma_1, \psi_{n+1})$ ). Then, by the same argument as in the proof of [7, Theorem (4.1)], both  $I(\tilde{\sigma})$  and  $I(\sigma_1)$  are ideals of  $\Delta_n$  and there exist elements  $V_1 \in \rho(\xi_n)V(\sigma, \psi_{n+1})$  and  $W_1 \in \rho(\xi_n)W(\sigma_1, \psi_{n+1})$  such that

$$\Xi(V_1, X_1, \dots, X_n) = \Xi(W_1, X_1, \dots, X_n) = \prod_{i=1}^n P_\sigma(X_i)(1 - X_i).$$

By Lemma 5, we have  $V_1 = W_1$ . Thus  $\rho(\xi_n)V(\sigma, \psi_{n+1})$  and  $\rho(\xi_n)W(\sigma_1, \psi_{n+1})$  have a nonzero intersection. Furthermore, if  $\sigma_1$  is irreducible, then we have

$$I(\tilde{\sigma}) = I(\sigma_1) = \Delta_n$$

(cf. [7, Theorem (4.1)]). This implies  $\rho(\xi_n)V(\sigma, \psi_{n+1}) = \rho(\xi_n)W(\sigma, \psi_{n+1})$ . We complete the proof of Theorem 1.

We note that if  $\sigma_1$  is reducible, then the assertion analogous to [7, Proposition (2.1)] for  $\sigma_1$  is false. Thus we cannot conclude that  $I(\sigma_1) = \Delta_n$  in this case.

**PROPOSITION 3.** *Let  $\sigma$  be an irreducible generic representation of  $G_n$ . Then  $\tilde{\sigma}$  never has a supercuspidal subquotient representation.*

**PROOF.** Suppose  $\tilde{\sigma}$  has an irreducible supercuspidal subquotient  $\tilde{\sigma}_c$ . Then, by [1, Proposition 3.30],  $\tilde{\sigma}_c$  is realized as a subrepresentation of  $\tilde{\sigma}$ . The representation space  $V(\sigma, \psi_{n+1})_c$  of  $\tilde{\sigma}_c$  in  $V(\sigma, \psi_{n+1})$  is a Whittaker model of  $\tilde{\sigma}_c$ . We set

$$I(\tilde{\sigma}_c) = \{ \Xi(V, X_1, \dots, X_n) \mid V \in \rho(\xi_n)V(\sigma, \psi_{n+1})_c \}.$$

Then, by the proof of [7, Theorem (4.1)], we have  $I(\tilde{\sigma}_c) = \Delta_n$ , and hence  $I(\sigma_1) \subset I(\tilde{\sigma}_c)$ . This implies  $\rho(\xi_n)V(\sigma, \psi_{n+1})_c = \rho(\xi_n)W(\sigma_1, \psi_{n+1})_c$ . Therefore,  $\tilde{\sigma}_c$  is realized as a subquotient of  $\sigma_1$ . This is a contradiction. ■

**PROPOSITION 4.** *Let  $\sigma$  be an irreducible generic representation of  $G_n$ . If  $\sigma_1$  is irreducible, then  $\tilde{\sigma}$  is an admissible representation of Whittaker type and  $\sigma_1$  is a unique irreducible subrepresentation of  $\tilde{\sigma}$ .*

**PROOF.** We note that any irreducible generic representation of  $G_{n+1}$  has a nonzero vector fixed by  $\rho(\xi_n)$ . If  $V'$  is an irreducible submodule of  $V(\sigma, \psi_{n+1})$ , then we have  $\rho(\xi_n)W(\sigma_1, \psi_{n+1}) \supset \rho(\xi_n)V' \neq 0$ , and hence  $W(\sigma_1, \psi_{n+1}) = V'$ . We prove that the dimension of  $\text{Hom}_{G_{n+1}}(V(\sigma, \psi_{n+1}), W(\psi_{n+1}))$  equals 1. Let  $L_0$  be the natural injection of  $V(\sigma, \psi_{n+1})$  to  $W(\psi_{n+1})$  and  $L \in \text{Hom}_{G_{n+1}}(V(\sigma, \psi_{n+1}), W(\psi_{n+1}))$  an arbitrary nonzero element. If  $\text{Ker } L$  is nonzero, then  $W(\sigma_1, \psi_{n+1}) \subset \text{Ker } L$  and  $V(\sigma, \psi_{n+1})/W(\sigma_1, \psi_{n+1})$  contains a nonzero generic irreducible subquotient. Therefore,  $V(\sigma, \psi_{n+1})/W(\sigma_1, \psi_{n+1})$  has a nonzero vector fixed by  $\rho(\xi_n)$ . This contradicts  $\rho(\xi_n)(V(\sigma, \psi_{n+1})/W(\sigma_1, \psi_{n+1})) = 0$ . Thus  $L$  must be injective. Then there exists a constant  $c$  such that  $L|_{W(\sigma_1, \psi_{n+1})} = cL_0|_{W(\sigma_1, \psi_{n+1})}$ . Since  $W(\sigma_1, \psi_{n+1}) \subset \text{Ker}(L - cL_0)$ , we obtain  $L - cL_0 = 0$ . ■

It is expected that  $\tilde{\sigma} = \sigma_1$  for any irreducible generic representation  $\sigma$ . In fact, this is the case if  $\sigma$  is an irreducible generic spherical representation. Namely, we have the following:

**PROPOSITION 5.** *Let  $z = (z_1, \dots, z_n) \in (\mathbb{C}^\times)^n$  and  $(z, 1) = (z_1, \dots, z_n, 1) \in (\mathbb{C}^\times)^{n+1}$ . If the spherical representation  $\pi_z$  is irreducible, then  $\tilde{\pi}_z = \pi_{(z, 1)}$ .*

PROOF. Since  $W_z(\psi_n^{-1}) \cong W_z(\psi_n)$ , we may substitute  $\psi_n^{-1}$  by  $\psi_n$  in the definitions of  $W_z$  and  $\pi_z$ . Let  $f^0 \in \mathcal{S}(M_{n,n+1}(F))$  be the characteristic function of  $M_{n,n+1}(\mathcal{O})$ . Then, from (2.4) and the fact that  $\omega(\xi_n)\mathcal{S}(M_{n,n+1}(F)) = \omega(\mathcal{H}_{n+1})f^0$  (cf. [5, Theorem 10.2]), it follows that  $V(\pi_z, \psi_{n+1})$  is generated by  $V_{(W_z, f^0)}$ . By calculation similar to that in the proof of Lemma 3, we obtain  $V_{(W_z, f^0)} = W_{(z,1)}$ . Therefore,  $V(\pi_z, \psi_{n+1})$  coincides with  $W_{(z,1)}(\psi_{n+1})$ . ■

## REFERENCES

- [1] I. N. BERNSTEIN AND A. V. ZELEVINSKY, Representations of the group  $GL(n, F)$  where  $F$  is a non-Archimedean local field, Russian Math. Surveys 31 (1976), 1–68.
- [2] I. N. BERNSTEIN AND A. V. ZELEVINSKY, Induced representations of reductive  $p$ -adic groups I., Ann. Sci. Ec. Norm. Sup. 10 (1977), 441–472.
- [3] W. CASSELMAN AND J. SHALIKA, The unramified principal series of  $p$ -adic groups II, Compositio Math. 41 (1980), 207–231.
- [4] R. GODEMENT AND H. JACQUET, Zeta Functions of Simple Algebras, Lecture Notes in Math. 260, Springer-Verlag, Berlin, Heidelberg, New York, 1972.
- [5] R. HOWE,  $\theta$ -series and invariant theory, Proc. Symp. Pure Math. Amer. Math. Soc., 33, Part I (1979), 275–285.
- [6] H. JACQUET, I. I. PIATETSKI-SHAPIRO AND J. SHALIKA, Automorphic forms on  $GL(3)$  I, II, Ann. Math. 109 (1979), 169–212, 213–258.
- [7] H. JACQUET, I. I. PIATETSKI-SHAPIRO AND J. SHALIKA, Conducteur des représentations du groupe linéaire, Math. Ann. 256 (1981), 199–214.
- [8] H. JACQUET, I. I. PIATETSKI-SHAPIRO AND J. SHALIKA, Rankin-Selberg convolutions, Amer. J. Math. 105 (1983), 367–464.
- [9] H. JACQUET AND J. SHALIKA, The Whittaker models of induced representations, Pacific J. Math. 109 (1983), 107–120.
- [10] H. JACQUET AND J. SHALIKA, A lemma of highly ramified  $\varepsilon$ -factors, Math. Ann. 271 (1985), 319–332.
- [11] H. JACQUET, Generic representations, in Non-commutative harmonic analysis, Lecture Notes in Math. 587, Springer-Verlag, Berlin Heidelberg, New York, 1976, pp. 91–100.
- [12] J.-S. LI, Some results on the unramified principal series of  $p$ -adic groups, Math. Ann. 292 (1992), 747–761.
- [13] C. MØGLIN, M.-F. VIGNÉRAS AND J.-L. WALDSPURGER, Correspondances de Howe sur un corps  $p$ -adique, Lecture Notes in Math. 1291, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1987.
- [14] D. PRASAD, Weil representation, Howe duality, and the theta correspondence, CRM Proc. and Lecture Notes 1, Centre de Recherches Math. (1993), 105–127.
- [15] S. RALLIS, Langlands’ functoriality and the Weil representation, Amer. J. Math. 104 (1982), 469–515.
- [16] T. SHINTANI, On an explicit formula for class-1 Whittaker functions on  $GL_n$  over  $p$ -adic fields, Proc. Japan Acad. 52 Ser. A. No. 4 (1976), 180–182.
- [17] T. WATANABE, The global theta correspondence of  $(GL_m, GL_n)$ , preprint.
- [18] A. ZELEVINSKI, Induced representations of  $p$ -adic reductive groups II, Ann. Sci. Ec. Norm. Sup. 13 (1980), 165–210.

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