

 Open access • Journal Article • DOI:10.1088/1361-6544/AB247C

## The local universality of Muttalib-Borodin biorthogonal ensembles with parameter $\theta = \frac{1}{2}$ — Source link

A. B. J. Kuijlaars, L. D. Molag

**Published on:** 01 Oct 2018 - arXiv: Classical Analysis and ODEs

**Topics:** Parametrix, Biorthogonal system and Method of steepest descent

Related papers:

- [Biorthogonal ensembles](#)
- [The local universality of Muttalib-Borodin biorthogonal ensembles with parameter  \$\theta = \frac{1}{2}\$](#)
- [Biorthogonal ensembles with two-particle interactions](#)
- [The local universality of Muttalib-Borodin ensembles when the parameter  \$\theta\$  is the reciprocal of an integer](#)
- [Random matrix models with additional interactions](#)

Share this paper:    

View more about this paper here: <https://typeset.io/papers/the-local-universality-of-muttalib-borodin-biorthogonal-30e2d9mfsq>

# The local universality of Muttalib-Borodin biorthogonal ensembles with parameter $\theta = \frac{1}{2}$

A. B. J. Kuijlaars and L. D. Molag

April 25, 2019

Katholieke Universiteit Leuven, Department of Mathematics,  
Celestijnenlaan 200B box 2400, 3001 Leuven, Belgium.  
E-mail: arno.kuijlaars@kuleuven.be, leslie.molag@kuleuven.be

## Abstract

The Muttalib-Borodin biorthogonal ensemble is a probability density function for  $n$  particles on the positive real line that depends on a parameter  $\theta$  and an external field  $V$ . For  $\theta = \frac{1}{2}$  we find the large  $n$  behavior of the associated correlation kernel with only few restrictions on  $V$ . The idea is to relate the ensemble to a type II multiple orthogonal polynomial ensemble that can in turn be related to a  $3 \times 3$  Riemann-Hilbert problem which we then solve with the Deift-Zhou steepest descent method. The main ingredient is the construction of the local parametrix at the origin, with the help of Meijer G-functions, and its matching condition with a global parametrix. We will present a new iterative technique to obtain the matching condition, which we expect to be applicable in more general situations as well.

## 1 Introduction and statement of results

### 1.1 The Muttalib-Borodin ensemble

The Muttalib-Borodin biorthogonal ensemble with parameter  $\theta > 0$  and weight function  $w$  is the following probability density function for the position of  $n$  particles on the positive half line  $[0, \infty)$

$$\frac{1}{Z_n} \prod_{j < k} (x_k - x_j)(x_k^\theta - x_j^\theta) \prod_{j=1}^n w(x_j). \quad (1.1)$$

We will consider an  $n$ -dependent weight function

$$w(x) = x^\alpha e^{-nV(x)} \quad (1.2)$$

with  $\alpha > -1$  and an external field  $V$  that has enough increase at infinity.

The model is named after Muttalib [36], who introduced it as a simplified model for disordered conductors in the metallic regime and Borodin [10], who obtained profound mathematical results for specific weights, in particular for Laguerre and Jacobi weights. Due to its relation to eigenvalue distributions of random matrix models [12, 25, 31] the model has attracted considerable attention in recent years.

In the large  $n$  limit the particles have an almost sure limiting empirical measure  $\mu^*$  which minimizes a corresponding equilibrium problem [9, 11, 21]. The equilibrium problem was studied in detail in [13]. In [27] it was pointed out that there is an equivalent vector equilibrium problem for vectors of  $q + r - 1$  measures, when  $\theta = \frac{q}{r}$  with  $q, r \in \mathbb{N}$  and  $q < r$ .

The Muttalib-Borodin ensemble is a determinantal point process (in fact a biorthogonal ensemble [10]) and as such it has a corresponding correlation kernel  $K_{V,n}^{\alpha,\theta}$ . The kernel is given by

$$K_{V,n}^{\alpha,\theta}(x, y) = w(y) \sum_{j=0}^{n-1} p_j(x) q_j(y^\theta) \quad (1.3)$$

where  $p_j, q_j$  are polynomials of degree  $j$  satisfying for  $j, k = 0, 1, \dots, n-1$ ,

$$\int_0^\infty p_j(x)q_k(x^\theta)w(x)dx = \delta_{j,k} \quad (1.4)$$

see [36]. Such polynomials exist uniquely (up to multiplicative constants) under certain conditions, as was shown in [13].

Borodin [10] computed the hard edge scaling limit for the Laguerre case, namely if  $V(x) = x$ , then

$$\lim_{n \rightarrow \infty} \frac{1}{n^{1+1/\theta}} K_{V,n}^{\alpha,\theta} \left( \frac{x}{n^{1+1/\theta}}, \frac{y}{n^{1+1/\theta}} \right) = \mathbb{K}^{(\alpha,\theta)}(x, y) \quad (1.5)$$

with limiting kernel

$$\mathbb{K}^{(\alpha,\theta)}(x, y) = \theta y^\alpha \int_0^1 J_{\frac{\alpha+1}{\theta}, \frac{1}{\theta}}(ux) J_{\alpha+1, \theta}((uy)^\theta) u^\alpha du \quad (1.6)$$

where

$$J_{a,b}(x) = \sum_{j=0}^{\infty} \frac{(-x)^j}{j! \Gamma(a + bj)}.$$

is Wright's generalization of the Bessel function. For  $\theta = 1$ , the limit (1.5) reduces to the well-known Bessel kernel in the theory of random matrices.

Several other expressions are known for the kernel (1.6). For example, Zhang [41, Theorem 1.2] gives a double contour integral, and if  $\theta$  or  $1/\theta$  is an integer, then (1.5) can be expressed in terms of Meijer G-functions [31]. These so-called Meijer G-kernels appear in singular value distributions for products of random matrices [2, 3] and in their hard edge scaling limit [4, 5, 23, 25, 26, 32]. See [1] for a survey paper. The bulk and soft edge scaling limits for singular values of Ginibre random matrices are the usual sine and Airy kernels [33], and these classical limits were also established for the Muttalib-Borodin model in the Laguerre case [41].

It is natural to expect that the limit (1.5) is not restricted to the case  $V(x) = x$ , but holds for much more general external fields. In this paper we consider  $\theta = \frac{1}{2}$  and we show that the hard edge scaling limit (1.5) indeed holds for a large class of external fields  $V$ .

## 1.2 Statement of main result

The main assumption on the external field  $V$  concerns the behavior of an equilibrium measure, that we describe first. We use

$$I(\mu) = \iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y), \quad I(\mu, \nu) = \iint \log \frac{1}{|x-y|} d\mu(x) d\nu(y),$$

to denote the logarithmic energy of the measure  $\mu$ , and the mutual energy of two measures  $\mu$  and  $\nu$ , respectively. We also write

$$I_\theta(\mu) = \iint \log \frac{1}{|x^\theta - y^\theta|} d\mu(x) d\mu(y).$$

The equilibrium problem that is relevant for (1.1) with weight (1.2) is the following. Minimize

$$I_{V,\theta}(\mu) = \frac{1}{2} I(\mu) + \frac{1}{2} I_\theta(\mu) + \int V(x) d\mu(x) \quad (1.7)$$

among all probability measures  $\mu$  on  $[0, \infty)$ . We assume that  $V$  is continuous and

$$\lim_{x \rightarrow +\infty} \frac{V(x)}{\log x} = +\infty.$$

Then existence and uniqueness of the minimizer can be shown with usual methods from logarithmic potential theory [15, 38] as the functional  $I_{V,\theta}$  is lower semi-continuous and strictly convex on the set of

all probability measures with finite logarithmic energy. The minimizer  $\mu_{V,\theta}^*$  is called the  $\theta$ -equilibrium measure in the presence of the external field  $V$ . If  $V$  and  $\theta$  are clear from the context then we simply say equilibrium measure.

It is known that the equilibrium measure has a compact support and it is characterized by the Euler-Lagrange variational conditions, see e.g. [13],

$$\int \log|x-s|d\mu_{V,\theta}^*(s) + \int \log|x^\theta - s^\theta|d\mu_{V,\theta}^*(s) \begin{cases} = V(x) + \ell, & x \in \text{supp}(\mu), \\ \leq V(x) + \ell, & x \in [0, \infty), \end{cases} \quad (1.8)$$

for some  $\ell \in \mathbb{R}$ . The equilibrium problem can be analyzed exactly in case  $V(x) = x$ , see [13, section 4.5.1] where this was done for  $\theta > 1$  and in particular  $\theta = 2$ , and see Proposition 2.5 below for  $\theta = \frac{1}{2}$ . The equilibrium measure has a support  $[0, q]$  for some  $q > 0$  with a density that has a square root behavior at  $q$  and behaves like  $s^{-1/(1+\theta)}$  as  $s \rightarrow 0+$ .

This is a generic behavior in a wider class of external fields, as was shown by Claeys and Romano [13, Theorem 1.8], see also Proposition 2.5 below, and our results apply to external fields that are one-cut  $\theta$ -regular in the following sense.

**Definition 1.1.** We call the external field  $V$  one-cut  $\theta$ -regular if the  $\theta$  equilibrium measure  $\mu_{V,\theta}^*$  in external field  $V$  is supported on one interval  $[0, q]$  for some  $q > 0$  with a density that is positive on  $(0, q)$  and satisfies

$$\frac{d\mu_{V,\theta}^*(s)}{ds} = \begin{cases} c_{0,V} s^{-\frac{1}{\theta+1}} (1 + o(1)), & \text{as } s \rightarrow 0+, \\ c_{1,V} (q-s)^{1/2} (1 + o(1)), & \text{as } s \rightarrow q-, \end{cases} \quad (1.9)$$

with positive constants  $c_{0,V}, c_{1,V} > 0$ , and, in addition, the inequality in (1.8) is strict for  $x > q$ .

The conditions of Definition 1.1 hold generically for external fields  $V$  that do not push the equilibrium measure away from 0. See Proposition 2.4 below for a sufficient condition on  $V$  in case  $\theta = \frac{1}{2}$ . For  $V(x) = x$  we are able to determine the density of  $\mu_{V,\frac{1}{2}}^*$  explicitly in Proposition 2.5. From this formula the conditions (1.9) can be verified directly.

The main result of our paper is the following theorem.

**Theorem 1.2.** *Let  $\alpha > -1$ ,  $\theta = \frac{1}{2}$ , and let  $V : [0, \infty) \rightarrow \mathbb{R}$  be a one-cut  $\theta$ -regular external field in the sense of Definition 1.1, which is real analytic on  $[0, \infty)$ . Then for  $x, y \in (0, \infty)$  we have*

$$\lim_{n \rightarrow \infty} \frac{1}{(c_V n)^3} K_{V,n}^{\alpha, \frac{1}{2}} \left( \frac{x}{(c_V n)^3}, \frac{y}{(c_V n)^3} \right) = \mathbb{K}^{(\alpha, \frac{1}{2})}(x, y) \quad (1.10)$$

uniformly on compact subsets of  $(0, \infty)$ , where  $c_V = \frac{2\pi}{\sqrt{3}} c_{0,V}$ , and  $c_{0,V} > 0$  is the constant from (1.9).

In the bulk and at the soft edge we find the usual sine and Airy kernels as the scaling limits of  $K_{V,n}^{\alpha, \frac{1}{2}}$  as  $n \rightarrow \infty$ , but we do not give the details for that.

**Remark 1.3.** The one-cut assumption is not essential, but is included for convenience only. Theorem 1.2 remains valid if the equilibrium measure is supported on several intervals, provided that one of the intervals is of the form  $[0, q]$  and the density of the equilibrium measure has the behavior (1.9) as  $s \rightarrow 0+$ . Similarly, we may relax the regularity assumption. Theorem 1.2 also holds if the density of the equilibrium measure vanishes somewhere in the interior of its support, or if there is a higher order vanishing at a non-zero endpoint of the support, or if the inequality (1.8) is an equality at a finite number of points outside of the support. These singular cases are also present in the classical case  $\theta = 1$  that is connected with orthogonal polynomials and  $2 \times 2$  matrix valued RH problems [16]. Each of these singular cases can be treated by means of special local parametrices, and this would work in a similar way in our situation.

In section 2 we discuss the main ingredients for the proof of Theorem 1.2. These include the equilibrium measure and the interpretation of the biorthogonality property (1.4) as multiple orthogonality in case  $\theta = 1/r$  with an integer  $r \in \mathbb{N}$ . The multiple orthogonality leads to a Riemann-Hilbert (RH)

problem [40] of size  $(r + 1) \times (r + 1)$ . To analyze the large  $n$  limit we are going to apply the Deift-Zhou steepest descent method [18] that was first applied to orthogonal polynomials in the influential papers [16, 17]. The Meijer G-functions appear in the construction of a local parametrix at 0. The model RH problem for Meijer G-functions is discussed in section 3.

The steepest descent analysis is in sections 4–6. The main technical difficulty is the matching of the local parametrix with the global parametrix. We deal with this issue in section 5. The matching is a serious problem for larger size RH problems as has been observed in other situations [6, 19, 20, 29]. As in [6] we develop an iterative technique to improve the matching step by step. The proof of Theorem 1.2 is in the final section 6.

**Remark 1.4.** We conjecture that Theorem 1.2 holds for any  $\theta > 0$ , with the exponent 3 replaced by  $1 + 1/\theta$  as in (1.5). Namely, for real analytic one-cut  $\theta$ -regular external fields  $V$  we expect that there is a constant  $c_V > 0$  such that

$$\lim_{n \rightarrow \infty} \frac{1}{(c_V n)^{1+1/\theta}} K_{V,n}^{\alpha,\theta} \left( \frac{x}{(c_V n)^{1+1/\theta}}, \frac{y}{(c_V n)^{1+1/\theta}} \right) = \mathbb{K}^{(\alpha,\theta)}(x, y). \quad (1.11)$$

It seems likely that the techniques of this paper extend to the case  $\theta = 1/r$  with  $r \in \mathbb{N}$ ,  $r \geq 3$ . As already mentioned, this case leads to a RH problem of size  $(r + 1) \times (r + 1)$ . The main ingredients that we are going to use (see next section) are available for any integer  $r$ . The challenge will be to establish the matching condition which becomes technically more involved.

The case  $\theta = 2$  is related to  $\theta = 1/2$  by a change of variables. Putting  $x_j = y_j^{1/2}$  for  $j = 1, \dots, n$  in (1.1) with  $\theta = 2$  gives a Muttalib-Borodin ensemble with  $\theta = 1/2$  and external field  $V(\sqrt{x})$ . Theorem 1.2 applies provided that  $x \mapsto V(\sqrt{x})$  is real analytic on  $[0, \infty)$ . This in turn means that (1.11) with  $\theta = 2$  holds provided that  $V$  is real analytic and even (and one-cut  $\theta$ -regular). The extension to arbitrary real analytic external fields remains open.

We use the assumption that  $V$  is real analytic at 0 in the proof of Lemma 5.3, and in particular for the identity (5.25).

Throughout the paper we use the principal branches of fractional powers, i.e., with a branch cut on the negative real axis and positive on the positive real axis.

## 2 Preliminaries

We discuss the main ingredients for the proof of Theorem 1.2.

### 2.1 Multiple orthogonal polynomials

The first step in the proof is the observation that the polynomials  $p_j$  that appear in the kernel (1.3) and that satisfy the biorthogonality (1.4) can be viewed as multiple orthogonal polynomials in case  $\theta = 1/r$  and  $r$  is an integer. Without loss of generality we can take these polynomials to be monic. Thus  $p_n$  is a polynomial of degree  $n$  that is characterized by the property that

$$\int_0^\infty p_n(x) x^{k\theta} w(x) dx = 0, \quad k = 0, 1, \dots, n-1. \quad (2.1)$$

**Lemma 2.1.** *Suppose  $\theta = 1/r$  with  $r$  an integer. Then  $p_n$  is the unique monic polynomial of degree  $n$  that satisfies*

$$\int_0^\infty p_n(x) x^k w_i(x) dx = 0, \quad i = 1, \dots, r, \quad k = 0, \dots, \lfloor \frac{n-i}{r} \rfloor, \quad (2.2)$$

where  $w_i(x) = x^{(i-1)\theta} w(x)$  for  $i = 1, \dots, r$ .

*Proof.* If  $i, k$  are as in (2.2) then  $rk + i - 1 \leq n - 1$ , and thus by (2.1) we have

$$\int_0^\infty p_n(x) x^{(rk+i-1)\theta} w(x) dx = 0$$

which reduces to (2.2) in view of the definition of  $w_i(x)$ .

Thus (2.1) implies (2.2) and it is easy to see that the converse holds as well.  $\square$

The relations (2.2) are multiple orthogonality conditions with respect to weight functions  $w_1, w_2, \dots, w_r$ , and with the multi-index  $(n_1, \dots, n_r)$  where  $n_i = \lfloor \frac{n-i}{r} \rfloor + 1$  for  $i = 1, \dots, r$ . While (2.2) is a simple reformulation of (1.4) it has the advantage of leading to a RH problem.

## 2.2 Riemann-Hilbert problem

We state the RH problem for the case  $r = 2$ . In this case the RH problem has size  $3 \times 3$ . For general  $r$  the size is  $(r+1) \times (r+1)$ , see [40].

### Riemann-Hilbert Problem 2.2.

**RH-Y1**  $Y : \mathbb{C} \setminus [0, \infty) \rightarrow \mathbb{C}^{3 \times 3}$  is analytic.

**RH-Y2**  $Y$  has boundary values for  $x \in (0, \infty)$ , denoted by  $Y_+(x)$  (from the upper half plane) and  $Y_-(x)$  (from the lower half plane), and

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x) & x^{\frac{1}{2}}w(x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x > 0. \quad (2.3)$$

**RH-Y3** As  $|z| \rightarrow \infty$

$$Y(z) = \left( \mathbb{I} + \mathcal{O}\left(\frac{1}{z}\right) \right) \begin{pmatrix} z^n & 0 & 0 \\ 0 & z^{-\lceil \frac{n}{2} \rceil} & 0 \\ 0 & 0 & z^{-\lfloor \frac{n}{2} \rfloor} \end{pmatrix}. \quad (2.4)$$

**RH-Y4** As  $z \rightarrow 0$

$$Y(z) = \mathcal{O} \begin{pmatrix} 1 & h_\alpha(z) & h_{\alpha+\frac{1}{2}}(z) \\ 1 & h_\alpha(z) & h_{\alpha+\frac{1}{2}}(z) \\ 1 & h_\alpha(z) & h_{\alpha+\frac{1}{2}}(z) \end{pmatrix} \quad \text{with } h_\alpha(z) = \begin{cases} |z|^\alpha, & \text{if } \alpha < 0, \\ \log |z|, & \text{if } \alpha = 0, \\ 1, & \text{if } \alpha > 0. \end{cases} \quad (2.5)$$

The  $\mathcal{O}$  condition in (2.4) and (2.5) is to be taken entry-wise.

The RH problem is the analogue of the Fokas-Its-Kitaev RH problem [22] for orthogonal polynomials. The condition **RH-Y4** is an endpoint condition that is analogous to the endpoint condition in [30].

There is a unique solution of the RH problem (since the polynomial  $p_n$  uniquely exists [13]) and

$$Y_{11}(z) = p_n(z), \quad Y_{12}(z) = \frac{1}{2\pi i} \int_0^\infty \frac{p_n(x)w(x)}{x-z} dx, \quad Y_{13}(z) = \frac{1}{2\pi i} \int_0^\infty \frac{p_n(x)x^{\frac{1}{2}}w(x)}{x-z} dx,$$

are the entries in the first row of the solution  $Y$ . The other two rows are built in the same way out of suitable polynomials of lower degree. Indeed,  $Y_{21}$  and  $Y_{31}$  are two polynomials of degree  $\leq n-1$ , that are expressible in terms of the biorthogonal polynomials  $p_{n-1}$  and  $p_{n-2}$  of degrees  $n-1$  and  $n-2$ , and for  $j = 2, 3$ ,

$$Y_{j2}(z) = \frac{1}{2\pi i} \int_0^\infty \frac{Y_{j1}(x)w(x)}{x-z} dx, \quad Y_{j3}(z) = \frac{1}{2\pi i} \int_0^\infty \frac{Y_{j1}(x)x^{\frac{1}{2}}w(x)}{x-z} dx.$$

**Lemma 2.3.** *The correlation kernel (1.3) with  $\theta = \frac{1}{2}$  is expressed in terms of the solution of the RH problem for  $Y$  as follows*

$$\begin{aligned} K_{V,n}^{\alpha, \frac{1}{2}}(x, y) &= \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & w(y) & y^{\frac{1}{2}}w(y) \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \frac{y^\beta e^{-nV(y)}}{2\pi i(x-y)} \begin{pmatrix} 0 & y^{-\frac{1}{4}} & y^{\frac{1}{4}} \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \beta = \alpha + \frac{1}{4}, \end{aligned} \quad (2.6)$$

for  $x, y > 0$ .

*Proof.* The first identity is due to [7]. See also [14] for the extension to  $r \geq 3$ . The second identity in (2.6) follows from the form (1.2) of the weight function  $w(y)$ .  $\square$

The first identity in (2.6) is a Christoffel-Darboux formula that expresses the kernel in terms of a finite number of biorthogonal polynomials. See also [13] for a Christoffel-Darboux formula for general rational values of  $\theta$ . For asymptotic analysis the expression (2.6) in terms of the RH problem is very convenient.

### 2.3 Vector equilibrium problem

The Deift-Zhou method of steepest descent consists of a number of explicit transformations of the RH problem. One of the transformations depends on the  $\theta$ -equilibrium measure  $\mu_{V,\theta}^*$  that minimizes the energy functional (1.7).

When  $\theta = 1/r$  with  $r$  an integer, the  $\theta$ -equilibrium measure can also be characterized as the first component of the minimizer of a vector equilibrium problem [27] and this will be important for us. We state it here for the case  $r = 2$ . In case  $r = 2$ , the vector equilibrium problem asks to minimize the energy functional

$$I(\mu) - I(\mu, \nu) + I(\nu) + \int V(x) d\mu(x) \quad (2.7)$$

among all pairs of measures  $(\mu, \nu)$  such that

- $\mu$  is a probability measure on  $[0, \infty)$ ,
- $\nu$  is a measure on  $(-\infty, 0]$  of total mass  $1/2$ .

For general  $\theta = 1/r$  the vector equilibrium problem has  $r$  measures.

It is shown in [27] that there is a unique minimizer  $(\mu^*, \nu^*)$  and the first component  $\mu^*$  coincides with the minimizer  $\mu_{V,\frac{1}{2}}^*$  of the  $\theta$ -energy functional (1.7) with  $\theta = \frac{1}{2}$ . In addition, the measure  $\nu^*$  has full support  $\text{supp}(\nu^*) = (-\infty, 0]$ , and the variational conditions

$$2 \int \log|x-s| d\mu^*(s) - \int \log|x-s| d\nu^*(s) \begin{cases} = V(x) + \ell, & x \in \text{supp}(\mu^*), \\ \leq V(x) + \ell, & x \in [0, \infty), \end{cases} \quad (2.8)$$

$$2 \int \log|x-s| d\nu^*(s) - \int \log|x-s| d\mu^*(s) = 0, \quad x \in (-\infty, 0], \quad (2.9)$$

are satisfied, with a constant  $\ell$  that is possibly different from the one appearing in (1.8). The identity (2.9) expresses that  $2\nu^*$  is the balayage of  $\mu^*$  onto the negative real axis.

We use the two measures from the vector equilibrium problem in the steepest descent analysis that will follow. We also use the vector equilibrium problem in the proof of the following sufficient condition for an external field to be one-cut regular.

**Proposition 2.4.** *Suppose  $V$  is twice differentiable on  $[0, \infty)$  such that  $xV'(x)$  is increasing for  $x > 0$ . Then  $V$  is one-cut  $\frac{1}{2}$ -regular in the sense of Definition 1.1.*

*Proof.* Because of (2.7) we have that  $\mu^* = \mu_{V,\frac{1}{2}}^*$  is the minimizer of the energy functional  $I(\mu) + \int \tilde{V} d\mu$  with the modified potential

$$\tilde{V}(x) = V(x) + \int \log(x-s) d\nu^*(s).$$

For every  $s < 0$ , we have that  $x \mapsto \frac{x}{x-s}$  is increasing on  $[0, \infty)$ . Since  $\nu^*$  is supported on  $(-\infty, 0]$ , it then easily follows from this and the assumption of the proposition that  $x \mapsto x\tilde{V}'(x) = xV'(x) + \int \frac{x}{x-s} d\nu^*(s)$  is strictly increasing for  $x \in [0, \infty)$ . It then follows by well-known results on equilibrium measures in external fields that  $\text{supp}(\mu^*) = [0, q]$  is an interval containing 0, see e.g. [38, Theorem IV.1.10 (c)].

Knowing that the support is an interval we can use [13, Theorem 1.11 and formula (1.28)] to conclude that the density of  $\mu^*$  is positive on  $(0, q)$  and has the required endpoint behavior (1.9). Actually, the result in [13] is stated for  $\theta > 1$ , but it also applies to  $0 < \theta < 1$ .

It remains to show that the inequality (1.8) is strict for  $x > q$ . From (2.8) we have

$$\tilde{V}(x) + \ell - 2 \int \log(x-s) d\mu^*(s) \geq 0, \quad x \geq q$$

with equality for  $x = q$ . Then the derivative is non-negative at  $x = q$ , which implies

$$x\tilde{V}'(x) - 2 \int \frac{x}{x-s} d\mu^*(s) \geq 0 \quad (2.10)$$

for  $x = q$ . The function in the left-hand side of (2.10) is strictly increasing for  $x \geq q$ . To see this we use that  $x\tilde{V}'(x)$  is strictly increasing and the fact that  $x \mapsto \frac{x}{x-s}$ ,  $x > q$ , is decreasing for every  $s \in [0, q]$ . It follows that strict inequality holds in (2.10) for  $x > q$ , which in turn leads to the property that (1.8) is strict for  $x > q$ . This concludes the proof of the proposition.  $\square$

For  $V(x) = x$  we can calculate the equilibrium measure explicitly. For this  $V$  the  $\theta$ -equilibrium measure with  $\theta = 2$  was computed in [34]. See also [24] for similar calculations.

**Proposition 2.5.** *When  $V(x) = x$  and  $\theta = \frac{1}{2}$ , the equilibrium measure of the Muttalib-Borodin ensemble is supported on  $[0, \frac{27}{8}]$  with density*

$$\frac{d\mu_{V, \frac{1}{2}}^*(s)}{ds} = \frac{\sqrt{3}}{4\pi s^{2/3}} \left[ \left( 1 - \frac{4s}{3} + \frac{8s^2}{27} + \sqrt{1 - \frac{8s}{27}} \right)^{1/3} + \left( -1 + \frac{4s}{3} - \frac{8s^2}{27} + \sqrt{1 - \frac{8s}{27}} \right)^{1/3} \right] \quad 0 < s < \frac{27}{8}. \quad (2.11)$$

In particular, (1.9) holds with constants  $q = \frac{27}{8}$ ,  $c_{0,V} = \frac{\sqrt{3}}{2^{5/3}\pi}$  and  $c_{1,V} = \frac{16\sqrt{2}}{81\pi}$ .

*Proof.* By Proposition 2.4 we have  $\text{supp}(\mu^*) = [0, q]$  for some  $q > 0$ . By differentiating the identities in (2.8)-(2.9) we see that the functions  $F_1(z) = \int \frac{d\mu^*(s)}{z-s}$  and  $F_2(z) = \int \frac{d\nu^*(s)}{z-s}$  satisfy

$$\begin{aligned} F_{1,+}(x) + F_{1,-}(x) - F_2(x) &= V'(x) = 1, & x \in (0, q), \\ F_{2,+}(x) + F_{2,-}(x) - F_1(x) &= 0, & x \in (-\infty, 0). \end{aligned} \quad (2.12)$$

Let  $\mathfrak{R}$  be the compact three sheeted Riemann surface with sheets  $\mathfrak{R}_0 = \overline{\mathbb{C}} \setminus [0, q]$ ,  $\mathfrak{R}_1 = \mathbb{C} \setminus (-\infty, q]$  and  $\mathfrak{R}_2 = \mathbb{C} \setminus (-\infty, 0]$  with sheet structure as in Figure 1. Then it follows from (2.12) that  $\Psi$  defined by

$$\Psi(z) = \begin{cases} \Psi_0(z) = 1 - F_1(z), & \text{if } z \in \mathfrak{R}_0, \\ \Psi_1(z) = F_1(z) - F_2(z), & \text{if } z \in \mathfrak{R}_1, \\ \Psi_2(z) = F_2(z), & \text{if } z \in \mathfrak{R}_2, \end{cases} \quad (2.13)$$

is meromorphic on  $\mathfrak{R}$ . We note the asymptotic behaviors

$$\Psi_0(z) = 1 - \frac{1}{z} + \mathcal{O}(z^{-2}), \quad \Psi_1(z) = \frac{1}{2z} + \mathcal{O}(z^{-3/2}), \quad \Psi_2(z) = \frac{1}{2z} + \mathcal{O}(z^{-3/2}), \quad (2.14)$$

as  $z \rightarrow \infty$ . Thus  $\Psi$  has a double zero at the point at infinity that is common to sheets  $\mathfrak{R}_1$  and  $\mathfrak{R}_2$ . From Proposition 2.4 we know that  $\mu^*$  has a density that behaves like  $c_{0,V} s^{-2/3}$  as  $s \rightarrow 0+$ , see (1.9). Then it can be shown that  $F_1(z) \sim -cz^{-2/3}$  as  $z \rightarrow 0$  with  $c = \frac{2\pi}{\sqrt{3}} c_{0,V}$ , which means that  $\Psi$  has a double pole at  $z = 0$ , as  $z = 0$  is a double branch point of the Riemann surface. There are no other poles and zeros of  $\Psi$ .

From (2.14) we obtain

$$\begin{aligned} \Psi_0(z) + \Psi_1(z) + \Psi_2(z) &= 1 + \mathcal{O}(z^{-3/2}), \\ \Psi_0(z)\Psi_1(z) + \Psi_0(z)\Psi_2(z) + \Psi_1(z)\Psi_2(z) &= \frac{1}{z} + \mathcal{O}(z^{-2}) \\ \Psi_0(z)\Psi_1(z)\Psi_2(z) &= \frac{1}{4z^2} + \mathcal{O}(z^{-3}) \end{aligned}$$



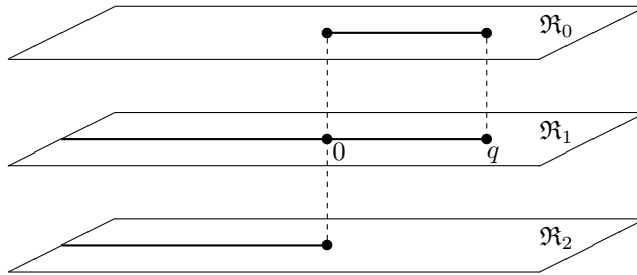


Figure 1: The Riemann surface  $\mathfrak{R}$  used in the proof of Proposition 2.5. The three sheets are defined by  $\mathfrak{R}_0 = \mathbb{C} \setminus [0, q]$ ,  $\mathfrak{R}_1 = \mathbb{C} \setminus (-\infty, q]$  and  $\mathfrak{R}_2 = \mathbb{C} \setminus (-\infty, 0]$ . The sheets are connected in the usual crosswise manner.

as  $z \rightarrow \infty$ . The three above functions are meromorphic in the full complex plane with a possible pole at  $z = 0$  only. Because  $\Psi_j(z) = O(z^{-2/3})$  as  $z \rightarrow 0$  for every  $j = 1, 2, 3$ , it follows that all  $\mathcal{O}$  terms vanish identically. Thus

$$\prod_{j=1}^3 (\zeta - \Psi_j(z)) = \zeta^3 - \zeta^2 + \frac{1}{z}\zeta - \frac{1}{4z^2}$$

and

$$z^2\zeta^3 - z^2\zeta^2 + z\zeta - \frac{1}{4} = 0 \quad (2.15)$$

is the algebraic equation satisfied by  $\Psi$ .

The discriminant of (2.15) is  $z^4 \frac{8z-27}{16}$  and it follows that there is a branch point at  $z = \frac{27}{8}$ . For real  $z > \frac{27}{8}$  there are three real solutions of (2.15), and for real  $z \in (0, \frac{27}{8})$  there is one real solution (namely  $\zeta = \Psi_2(z)$ ) and two complex conjugate non-real solutions  $\zeta = \Psi_j(z)$ ,  $j = 0, 1$ .

The density of  $\mu^*$  satisfies

$$\frac{d\mu^*}{ds} = -\frac{1}{\pi} \operatorname{Im} F_{1,+}(s) = \frac{1}{\pi} \operatorname{Im} \Psi_{0,+}(s) \quad (2.16)$$

because of (2.13) and the Stieltjes inversion formula. Thus  $\mu^*$  is supported on  $[0, \frac{27}{8}]$ . With Cardano's method for solving a cubic equation we find the solutions of (2.15) for  $z = s \in (0, \frac{27}{8})$ .  $\Psi_0(s)$  is the unique solution with positive imaginary part and the calculations result in the formula (2.11).  $\square$

## 2.4 Meijer G-functions

The final main ingredient concerns the Meijer G-functions, that will be used in the construction of a local parametrix at 0. In [32] the limiting kernel  $\mathbb{K}^{(\alpha, \frac{1}{2})}(x, y)$  from our main result Theorem 1.2 was expressed in terms of Meijer G-functions, see also Theorem 2.8 below.

The appearance of Meijer G-functions, or equivalently, generalized hypergeometric functions, can also be expected from the recent paper [39] on multiple Laguerre polynomials. In case  $w(x) = x^\alpha e^{-nx}$  the multiple orthogonal polynomials from (2.2) are such multiple Laguerre polynomials, and it was shown in [39, Theorem 3] that in a suitable scaling regime these polynomials tend to a generalized hypergeometric function

$${}_0F_r \left( \begin{matrix} - \\ 1 + \alpha_1, \dots, 1 + \alpha_r \end{matrix}; -z \right) = \sum_{k=0}^{\infty} \frac{1}{(1 + \alpha_1)_k \cdots (1 + \alpha_r)_k} \frac{(-z)^k}{k!}$$

where  $\alpha_j = \alpha + \frac{j-1}{r}$  for  $j = 1, \dots, r$ . This is the so-called Mehler-Heine asymptotics and it captures the behavior of the polynomials near the origin. The case  $r = 2$  is relevant for the present paper and then we obtain

$$\phi_0(z) := {}_0F_2 \left( \begin{matrix} - \\ 1 + \alpha, \frac{3}{2} + \alpha \end{matrix}; -z \right) \quad (2.17)$$

which is a solution of the third order differential equation (we use  $\vartheta = z \frac{d}{dz}$  as in [37], for example),

$$\vartheta(\vartheta + \alpha)(\vartheta + \alpha + \frac{1}{2})\phi + z\phi = 0, \quad \vartheta = z \frac{d}{dz}. \quad (2.18)$$

In case  $2\alpha \notin \mathbb{Z}$ , other solutions of (2.18) are  $z^{-\alpha} {}_0F_2 \left( \begin{matrix} - \\ 1 - \alpha, \frac{3}{2} \end{matrix}; -z \right)$ , and  $z^{-\alpha - \frac{1}{2}} {}_0F_2 \left( \begin{matrix} - \\ \frac{1}{2} - \alpha, \frac{1}{2} \end{matrix}; -z \right)$ . Together with  $\phi_0$  they are a basis of all solutions as they come from applying the Frobenius method to (2.18). In case  $2\alpha \in \mathbb{Z}$  (the resonant case), this method produces series solutions with additional log terms. The Frobenius method provides a basis that is suitable for the study of solutions near  $z = 0$ .

We could have worked with the above  ${}_0F_2$  hypergeometric functions. However, we prefer to use instead a representation of solutions of (2.18) in terms of Meijer G-functions, since this representation is more convenient to describe the behavior of solutions as  $z \rightarrow \infty$ .

The general Meijer G-function is defined by the following contour integral:

$$G_{p,q}^{m,n} \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z \right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j + s) \prod_{j=1}^n \Gamma(1 - a_j - s)}{\prod_{j=m+1}^q \Gamma(1 - b_j - s) \prod_{j=n+1}^p \Gamma(a_j + s)} z^{-s} ds, \quad (2.19)$$

where  $\Gamma$  denotes the gamma function and empty products in (2.19) should be interpreted as 1, as usual. The numbers  $m, n, p$ , and  $q$  are integers with  $0 \leq m \leq q$  and  $0 \leq n \leq p$ . See [35, section 5.2] for conditions on the parameters  $a_1, \dots, a_p$  and  $b_1, \dots, b_q$  and the contour  $L$ .

By [37, formula 16.18.1] and (2.17) we have

$$\begin{aligned} \phi_0(z) &= \Gamma(1 + \alpha) \Gamma(\frac{3}{2} + \alpha) G_{0,3}^{1,0} \left( \begin{matrix} - \\ 0, -\alpha, -\alpha - \frac{1}{2} \end{matrix} \middle| z \right) \\ &= \frac{\Gamma(1 + \alpha) \Gamma(\frac{3}{2} + \alpha)}{2\pi i} \int_L \frac{\Gamma(s)}{\Gamma(1 + \alpha - s) \Gamma(\frac{3}{2} + \alpha - s)} z^{-s} ds \end{aligned} \quad (2.20)$$

with a Hankel contour  $L$  that encircles the negative real axis. Another solution of (2.18), see [37, formula 16.21.1], is

$$G_{0,3}^{3,0} \left( \begin{matrix} - \\ 0, -\alpha, -\alpha - \frac{1}{2} \end{matrix} \middle| z \right) = \frac{1}{2\pi i} \int_L \Gamma(s) \Gamma(s - \alpha) \Gamma(s - \alpha - \frac{1}{2}) z^{-s} ds, \quad (2.21)$$

where now  $L$  encircles the interval  $(-\infty, \max(0, \alpha + \frac{1}{2})]$  in the complex  $s$ -plane (or alternatively it could be a vertical line  $\operatorname{Re} s = c > \max(0, \alpha + \frac{1}{2})$  in the complex  $s$ -plane).

This solution behaves like  $e^{-3z^{\frac{1}{3}}}$  as  $z \rightarrow \infty$  with  $-4\pi < \arg z < 4\pi$ . In particular, it is the recessive solution of (2.18) in  $-\pi < \arg z < \pi$ . More precise asymptotics are given in (3.3) below. Analytic continuations of (2.21) to other Riemann sheets provide other solutions.

Besides  $\phi_0$  we will use the following four solutions of (2.18), each defined in the sector  $-\pi < \arg z < \pi$ .

$$\phi_1(z) = ie^{2\pi i \alpha} G_{0,3}^{3,0} \left( \begin{matrix} - \\ 0, -\alpha, -\alpha - \frac{1}{2} \end{matrix} \middle| ze^{2\pi i} \right), \quad (2.22)$$

$$\phi_2(z) = -ie^{-2\pi i \alpha} G_{0,3}^{3,0} \left( \begin{matrix} - \\ 0, -\alpha, -\alpha - \frac{1}{2} \end{matrix} \middle| ze^{-2\pi i} \right), \quad (2.23)$$

$$\phi_3(z) = G_{0,3}^{3,0} \left( \begin{matrix} - \\ 0, -\alpha, -\alpha - \frac{1}{2} \end{matrix} \middle| z \right), \quad (2.24)$$

$$\phi_4(z) = \phi_1(z) + \phi_2(z). \quad (2.25)$$

Here the notation  $ze^{2\pi i}$  means that we have analytically continued  $\phi_3$  along a counterclockwise loop around the origin. Likewise, the notation  $ze^{-2\pi i}$  means that we have analytically continued  $\phi_3$  along a clockwise loop around the origin. Indeed, these expressions are also given by (2.21) by picking the argument of  $z$  appropriately.

**Lemma 2.6.** *We have*

$$\phi_4(z) = -\frac{4\pi^2}{\Gamma(1 + \alpha) \Gamma(\frac{3}{2} + \alpha)} \phi_0(z) \quad (2.26)$$

*Proof.* We use the reflection formula  $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$  to write the combination of gamma functions appearing in (2.20) as

$$\begin{aligned} \frac{\Gamma(s)}{\Gamma(1+\alpha-s)\Gamma(\frac{3}{2}+\alpha-s)} &= \Gamma(s)\Gamma(s-\alpha)\Gamma(s-\alpha-\frac{1}{2}) \frac{\sin \pi(s-\alpha) \sin \pi(s-\alpha-\frac{1}{2})}{\pi^2} \\ &= \Gamma(s)\Gamma(s-\alpha)\Gamma(s-\alpha-\frac{1}{2}) \frac{1}{(2\pi i)^2} \left( e^{2\pi i(s-\beta)} + e^{-2\pi i(s-\beta)} \right). \end{aligned}$$

Using this in (2.20) and comparing with (2.21), (2.22), (2.23), we see that (2.20) is equal to  $-\frac{1}{4\pi^2}(\phi_1(z) + \phi_2(z))$ . Then (2.26) follows by (2.17), (2.25), and (2.20).  $\square$

We collect the solutions into a matrix valued function that is piecewise analytic in the complex plane.

**Definition 2.7.** We define (where  $\vartheta = z \frac{d}{dz}$  as in (2.18))

$$\Phi_\alpha(z) = \begin{cases} \begin{pmatrix} \phi_1(z) & \phi_2(z) & \phi_3(z) \\ \vartheta\phi_1(z) & \vartheta\phi_2(z) & \vartheta\phi_3(z) \\ \vartheta^2\phi_1(z) & \vartheta^2\phi_2(z) & \vartheta^2\phi_3(z) \end{pmatrix}, & 0 < \arg(z) < \frac{\pi}{2}, \\ \begin{pmatrix} \phi_4(z) & \phi_2(z) & \phi_3(z) \\ \vartheta\phi_4(z) & \vartheta\phi_2(z) & \vartheta\phi_3(z) \\ \vartheta^2\phi_4(z) & \vartheta^2\phi_2(z) & \vartheta^2\phi_3(z) \end{pmatrix}, & \frac{\pi}{2} < \arg(z) < \pi, \\ \begin{pmatrix} \phi_2(z) & -\phi_1(z) & \phi_3(z) \\ \vartheta\phi_2(z) & -\vartheta\phi_1(z) & \vartheta\phi_3(z) \\ \vartheta^2\phi_2(z) & -\vartheta^2\phi_1(z) & \vartheta^2\phi_3(z) \end{pmatrix}, & -\frac{\pi}{2} < \arg(z) < 0, \\ \begin{pmatrix} \phi_4(z) & -\phi_1(z) & \phi_3(z) \\ \vartheta\phi_4(z) & -\vartheta\phi_1(z) & \vartheta\phi_3(z) \\ \vartheta^2\phi_4(z) & -\vartheta^2\phi_1(z) & \vartheta^2\phi_3(z) \end{pmatrix}, & -\pi < \arg(z) < -\frac{\pi}{2}. \end{cases} \quad (2.27)$$

The definition is such that  $\Phi_\alpha$  satisfies a convenient RH problem that will be discussed in the next section.

We conclude this preliminary section by announcing the following main result. It expresses the limiting kernel  $\mathbb{K}^{(\alpha, \frac{1}{2})}$ , see (1.6) and (1.10), in terms of  $\Phi_\alpha$  and its inverse matrix  $\Phi_\alpha^{-1}$ .

**Theorem 2.8.** *We have for  $x, y > 0$ ,*

$$\mathbb{K}^{(\alpha, \frac{1}{2})}(x, y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} \Phi_{\alpha,+}^{-1}(y) \Phi_{\alpha,+}(x) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad (2.28)$$

We prove Theorem 2.8 in the next section.

Theorem 1.2 follows from a steepest descent analysis of the RH problem for  $Y$ , where  $\Phi_\alpha$  is used to construct a local parametrix near 0. In the appropriate scaling the expression (2.6) for the correlation kernel tends to the kernel (2.28) as given in terms of  $\Phi_\alpha$ .

### 3 RH problem for $\Phi_\alpha$ and proof of Theorem 2.8

We discuss the RH problem satisfied by  $\Phi_\alpha$ .

#### 3.1 Jump conditions

The definition (2.27) is such that  $\Phi_\alpha$  has the following jumps along each of the contours in  $\Sigma_\Phi = \mathbb{R} \cup i\mathbb{R}$ ,

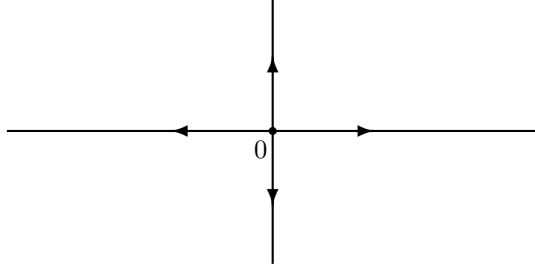


Figure 2: The directions of the contour  $\Sigma_\Phi = \mathbb{R} \cup i\mathbb{R}$  used in Lemma 3.1 and in the model RH problem.

**Lemma 3.1.**  $\Phi_\alpha$  is analytic in each of the four quadrants with constant jumps (all contours are oriented away from the origin, see Figure 2)

$$\Phi_{\alpha,+}(z) = \Phi_{\alpha,-}(z) \times \begin{cases} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \mathbb{R}^+, \\ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in i\mathbb{R}^\pm, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & ie^{2\pi i\alpha} \\ 0 & -ie^{2\pi i\alpha} & 0 \end{pmatrix}, & z \in \mathbb{R}^- \end{cases} \quad (3.1)$$

*Proof.* We only have to verify the jump conditions for the first row, since the jump matrices are constant along each of the four contours. Indeed, once we have the jump conditions for the first row, the conditions for the other rows will follow by simple differentiation and multiplying by  $z$ .

The jumps on  $\mathbb{R}^+$  and on  $i\mathbb{R}^\pm$  follow immediately from the definition (2.27) combined with (2.25). To check the jump on  $\mathbb{R}^-$ , we note that  $\phi_4$  is entire because by (2.26) and (2.17) it is a multiple of the  ${}_0F_2$  hypergeometric function, which is entire. Thus  $\phi_{4,+} = \phi_{4,-}$  on  $\mathbb{R}^-$ . The identities  $\phi_{1,+} = e^{2\pi i\beta}\phi_{3,-}$  and  $\phi_{3,+} = e^{2\pi i\beta}\phi_{2,-}$  are almost immediate from (2.22)–(2.25). For example if  $z = -x$  with  $x > 0$ , then by (2.22) and (2.24)

$$\phi_{1,+}(z) = \phi_{1,+}(xe^{-\pi i}) = e^{2\pi i\beta}G_\alpha(xe^{\pi i}) = e^{2\pi i\beta}\phi_{3,-}(z).$$

This shows that the jump conditions (3.1) on  $\mathbb{R}^-$  are indeed satisfied.  $\square$

### 3.2 Behavior as $z \rightarrow 0$

**Lemma 3.2.** As  $z \rightarrow 0$  we have the behavior

$$\Phi_\alpha(z) = \begin{cases} \mathcal{O} \begin{pmatrix} h_{-\alpha-\frac{1}{2}}(z) & h_{-\alpha-\frac{1}{2}}(z) & h_{-\alpha-\frac{1}{2}}(z) \\ h_{-\alpha-\frac{1}{2}}(z) & h_{-\alpha-\frac{1}{2}}(z) & h_{-\alpha-\frac{1}{2}}(z) \\ h_{-\alpha-\frac{1}{2}}(z) & h_{-\alpha-\frac{1}{2}}(z) & h_{-\alpha-\frac{1}{2}}(z) \end{pmatrix}, & \text{for } \operatorname{Re} z > 0, \\ \mathcal{O} \begin{pmatrix} 1 & h_{-\alpha-\frac{1}{2}}(z) & h_{-\alpha-\frac{1}{2}}(z) \\ 1 & h_{-\alpha-\frac{1}{2}}(z) & h_{-\alpha-\frac{1}{2}}(z) \\ 1 & h_{-\alpha-\frac{1}{2}}(z) & h_{-\alpha-\frac{1}{2}}(z) \end{pmatrix}, & \text{for } \operatorname{Re} z < 0, \end{cases} \quad (3.2)$$

where  $h_{-\alpha-\frac{1}{2}}$  is as in (2.5).

*Proof.* Because of (2.26) and (2.17) the solution  $\phi_4$  is entire and thus bounded at 0. By the definition (2.27) the function  $\phi_4$  appears in the first column of  $\Phi_\alpha$  in the left half-plane. This accounts for the  $\mathcal{O}(1)$  terms in (3.2) as  $z \rightarrow 0$  with  $\operatorname{Re} z < 0$ .

The remaining terms come from the behavior of the functions  $\phi_j$ ,  $j = 1, 2, 3$  as  $z \rightarrow 0$ . These are solutions of the linear differential equation (2.18), which can be solved by the method of Frobenius. The indicial equation has roots  $0$ ,  $-\alpha$  and  $-\alpha - \frac{1}{2}$ . If  $\alpha > -\frac{1}{2}$  then all solutions behave as  $\mathcal{O}(z^{-\alpha-\frac{1}{2}})$  as  $z \rightarrow 0$ . Also  $\vartheta\phi_j$  and  $\vartheta^2\phi_j$  have the same behavior and (3.2) follows.

Similar considerations give (3.2) in case  $\alpha = -\frac{1}{2}$ , in which case there is a generic  $\mathcal{O}(\log z)$  behavior as  $z \rightarrow 0$ , or  $-1 < \alpha < -\frac{1}{2}$ , in which case solutions remain bounded as  $z \rightarrow 0$ .  $\square$

**Remark 3.3.** The behavior (3.2) suffices for the purposes of this paper. However it is possible to obtain more precise information on the behavior near 0 by means of connection matrices  $C_j$ ,  $j = 1, 2, 3, 4$ , that are such that

$$\Phi_\alpha(z)C_j = \mathcal{O} \begin{pmatrix} 1 & z^{-\alpha} & z^{-\alpha-\frac{1}{2}} \\ 1 & z^{-\alpha} & z^{-\alpha-\frac{1}{2}} \\ 1 & z^{-\alpha} & z^{-\alpha-\frac{1}{2}} \end{pmatrix}$$

as  $z \rightarrow 0$  in the  $j$ th quadrant. For  $\alpha = 0$  or  $\alpha = -\frac{1}{2}$  there are additional log terms in either the second or the third columns.

### 3.3 Behavior as $z \rightarrow \infty$

The behavior of  $\Phi_\alpha(z)$  as  $z \rightarrow \infty$  will be deduced from the known asymptotic behavior of Meijer G-functions given in [35, Theorem 5, page 179]. It gives us

$$G_{0,3}^{3,0} \left( 0, -\alpha, -\alpha - \frac{1}{2} \middle| z \right) \sim \frac{2\pi}{\sqrt{3}} z^{-\gamma} e^{-3z^{\frac{1}{3}}} \left( 1 + \sum_{k=1}^{\infty} M_k z^{-\frac{k}{3}} \right), \quad -4\pi < \arg z < 4\pi \quad (3.3)$$

as  $z \rightarrow \infty$  with explicit constants (with  $\beta = \alpha + \frac{1}{4}$  as in (2.6))

$$\gamma = \frac{2}{3}\alpha + \frac{1}{2} = \frac{2}{3}\beta + \frac{1}{3}, \quad (3.4)$$

$$M_1 = \frac{1}{3}\alpha^2 + \frac{1}{6}\alpha - \frac{1}{36}, \quad (3.5)$$

$$M_2 = \frac{1}{18}\alpha^4 + \frac{1}{54}\alpha^3 - \frac{17}{216}\alpha^2 - \frac{1}{54}\alpha + \frac{25}{2592}, \quad (3.6)$$

and other  $M_k$  can be calculated, if necessary.

**Lemma 3.4.** Let  $\beta = \alpha + \frac{1}{4}$  and  $\omega = e^{2\pi i/3}$ . Then we have as  $z \rightarrow \infty$ ,

$$T_\alpha \Phi_\alpha(z) = \frac{2\pi}{\sqrt{3}} z^{-\frac{2\beta}{3}} (\mathbb{I} + A_\alpha z^{-1} + \mathcal{O}(z^{-2})) L_\alpha(z) \times \begin{cases} \begin{pmatrix} e^{-3\omega z^{1/3}} & 0 & 0 \\ 0 & e^{-3\omega^2 z^{1/3}} & 0 \\ 0 & 0 & e^{-3z^{1/3}} \end{pmatrix}, & \text{Im}(z) > 0, \\ \begin{pmatrix} e^{-3\omega^2 z^{1/3}} & 0 & 0 \\ 0 & e^{-3\omega z^{1/3}} & 0 \\ 0 & 0 & e^{-3z^{1/3}} \end{pmatrix}, & \text{Im}(z) < 0, \end{cases} \quad (3.7)$$

where

$$L_\alpha(z) = \begin{pmatrix} z^{-\frac{1}{3}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{\frac{1}{3}} \end{pmatrix} \times \begin{cases} \begin{pmatrix} \omega^2 & \omega & 1 \\ 1 & 1 & 1 \\ \omega & \omega^2 & 1 \end{pmatrix} \begin{pmatrix} e^{\frac{2\pi i\beta}{3}} & 0 & 0 \\ 0 & e^{-\frac{2\pi i\beta}{3}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \text{Im}(z) > 0, \\ \begin{pmatrix} \omega & -\omega^2 & 1 \\ 1 & -1 & 1 \\ \omega^2 & -\omega & 1 \end{pmatrix} \begin{pmatrix} e^{-\frac{2\pi i\beta}{3}} & 0 & 0 \\ 0 & e^{\frac{2\pi i\beta}{3}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \text{Im}(z) < 0 \end{cases} \quad (3.8)$$

$A_\alpha$  is a certain constant matrix and

$$T_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ t_1 & -1 & 0 \\ t_2 & t_3 & 1 \end{pmatrix} \quad (3.9)$$

is a lower triangular matrix with entries

$$t_1 = -\gamma - M_1, \quad t_2 = \gamma(\gamma - \frac{1}{3}) + M_1(M_1 + \gamma - \frac{2}{3}) - M_2, \quad t_3 = 2\gamma - \frac{1}{3} + M_1 \quad (3.10)$$

where  $\gamma$ ,  $M_1$  and  $M_2$  are given in (3.4), (3.5), (3.6).

*Proof.* Let us focus on the last column of  $\Phi_\alpha$ , see (2.24) and (2.27), with  $-\pi < \arg z < \pi$ . Then by (2.24) and (3.3) we have

$$\phi_3(z) = \frac{2\pi}{\sqrt{3}} z^{-\gamma} e^{-3z^{\frac{1}{3}}} \left( 1 + M_1 z^{-\frac{1}{3}} + M_2 z^{-\frac{2}{3}} + \mathcal{O}(z^{-1}) \right) \quad (3.11)$$

The expansion (3.3) can be differentiated termwise and we find

$$-\vartheta\phi_3(z) = \frac{2\pi}{\sqrt{3}} z^{-\gamma} e^{-3z^{\frac{1}{3}}} \left( z^{\frac{1}{3}} + (M_1 + \gamma) + (M_2 + (\gamma + \frac{1}{3})M_1) z^{-\frac{1}{3}} + \mathcal{O}(z^{-\frac{2}{3}}) \right) \quad (3.12)$$

$$\vartheta^2\phi_3(z) = \frac{2\pi}{\sqrt{3}} z^{-\gamma} e^{-3z^{\frac{1}{3}}} \left( z^{\frac{2}{3}} + (M_1 + 2\gamma - \frac{1}{3}) z^{\frac{1}{3}} + (M_2 + (2\gamma + \frac{1}{3})M_1 + \gamma^2) + \mathcal{O}(z^{-\frac{1}{3}}) \right). \quad (3.13)$$

If we choose constants  $t_1, t_2, t_3$  as in (3.10) then we get from (3.11), (3.12), (3.13),

$$t_1\phi_3(z) - \vartheta\phi_3(z) = \frac{2\pi}{\sqrt{3}} z^{-\gamma} e^{-3z^{\frac{1}{3}}} \left( z^{\frac{1}{3}} + \mathcal{O}(z^{-\frac{1}{3}}) \right), \quad (3.14)$$

$$t_2\phi_3(z) + t_3\vartheta\phi_3(z) + \vartheta^2\phi_3(z) = \frac{2\pi}{\sqrt{3}} z^{-\gamma} e^{-3z^{\frac{1}{3}}} \left( z^{\frac{2}{3}} + \mathcal{O}(z^{-\frac{1}{3}}) \right). \quad (3.15)$$

With the matrix  $T_\alpha$  from (3.9) we write (3.11), (3.14) and (3.15) as

$$T_\alpha \begin{pmatrix} \phi_3(z) \\ \vartheta\phi_3(z) \\ \vartheta^2\phi_3(z) \end{pmatrix} = \frac{2\pi}{\sqrt{3}} z^{-\gamma + \frac{1}{3}} e^{-3z^{\frac{1}{3}}} \begin{pmatrix} z^{-\frac{1}{3}} + \mathcal{O}(z^{-\frac{2}{3}}) \\ 1 + \mathcal{O}(z^{-\frac{2}{3}}) \\ z^{\frac{1}{3}} + \mathcal{O}(z^{-\frac{2}{3}}) \end{pmatrix},$$

which may also be written as

$$T_\alpha \Phi_\alpha(z) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{2\pi}{\sqrt{3}} z^{-\gamma + \frac{1}{3}} (\mathbb{I} + \mathcal{O}(z^{-1})) L_\alpha(z) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-3z^{\frac{1}{3}}} \quad (3.16)$$

by the definitions (3.8) of  $L_\alpha$  and (2.27) of  $\Phi_\alpha$ . This is the equality (3.7) for the third column with  $\mathbb{I} + A_\alpha z^{-1} + \mathcal{O}(z^{-2})$  replaced by  $\mathbb{I} + \mathcal{O}(z^{-1})$ . The equality of the other two columns follows in a similar manner, and with the same matrix  $T_\alpha$ , although the details are a bit more involved. We apply (3.3) with  $ze^{2\pi i}$  and  $ze^{-2\pi i}$  instead of  $z$ , and this leads to the various factors of  $\omega$  and  $\omega^2$  in (3.7) and (3.8). The result is that (3.7) holds with  $\mathbb{I} + \mathcal{O}(z^{-1})$  instead of  $\mathbb{I} + A_\alpha z^{-1} + \mathcal{O}(z^{-2})$ .

To obtain the sharper order estimate, we consider

$$\widehat{L}_\alpha(z) = \frac{2\pi}{\sqrt{3}} z^{-\frac{2\beta}{3}} L_\alpha(z) \times \begin{cases} \begin{pmatrix} e^{-3\omega z^{1/3}} & 0 & 0 \\ 0 & e^{-3\omega^2 z^{1/3}} & 0 \\ 0 & 0 & e^{-3z^{1/3}} \end{pmatrix}, & \text{Im}(z) > 0, \\ \begin{pmatrix} e^{-3\omega^2 z^{1/3}} & 0 & 0 \\ 0 & e^{-3\omega z^{1/3}} & 0 \\ 0 & 0 & e^{-3z^{1/3}} \end{pmatrix}, & \text{Im}(z) < 0, \end{cases} \quad (3.17)$$

and we verify that  $\widehat{L}_\alpha$  has the same jumps as  $\Phi_\alpha$  has on  $\mathbb{R}^\pm$ . The details of this calculation are similar to what we will do in the the proof of Lemma 5.10(a) below. This lemma is about a slightly different function,  $F_n(z)$ , but the arguments are essentially the same. It follows that

$$R_\alpha = T_\alpha \Phi_\alpha \widehat{L}_\alpha^{-1} = \mathbb{I} + \mathcal{O}(z^{-1}) \quad (3.18)$$

has no jumps on  $\mathbb{R}^\pm$ , and therefore is analytic in  $\mathbb{C} \setminus i\mathbb{R}$ .

Note that  $\widehat{L}_\alpha$  is analytic across the imaginary axis, while  $\Phi_\alpha$  has the jump (3.1) there. For  $R_\alpha$  we then have the jump matrix

$$R_{\alpha,-}^{-1}(z)R_{\alpha,+}(z) = \widehat{L}_\alpha(z) \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\widehat{L}_\alpha(z)\right)^{-1}, \quad z \in i\mathbb{R},$$

which by (3.17) is equal to

$$\begin{aligned} R_{\alpha,-}^{-1}(z)R_{\alpha,+}(z) &= L_\alpha(z) \begin{pmatrix} 1 & 0 & 0 \\ e^{\pm 3(\omega - \omega^2)z^{1/3}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} L_\alpha^{-1}(z), \quad z \in i\mathbb{R}^\pm, \\ &= L_\alpha(z) \begin{pmatrix} 1 & 0 & 0 \\ e^{-\frac{3\sqrt{3}}{2}|z|^{1/3}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} L_\alpha^{-1}(z). \end{aligned}$$

The entries of  $L_\alpha(z)$  and  $L_\alpha^{-1}(z)$  are  $\mathcal{O}(z^{1/3})$  as  $z \rightarrow \infty$ , as can be seen from (3.8). Hence there is a constant  $c > 0$  such that  $R_{\alpha,+}(z) = R_{\alpha,-}(z) \left(\mathbb{I} + \mathcal{O}(e^{-c|z|^{1/3}})\right)$  as  $z \rightarrow \infty$  along the imaginary axis.

It follows from this and (3.18) that the  $\mathcal{O}$  term in (3.18) can be written as an asymptotic series in powers of  $z^{-1}$  as  $z \rightarrow \infty$ . In particular we find that there is a constant matrix  $A_\alpha$  such that  $R_\alpha(z) = \mathbb{I} + A_\alpha z^{-1} + \mathcal{O}(z^{-2})$  and the lemma follows.  $\square$

### 3.4 RH problem for $\Phi_\alpha$

We combine the above to conclude that  $\Phi_\alpha$  satisfies the following RH problem.

#### Riemann-Hilbert Problem 3.5.

**RH- $\Phi$ 1**  $\Phi_\alpha$  is analytic in each of the four quadrants.

**RH- $\Phi$ 2**  $\Phi_\alpha$  has the constant jumps (3.1) on  $\Sigma_\Phi = \mathbb{R} \cup i\mathbb{R}$ .

**RH- $\Phi$ 3**  $\Phi_\alpha$  has the asymptotic behavior (3.7) as  $z \rightarrow \infty$ .

**RH- $\Phi$ 4**  $\Phi_\alpha$  has the behavior (3.2) as  $z \rightarrow 0$ .

With standard RH arguments it can be shown that the solution to the RH problem is unique (and thus is given by (2.27)), and that

$$\det \Phi_\alpha(z) = - \left(\frac{2\pi}{\sqrt{3}}\right)^3 z^{-2\beta},$$

where  $\beta = \alpha + \frac{1}{4}$  as before.

**Remark 3.6.** In their analysis of coupled random matrices in a  $p$ -chain, Bertola and Bothner [6] also construct a model RH problem for Meijer G-functions. It is called Bare Meijer G-parametrix in [6, section 4.2.1]. For  $p = 2$  the RH problem as well as its solution in terms of Mellin-Barnes type integrals shows great similarity with RH problem 3.5, although there does not seem to be a direct way to connect the two RH problems.

### 3.5 The inverse of $\Phi_\alpha$

The inverse of  $\Phi_\alpha$  also appears in the formula for the limiting kernel  $\mathbb{K}^{(\alpha, \frac{1}{2})}(x, y)$  in Theorem 2.8. The inverse contains the Meijer G-functions  $G_{0,3}^{3,0}\left(0, \alpha, \alpha + \frac{1}{2} \middle| -z\right)$ . We define, for  $-\pi < \arg(z) < \pi$ ,

$$\psi_1(z) = G_{0,3}^{3,0}\left(0, \alpha, \alpha + \frac{1}{2} \middle| ze^{-\pi i}\right), \quad (3.19)$$

$$\psi_2(z) = G_{0,3}^{3,0}\left(0, \alpha, \alpha + \frac{1}{2} \middle| ze^{\pi i}\right), \quad (3.20)$$

$$\psi_3(z) = -ie^{2\pi i\alpha} G_{0,3}^{3,0}\left(0, \alpha, \alpha + \frac{1}{2} \middle| ze^{-\pi i}\right) + ie^{2\pi i\alpha} G_{0,3}^{3,0}\left(0, \alpha, \alpha + \frac{1}{2} \middle| ze^{-3\pi i}\right), \quad (3.21)$$

$$= ie^{-2\pi i\alpha} G_{0,3}^{3,0}\left(0, \alpha, \alpha + \frac{1}{2} \middle| ze^{\pi i}\right) - ie^{-2\pi i\alpha} G_{0,3}^{3,0}\left(0, \alpha, \alpha + \frac{1}{2} \middle| ze^{3\pi i}\right), \quad (3.22)$$

$$\psi_4(z) = \psi_2(z) - \psi_1(z). \quad (3.23)$$

The functions  $\psi_j$  are solutions of the third order differential equation (2.18), but with  $\alpha$  replaced by  $-\alpha - \frac{1}{2}$ , and  $z$  replaced by  $-z$ , i.e.,

$$\vartheta(\vartheta - \alpha)(\vartheta - \alpha - \frac{1}{2})\psi - z\psi = 0. \quad (3.24)$$

This differential equation has Frobenius indices  $0, \alpha$  and  $\alpha + \frac{1}{2}$ . When  $2\alpha \notin \mathbb{Z}$ , (3.24) has a basis of solutions around  $z = 0$  formed by the functions

$${}_0F_2\left(1 - \alpha, \frac{3}{2} - \alpha; z\right), z^\alpha {}_0F_2\left(1 + \alpha, \frac{3}{2}; z\right) \text{ and } z^{\alpha + \frac{1}{2}} {}_0F_2\left(\frac{1}{2} + \alpha, \frac{1}{2}; z\right). \quad (3.25)$$

In particular, there are (real) constants  $c_1, c_2, c_3$  such that for  $-\pi < \arg(z) < \pi$

$$\psi_1(z) = c_1 {}_0F_2\left(1 - \alpha, \frac{3}{2} - \alpha; z\right) + c_2 e^{-\pi i\alpha} z^\alpha {}_0F_2\left(1 + \alpha, \frac{3}{2}; z\right) - c_3 i e^{-\pi i\alpha} z^{\alpha + \frac{1}{2}} {}_0F_2\left(\frac{1}{2} + \alpha, \frac{1}{2}; z\right).$$

By analytically continuing this expression along circles we can write the other  $\psi_j$  in terms of the Frobenius basis. Note that, regardless of whether we use (3.21) or (3.22), the Frobenius basis (3.25) yields that

$$\psi_3(z) = 2c_2 \sin(\pi\alpha) z^\alpha {}_0F_2\left(1 + \alpha, \frac{3}{2}; z\right) - 2c_3 \cos(\pi\alpha) z^{\alpha + \frac{1}{2}} {}_0F_2\left(\frac{1}{2} + \alpha, \frac{1}{2}; z\right), \quad (3.26)$$

showing that the two definitions (3.21) and (3.22) are indeed the same. By continuity, the equality also holds true if  $2\alpha \in \mathbb{Z}$ .

**Definition 3.7.** We define, with  $\vartheta = z \frac{d}{dz}$  as before,

$$\Psi_\alpha(z) = \begin{cases} \begin{pmatrix} \vartheta^2 \psi_1(z) & \vartheta^2 \psi_2(z) & \vartheta^2 \psi_3(z) \\ \vartheta \psi_1(z) & \vartheta \psi_2(z) & \vartheta \psi_3(z) \\ \psi_1(z) & \psi_2(z) & \psi_3(z) \end{pmatrix}, & 0 < \arg(z) < \frac{\pi}{2}, \\ \begin{pmatrix} \vartheta^2 \psi_1(z) & \vartheta^2 \psi_4(z) & \vartheta^2 \psi_3(z) \\ \vartheta \psi_1(z) & \vartheta \psi_4(z) & \vartheta \psi_3(z) \\ \psi_1(z) & \psi_4(z) & \psi_3(z) \end{pmatrix}, & \frac{\pi}{2} < \arg(z) < \pi, \\ \begin{pmatrix} \vartheta^2 \psi_2(z) & -\vartheta^2 \psi_1(z) & \vartheta^2 \psi_3(z) \\ \vartheta \psi_2(z) & -\vartheta \psi_1(z) & \vartheta \psi_3(z) \\ \psi_2(z) & -\psi_1(z) & \psi_3(z) \end{pmatrix}, & -\frac{\pi}{2} < \arg(z) < 0, \\ \begin{pmatrix} \vartheta^2 \psi_2(z) & \vartheta^2 \psi_4(z) & \vartheta^2 \psi_3(z) \\ \vartheta \psi_2(z) & \vartheta \psi_4(z) & \vartheta \psi_3(z) \\ \psi_2(z) & \psi_4(z) & \psi_3(z) \end{pmatrix}, & -\pi < \arg(z) < -\frac{\pi}{2}. \end{cases} \quad (3.27)$$



The main property of  $\Psi_\alpha$  is contained in

**Lemma 3.8.** *We have*

$$\Phi_\alpha \Psi_\alpha^T = -4\pi^2 \begin{pmatrix} 1 & 0 & 0 \\ -2\alpha - \frac{1}{2} & -1 & 0 \\ \alpha(\alpha + \frac{1}{2}) & 2\alpha + \frac{1}{2} & 1 \end{pmatrix}^{-1} \quad (3.28)$$

*Proof.* It is clear that  $\Psi_\alpha$  is analytic in each of the four quadrants with jumps

$$\Psi_{\alpha,+}(z) = \Psi_{\alpha,-}(z) \times \begin{cases} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \mathbb{R}^+, \\ \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in i\mathbb{R}^\pm, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -ie^{-2\pi i\alpha} \\ 0 & ie^{-2\pi i\alpha} & 0 \end{pmatrix}, & z \in \mathbb{R}^-. \end{cases} \quad (3.29)$$

Indeed, the jumps on  $\mathbb{R}^+$  and  $i\mathbb{R}^\pm$  are immediate from (3.27). To verify the jump on  $\mathbb{R}^-$  we have to check that on  $\mathbb{R}^-$

$$\psi_{2,+} = \psi_{1,-}, \quad \psi_{3,+} = ie^{-2\pi i\alpha} \psi_{4,-}, \quad \psi_{4,+} = -ie^{-2\pi i\alpha} \psi_{3,-}.$$

These identities are indeed satisfied because of the definitions (3.19)–(3.23). We use that  $z_- = z_+ e^{2\pi i}$  for  $z \in \mathbb{R}^-$ , since the negative real axis is oriented from right to left.

The jump matrices in (3.29) are the inverse transposes of those in (3.1). It follows that  $\Phi_\alpha \Psi_\alpha^T$  is analytic across all contours, and so is analytic in  $\mathbb{C} \setminus \{0\}$ . The isolated singularity at 0 is at most a pole, since all functions involved in  $\Phi_\alpha$  and  $\Psi_\alpha$  have a power law behavior near 0.

The functions  $\psi_j$  are solutions of (3.24) which has Frobenius indices 0,  $\alpha$  and  $\alpha + \frac{1}{2}$  at 0. As such they are linear combinations of the functions (3.25) when  $2\alpha \notin \mathbb{Z}$ . If  $\alpha < 0$  then all solutions are  $\mathcal{O}(z^\alpha)$  as  $z \rightarrow 0$ , and it follows from (3.27) that  $\Psi_\alpha(z) = \mathcal{O}(z^\alpha)$ . By (3.2) one has  $\Phi_\alpha(z) = \mathcal{O}(z^{-\alpha - \frac{1}{2}})$  for  $-\frac{1}{2} < \alpha < 0$  or  $\Phi_\alpha = \mathcal{O}(1)$  for  $\alpha < -\frac{1}{2}$ . Thus

$$\Phi_\alpha(z) \Psi_\alpha^T(z) = \mathcal{O}\left(z^{\min(\alpha, -\frac{1}{2})}\right) \quad \text{if } \alpha \in (-1, 0) \setminus \{-\frac{1}{2}\}. \quad (3.30)$$

For  $\alpha = -\frac{1}{2}$  or  $\alpha = 0$  there is an additional log term in the  $\mathcal{O}$  term in (3.30). Since  $\alpha > -1$ , we see from (3.30) that the isolated singularity is removable in case  $\alpha \leq 0$ .

For  $\alpha > 0$  we have to be a little more careful. Now all solutions are bounded at 0, but we use that  $\psi_3$  and  $\psi_4$  are solutions of (3.24) which are  $\mathcal{O}(z^\alpha)$  as  $z \rightarrow 0$ . This fact can be checked by writing the solutions out in the Frobenius basis around  $z = 0$ . We already did this for  $\psi_3$  in (3.26) and the situation for  $\psi_4$  is analogous. Then it follows from (3.27) that

$$\Psi_\alpha(z) = \mathcal{O} \begin{pmatrix} 1 & z^\alpha & z^\alpha \\ 1 & z^\alpha & z^\alpha \\ 1 & z^\alpha & z^\alpha \end{pmatrix} \quad (3.31)$$

as  $z \rightarrow 0$  with  $\text{Re } z < 0$ . Thus

$$\Phi_\alpha(z) \Psi_\alpha^T(z) = \mathcal{O} \begin{pmatrix} 1 & z^{-\alpha - \frac{1}{2}} & z^{-\alpha - \frac{1}{2}} \\ 1 & z^{-\alpha - \frac{1}{2}} & z^{-\alpha - \frac{1}{2}} \\ 1 & z^{-\alpha - \frac{1}{2}} & z^{-\alpha - \frac{1}{2}} \end{pmatrix} \mathcal{O} \begin{pmatrix} 1 & 1 & 1 \\ z^\alpha & z^\alpha & z^\alpha \\ z^\alpha & z^\alpha & z^\alpha \end{pmatrix} = \mathcal{O} \begin{pmatrix} z^{-\frac{1}{2}} & z^{-\frac{1}{2}} & z^{-\frac{1}{2}} \\ z^{-\frac{1}{2}} & z^{-\frac{1}{2}} & z^{-\frac{1}{2}} \\ z^{-\frac{1}{2}} & z^{-\frac{1}{2}} & z^{-\frac{1}{2}} \end{pmatrix}$$

as  $z \rightarrow 0$  with  $\text{Re } z < 0$ . The singularity of  $\Phi_\alpha \Psi_\alpha^T$  at 0 is at most a pole, and thus it must be removable, since if it were a pole the absolute value of some entry would be  $\geq \frac{C}{|z|}$  for some  $C > 0$  and all  $z$  close enough to 0. We conclude that  $\Phi_\alpha \Psi_\alpha^T$  is entire.

Next we investigate the behavior as  $z \rightarrow \infty$ . The asymptotic behavior of  $\Phi_\alpha$  is given in (3.7). With similar arguments we find the behavior of  $\Psi_\alpha$ . We use (3.3), (3.4), (3.5), (3.6) with  $\alpha$  replaced by  $-\alpha - \frac{1}{2}$  and find

$$G_{0,3}^{3,0} \left( 0, \alpha, \alpha + \frac{1}{2} \middle| z \right) \sim \frac{2\pi}{\sqrt{3}} e^{-3z^{\frac{1}{3}}} z^{-\tilde{\gamma}} \left( 1 + \sum_{k=1}^{\infty} \tilde{M}_k z^{-\frac{k}{3}} \right), \quad -4\pi < \arg z < 4\pi \quad (3.32)$$

with

$$\tilde{\gamma} = -\frac{2}{3}\alpha + \frac{1}{6} = -\frac{2}{3}\beta + \frac{1}{3}, \quad (3.33)$$

$$\tilde{M}_1 = \frac{1}{3}\alpha^2 + \frac{1}{6}\alpha - \frac{1}{3}, \quad (3.34)$$

$$\tilde{M}_2 = \frac{1}{18}\alpha^4 + \frac{5}{54}\alpha^3 - \frac{5}{216}\alpha^2 - \frac{5}{108}\alpha + \frac{1}{2592}. \quad (3.35)$$

We then have for  $\psi_3$

$$\psi_3(z) = -\frac{2\pi}{\sqrt{3}} z^{-\tilde{\gamma}} e^{3z^{\frac{1}{3}}} \left( 1 - \tilde{M}_1 z^{-\frac{1}{3}} + \tilde{M}_2 z^{-\frac{2}{3}} + \mathcal{O}(z^{-1}) \right), \quad (3.36)$$

$$\vartheta\psi_3(z) = -\frac{2\pi}{\sqrt{3}} z^{-\tilde{\gamma}} e^{3z^{\frac{1}{3}}} \left( z^{\frac{1}{3}} - (\tilde{M}_1 + \tilde{\gamma}) + (\tilde{M}_2 + (\frac{1}{3} + \tilde{\gamma})\tilde{M}_1) z^{-\frac{1}{3}} + \mathcal{O}(z^{-\frac{2}{3}}) \right), \quad (3.37)$$

$$\vartheta^2\psi_3(z) = -\frac{2\pi}{\sqrt{3}} z^{-\tilde{\gamma}} e^{3z^{\frac{1}{3}}} \left( z^{\frac{2}{3}} + (-2\tilde{\gamma} + \frac{1}{3} - \tilde{M}_1) z^{\frac{1}{3}} + (\tilde{M}_2 + (\frac{1}{3} + 2\tilde{\gamma})\tilde{M}_1 + \tilde{\gamma}^2) + \mathcal{O}(z^{-\frac{1}{3}}) \right). \quad (3.38)$$

These calculations come from applying (3.32) to the second term in either (3.21) or (3.22), since this term is dominant in the full range  $-\pi < \arg z < \pi$ .

Then with numbers analogous to (3.10)

$$\tilde{t}_1 = \tilde{\gamma} + \tilde{M}_1, \quad \tilde{t}_2 = \tilde{\gamma}(\tilde{\gamma} - \frac{1}{3}) + \tilde{M}_1(\tilde{M}_1 + \tilde{\gamma} - \frac{2}{3}) - \tilde{M}_2, \quad \tilde{t}_3 = 2\tilde{\gamma} - \frac{1}{3} + \tilde{M}_1, \quad (3.39)$$

and

$$\tilde{T}_\alpha = \begin{pmatrix} 1 & \tilde{t}_3 & \tilde{t}_2 \\ 0 & 1 & \tilde{t}_1 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.40)$$

we have

$$\tilde{T}_\alpha \begin{pmatrix} \vartheta^2\psi_3(z) \\ \vartheta\psi_3(z) \\ \psi_3(z) \end{pmatrix} = -\frac{2\pi}{\sqrt{3}} z^{\frac{2\beta}{3}} e^{3z^{\frac{1}{3}}} \begin{pmatrix} z^{\frac{1}{3}} + \mathcal{O}(z^{-\frac{2}{3}}) \\ 1 + \mathcal{O}(z^{-\frac{2}{3}}) \\ z^{-\frac{1}{3}} + \mathcal{O}(z^{-\frac{2}{3}}) \end{pmatrix}.$$

After similar calculations for  $\psi_1, \psi_2, \psi_4$  we find

$$\tilde{T}_\alpha \Psi_\alpha(z) = -\frac{2\pi}{\sqrt{3}} z^{\frac{2\beta}{3}} (\mathbb{I} + \mathcal{O}(z^{-1})) \tilde{L}_\alpha(z) \times \begin{cases} \begin{pmatrix} e^{3\omega z^{1/3}} & 0 & 0 \\ 0 & e^{3\omega^2 z^{1/3}} & 0 \\ 0 & 0 & e^{3z^{1/3}} \end{pmatrix}, & \text{Im}(z) > 0, \\ \begin{pmatrix} e^{3\omega^2 z^{1/3}} & 0 & 0 \\ 0 & e^{3\omega z^{1/3}} & 0 \\ 0 & 0 & e^{3z^{1/3}} \end{pmatrix}, & \text{Im}(z) < 0, \end{cases} \quad (3.41)$$

with

$$\tilde{L}_\alpha(z) = \begin{pmatrix} z^{\frac{1}{3}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-\frac{1}{3}} \end{pmatrix} \times \begin{cases} \begin{pmatrix} \omega & \omega^2 & 1 \\ 1 & 1 & 1 \\ \omega^2 & \omega & 1 \end{pmatrix} \begin{pmatrix} e^{-\frac{2\pi i\beta}{3}} & 0 & 0 \\ 0 & e^{\frac{2\pi i\beta}{3}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \text{Im}(z) > 0, \\ \begin{pmatrix} \omega^2 & -\omega & 1 \\ 1 & -1 & 1 \\ \omega & -\omega^2 & 1 \end{pmatrix} \begin{pmatrix} e^{\frac{2\pi i\beta}{3}} & 0 & 0 \\ 0 & e^{-\frac{2\pi i\beta}{3}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \text{Im}(z) < 0. \end{cases} \quad (3.42)$$

For  $z \rightarrow \infty$  we then have from (3.7) and (3.41),

$$\begin{aligned} T_\alpha \Phi_\alpha(z) \Psi_\alpha^T(z) \tilde{T}_\alpha^T &= T_\alpha \Phi_\alpha(z) \left( \tilde{T}_\alpha \Psi_\alpha(z) \right)^T \\ &= - \left( \frac{2\pi}{\sqrt{3}} \right)^2 \left( \mathbb{I} + \mathcal{O}(z^{-\frac{1}{3}}) \right) L_\alpha(z) \tilde{L}_\alpha^T(z) \left( \mathbb{I} + \mathcal{O}(z^{-\frac{1}{3}}) \right)^T \end{aligned} \quad (3.43)$$

By multiplying (3.8) and (3.42) we get

$$L_\alpha(z) \tilde{L}_\alpha^T(z) = 3\mathbb{I}. \quad (3.44)$$

Thus (3.43) remains bounded as  $z \rightarrow \infty$ . Since  $\Phi_\alpha(z) \Psi_\alpha^T(z)$  is entire, (3.43) is a constant matrix by Liouville's theorem, and using (3.44) we conclude

$$T_\alpha \Phi_\alpha(z) \Psi_\alpha^T(z) \tilde{T}_\alpha^T = -4\pi^2 \mathbb{I} \quad (3.45)$$

for all  $z \in \mathbb{C}$ . A final computation (that we checked with Maple), based on (3.9), (3.40), and (3.10), (3.39) shows that

$$\tilde{T}_\alpha^T T_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ \tilde{t}_3 & 1 & 0 \\ \tilde{t}_2 & \tilde{t}_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ t_1 & -1 & 0 \\ t_2 & t_3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2\alpha - \frac{1}{2} & -1 & 0 \\ \alpha(\alpha + \frac{1}{2}) & 2\alpha + \frac{1}{2} & 1 \end{pmatrix}$$

which by (3.45) leads to the identity in (3.28).  $\square$

### 3.6 Proof of Theorem 2.8

*Proof.* We are going to use [32, Proposition 5.4] with parameters  $M = 2$  and  $\nu_0 = 0, \nu_1 = \alpha$ , and  $\nu_2 = \alpha + \frac{1}{2}$ . In the paper [32] the parameters  $\nu_j$  are supposed to be integers, but this fact does not play a role in the proof of the formula (5.13) of [32], and we can use it in our situation. The result is that

$$\mathbb{K}^{(\alpha, \frac{1}{2})}(x, y) = \frac{-1}{x-y} \sum_{j=0}^2 \sum_{i=0}^{2-j} (-1)^j a_{i+j} (\vartheta_x^j f(x)) (\vartheta_y^i g(y)) \quad (3.46)$$

with  $a_0 = \alpha(\alpha + \frac{1}{2})$ ,  $a_1 = -2\alpha - \frac{1}{2}$ ,  $a_2 = 1$ , and

$$f(x) = G_{0,3}^{1,0} \left( 0, -\alpha, -\alpha - \frac{1}{2} \middle| x \right), \quad g(y) = G_{0,3}^{2,0} \left( \alpha, \alpha + \frac{1}{2}, 0 \middle| y \right).$$

Thus

$$\mathbb{K}^{(\alpha, \frac{1}{2})}(x, y) = \frac{-1}{x-y} \begin{pmatrix} \vartheta_y^2 g(y) & \vartheta_y g(y) & g(y) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2\alpha - \frac{1}{2} & 1 & 0 \\ \alpha(\alpha + \frac{1}{2}) & -2\alpha - \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} f(x) \\ -\vartheta_x f(x) \\ \vartheta_x^2 f(x) \end{pmatrix} \quad (3.47)$$

By (2.20), (2.26), and (2.25) we have  $f(x) = -\frac{1}{4\pi^2}(\phi_1(x) + \phi_2(x))$ . Also, by [35, page 151] we have

$$2\pi i G_{0,3}^{2,0} \left( \alpha, \alpha + \frac{1}{2}, 0 \middle| z \right) = G_{0,3}^{3,0} \left( 0, \alpha, \alpha + \frac{1}{2} \middle| ze^{-\pi i} \right) - G_{0,3}^{3,0} \left( 0, \alpha, \alpha + \frac{1}{2} \middle| ze^{\pi i} \right). \quad (3.48)$$

so that  $g(y) = \frac{1}{2\pi i}(\psi_1(y) - \psi_2(y))$  by (3.19) and (3.20). Thus in view of (2.27) and (3.27),

$$\begin{pmatrix} f(x) \\ \vartheta_x f(x) \\ \vartheta_x^2 f(x) \end{pmatrix} = -\frac{1}{4\pi^2} \Phi_{\alpha,+}(x) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \vartheta_y^2 g(y) \\ \vartheta_y g(y) \\ g(y) \end{pmatrix} = \frac{1}{2\pi i} \Psi_{\alpha,+}(y) \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

with  $x, y > 0$ . Thus (3.47) gives us

$$\mathbb{K}^{(\alpha, \frac{1}{2})}(x, y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \frac{1}{4\pi^2} \Psi_{\alpha,+}^T(y) \begin{pmatrix} 1 & 0 & 0 \\ -2\alpha - \frac{1}{2} & -1 & 0 \\ \alpha(\alpha + \frac{1}{2}) & 2\alpha + \frac{1}{2} & 1 \end{pmatrix} \Phi_{\alpha,+}(x) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad (3.49)$$

In view of (3.28) we arrive at (2.28).  $\square$

## 4 Transformations of the Riemann-Hilbert problem

In sections 4-6 we apply the Deift-Zhou steepest descent method [18] to the RH problem 2.2 for  $Y$ . The method consists of a sequence of explicit transformations  $Y \mapsto X \mapsto T \mapsto S \mapsto R$ .

### 4.1 First transformation $Y \mapsto X$

We start with a preliminary transformation that simplifies the jump condition in the RH problem on the positive real axis. It also introduces a jump on the negative real axis.

**Definition 4.1.** If  $n$  is even then

$$X(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix} Y(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & z^{\frac{1}{4}} & 0 \\ 0 & 0 & z^{-\frac{1}{4}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}, \quad (4.1)$$

with  $z \in \mathbb{C} \setminus \mathbb{R}$ , while if  $n$  is odd then

$$X(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} Y(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & z^{-\frac{1}{4}} & 0 \\ 0 & 0 & z^{\frac{1}{4}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}. \quad (4.2)$$

The RH problem for  $X$  follows from the definitions (4.1)-(4.2) and the RH problem 2.2 for  $Y$ . While there is a different definition for  $X$  in the cases  $n$  even and  $n$  odd, the RH problem for  $X$  sees the two cases only in the jump (4.5) on the negative real axis. We recall

$$\beta = \alpha + \frac{1}{4}. \quad (4.3)$$

#### Riemann-Hilbert Problem 4.2.

**RH-X1**  $X : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}^{3 \times 3}$  is analytic.

**RH-X2**  $X$  has boundary values for  $x \in (-\infty, 0) \cup (0, \infty)$ , denoted by  $X_+(x)$  and  $X_-(x)$ , and

$$X_+(x) = X_-(x) \begin{pmatrix} 1 & \sqrt{2}x^\beta e^{-nV(x)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x > 0, \quad (4.4)$$

$$X_+(x) = X_-(x) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & (-1)^{n+1} \\ 0 & (-1)^n & 0 \end{pmatrix}, \quad x < 0, \quad (4.5)$$

**RH-X3** As  $|z| \rightarrow \infty$

$$X(z) = \left( \mathbb{I} + \mathcal{O}\left(\frac{1}{z}\right) \right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & z^{\frac{1}{4}} & 0 \\ 0 & 0 & z^{-\frac{1}{4}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} z^n & 0 & 0 \\ 0 & z^{-\frac{n}{2}} & 0 \\ 0 & 0 & z^{-\frac{n}{2}} \end{pmatrix}. \quad (4.6)$$

**RH-X4** As  $z \rightarrow 0$

$$X(z) = \mathcal{O} \begin{pmatrix} 1 & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) \\ 1 & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) \\ 1 & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) \end{pmatrix}. \quad (4.7)$$

While the RH problem 4.2 follows in principle by straightforward calculations, it may be good to point out a few of the features leading to it.

First we may define  $\tilde{Y} = Y$  in case  $n$  is even and

$$\tilde{Y}(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} Y(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & z^{-1/2} & 0 \\ 0 & 0 & z^{1/2} \end{pmatrix} \quad (4.8)$$

in case  $n$  is odd. Then  $\tilde{Y}$  satisfies of course the RH problem for  $Y$  in case  $n$  is even. For odd  $n$ , the RH problem is very close to that of  $Y$ . Namely the jump on the positive real axis remains the same with the jump matrix  $\begin{pmatrix} 1 & w(x) & x^{1/2}w(x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  as before. For odd  $n$ , there is an additional jump on the negative real axis, but if we write the jump as

$$\tilde{Y}_+ = \tilde{Y}_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & (-1)^n \end{pmatrix},$$

then it applies to both cases  $n$  even and  $n$  odd.

In case  $n$  is odd we obtain the asymptotic behavior of  $\tilde{Y}$  from (2.4) and (4.8). It gives us

$$\begin{aligned} \tilde{Y}(z) &= (\mathbb{I} + \mathcal{O}(z^{-1})) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} z^n & 0 & 0 \\ 0 & z^{-\frac{n+1}{2}} & 0 \\ 0 & 0 & z^{-\frac{n-1}{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & z^{-\frac{1}{2}} & 0 \\ 0 & 0 & z^{\frac{1}{2}} \end{pmatrix} \\ &= (\mathbb{I} + \mathcal{O}(z^{-1})) \begin{pmatrix} z^n & 0 & 0 \\ 0 & z^{-\frac{n}{2}} & 0 \\ 0 & 0 & z^{-\frac{n}{2}} \end{pmatrix} \end{aligned} \quad (4.9)$$

and this is exactly the same as the asymptotic behavior (2.4) for  $n$  even.

The behavior near 0 follows from (2.5) and (4.8). In case  $n$  is odd it is

$$\tilde{Y}(z) = \mathcal{O} \begin{pmatrix} 1 & z^{-1/2}h_{\alpha+\frac{1}{2}} & z^{1/2}h_{\alpha}(z) \\ 1 & z^{-1/2}h_{\alpha+\frac{1}{2}} & z^{1/2}h_{\alpha}(z) \\ 1 & z^{-1/2}h_{\alpha+\frac{1}{2}} & z^{1/2}h_{\alpha}(z) \end{pmatrix}. \quad (4.10)$$

From (4.1)-(4.2) and (4.8) we have that in both case  $n$  even and  $n$  odd,

$$X(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix} \tilde{Y}(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & z^{\frac{1}{4}} & 0 \\ 0 & 0 & z^{-\frac{1}{4}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}. \quad (4.11)$$

To come from (4.11) to the RH problem 4.2 requires some more calculations but these are more routine. We only note that for the asymptotic behavior near 0, we use either (2.5) or (4.10) in (4.11) and (4.7) follows in both cases since

$$z^{\frac{1}{4}}h_{\alpha}(z) = \mathcal{O} \left( z^{-\frac{1}{4}}h_{\alpha+\frac{1}{2}}(z) \right) \quad \text{as } z \rightarrow 0,$$

as can be checked from the definition of  $h_{\alpha}$  in (2.5).

## 4.2 Second transformation $X \mapsto T$

The solution to the vector equilibrium problem [27], provides us with two measures  $\mu^*$  and  $\nu^*$  satisfying the variational conditions (2.8) and (2.9). Here  $\mu^* = \mu_{V, \frac{1}{2}}^*$  is the  $\theta = \frac{1}{2}$ -equilibrium measure in the external field  $V$ , which has a compact support

$$\text{supp}(\mu^*) = \overline{\Delta_1}, \quad \Delta_1 = (0, q) \quad (4.12)$$

by our assumption on the external field  $V$  in Theorem 1.2. The measure  $\nu^*$  has support

$$\text{supp}(\nu^*) = \overline{\Delta_2}, \quad \Delta_2 = (-\infty, 0). \quad (4.13)$$

Writing

$$U^{\mu^*}(z) = \int \log \frac{1}{|z-s|} d\mu^*(s), \quad U^{\nu^*}(z) = \int \log \frac{1}{|z-s|} d\nu^*(s),$$

for the logarithmic potentials, we obtain from (2.8) and (2.9),

$$\begin{aligned} 2U^{\mu^*} - U^{\nu^*} + V &\begin{cases} = -\ell & \text{on } [0, q], \\ > -\ell & \text{on } (q, \infty), \end{cases} \\ 2U^{\nu^*} = U^{\mu^*} &\quad \text{on } (-\infty, 0]. \end{aligned} \quad (4.14)$$

The strict inequality on  $(q, \infty)$  comes from the one cut  $\theta$ -regular assumption in Theorem 1.2, see Definition 1.1. Indeed the strict inequality that is assumed in (2.9) for  $x > q$ , translates into the strict inequality in (4.14) for the minimizer of the vector equilibrium problem.

We define two  $g$ -functions by

$$g_1(z) = \int_0^q \log(z-s) d\mu^*(s), \quad z \in \mathbb{C} \setminus (-\infty, q], \quad (4.15)$$

$$g_2(z) = \int_{-\infty}^0 \log(z-s) d\nu^*(s), \quad z \in \mathbb{C} \setminus (-\infty, 0], \quad (4.16)$$

with principal branches of the logarithm. From the conditions (4.14) and definitions (4.15)-(4.16) it follows that for  $x > 0$

$$\begin{aligned} g_{1+}(x) - g_{1-}(x) &= 2\pi i \mu^*([x, \infty)), \\ g_{1-}(x) + g_{1+}(x) - g_2(x) - V(x) &\begin{cases} = \ell & \text{on } [0, q], \\ < \ell & \text{on } (q, \infty), \end{cases} \end{aligned} \quad (4.17)$$

and for  $x < 0$

$$\begin{aligned} g_{1+}(x) - g_{1-}(x) &= 2\pi i, \\ g_{2+}(x) - g_{2-}(x) &= 2\pi i \nu^*([x, 0]), \\ -g_{1-}(x) + g_{2-}(x) + g_{2+}(x) &= \pi i. \end{aligned} \quad (4.18)$$

As before, the plus (minus) sign indicates that we use the limiting values from the upper (lower) half-plane.

**Proposition 4.3.** *For the behavior as  $z \rightarrow \infty$  we have*

$$g_1(z) = \log z - \frac{m_1}{z} + \mathcal{O}(z^{-2}) \quad (4.19)$$

$$g_2(z) = \frac{1}{2} \log z + \frac{m_{\frac{1}{2}}}{\sqrt{z}} - \frac{m_1}{2z} + \mathcal{O}(z^{-\frac{3}{2}}), \quad (4.20)$$

where

$$m_1 = \int_0^q s d\mu^*(s) \quad \text{and} \quad m_{\frac{1}{2}} = \int_0^q \sqrt{s} d\mu^*(s). \quad (4.21)$$

*Proof.* The expansion (4.19) is immediate from (4.15), since  $\mu^*$  is a probability measure with compact support. The first term of the expansion (4.20) for  $g_2$  comes from the fact that  $\nu$  has total mass  $1/2$ . The full expansion can be obtained from the proofs of Propositions 3.1 and 3.2 in [27]. Namely, in the proof of Proposition 3.2 we find the relation

$$\nu^* = \int_0^q \nu_t d\mu^*(t), \quad (4.22)$$

where  $\nu_t$  for  $t > 0$  is the minimizer of the energy functional  $I(\nu) + \int \log |s - t| d\nu(s)$  among measures on  $(-\infty, 0]$  with total mass  $\frac{1}{2}$ . The proof of [27, Proposition 3.1] yields

$$\int_{-\infty}^0 \log(z - s) d\nu_t(s) = \log(z - t) - \log\left(z^{\frac{1}{2}} - \sqrt{t}\right) = \log\left(z^{\frac{1}{2}} + \sqrt{t}\right).$$

We therefore obtain from (4.16) and the above

$$g_2(z) = \int_0^q \left( \int_{-\infty}^0 \log(z - s) d\nu_t(s) \right) d\mu^*(t) = \int_0^q \log\left(z^{\frac{1}{2}} + \sqrt{t}\right) d\mu^*(t). \quad (4.23)$$

From the alternative expression (4.23) for  $g_2$  we find as  $z \rightarrow \infty$

$$\begin{aligned} g_2(z) &= \log z^{\frac{1}{2}} + \int_0^q \log\left(1 + \frac{\sqrt{t}}{z^{\frac{1}{2}}}\right) d\mu^*(t) \\ &= \frac{1}{2} \log z + \int_0^q \left( \frac{\sqrt{t}}{z^{\frac{1}{2}}} - \frac{t}{2z} + \mathcal{O}\left(z^{-\frac{3}{2}}\right) \right) d\mu^*(t), \end{aligned}$$

which indeed yields the required expansion (4.20) as  $z \rightarrow \infty$ .  $\square$

The  $g$  functions are used in the transformation  $X \mapsto T$ .

**Definition 4.4.**  $T$  is defined by

$$T(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & inm_{\frac{1}{2}} \\ 0 & 0 & 1 \end{pmatrix} L^{-1} X(z) \begin{pmatrix} e^{-ng_1(z)} & 0 & 0 \\ 0 & e^{n(g_1(z) - g_2(z))} & 0 \\ 0 & 0 & e^{ng_2(z)} \end{pmatrix} L, \quad (4.24)$$

where

$$L = \begin{pmatrix} \sqrt{2}e^{\frac{2n\ell}{3}} & 0 & 0 \\ 0 & e^{-\frac{n\ell}{3}} & 0 \\ 0 & 0 & e^{-\frac{n\ell}{3}} \end{pmatrix}. \quad (4.25)$$

In the RH problem for  $T$  it will be convenient to use a function that we call  $\varphi$ . To define it, we recall the assumption that  $V$  is real analytic on  $[0, \infty)$ . Thus  $V$  has an analytic continuation to an open neighborhood  $O_V$  of  $[0, \infty)$  in the complex plane. Then

$$\varphi(z) = -g_1(z) + \frac{1}{2}g_2(z) + \frac{1}{2}(V(z) + \ell), \quad (4.26)$$

is defined and analytic for  $z \in O_V \setminus (-\infty, q]$ . By (4.17) and (4.26),

$$\varphi_{\pm}(x) = \mp \pi i \mu^*([x, \infty)) = \pm \pi i \mu^*([0, x]) \mp \pi i, \quad x \in [0, q]. \quad (4.27)$$

From the variational inequality in (4.17) it follows that

$$\varphi(x) > 0 \quad \text{for real } x > q. \quad (4.28)$$

Then  $T$  satisfies the following RH problem.

**Riemann-Hilbert Problem 4.5.**

**RH-T1**  $T : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}^{3 \times 3}$  is analytic.

**RH-T2**  $T$  has boundary values for  $x \in \Delta_2 \cup \Delta_1 \cup (q, \infty)$  and

$$T_+(x) = T_-(x) \begin{pmatrix} e^{2n\varphi_+(x)} & x^\beta & 0 \\ 0 & e^{2n\varphi_-(x)} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x \in \Delta_1 = (0, q), \quad (4.29)$$

$$T_+(x) = T_-(x) \begin{pmatrix} 1 & x^\beta e^{-2n\varphi(x)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x \in (q, \infty), \quad (4.30)$$

$$T_+(x) = T_-(x) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad x \in \Delta_2 = (-\infty, 0). \quad (4.31)$$

**RH-T3** As  $z \rightarrow \infty$

$$T(z) = \left( \mathbb{I} + \mathcal{O}\left(\frac{1}{z}\right) \right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & z^{\frac{1}{4}} & 0 \\ 0 & 0 & z^{-\frac{1}{4}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (4.32)$$

**RH-T4** As  $z \rightarrow 0$

$$T(z) = \mathcal{O} \begin{pmatrix} 1 & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) \\ 1 & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) \\ 1 & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) \end{pmatrix}. \quad (4.33)$$

The behavior (4.33) arises from (4.7) since  $g_1$  and  $g_2$  remain bounded near  $z$ . The jump in (4.29) follows from the variational conditions (4.17)-(4.18) and (4.27). For  $x < 0$ , we have from (4.24) and (4.5)

$$\begin{aligned} T_-^{-1}(x)T_+(x) &= L^{-1} \begin{pmatrix} e^{ng_{1,-}(x)} & 0 & 0 \\ 0 & e^{-n(g_{1,-}(x)-g_{2,-}(x))} & 0 \\ 0 & 0 & e^{-ng_{2,-}(x)} \end{pmatrix} \\ &\quad \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -(-1)^n \\ 0 & (-1)^n & 0 \end{pmatrix} \begin{pmatrix} e^{-ng_{1,+}(x)} & 0 & 0 \\ 0 & e^{n(g_{1,+}(x)-g_{2,+}(x))} & 0 \\ 0 & 0 & e^{ng_{2,+}(x)} \end{pmatrix} L \end{aligned} \quad (4.34)$$

The matrices  $L^{-1}$  and  $L$  commute with all other factors and thus cancel out because of the special diagonal form (4.25) of  $L$ . Then (4.31) follows since by (4.18)

$$e^{n(g_{1,+}(x)-g_{1,-}(x))} = 1 \quad \text{and} \quad e^{n(-g_{1,-}(x)+g_{2,+}(x)+g_{2,-}(x))} = (-1)^n$$

for  $x < 0$ .

Because of (4.28) the jump matrix in (4.30) tends to the identity matrix as  $n \rightarrow \infty$ .

The distinction between the cases  $n$  even and  $n$  odd has disappeared in the RH problem for  $T$ .

### 4.3 Third transformation $T \mapsto S$ : opening of lenses

In the next transformation  $T \mapsto S$  we open a lense around  $\Delta_1 = (0, q)$  as in Figure 3. The lense is contained in the neighborhood  $O_V$  around  $[0, \infty)$  where  $V$  is analytic. We denote the upper and lower lips of the lense by  $\Delta_1^+$  and  $\Delta_1^-$ , respectively, with orientation from 0 to  $q$ . The lips start from 0 in a vertical direction. However, they do not necessarily follow the imaginary axis exactly, contrary to how it is shown in the figure. Later we will construct a conformal map  $f$  in a neighborhood of 0 with  $f(0) = 0$ . The lips of the lense near 0 should be such that they are mapped by  $f$  to the imaginary axis.

**Definition 4.6.**  $S$  is defined by

$$S(z) = T(z) \begin{pmatrix} 1 & 0 & 0 \\ -z^{-\beta} e^{2n\varphi(z)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{for } z \text{ in the upper part of the lense,} \quad (4.35)$$

$$S(z) = T(z) \begin{pmatrix} 1 & 0 & 0 \\ z^{-\beta} e^{2n\varphi(z)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{for } z \text{ in the lower part of the lense,} \quad (4.36)$$

$$S(z) = T(z), \quad \text{elsewhere.} \quad (4.37)$$

The corresponding RH problem is

#### Riemann-Hilbert Problem 4.7.

**RH-S1**  $S : \mathbb{C} \setminus \Sigma_S \rightarrow \mathbb{C}^{3 \times 3}$  is analytic (see Figure 3 for the contour  $\Sigma_S$ ).



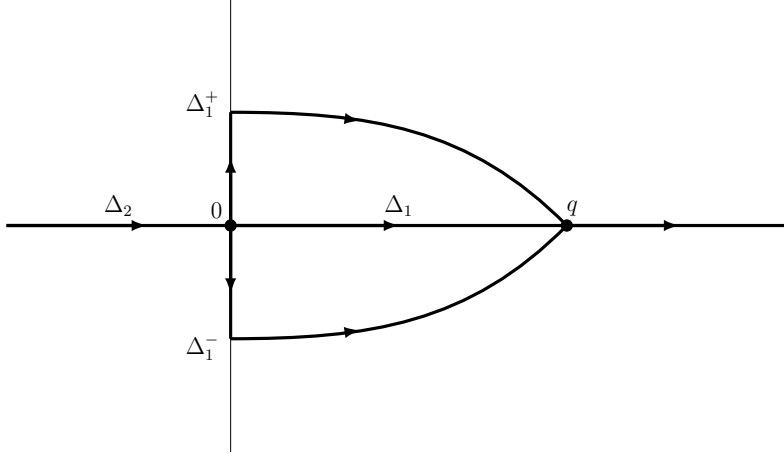


Figure 3: Contour  $\Sigma_S = \mathbb{R} \cup \Delta_1^\pm$  for the RH problem for  $S$ . The lense around  $\Delta_1$  is contained in the domain  $O_V$  where  $V$  is analytic.

**RH-S2** On  $\Sigma_S$  we have the jumps

$$S_+(x) = S_-(x) \begin{pmatrix} 0 & x^\beta & 0 \\ -x^{-\beta} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x \in \Delta_1, \quad (4.38)$$

$$S_+(z) = S_-(z) \begin{pmatrix} 1 & 0 & 0 \\ z^{-\beta} e^{2n\varphi(z)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z \in \Delta_1^+ \cup \Delta_1^-, \quad (4.39)$$

$$S_+(x) = S_-(x) \begin{pmatrix} 1 & x^\beta e^{-2n\varphi(x)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x \in (q, \infty), \quad (4.40)$$

$$S_+(x) = S_-(x) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad x \in \Delta_2, \quad (4.41)$$

**RH-S3** As  $|z| \rightarrow \infty$

$$S(z) = \left( \mathbb{I} + \mathcal{O}\left(\frac{1}{z}\right) \right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & z^{\frac{1}{4}} & 0 \\ 0 & 0 & z^{-\frac{1}{4}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (4.42)$$

**RH-S4** As  $z \rightarrow 0$

$$S(z) = \mathcal{O} \begin{pmatrix} z^{-\alpha-\frac{1}{2}} h_{\alpha+\frac{1}{2}}(z) & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) \\ z^{-\alpha-\frac{1}{2}} h_{\alpha+\frac{1}{2}}(z) & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) \\ z^{-\alpha-\frac{1}{2}} h_{\alpha+\frac{1}{2}}(z) & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) \end{pmatrix} \quad \text{for } z \text{ inside the lense,} \quad (4.43)$$

$$S(z) = \mathcal{O} \begin{pmatrix} 1 & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) \\ 1 & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) \\ 1 & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) \end{pmatrix} \quad \text{for } z \text{ outside the lense.} \quad (4.44)$$

The asymptotic behavior (4.43) for  $z \rightarrow 0$  inside the lense, comes from (4.33) combined with the factor  $z^{-\beta}$  that we have in the definitions (4.35)-(4.36). Note that  $\varphi$  remains bounded as  $z \rightarrow 0$ .

From the Cauchy-Riemann equations and (4.27) one can show that the real part of  $\varphi(z)$  is strictly negative for  $z$  inside the lense with  $\operatorname{Re} z \in (0, q)$  provided we take the lense close enough to  $[0, q]$ . In section 5 we will do a closer analysis of  $\varphi$  and related functions near 0. As a consequence of the assumed one-cut  $\theta$ -regularity we have (5.30) and together with (5.13), (5.34) it implies that

$$\varphi(z) = \begin{cases} -\pi i + 3\pi i c_{0,V} z^{\frac{1}{3}} + \mathcal{O}\left(z^{\frac{2}{3}}\right), & \text{for } z \text{ in the upper part of the lense,} \\ \pi i - 3\pi i c_{0,V} z^{\frac{1}{3}} + \mathcal{O}\left(z^{\frac{2}{3}}\right), & \text{for } z \text{ in the lower part of the lense,} \end{cases} \quad (4.45)$$

as  $z \rightarrow 0$ . Thus for  $z$  close to the origin,

$$\operatorname{Re} \varphi(z) \approx -3\pi c_{0,V} \sin\left(\frac{1}{3}|\arg(z)|\right) |z|^{\frac{1}{3}} \quad (4.46)$$

which is negative when  $z$  is not on the positive real line. In particular,  $\operatorname{Re} \varphi(z) < 0$  for  $z$  on the parts of  $\Delta_1^\pm$  that are close to 0.

By taking a smaller lense if necessary, we conclude that  $\operatorname{Re} \varphi(z) < 0$  on the lips of the lense. Thus the jump matrices on  $\Delta_1^\pm$  and on  $(q, \infty)$  (this is due to (4.28)) in the RH problem for  $S$  tend to the identity matrix as  $n \rightarrow \infty$ .

#### 4.4 Global parametrix

We expect that  $S$  is close to the solution of the following RH problem, at least away from the end points 0 and  $q$ :

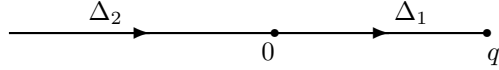


Figure 4: Contour  $\Sigma_N = (-\infty, q]$  for the RH problem for  $N$ .

##### Riemann-Hilbert Problem 4.8.

**RH-N1**  $N : \mathbb{C} \setminus (-\infty, q] \rightarrow \mathbb{C}^{3 \times 3}$  is analytic.

**RH-N2** On  $\Sigma_N$  (see Figure 4) we have the jumps

$$N_+(x) = N_-(x) \begin{pmatrix} 0 & x^\beta & 0 \\ -x^{-\beta} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{for } x \in (0, q), \quad (4.47)$$

$$N_+(x) = N_-(x) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{for } x < 0. \quad (4.48)$$

**RH-N3** As  $z \rightarrow \infty$

$$N(z) = \left( \mathbb{I} + \mathcal{O}\left(\frac{1}{z}\right) \right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & z^{\frac{1}{4}} & 0 \\ 0 & 0 & z^{-\frac{1}{4}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (4.49)$$

In [29, section 4] a global parametrix  $N_\alpha$  is constructed that is analytic in  $\mathbb{C} \setminus (-\infty, q]$  with jumps

$$N_{\alpha,+}(x) = N_{\alpha,-}(x) \begin{pmatrix} 0 & x^\alpha & 0 \\ -x^{-\alpha} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x \in (0, q),$$

$$N_{\alpha,+}(x) = N_{\alpha,-}(x) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -|x|^{-\alpha} \\ 0 & |x|^\alpha & 0 \end{pmatrix}, \quad x < 0,$$

and asymptotic behavior

$$N_\alpha(z) = \left( \mathbb{I} + \mathcal{O}\left(\frac{1}{z}\right) \right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & z^{\frac{1}{4}} & 0 \\ 0 & 0 & z^{-\frac{1}{4}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & z^{\alpha/2} & 0 \\ 0 & 0 & z^{-\alpha/2} \end{pmatrix} \quad \text{as } z \rightarrow \infty.$$

It is straightforward to check that

$$N(z) = N_{2\beta}(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & z^{-\beta} & 0 \\ 0 & 0 & z^\beta \end{pmatrix} \quad (4.50)$$

satisfies the conditions in the RH problem 4.8.

From (4.50) and formulas (4.3) and (4.4) in [29] we also find the following behaviors near  $z = 0$  and  $z = q$ .

**RH-N4**

$$N(z) \begin{pmatrix} z^{\frac{2\beta}{3}} & 0 & 0 \\ 0 & z^{-\frac{\beta}{3}} & 0 \\ 0 & 0 & z^{-\frac{\beta}{3}} \end{pmatrix} = \mathcal{O} \begin{pmatrix} z^{-\frac{1}{3}} & z^{-\frac{1}{3}} & z^{-\frac{1}{3}} \\ z^{-\frac{1}{3}} & z^{-\frac{1}{3}} & z^{-\frac{1}{3}} \\ z^{-\frac{1}{3}} & z^{-\frac{1}{3}} & z^{-\frac{1}{3}} \end{pmatrix} \quad \text{as } z \rightarrow 0, \quad (4.51)$$

$$N(z) = \mathcal{O} \begin{pmatrix} (z-q)^{-\frac{1}{4}} & (z-q)^{-\frac{1}{4}} & 1 \\ (z-q)^{-\frac{1}{4}} & (z-q)^{-\frac{1}{4}} & 1 \\ (z-q)^{-\frac{1}{4}} & (z-q)^{-\frac{1}{4}} & 1 \end{pmatrix} \quad \text{as } z \rightarrow q.$$

## 4.5 Local parametrix at the soft edge

We consider an open disk  $D(q, r_q)$  around  $q$  with some small radius  $r_q > 0$ , so that  $D(q, r_q)$  is contained in the region  $O_V$  where  $V$  is analytic. The local parametrix problem around  $q$  is of the form:

**Riemann-Hilbert Problem 4.9.**

**RH-Q1**  $Q$  is analytic on  $D(q, r_q) \setminus (\Delta_1 \cup \Delta_{1+} \cup \Delta_{1-} \cup [q, \infty))$ .

**RH-Q2**  $Q$  has the same jumps as  $S$  has on  $(\Delta_1 \cup \Delta_{1+} \cup \Delta_{1-} \cup [q, \infty)) \cap D(q, r_q)$ .

**RH-Q3**  $Q$  has the same behavior as  $z \rightarrow q$  as  $S$  has on  $D(q, r_q)$ .

**RH-Q4**  $Q$  matches with  $N$  on the boundary:

$$Q(z)N^{-1}(z) = \mathbb{I} + \mathcal{O}(n^{-1}) \quad \text{as } n \rightarrow \infty, \quad (4.52)$$

uniformly for  $z \in \partial D(q, r_q)$ .

The construction of  $Q$  can be done by a standard procedure using Airy functions [15, 16, 17]. See [28] for a similar construction in a  $3 \times 3$  matrix valued context. We omit the details.

## 5 Local parametrix at the hard edge 0

Our main task will be the construction of the local parametrix  $P$  around the origin with the help of Meijer G-functions. Usually such a local parametrix is constructed in a (possibly shrinking) disk with a matching condition on the boundary of the disk. We deviate from this by constructing the local parametrix on two concentric disks, which leads to a matching condition on two circles. We introduce a new iterative technique to improve the matching.

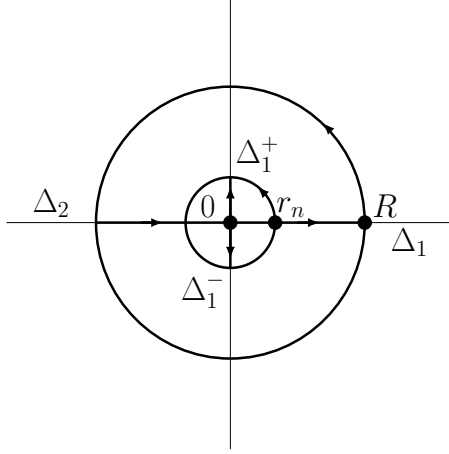


Figure 5: The contours for the local parametrix  $P$ .

### 5.1 Statement of the RH problem for $P$

We consider two concentric disks with radii  $r_n$  and  $R$  which are centered at the origin, see Figure 5. For our purposes we will take

$$r_n = n^{-\frac{3}{2}} \quad (5.1)$$

and  $R$  is fixed, but small enough so that the boundary of the disk  $D(0, R)$  is contained in the region  $O_V$  where  $V$  is analytic, and such that the boundary of the disk meets the lips of the lens in the (almost) vertical parts of  $\Delta_1^\pm$ , see Figure 3. We also want  $R < q$ . For large enough  $n$  we of course have  $r_n < R$  and this is what we assume without further notice. We use

$$A(0; r_n, R) = \{z \in \mathbb{C} \mid r_n < |z| < R\} \quad (5.2)$$

to denote the annular region bounded by the two circles of radius  $r_n$  and  $R$ . Our aim is to construct a local parametrix  $P$  in the neighborhood  $D(0, R)$  of 0 that also has a jump on the circle  $|z| = r_n$ .

#### Riemann-Hilbert Problem 5.1.

**RH-P1**  $P : D(0, R) \setminus (\partial D(0, r_n) \cup \Sigma_S) \rightarrow \mathbb{C}^{3 \times 3}$  is analytic.

**RH-P2a**  $P$  has the same jumps as  $S$  has on  $\Sigma_S \cap D(0, r_n)$ , see **RH-S2** in the RH problem 4.7 for  $S$ . That is, we have

$$P_+(x) = P_-(x) \begin{pmatrix} 0 & x^\beta & 0 \\ -x^{-\beta} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x \in \Delta_1 \cap D(0, r_n), \quad (5.3)$$

$$P_+(z) = P_-(z) \begin{pmatrix} 1 & 0 & 0 \\ z^{-\beta} e^{2n\varphi(z)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z \in (\Delta_1^+ \cup \Delta_2^+) \cap D(0, r_n), \quad (5.4)$$

$$P_+(x) = P_-(x) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad x \in \Delta_2 \cap D(0, r_n), \quad (5.5)$$

where all contours are oriented as in Figure 5.

**RH-P2b**  $P$  has the same jumps as  $N$  has on  $\Sigma_N \cap A(0; r_n, R)$ , see **RH-N2** in the RH problem 4.8 for

$N$ , and no jump on  $(\Delta_1^+ \cup \Delta_1^-) \cap A(0; r_n, R)$ . That is, we have

$$P_+(x) = P_-(x) \begin{pmatrix} 0 & x^\beta & 0 \\ -x^{-\beta} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x \in \Delta_1 \cap A(0; r_n, R), \quad (5.6)$$

$$P_+(x) = P_-(x) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad x \in \Delta_2 \cap A(0; r_n, R), \quad (5.7)$$

$$P_+(z) = P_-(z), \quad z \in \Delta_1^\pm \cap A(0; r_n, R). \quad (5.8)$$

**RH-P3** As  $z \rightarrow 0$ ,  $P$  has the same behavior as  $S$  has, see **RH-S4** in the RH problem for  $S$ . That is, we have

$$P(z) = \mathcal{O} \begin{pmatrix} z^{-\alpha-\frac{1}{2}} h_{\alpha+\frac{1}{2}}(z) & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) \\ z^{-\alpha-\frac{1}{2}} h_{\alpha+\frac{1}{2}}(z) & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) \\ z^{-\alpha-\frac{1}{2}} h_{\alpha+\frac{1}{2}}(z) & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) \end{pmatrix} \quad \text{for } z \text{ to the right of } \Delta_1^\pm \cap D(0, R), \quad (5.9)$$

$$P(z) = \mathcal{O} \begin{pmatrix} 1 & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) \\ 1 & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) \\ 1 & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) & z^{-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) \end{pmatrix} \quad \text{for } z \text{ to the left of } \Delta_1^\pm \cap D(0, R). \quad (5.10)$$

**RH-P4** (Matching condition) We have

$$P_+(z)P_-^{-1}(z) = \mathbb{I} + \mathcal{O}(n^{-1}) \quad \text{as } n \rightarrow \infty, \quad \text{uniformly for } z \in \partial D(0, r_n), \quad (5.11)$$

$$P(z)N^{-1}(z) = \mathbb{I} + \mathcal{O}(n^{-1}) \quad \text{as } n \rightarrow \infty, \quad \text{uniformly for } z \in \partial D(0, R). \quad (5.12)$$

## 5.2 Ansatz for $P$

We will look for  $P$  in a form that contains the matrix valued function  $\Phi_\alpha$  defined in (2.27) and that is built out of Meijer G-functions. It has the piecewise constant jumps given in (3.1). We will combine it with certain functions  $\varphi_1$  and  $\varphi_2$  to make sure that  $P$  has the correct jumps from (5.3)-(5.8). Of course, we also have to take care of the behavior (5.9)-(5.10) as  $z \rightarrow 0$  and the matching condition (5.11)-(5.12).

Recall that  $\varphi$  is given in (4.26). We are going to use the slightly modified version

$$\varphi_1(z) = \varphi(z) + \begin{cases} \pi i, & \text{if } z \in O_V, \operatorname{Im}(z) > 0, \\ -\pi i, & \text{if } z \in O_V, \operatorname{Im}(z) < 0. \end{cases} \quad (5.13)$$

We also introduce

$$\varphi_2(z) = -g_2(z) + \frac{1}{2}g_1(z) + \begin{cases} -\frac{\pi i}{2}, & \text{if } \operatorname{Im}(z) > 0, \\ \frac{\pi i}{2}, & \text{if } \operatorname{Im}(z) < 0, \end{cases} \quad (5.14)$$

where  $g_1$  and  $g_2$  are the functions from (4.15), (4.16). The  $\varphi_1$  and  $\varphi_2$  functions appear in the following diagonal matrix

$$D_0(z) = \begin{pmatrix} e^{\frac{2}{3}(2\varphi_1(z)+\varphi_2(z))} & 0 & 0 \\ 0 & e^{\frac{2}{3}(\varphi_2(z)-\varphi_1(z))} & 0 \\ 0 & 0 & e^{-\frac{2}{3}(\varphi_1(z)+2\varphi_2(z))} \end{pmatrix}. \quad (5.15)$$

We will look for  $P$  in the following form.

**Definition 5.2.** Suppose  $f$  is a conformal map and  $E_{in}$  and  $E_{out}$  are non-singular analytic matrix-valued functions on  $D(0, r_n)$  and  $A(0; r_n, R)$  respectively. Then we define

$$P(z) = E_{in}(z)\Phi_\alpha(n^3 f(z))D_0^n(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & z^\beta & 0 \\ 0 & 0 & z^\beta \end{pmatrix}, \quad \text{for } z \in D(0, r_n), \quad (5.16)$$

$$P(z) = E_{out}(z)N(z), \quad \text{for } z \in A(0; r_n, R), \quad (5.17)$$

where  $D_0$  is given by (5.15).

**Lemma 5.3.** *Suppose that  $f$  is a conformal map in  $D(0, R)$  that maps the positive (negative) real line inside  $D(0, R)$  to part of the positive (negative) real line. Suppose also that the lense is opened such that  $\Delta_1^+ \cap D(0, R)$  is mapped to the positive imaginary axis and  $\Delta_1^- \cap D(0, R)$  is mapped to the negative imaginary axis. Then for any analytic non-singular  $3 \times 3$  matrix valued functions  $E_{in}$  and  $E_{out}$  the function  $P(z)$  defined in (5.16) and (5.17) satisfies the conditions **RH-P1**, **RH-P2a**, **RH-P2b**, and **RH-P3** of the RH problem for  $P$ .*

*Proof.* The analyticity condition **RH-P1** is clearly satisfied. The condition **RH-P2b** is also immediate since the jumps in (5.6)-(5.8) are satisfied by  $N$ , and they do not change if we multiply by an analytic prefactor  $E_{out}$  as in (5.17). For **RH-P3** we note that by the definition of  $h_\alpha$  in (2.5) we check that  $h_{-\alpha-\frac{1}{2}}(z) = z^{-\alpha-\frac{1}{2}}h_{\alpha+\frac{1}{2}}(z)$ . Then we use (3.2) and (5.16) where we note that  $\varphi_1(z)$  and  $\varphi_2(z)$  remain bounded as  $z \rightarrow 0$ . Also  $E_{out}(z)$  is bounded as  $z \rightarrow 0$  and the conditions (5.9) and (5.10) of **RH-P3** follow. What remains is to prove **RH-P2a**.

We first check the jumps on the imaginary axis. The functions  $z \mapsto z^\beta$ ,  $\varphi_1$  and  $\varphi_2$  are analytic on the imaginary axis. Then by (3.1) and (5.16), we have for  $z \in \Delta_1 \pm \cap D(0, r_n)$ ,

$$\begin{aligned} P_-^{-1}(z)P_+(z) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & z^{-\beta} & 0 \\ 0 & 0 & z^{-\beta} \end{pmatrix} D_0^{-n}(z) \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} D_0^n(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & z^\beta & 0 \\ 0 & 0 & z^\beta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ z^{-\beta} e^{2n\varphi_1(z)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \quad (5.18)$$

Because of (5.13) we may replace  $\varphi_1$  by  $\varphi$  in (5.18), since  $n$  is an integer. Thus (5.4) holds.

When checking the jumps on the real line, we have to take into account the fact that  $\varphi_1$  and  $\varphi_2$  have jumps there. Also  $z \mapsto z^\beta$  is defined as the principal branch and so it has a jump on the negative real axis (unless  $\beta$  is an integer). For  $x \in \Delta_1$ , we find by (4.27) that

$$\varphi_{1,\pm}(x) = \pm\pi i \mu^*([0, x]) = \pm\pi i (1 - \mu^*([x, \infty))) \quad (5.19)$$

and therefore

$$\varphi_{1,-}(x) = -\varphi_{1,+}(x), \quad x \in \Delta_1. \quad (5.20)$$

From (5.14) and (4.17), we find (since  $g_2$  is analytic across  $\Delta_1$ )

$$\varphi_{2,+}(x) - \varphi_{2,-}(x) = \frac{1}{2}(g_{1,+}(x) - g_{1,-}(x)) - \pi i = \pi i \mu^*([x, \infty)) - \pi i$$

which by (5.19) leads to

$$\varphi_{2,-}(x) = \varphi_{1,+}(x) + \varphi_{2,+}(x), \quad x \in \Delta_1. \quad (5.21)$$

The two jump relations (5.20) and (5.21) are used in the calculation of the jump of  $P$  across  $\Delta_1$ .

By (3.1) and (5.16), we have for  $x \in \Delta_1 \cap D(0, r_n)$ ,

$$P_-^{-1}(x)P_+(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x^{-\beta} & 0 \\ 0 & 0 & x^{-\beta} \end{pmatrix} (D_0^{-n})_-(z) \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} (D_0^n)_+(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & x^\beta & 0 \\ 0 & 0 & x^\beta \end{pmatrix}. \quad (5.22)$$

If we do the multiplication we find three non-zero entries with an exponential factor containing certain linear combinations of  $\varphi_{1,\pm}(x)$  and  $\varphi_{2,\pm}(x)$ . Because of (5.20) and (5.21) all three linear combinations turn out to vanish identically for  $x \in \Delta_1$ . The factors  $x^\beta$  and  $x^{-\beta}$  end up in the 12- and 21-entries, respectively, and the jump condition (5.3) follows.

Something similar happens on  $\Delta_2$ . From (5.14) and (4.15), (4.16) we deduce

$$\varphi_{2,\pm}(x) = \mp \pi i \nu^*([x, 0]), \quad x \in \Delta_2, \quad (5.23)$$

and therefore

$$\varphi_{2,-}(x) = -\varphi_{2,+}(x), \quad x \in \Delta_2, \quad (5.24)$$

Also from (4.18), (4.26), (5.13), and the fact that  $V$  is analytic in  $O_V$ , we get for  $x \in \Delta_2 \cap O_V$ ,

$$\varphi_{1,+}(x) - \varphi_{1,-}(x) = -g_{1,+}(x) + g_{1,-}(x) - \frac{1}{2}(g_{2,+}(x) - g_{2,-}(x)) + 2\pi i = \pi i \nu^*([x, 0]),$$

which by (5.23) leads to

$$\varphi_{1,-}(x) = \varphi_{1,+}(x) + \varphi_{2,+}(x), \quad x \in \Delta_2 \cap O_V. \quad (5.25)$$

By (3.1) and (5.16), we then have for  $x \in \Delta_2 \cap D(0, r_n)$ ,

$$\begin{aligned} & P_-^{-1}(x)P_+(x) \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & x_-^{-\beta} & 0 \\ 0 & 0 & x_-^{-\beta} \end{pmatrix} (D_0^{-n})_-(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & ie^{2\pi i \alpha} \\ 0 & -ie^{2\pi i \alpha} & 0 \end{pmatrix}^{-1} (D_0^n)_+(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & x_+^\beta & 0 \\ 0 & 0 & x_+^\beta \end{pmatrix}. \end{aligned} \quad (5.26)$$

Note that the inverse of the jump matrix in (3.1) appears in (5.26), since the orientation on the negative real axis is from left to right, while in (3.1) it is from right to left.

Multiplying the matrices in (5.26) we get a jump matrix with three non-zero entries. Each entry has an exponential factor containing a linear combination of  $\varphi_{1,\pm}(x)$  and  $\varphi_{2,\pm}(x)$ . Each of these three linear combinations vanishes because of the identities (5.25) and (5.24). What we are left with from (5.26) then is

$$P_-^{-1}(x)P_+(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & ie^{-2\pi i \alpha} x_-^{-\beta} x_+^\beta \\ 0 & -ie^{2\pi i \alpha} x_-^{-\beta} x_+^\beta & 0 \end{pmatrix}, \quad x \in \Delta_2 \cap O_V. \quad (5.27)$$

Finally, we also use that  $x_-^{-\beta} = |x|^{-\beta} e^{\pi i \beta}$  and  $x_+^\beta = |x|^\beta e^{\pi i \beta}$ , so that  $x_-^{-\beta} x_+^\beta = e^{2\pi i \beta} = ie^{2\pi i \alpha}$  for  $x \in \Delta_2$ , where we recall  $\beta = \alpha + \frac{1}{4}$ , see (4.3). Then (5.27) reduces to (5.4), as claimed.  $\square$

In view of Lemma 5.3 our remaining task is to construct a specific conformal map  $f$  and analytic prefactors  $E_{in}$  and  $E_{out}$  such that the matching condition **RH-P4** is satisfied as well.

### 5.3 The conformal map $f$

Remember that  $V$  can be analytically continued to an open neighborhood  $O_V$  of  $[0, \infty)$ , and that  $\overline{D(0, R)} \subset O_V$ .

We let  $O_V^0$  be such that  $\overline{D(0, R)} \subset O_V^0 \subset O_V$  and  $O_V^0$  is slightly larger than  $\overline{D(0, R)}$ . We make sure that  $\operatorname{Re} z < q$  for  $z \in O_V^0$ . On  $O_V^0 \setminus \mathbb{R}$  we define the auxiliary functions

$$f_1(z) = -z^{-\frac{1}{3}} \times \begin{cases} -\omega^2 \varphi_1(z) + \varphi_2(z), & \text{for } z \in O_V^0, \operatorname{Im}(z) > 0, \\ -\omega \varphi_1(z) + \varphi_2(z), & \text{for } z \in O_V^0, \operatorname{Im}(z) < 0, \end{cases} \quad (5.28)$$

$$f_2(z) = -z^{-\frac{2}{3}} \times \begin{cases} -\omega \varphi_1(z) + \varphi_2(z), & \text{for } z \in O_V^0, \operatorname{Im}(z) > 0, \\ -\omega^2 \varphi_1(z) + \varphi_2(z), & \text{for } z \in O_V^0, \operatorname{Im}(z) < 0. \end{cases} \quad (5.29)$$

**Proposition 5.4.** *The functions  $f_1$  and  $f_2$  have analytic continuation to  $O_V^0$ , and*

$$f_1(0) = 3\sqrt{3}\pi c_{0,V} > 0. \quad (5.30)$$

*Proof.* It follows from (5.20), (5.23), (5.25) and (5.21) that  $f_1$  and  $f_2$  have no jumps on  $O_V^0 \cap \mathbb{R}$ . For example for  $x \in \Delta_2 \cap O_V^0$  we have

$$\begin{aligned} f_{1-}(x) &= -e^{\frac{\pi i}{3}} |x|^{-\frac{1}{3}} (-\omega\varphi_{1-}(x) + \varphi_{2-}(x)) \\ &= -e^{\frac{\pi i}{3}} |x|^{-\frac{1}{3}} (-\omega(\varphi_{1+}(x) + \varphi_{2+}(x)) + \varphi_{2-}(x)) \\ &= -e^{\frac{\pi i}{3}} |x|^{-\frac{1}{3}} (-\omega\varphi_{1+}(x) - (1 + \omega)\varphi_{2+}(x)) \\ &= -e^{-\frac{\pi i}{3}} |x|^{-\frac{1}{3}} (-\omega^2\varphi_{1+}(x) + \varphi_{2+}(x)) = f_{1+}(x), \end{aligned} \quad (5.31)$$

where we have used (5.25) in the second line and (5.23) in the third line. We conclude that  $f_1$  and  $f_2$  have a Laurent series around  $z = 0$ . In the upper half plane we have

$$z^{1/3}f_1(z) + z^{2/3}f_2(z) = -\varphi_1(z) - 2\varphi_2(z) \quad (5.32)$$

$$\omega^2 z^{1/3}f_1(z) + \omega z^{2/3}f_2(z) = \varphi_2(z) - \varphi_1(z). \quad (5.33)$$

It follows that (where we use  $1 + 2\omega = i\sqrt{3}$  and  $1 + 2\omega^2 = -i\sqrt{3}$ ),

$$i\sqrt{3}z^{1/3}f_1(z) - i\sqrt{3}z^{2/3}f_2(z) = 3\varphi_1(z), \quad \text{Im } z > 0. \quad (5.34)$$

Thus using the behavior (1.9) of the density of the equilibrium measure  $\mu^*$  we have

$$\sqrt{3}ix^{1/3}f_1(x) - \sqrt{3}ix^{2/3}f_2(x) = 3\varphi_{1+}(x) = 3\pi i\mu^*([0, x]) \sim 9\pi ic_{0,V}x^{1/3} \quad \text{as } x \downarrow 0. \quad (5.35)$$

This is only possible if  $f_1$  and  $f_2$  do not have negative powers in their respective Laurent series. We conclude that  $f_1$  and  $f_2$  can be analytically continued to  $O_V^0$ .

The value (5.30) follows from (5.35).  $\square$

Inverting the relations (5.28), (5.29), we see that for  $z \in O_V^0$ ,  $\text{Im } z > 0$ ,

$$\begin{aligned} 2\varphi_1(z) + \varphi_2(z) &= \omega z^{1/3}f_1(z) + \omega^2 z^{2/3}f_2(z), \\ \varphi_2(z) - \varphi_1(z) &= \omega^2 z^{1/3}f_1(z) + \omega z^{2/3}f_2(z), \\ -\varphi_1(z) - 2\varphi_2(z) &= z^{1/3}f_1(z) + z^{2/3}f_2(z), \end{aligned} \quad (5.36)$$

A similar relation, with  $\omega$  and  $\omega^2$  interchanged, holds for  $\text{Im } z < 0$ . Observe that the combinations (5.36) appear in the diagonal matrix  $D_0$ , see (5.15).

Our choice of conformal map is dictated by the desire that  $D_1(n^3f(z))$  should cancel with  $D_0^n(z)$  in (5.16) as much as possible, where

$$D_1(z) = \begin{cases} \begin{pmatrix} e^{-3\omega z^{1/3}} & 0 & 0 \\ 0 & e^{-3\omega^2 z^{1/3}} & 0 \\ 0 & 0 & e^{-3z^{1/3}} \end{pmatrix}, & \text{for } \text{Im } z > 0, \\ \begin{pmatrix} e^{-3\omega^2 z^{1/3}} & 0 & 0 \\ 0 & e^{-3\omega z^{1/3}} & 0 \\ 0 & 0 & e^{-3z^{1/3}} \end{pmatrix}, & \text{for } \text{Im } z < 0, \end{cases} \quad (5.37)$$

is the exponential part of the asymptotic formula for  $\Phi_\alpha$ . Ideally, we would have  $D_1(n^3f(z))D_0^n(z) = \mathbb{I}$ , but this is not possible.

**Definition 5.5.** Let  $f_1$  be given by (5.28). We define for  $z \in O_V^0$ ,

$$\begin{aligned} f(z) &= \frac{8}{729} z f_1^3(z) \\ &= \frac{8}{729} \times \begin{cases} (\omega^2\varphi_1(z) - \varphi_2(z))^3, & \text{Im } z > 0, \\ (\omega\varphi_1(z) - \varphi_2(z))^3, & \text{Im } z < 0. \end{cases} \end{aligned} \quad (5.38)$$



The second equality in (5.38) follows from (5.28).

**Proposition 5.6.**  *$f$  is a conformal map near 0 that maps positive and negative numbers to positive and negative numbers, respectively.*

*Proof.* From (5.38) and (5.30) we learn that  $f'(0) = \frac{8}{729}f_1(0)^3 = \frac{8\pi^3}{3\sqrt{3}}c_{0,v}^3 > 0$ . Thus  $f$  is conformal in some small enough neighbourhood of 0. Now let  $x$  be a real number in this open neighbourhood, then

$$\overline{f(x)} = \frac{8}{729} \left( \overline{\omega\varphi_{1+}(x)} - \overline{\varphi_{2+}(x)} \right)^3 = \frac{8}{729} (\omega\varphi_{1-}(x) - \varphi_{2-}(x))^3 = f(x). \quad (5.39)$$

Here we have used that  $\overline{\varphi_{1+}(x)} = \varphi_{1-}(x)$  and  $\overline{\varphi_{2+}(x)} = \varphi_{2-}(x)$ , which follows from the definitions (5.13), (5.14) and the fact that  $\overline{g_{1+}(x)} = g_{1-}(x)$  and  $\overline{g_{2+}(x)} = g_{2-}(x)$ , which is immediate from the definitions of the  $g$ -functions. It follows that real numbers are mapped to real numbers. Since  $f(0) = 0$  and  $f'(0) > 0$  it then follows that positive numbers are mapped to positive numbers, and negative numbers are mapped to negative numbers.  $\square$

**Remark 5.7.** From now on we adjust the construction so that the conditions regarding the conformal map in Lemma 5.3 are satisfied. That is, we let  $f$  be as in (5.38), and if necessary we decrease  $R$  so that  $f$  is a conformal map in  $D(0, R)$ . We open the lense so that  $f$  maps

$$f(\Delta_1^\pm \cap D(0, R)) \subset i\mathbb{R}^\pm.$$

**Proposition 5.8.** *The conformal map  $f$  is such that  $D_1(f(z))D_0(z) = D_2(z)$ ,  $z \in O_V^0$ , with*

$$D_2(z) = \begin{cases} \begin{pmatrix} e^{\frac{2}{3}\omega z^{2/3}f_2(z)} & 0 & 0 \\ 0 & e^{\frac{2}{3}\omega z^{2/3}f_2(z)} & 0 \\ 0 & 0 & e^{\frac{2}{3}z^{2/3}f_2(z)} \end{pmatrix}, & \text{for } \text{Im } z > 0, \\ \begin{pmatrix} e^{\frac{2}{3}\omega z^{2/3}f_2(z)} & 0 & 0 \\ 0 & e^{\frac{2}{3}\omega z^{2/3}f_2(z)} & 0 \\ 0 & 0 & e^{\frac{2}{3}z^{2/3}f_2(z)} \end{pmatrix}, & \text{for } \text{Im } z < 0, \end{cases} \quad (5.40)$$

*Proof.* We combine (5.36) with the definitions (5.15), (5.37), and (5.38).  $\square$

Observe that for  $|z| = r_n = n^{-3/2}$  the diagonal entries of  $D_2^n(z)$  remain bounded and bounded away from 0 as  $n \rightarrow \infty$ , since  $f_2$  is analytic by Proposition 5.4.

## 5.4 First step towards the matching

From the asymptotic behavior of  $\Phi_\alpha$  in (3.7) and Proposition 5.8 we obtain

$$T_\alpha \Phi_\alpha(n^3 f(z)) D_0^n(z) = \frac{2\pi}{\sqrt{3}} (n^3 f(z))^{-\frac{2\beta}{3}} \left( \mathbb{I} + \frac{A_\alpha}{n^3 f(z)} + O\left(\frac{1}{n^6 f(z)^2}\right) \right) L_\alpha(n^3 f(z)) D_2^n(z) \quad (5.41)$$

with a constant matrix  $A_\alpha$ . In the first step towards the matching condition, we forget about the terms  $\frac{A_\alpha}{n^3 f(z)}$  and  $O\left(\frac{1}{n^6 f(z)^2}\right)$  and in view of (5.16)-(5.17), we aim to find  $E_{in}^{(1)}$  and  $E_{out}^{(1)}$  such that

$$E_{in}^{(1)}(z) T_\alpha^{-1} \frac{2\pi}{\sqrt{3}} (n^3 f(z))^{-\frac{2\beta}{3}} L_\alpha(n^3 f(z)) D_2^n(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & z^\beta & 0 \\ 0 & 0 & z^\beta \end{pmatrix} = E_{out}^{(1)}(z) N(z), \quad |z| = r_n, \quad (5.42)$$

To that end we introduce the following definitions.

**Definition 5.9.** We define

$$F_n(z) = L_\alpha(n^3 f(z))D_2^n(z), \quad (5.43)$$

$$E_n(z) = N(z) \begin{pmatrix} z^{\frac{2\beta}{3}} & 0 & 0 \\ 0 & z^{-\frac{\beta}{3}} & 0 \\ 0 & 0 & z^{-\frac{\beta}{3}} \end{pmatrix} F_n^{-1}(z)T_\alpha, \quad (5.44)$$

$$E_{in}^{(1)}(z) = \frac{\sqrt{3}}{2\pi} n^{2\beta} \left( \frac{f(z)}{z} \right)^{\frac{2\beta}{3}} E_n(z), \quad (5.45)$$

$$E_{out}^{(1)}(z) = \mathbb{I}. \quad (5.46)$$

With these definitions (5.42) holds. In (5.45) we separated the scalar factors from the matrix valued factors which are in  $E_n$ . We need that  $E_{in}^{(1)}$  is analytic in a neighborhood of 0, which in view of (5.45) means that we have to show that  $E_n$  is analytic.

**Lemma 5.10.** (a)  $N(z) \begin{pmatrix} z^{\frac{2\beta}{3}} & 0 & 0 \\ 0 & z^{-\frac{\beta}{3}} & 0 \\ 0 & 0 & z^{-\frac{\beta}{3}} \end{pmatrix}$ ,  $L_\alpha(z)$  and  $F_n(z)$  all satisfy the same jump conditions near  $z = 0$ . Namely, all three matrix valued functions are analytic in  $O_V^0 \setminus \mathbb{R}$ , with jump matrix  $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  for  $x > 0$ , and  $\begin{pmatrix} e^{\frac{4\pi i\beta}{3}} & 0 & 0 \\ 0 & 0 & -e^{-\frac{2\pi i\beta}{3}} \\ 0 & e^{-\frac{2\pi i\beta}{3}} & 0 \end{pmatrix}$  for  $x < 0$ .

(b) In addition, all three matrix valued functions are  $\mathcal{O}(z^{-1/3})$  as  $z \rightarrow 0$ .

(c)  $E_n(z)$  is an invertible and analytic matrix valued function in a neighborhood of  $z = 0$ .

*Proof.* (a) Using **RH-N2** we obtain for  $x \in (0, q)$ ,

$$\begin{aligned} & \begin{pmatrix} x_-^{\frac{2\beta}{3}} & 0 & 0 \\ 0 & x_-^{-\frac{\beta}{3}} & 0 \\ 0 & 0 & x_-^{-\frac{\beta}{3}} \end{pmatrix}^{-1} N_-^{-1}(x)N_+(x) \begin{pmatrix} x_+^{\frac{2\beta}{3}} & 0 & 0 \\ 0 & x_+^{-\frac{\beta}{3}} & 0 \\ 0 & 0 & x_+^{-\frac{\beta}{3}} \end{pmatrix} \\ &= \begin{pmatrix} x_-^{-\frac{2\beta}{3}} & 0 & 0 \\ 0 & x^{\frac{\beta}{3}} & 0 \\ 0 & 0 & x^{\frac{\beta}{3}} \end{pmatrix} \begin{pmatrix} 0 & x^\beta & 0 \\ -x^{-\beta} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^{\frac{2\beta}{3}} & 0 & 0 \\ 0 & x^{-\frac{\beta}{3}} & 0 \\ 0 & 0 & x^{-\frac{\beta}{3}} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (5.47) \end{aligned}$$

and for  $x < 0$  we get

$$\begin{aligned} & \begin{pmatrix} x_-^{\frac{2\beta}{3}} & 0 & 0 \\ 0 & x_-^{-\frac{\beta}{3}} & 0 \\ 0 & 0 & x_-^{-\frac{\beta}{3}} \end{pmatrix}^{-1} N_-^{-1}(x)N_+(x) \begin{pmatrix} x_+^{\frac{2\beta}{3}} & 0 & 0 \\ 0 & x_+^{-\frac{\beta}{3}} & 0 \\ 0 & 0 & x_+^{-\frac{\beta}{3}} \end{pmatrix} \\ &= \begin{pmatrix} |x|^{-\frac{2\beta}{3}} e^{\frac{2\pi i\beta}{3}} & 0 & 0 \\ 0 & |x|^{\frac{\beta}{3}} e^{-\frac{\pi i\beta}{3}} & 0 \\ 0 & 0 & |x|^{\frac{\beta}{3}} e^{\frac{\pi i\beta}{3}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} |x|^{\frac{2\beta}{3}} e^{\frac{2\pi i\beta}{3}} & 0 & 0 \\ 0 & |x|^{-\frac{\beta}{3}} e^{-\frac{\pi i\beta}{3}} & 0 \\ 0 & 0 & |x|^{-\frac{\beta}{3}} e^{-\frac{\pi i\beta}{3}} \end{pmatrix} \\ &= \begin{pmatrix} e^{\frac{4\pi i\beta}{3}} & 0 & 0 \\ 0 & 0 & -e^{-\frac{2\pi i\beta}{3}} \\ 0 & e^{-\frac{2\pi i\beta}{3}} & 0 \end{pmatrix}. \quad (5.48) \end{aligned}$$

Next we notice using (3.8) that for  $x > 0$

$$\begin{aligned} L_{\alpha,-}^{-1}(x)L_{\alpha,+}(x) &= \begin{pmatrix} e^{-\frac{2\pi i\beta}{3}} & 0 & 0 \\ 0 & e^{\frac{2\pi i\beta}{3}} & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \omega & -\omega^2 & 1 \\ 1 & -1 & 1 \\ \omega^2 & -\omega & 1 \end{pmatrix}^{-1} \begin{pmatrix} \omega^2 & \omega & 1 \\ 1 & 1 & 1 \\ \omega & \omega^2 & 1 \end{pmatrix} \begin{pmatrix} e^{\frac{2\pi i\beta}{3}} & 0 & 0 \\ 0 & e^{-\frac{2\pi i\beta}{3}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} e^{\frac{2\pi i\beta}{3}} & 0 & 0 \\ 0 & e^{-\frac{2\pi i\beta}{3}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\frac{2\pi i\beta}{3}} & 0 & 0 \\ 0 & e^{-\frac{2\pi i\beta}{3}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned} \quad (5.49)$$

and for  $x < 0$  we obtain

$$\begin{aligned} L_{\alpha,-}^{-1}(x)L_{\alpha,+}(x) &= \begin{pmatrix} e^{-\frac{2\pi i\beta}{3}} & 0 & 0 \\ 0 & e^{\frac{2\pi i\beta}{3}} & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \omega & -\omega^2 & 1 \\ 1 & -1 & 1 \\ \omega^2 & -\omega & 1 \end{pmatrix}^{-1} \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix} \begin{pmatrix} \omega^2 & \omega & 1 \\ 1 & 1 & 1 \\ \omega & \omega^2 & 1 \end{pmatrix} \begin{pmatrix} e^{\frac{2\pi i\beta}{3}} & 0 & 0 \\ 0 & e^{-\frac{2\pi i\beta}{3}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} e^{\frac{2\pi i\beta}{3}} & 0 & 0 \\ 0 & e^{-\frac{2\pi i\beta}{3}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e^{\frac{2\pi i\beta}{3}} & 0 & 0 \\ 0 & e^{-\frac{2\pi i\beta}{3}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{\frac{4\pi i\beta}{3}} & 0 & 0 \\ 0 & 0 & -e^{-\frac{2\pi i\beta}{3}} \\ 0 & e^{-\frac{2\pi i\beta}{3}} & 0 \end{pmatrix}. \end{aligned} \quad (5.50)$$

Since  $f$  is a conformal map that maps the positive (negative) real line near 0 to the positive (negative) real line near 0, we find that  $L_{\alpha}(n^3 f(z))$  has the same jump matrices (5.49)-(5.50) as  $L_{\alpha}(z)$ . Then by the definition (5.43) of  $F_n$ , we obtain for  $x \in O_V^0 \cap \mathbb{R}$ ,

$$F_{n,-}^{-1}(x)F_{n,+}(x) = (D_2^{-n})_-(x) \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} (D_2^n)_+(x), \quad x > 0, \quad (5.51)$$

$$F_{n,-}^{-1}(x)F_{n,+}(x) = (D_2^{-n})_-(x) \begin{pmatrix} e^{\frac{4\pi i\beta}{3}} & 0 & 0 \\ 0 & 0 & -e^{-\frac{2\pi i\beta}{3}} \\ 0 & e^{-\frac{2\pi i\beta}{3}} & 0 \end{pmatrix} (D_2^n)_+(x), \quad x < 0. \quad (5.52)$$

The definition (5.40) of  $D_2$  is such that (where we recall  $f_2$  is analytic in a neighborhood of 0)

$$\begin{aligned} D_{2,+}(x) &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} D_{2,-}(x) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x > 0, \\ D_{2,+}(x) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} D_{2,-}(x) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad x < 0. \end{aligned}$$

Combining this with (5.51)-(5.52) and using the fact that  $D_2$  is a diagonal matrix, we obtain that  $F_n$  has the correct jumps as claimed in part (a).

(b) Part (b) is a direct consequence of **RH-N4**, (3.8) and (5.38).

(c) It follows from (a) and the definition (5.44) that  $E_n$  has the identity jump on  $O_V^0 \cap (\mathbb{R} \setminus \{0\})$  and therefore  $E_n$  is analytic in  $O_V^0 \setminus \{0\}$ . By (b) it is  $\mathcal{O}(z^{-2/3})$  as  $z \rightarrow 0$  and we conclude that the singularity at  $z = 0$  is removable, and thus  $E_n$  is analytic in  $O_V^0$ . Furthermore, since all the factors in the definition on  $E_n$  are invertible,  $E_n$  itself is also invertible.  $\square$

As in (5.16)-(5.17) we now make the first attempt in defining  $P$ .

**Definition 5.11.** We define

$$P^{(1)}(z) = \begin{cases} E_{in}^{(1)}(z)\Phi_{\alpha}(n^3 f(z))D_0^n(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & z^{\beta} & 0 \\ 0 & 0 & z^{\beta} \end{pmatrix}, & \text{for } z \in D(0, r_n) \\ E_{out}^{(1)}(z)N(z) = N(z), & \text{for } z \in A(0; r_n, R), \end{cases} \quad (5.53)$$

with  $E_{in}^{(1)}(z)$  and  $E_{out}^{(1)}(z)$  as in (5.45)-(5.46).

In what follows we will give  $\partial D(0, r_n)$  a positive orientation.

**Corollary 5.12.** For  $|z| = r_n$ ,

$$P_+^{(1)}(z)P_-^{(1)}(z)^{-1} = \mathbb{I} + \frac{A_n^{(1)}(z)}{n^3 z} + E_n(z)\mathcal{O}(n^{-3})E_n^{-1}(z) \quad (5.54)$$

where

$$A_n^{(1)}(z) = \frac{1}{f'(0)}E_n(z)T_\alpha^{-1}A_\alpha T_\alpha E_n^{-1}(z). \quad (5.55)$$

*Proof.* Let  $|z| = r_n$ . Then we have  $\frac{1}{n^6 f(z)^2} = \mathcal{O}(n^{-3})$  and  $\frac{1}{n^3 f(z)} = \frac{1}{n^3 f'(0)z} + \mathcal{O}(n^{-3})$ . Plugging this into (5.41) and using (5.43) gives us

$$T_\alpha \Phi_\alpha(n^3 f(z))D_0^n(z) = \frac{2\pi}{\sqrt{3}}(n^3 f(z))^{-\frac{2\beta}{3}} \left( \mathbb{I} + \frac{A_\alpha}{n^3 f'(0)z} + \mathcal{O}(n^{-3}) \right) F_n(z), \quad (5.56)$$

Now using (5.44) and (5.56) we get

$$\begin{aligned} T_\alpha \Phi_\alpha(n^3 f(z))D_0^n(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & z^\beta & 0 \\ 0 & 0 & z^\beta \end{pmatrix} N^{-1}(z) \\ = \frac{2\pi}{\sqrt{3}} \left( \frac{n^3 f(z)}{z} \right)^{-\frac{2\beta}{3}} \left( \mathbb{I} + \frac{A_\alpha}{n^3 f'(0)z} + \mathcal{O}(n^{-3}) \right) T_\alpha E_n^{-1}(z). \end{aligned}$$

Finally, from this and (5.53), (5.45), (5.46), and (5.44) we arrive at

$$\begin{aligned} P_+^{(1)}(z)P_-^{(1)}(z)^{-1} &= E_{in}^{(1)}(z)T_\alpha^{-1} \frac{2\pi}{\sqrt{3}} \left( \frac{n^3 f(z)}{z} \right)^{-\frac{2\beta}{3}} \left( \mathbb{I} + \frac{A_\alpha}{n^3 f'(0)z} + \mathcal{O}(n^{-3}) \right) T_\alpha E_n^{-1}(z) \\ &= E_n(z)T_\alpha^{-1} \left( \mathbb{I} + \frac{A_\alpha}{n^3 f'(0)z} + \mathcal{O}(n^{-3}) \right) T_\alpha E_n^{-1}(z) \\ &= \mathbb{I} + \frac{A_n^{(1)}(z)}{n^3 z} + E_n(z)\mathcal{O}(n^{-3})E_n^{-1}(z), \end{aligned}$$

where we have used (5.55) in the last line.  $\square$

Unfortunately, (5.54) is not  $\mathbb{I} + \text{"small"}$  since both  $E_n(z)$  and  $E_n^{-1}(z)$  are  $\mathcal{O}(n)$  as  $n \rightarrow \infty$  with  $|z| = r_n$ , as we prove in the next subsection. Then it follows from (5.55) that  $A_n^{(1)}(z) = \mathcal{O}(n^2)$  and thus  $\frac{A_n^{(1)}(z)}{n^3 z} = \mathcal{O}(n^{1/2})$  as  $n \rightarrow \infty$  with  $|z| = r_n = n^{-\frac{3}{2}}$ .

## 5.5 Estimates on $E_n$

The following result will turn out to be key to obtain the matching.

**Lemma 5.13.** We have  $E_n(z) = \mathcal{O}(n)$  and  $E_n^{-1}(z) = \mathcal{O}(n)$  as  $n \rightarrow \infty$ , uniformly for  $z \in \overline{D(0, r_n)}$ . Furthermore, we have

$$E_n^{-1}(z_1)E_n(z_2) = \mathbb{I} + \mathcal{O}\left(n^{\frac{5}{2}}(z_1 - z_2)\right) \quad (5.57)$$

as  $n \rightarrow \infty$ , uniformly for  $z_1, z_2 \in \overline{D(0, r_n)}$ .

*Proof.* We write (5.44) with  $F_n$  given in (5.43) as

$$E_n(z) = M_\alpha(z) \begin{pmatrix} n^{\frac{1}{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & n^{-\frac{1}{2}} \end{pmatrix} L_\alpha(n^{3/2}z) D_2^{-n}(z) L_\alpha^{-1}(n^{3/2}f(z)) \begin{pmatrix} n^{\frac{1}{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & n^{-\frac{1}{2}} \end{pmatrix} T_\alpha \quad (5.58)$$

with

$$M_\alpha(z) = N(z) \begin{pmatrix} z^{\frac{2\beta}{3}} & 0 & 0 \\ 0 & z^{-\frac{\beta}{3}} & 0 \\ 0 & 0 & z^{-\frac{\beta}{3}} \end{pmatrix} L_\alpha^{-1}(z).$$

To obtain (5.58) we also used the identities

$$L_\alpha(n^{3/2}z) = \begin{pmatrix} n^{-\frac{1}{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & n^{\frac{1}{2}} \end{pmatrix} L_\alpha(z), \quad L_\alpha(n^{3/2}f(z)) = \begin{pmatrix} n^{\frac{1}{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & n^{-\frac{1}{2}} \end{pmatrix} L_\alpha(nf(z))$$

that follow from the definition (3.8) of  $L_\alpha$ .

Then  $M_\alpha(z)$  is analytic by Lemma 5.10 and does not depend on  $n$ . Therefore it is certainly uniformly bounded for  $z \in \overline{D(0, r_n)}$ .

Each of the factors in  $L_\alpha(n^{3/2}z) D_2^{-n}(z) L_\alpha^{-1}(n^{3/2}f(z))$  remains bounded for  $|z| = r_n$ , and since the product is analytic (which follows from Lemma 5.10 and (5.43)), the product also remains bounded for  $D(0, r_n)$ , uniformly in  $n$ , by the maximum modulus principle. So the only factors in (5.58) that grow are the two diagonal matrices and they are  $\mathcal{O}(n^{1/2})$ , which in total leads to  $E_n(z) = \mathcal{O}(n)$ . Similarly  $E_n^{-1}(z) = \mathcal{O}(n)$ .

To prove (5.57), let us denote

$$\mathcal{L}_n(z) := L_\alpha(n^{3/2}z) D_2^{-n}(z) L_\alpha^{-1}(n^{3/2}f(z)). \quad (5.59)$$

We already observed that  $\mathcal{L}_n$  is an analytic function that is bounded by an  $n$ -independent constant on  $\overline{D(0, r_n)}$ . The uniform bound also works on  $\overline{D(0, 2r_n)}$ . Cauchy's integral formula then gives us

$$\mathcal{L}_n(z_2) - \mathcal{L}_n(z_1) = \frac{z_1 - z_2}{2\pi i} \oint_{|z|=2r_n} \frac{\mathcal{L}_n(z)}{(z - z_1)(z - z_2)} dz = \mathcal{O}\left(n^{\frac{3}{2}}(z_1 - z_2)\right) \quad (5.60)$$

uniformly for  $z_1, z_2 \in \overline{D(0, r_n)}$ . Combining this with  $\mathcal{L}_n^{-1}(z_1) = \mathcal{O}(1)$ , we get

$$\mathcal{L}_n^{-1}(z_1) \mathcal{L}_n(z_2) = \mathbb{I} + \mathcal{L}_n^{-1}(z_1) (\mathcal{L}_n(z_2) - \mathcal{L}_n(z_1)) = \mathbb{I} + \mathcal{O}\left(n^{\frac{3}{2}}(z_1 - z_2)\right). \quad (5.61)$$

uniformly for  $z_1, z_2 \in \overline{D(0, r_n)}$ .

We also have  $M_\alpha^{-1}(z_1) M_\alpha(z_2) = \mathbb{I} + \mathcal{O}(z_1 - z_2)$  if  $z_1, z_2 \in \overline{D(0, r_n)}$ . Applying these estimates and using (5.58) yields for  $z_1, z_2 \in \overline{D(0, r_n)}$ ,

$$\begin{aligned} E_n^{-1}(z_1) E_n(z_2) &= T_\alpha^{-1} \begin{pmatrix} n^{-\frac{1}{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & n^{\frac{1}{2}} \end{pmatrix} \mathcal{L}_n^{-1}(z_1) \begin{pmatrix} n^{-\frac{1}{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & n^{\frac{1}{2}} \end{pmatrix} (\mathbb{I} + \mathcal{O}(z_1 - z_2)) \\ &\quad \times \begin{pmatrix} n^{\frac{1}{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & n^{-\frac{1}{2}} \end{pmatrix} \mathcal{L}_n(z_2) \begin{pmatrix} n^{\frac{1}{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & n^{-\frac{1}{2}} \end{pmatrix} T_\alpha \\ &= T_\alpha^{-1} \begin{pmatrix} n^{-\frac{1}{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & n^{\frac{1}{2}} \end{pmatrix} \mathcal{L}_n^{-1}(z_1) (\mathbb{I} + \mathcal{O}(n(z_1 - z_2))) \mathcal{L}_n(z_2) \begin{pmatrix} n^{\frac{1}{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & n^{-\frac{1}{2}} \end{pmatrix} T_\alpha \\ &= T_\alpha^{-1} \begin{pmatrix} n^{-\frac{1}{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & n^{\frac{1}{2}} \end{pmatrix} (\mathbb{I} + \mathcal{O}(n^{\frac{3}{2}}(z_1 - z_2))) \begin{pmatrix} n^{\frac{1}{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & n^{-\frac{1}{2}} \end{pmatrix} T_\alpha \\ &= \mathbb{I} + \mathcal{O}\left(n^{\frac{5}{2}}(z_1 - z_2)\right). \end{aligned}$$

as claimed in the lemma.  $\square$

Note that (5.57) implies that for  $z_1, z_2 \in \overline{D(0, r_n)}$  we have

$$E_n^{-1}(z_1)E_n(z_2) = \mathcal{O}(n)$$

while for  $z_1 = \frac{x}{(c\nu n)^3}$ ,  $z_2 = \frac{y}{(c\nu n)^3}$  with fixed  $x, y > 0$  we have

$$E_n^{-1}(z_1)E_n(z_2) = \mathbb{I} + \mathcal{O}\left(n^{-\frac{1}{2}}(x-y)\right).$$

We find from Lemma 5.13 the following estimates for the terms in (5.54).

**Corollary 5.14.** *Uniformly for  $|z| = r_n$ , we have  $\frac{A_n^{(1)}(z)}{n^3 z} = \mathcal{O}(n^{\frac{1}{2}})$  and  $E_n(z)\mathcal{O}(n^{-3})E_n^{-1}(z) = \mathcal{O}(n^{-1})$  as  $n \rightarrow \infty$ .*

## 5.6 Estimates on $A_n^{(1)}(z)$

After the first step towards the matching condition we obtained  $E_{in}^{(1)}$  and  $E_{out}^{(1)}$  and a corresponding  $P^{(1)}$ , as in (5.45), (5.46) and (5.53), such that (5.54) holds for  $|z| = r_n$ . To proceed we need estimates on  $A_n^{(1)}(z)$ .

**Proposition 5.15.** *Let  $k = 1, 2, 3, \dots$*

(a) *We have uniformly for  $z \in \overline{D(0, r_n)}$  that*

$$\left(A_n^{(1)}(z)\right)^k = \mathcal{O}(n^2). \quad (5.62)$$

(b) *We have uniformly for  $z_1, z_2, \dots, z_k \in \overline{D(0, r_n)}$  that*

$$A_n^{(1)}(z_1) \cdots A_n^{(1)}(z_k) = \mathcal{O}(n^{k+1}). \quad (5.63)$$

*When  $\ell$  indices  $j \in \{1, 2, \dots, k-1\}$  are such that  $z_j = z_{j+1}$  then this is replaced by*

$$A_n^{(1)}(z_1) \cdots A_n^{(1)}(z_k) = \mathcal{O}(n^{k+1-\ell}). \quad (5.64)$$

(c) *We have uniformly for  $z_1, z_2 \in \overline{D(0, r_n)}$  that*

$$A_n^{(1)}(z_1)E_n(z_2) = \mathcal{O}(n^2) \quad (5.65)$$

$$E_n^{-1}(z_1)A_n^{(1)}(z_2) = \mathcal{O}(n^2). \quad (5.66)$$

*Proof.* To prove (a) we notice using definition (5.55) that

$$\left(A_n^{(1)}(z)\right)^k = \frac{1}{f'(0)^k} E_n(z) T_\alpha^{-1} A_\alpha^k T_\alpha E_n^{-1}(z). \quad (5.67)$$

The only  $n$ -dependent factors here are  $E_n(z)$  and  $E_n^{-1}(z)$  which are both  $\mathcal{O}(n)$  for  $z \in \overline{D(0, r_n)}$  by Lemma 5.13, leading to an overall  $\mathcal{O}(n^2)$  behavior.

For (b) we first remark that by Lemma 5.13 we have for  $j = 1, 2, \dots, k-1$  that

$$E_n^{-1}(z_j)E_n(z_{j+1}) = \mathcal{O}(n) \quad (5.68)$$

uniformly on  $\overline{D(0, r_n)}$ . This implies by (5.55) that

$$\begin{aligned} A_n^{(1)}(z_1) \cdots A_n^{(1)}(z_k) &= \frac{1}{f'(0)^k} E_n(z_1) \left( \prod_{j=1}^{k-1} T_\alpha^{-1} A_\alpha T_\alpha E_n^{-1}(z_j) E_n(z_{j+1}) \right) T_\alpha^{-1} A_\alpha T_\alpha E_n^{-1}(z_k) \\ &= E_n(z_1) \mathcal{O}(n^{k-1}) E_n^{-1}(z_k). \end{aligned} \quad (5.69)$$

The factors  $E_n(z_1)$  and  $E_n^{-1}(z_k)$  are  $\mathcal{O}(n)$  uniformly by Lemma 5.13 and this leads to the  $\mathcal{O}(n^{k+1})$  behavior.

A combination of this argument and the proof of part (a) gives the second statement of (b).

To prove (c) we notice that

$$A_n^{(1)}(z_1)E_n(z_2) = \frac{1}{f'(0)} E_n(z_1) T_\alpha^{-1} A_\alpha T_\alpha (E_n^{-1}(z_1)E_n(z_2)) \quad (5.70)$$

and this is indeed  $\mathcal{O}(n^2)$  by Lemma 5.13. A similar argument gives us (5.66).  $\square$

## 5.7 Second step towards the matching

To improve the matching we now define

**Definition 5.16.**

$$E_{in}^{(2)}(z) = \left( \mathbb{I} - \frac{A_n^{(1)}(z) - A_n^{(1)}(0)}{n^3 z} \right) E_{in}^{(1)}(z), \quad z \in D(0, r_n) \quad (5.71)$$

$$E_{out}^{(2)}(z) = \left( \mathbb{I} - \frac{A_n^{(1)}(0)}{n^3 z} \right)^{-1}, \quad z \in A(0; r_n, R), \quad (5.72)$$

and

$$P^{(2)}(z) = \begin{cases} \left( \mathbb{I} - \frac{A_n^{(1)}(z) - A_n^{(1)}(0)}{n^3 z} \right) P^{(1)}(z), & z \in D(0, r_n), \\ \left( \mathbb{I} - \frac{A_n^{(1)}(0)}{n^3 z} \right)^{-1} P^{(1)}(z), & z \in A(0; r_n, R). \end{cases} \quad (5.73)$$

Note that  $E_{in}^{(2)}(z)$  is analytic at  $z = 0$ .

For  $n$  large enough, the inverse  $\left( \mathbb{I} - \frac{A_n^{(1)}(0)}{n^3 z} \right)^{-1}$  in (5.72) does indeed exist. The reason for this is that  $\left( \frac{A_n^{(1)}(0)}{n^3 z} \right)^2 = \mathcal{O}(n^{-1})$  for  $|z| \geq r_n$  by (5.62), and this implies that for sufficiently large  $n$ ,  $\frac{A_n^{(1)}(0)}{n^3 z}$  does not have an eigenvalue equal to 1.

**Lemma 5.17.** *On  $\partial D(0, r_n)$  we have the jump*

$$P_+^{(2)}(z) \left( P_-^{(2)}(z) \right)^{-1} = \mathbb{I} + \frac{A_n^{(2)}(z)}{n^6 z} - \frac{A_n^{(1)}(0)A_n^{(1)}(z)A_n^{(1)}(0)}{n^9 z^3} + \mathcal{O}(n^{-1}) \quad (5.74)$$

with

$$A_n^{(2)}(z) = \frac{A_n^{(1)}(0) \left( A_n^{(1)}(z) - A_n^{(1)}(0) \right)}{z} \quad (5.75)$$

The terms in (5.74) satisfy, for  $|z| = r_n$ ,

$$\frac{A_n^{(2)}(z)}{n^6 z} = \mathcal{O}(1), \quad \text{and} \quad \frac{A_n^{(1)}(0)A_n^{(1)}(z)A_n^{(1)}(0)}{n^9 z^3} = \mathcal{O}(n^{-\frac{1}{2}}). \quad (5.76)$$

*Proof.* On  $\overline{D(0, r_n)}$  we have by (5.54) and (5.73),

$$\begin{aligned} P_+^{(2)}(z) \left( P_-^{(2)}(z) \right)^{-1} &= \left( \mathbb{I} - \frac{A_n^{(1)}(z) - A_n^{(1)}(0)}{n^3 z} \right) \left( \mathbb{I} + \frac{A_n^{(1)}(z)}{n^3 z} + E_n(z) \mathcal{O}(n^{-3}) E_n^{-1}(z) \right) \left( \mathbb{I} - \frac{A_n^{(1)}(0)}{n^3 z} \right) \end{aligned}$$

We first expand the product of the second and third factors on the right to obtain

$$\begin{aligned} P_+^{(2)}(z) \left( P_-^{(2)}(z) \right)^{-1} &= \left( \mathbb{I} - \frac{A_n^{(1)}(z) - A_n^{(1)}(0)}{n^3 z} \right) \\ &\quad \times \left( \mathbb{I} + \frac{A_n^{(1)}(z) - A_n^{(1)}(0)}{n^3 z} - \frac{A_n^{(1)}(z)A_n^{(1)}(0)}{n^6 z^2} + E_n(z) \mathcal{O}(n^{-3}) E_n^{-1}(z) + \mathcal{O}(n^{-\frac{3}{2}}) \right) \end{aligned}$$

The  $\mathcal{O}(n^{-\frac{3}{2}})$  term comes from

$$E_n(z) \mathcal{O}(n^{-3}) E_n^{-1}(z) \frac{A_n^{(1)}(0)}{n^3 z} = \mathcal{O}(n^{-\frac{3}{2}}), \quad |z| = r_n, \quad (5.77)$$

where we used Lemma 5.13 and (5.66). When we multiply this with the remaining factor on the left we get

$$\begin{aligned} P_+^{(2)}(z) \left( P_-^{(2)}(z) \right)^{-1} &= \\ \mathbb{I} - \frac{A_n^{(1)}(z)A_n^{(1)}(0)}{n^6 z^2} - \left( \frac{A_n^{(1)}(z) - A_n^{(1)}(0)}{n^3 z} \right)^2 &+ \frac{(A_n^{(1)}(z) - A_n^{(1)}(0))A_n^{(1)}(z)A_n^{(1)}(0)}{n^9 z^3} + \mathcal{O}\left(\frac{1}{n}\right). \end{aligned} \quad (5.78)$$

Here we again used (5.57) and (5.62) and (5.65) to obtain the  $\mathcal{O}\left(\frac{1}{n}\right)$  behavior.

By (5.62) and (5.64) respectively we get  $\frac{A_n^{(1)}(z)^2}{n^6 z^2} = \mathcal{O}\left(\frac{1}{n}\right)$  and  $\frac{A_n^{(1)}(z)^2 A_n^{(1)}(0)}{n^9 z^3} = \mathcal{O}\left(n^{-\frac{3}{2}}\right)$  and we obtain (5.74). The estimates in (5.76) follow directly from (5.63).  $\square$

## 5.8 Third step towards the matching

In the next step we define

**Definition 5.18.**

$$E_{in}^{(3)}(z) = \left( \mathbb{I} - \frac{A_n^{(2)}(z) - A_n^{(2)}(0)}{n^6 z} \right) E_{in}^{(2)}(z), \quad z \in D(0, r_n) \quad (5.79)$$

$$E_{out}^{(3)}(z) = \left( \mathbb{I} - \frac{A_n^{(2)}(0)}{n^6 z} \right)^{-1} E_{out}^{(2)}(z), \quad z \in A(0; r_n, R), \quad (5.80)$$

and

$$P^{(3)}(z) = \begin{cases} \left( \mathbb{I} - \frac{A_n^{(2)}(z) - A_n^{(2)}(0)}{n^6 z} \right) P^{(2)}(z), & z \in D(0, r_n) \\ \left( \mathbb{I} - \frac{A_n^{(2)}(0)}{n^6 z} \right)^{-1} P^{(2)}(z), & z \in A(0; r_n, R), \end{cases} \quad (5.81)$$

By (5.63) we have

$$\left( \frac{A_n^{(2)}(0)}{n^6 z} \right)^2 = \frac{A_n^{(1)}(0)(A_n^{(1)}(z) - A_n^{(1)}(0))A_n^{(1)}(0)(A_n^{(1)}(z) - A_n^{(1)}(0))}{n^{12} z^4} = \mathcal{O}\left(\frac{1}{n}\right) \quad (5.82)$$

and we may use this to conclude that for  $n$  large enough,  $\frac{A_n^{(2)}(0)}{n^6 z}$  does not have eigenvalue 1 for  $|z| \geq r_n$ . Therefore the inverse in (5.80) and (5.81) is well-defined.

**Lemma 5.19.** *On  $\partial D(0, r_n)$  we have the jump*

$$P_+^{(3)}(z) \left( P_-^{(3)}(z) \right)^{-1} = \mathbb{I} + \frac{A_n^{(3)}(z)}{n^9 z} + \mathcal{O}(n^{-1}) \quad (5.83)$$

with

$$A_n^{(3)}(z) = - \frac{\left( A_n^{(1)}(z) - A_n^{(1)}(0) \right) A_n^{(1)}(z) \left( A_n^{(1)}(z) - A_n^{(1)}(0) \right)}{z^2}, \quad (5.84)$$

and for  $|z| = r_n$ ,

$$\frac{A_n^{(3)}(z)}{n^9 z} = \mathcal{O}\left(n^{-\frac{1}{2}}\right). \quad (5.85)$$



*Proof.* By (5.74) and (5.81) we have

$$P_+^{(3)}(z) \left( P_-^{(3)}(z) \right)^{-1} = \left( \mathbb{I} - \frac{A_n^{(2)}(z) - A_n^{(2)}(0)}{n^6 z} \right) \times \left( \mathbb{I} + \frac{A_n^{(2)}(z)}{n^6 z} + \frac{A_n^{(1)}(0) A_n^{(1)}(z) A_n^{(1)}(0)}{n^9 z^3} + \mathcal{O}(n^{-1}) \right) \left( \mathbb{I} - \frac{A_n^{(2)}(0)}{n^6 z} \right). \quad (5.86)$$

By counting the number of  $A_n^{(1)}$  factors it follows from (5.63) and (5.75) that for  $|z| = r_n$ ,

$$\begin{aligned} \frac{A_n^{(2)}(z) A_n^{(2)}(0)}{n^6 z n^6 z} &= \mathcal{O}(n^{-1}) \\ \frac{A_n^{(1)}(0) A_n^{(1)}(z) A_n^{(1)}(0) A_n^{(2)}(0)}{n^9 z^3 n^6 z} &= \mathcal{O}(n^{-\frac{3}{2}}) \\ \frac{A_n^{(2)}(0)}{n^6 z} &= \mathcal{O}(1). \end{aligned}$$

Hence the product of the middle and the right factors in (5.86) equals

$$\mathbb{I} + \frac{A_n^{(2)}(z) - A_n^{(2)}(0)}{n^6 z} + \frac{A_n^{(1)}(0) A_n^{(1)}(z) A_n^{(1)}(0)}{n^9 z^3} + \mathcal{O}(n^{-1}).$$

To also incorporate the factor on the left, we again count the number of  $A_n^{(1)}$  factors and use (5.63) and (5.75) to get

$$\begin{aligned} \left( \frac{A_n^{(2)}(z) - A_n^{(2)}(0)}{n^6 z} \right)^2 &= \mathcal{O}(n^{-1}) \\ \frac{A_n^{(2)}(z) - A_n^{(2)}(0)}{n^6 z} \frac{A_n^{(1)}(0) A_n^{(1)}(z) A_n^{(1)}(0)}{n^9 z^3} &= \mathcal{O}(n^{-\frac{3}{2}}) \end{aligned}$$

We conclude that

$$P_+^{(3)}(z) \left( P_-^{(3)}(z) \right)^{-1} = \mathbb{I} - \frac{A_n^{(1)}(0) A_n^{(1)}(z) A_n^{(1)}(0)}{n^9 z^3} + \mathcal{O}(n^{-1}). \quad (5.87)$$

Note that by (5.84) and (5.64)

$$- \frac{A_n^{(1)}(0) A_n^{(1)}(z) A_n^{(1)}(0)}{n^9 z^3} = \frac{A_n^{(3)}(z)}{n^9 z^3} + \mathcal{O}(n^{-\frac{3}{2}}) \quad (5.88)$$

for  $|z| = r_n$ . We use (5.88) in (5.87) and the result (5.83) follows. The estimate (5.85) follows from (5.76) and (5.88).  $\square$

Note that  $A_n^{(3)}(z)$  is analytic at  $z = 0$ .

We now have by (5.83) and (5.85) that uniformly on  $\partial D(0, r_n)$

$$P_+^{(3)}(z) \left( P_-^{(3)}(z) \right)^{-1} = \mathbb{I} + \mathcal{O}(n^{-\frac{1}{2}})$$

as  $n \rightarrow \infty$ . This is a weaker form of the matching condition (5.11), with  $\mathcal{O}(n^{-1})$  replaced by  $\mathcal{O}(n^{-\frac{1}{2}})$ . The weaker form is not sufficient to prove the scaling limit of the correlation kernel. Therefore we make another step.

## 5.9 Final step towards the matching

The final step takes a by now familiar form.

**Definition 5.20.** We define

$$E_{in}^{(4)}(z) = \left( \mathbb{I} - \frac{A_n^{(3)}(z) - A_n^{(3)}(0)}{n^9 z} \right) E_{in}^{(3)}(z) \quad (5.89)$$

$$E_{out}^{(4)}(z) = \left( \mathbb{I} - \frac{A_n^{(3)}(0)}{n^9 z} \right)^{-1} E_{out}^{(3)}(z) \quad (5.90)$$

and

$$P^{(4)}(z) = \begin{cases} \left( \mathbb{I} - \frac{A_n^{(3)}(z) - A_n^{(3)}(0)}{n^9 z} \right) P^{(3)}(z), & z \in D(0, r_n) \\ \left( \mathbb{I} - \frac{A_n^{(3)}(0)}{n^9 z} \right)^{-1} P^{(3)}(z), & z \in A(0; r_n, R), \end{cases} \quad (5.91)$$

Again, we argue in similar fashion as before that the inverse is well-defined provided  $n$  is large enough.

**Lemma 5.21.** *On  $\partial D(0, r_n)$  we have the jump*

$$P_+^{(4)}(z) \left( P_-^{(4)}(z) \right)^{-1} = \mathbb{I} + \mathcal{O}(n^{-1}) \quad (5.92)$$

*Proof.* By (5.83) and (5.91) the jump is given by

$$\begin{aligned} P_+^{(4)}(z) \left( P_-^{(4)}(z) \right)^{-1} &= \left( \mathbb{I} - \frac{A_n^{(3)}(z) - A_n^{(3)}(0)}{n^9 z} \right) \left( \mathbb{I} - \frac{A_n^{(1)}(0)A_n^{(1)}(z)A_n^{(1)}(0)}{n^9 z^3} + \mathcal{O}(n^{-1}) \right) \left( \mathbb{I} - \frac{A_n^{(3)}(0)}{n^9 z} \right) \\ &= \left( \mathbb{I} - \frac{A_n^{(3)}(z) - A_n^{(3)}(0)}{n^9 z} \right) \left( \mathbb{I} + \frac{A_n^{(3)}(z)}{n^9 z} + \mathcal{O}(n^{-1}) \right) \left( \mathbb{I} - \frac{A_n^{(3)}(0)}{n^9 z} \right). \end{aligned} \quad (5.93)$$

By counting the number of  $A_n^{(1)}$  factors and applying (5.63) we obtain

$$\frac{A_n^{(3)}(z)A_n^{(3)}(0)}{n^9 z} = \mathcal{O}(n^{-2}) \quad \text{and} \quad \frac{A_n^{(3)}(0)}{n^9 z} = \mathcal{O}(n^{-\frac{1}{2}}).$$

Hence the product of the middle and right factor in (5.93) equals

$$\mathbb{I} + \frac{A_n^{(3)}(z) - A_n^{(3)}(0)}{n^9 z} + \mathcal{O}(n^{-1}).$$

Again by counting the number of  $A_n^{(1)}$  factors and applying (5.63) we obtain

$$\left( \frac{A_n^{(3)}(z) - A_n^{(3)}(0)}{n^9 z} \right)^2 = \mathcal{O}(n^{-2})$$

and we arrive at (5.92). □

We are ready to fix the definitions of  $E_{in}$  and  $E_{out}$ .

**Definition 5.22.** We set  $E_{in} = E_{in}^{(4)}$  and  $E_{out} = E_{out}^{(4)}$ , where  $E_{in}^{(4)}$  and  $E_{out}^{(4)}$  are as in (5.89) and (5.90).

**Theorem 5.23.**  *$P$ , as defined in (5.16) and (5.17) with  $E_{in} = E_{in}^{(4)}$  and  $E_{out} = E_{out}^{(4)}$ , satisfies the matching condition. That is, we have*

$$P_+(z)P_-^{-1}(z) = \mathbb{I} + \mathcal{O}(n^{-1}) \quad \text{as } n \rightarrow \infty, \quad \text{uniformly for } z \in \partial D(0, r_n), \quad (5.94)$$

$$P(z)N^{-1}(z) = \mathbb{I} + \mathcal{O}(n^{-1}) \quad \text{as } n \rightarrow \infty, \quad \text{uniformly for } z \in \partial D(0, R). \quad (5.95)$$

*Proof.* The matching condition (5.94) on  $\partial D(0, r_n)$  was just proved in (5.92). To prove the matching condition (5.95) on  $\partial D(0, R)$ , we notice that  $P = P^{(4)}$  satisfies, by (5.91), (5.81), (5.73), (5.53)

$$P(z) = \left( \mathbb{I} - \frac{A_n^{(3)}(0)}{n^9 z} \right)^{-1} \left( \mathbb{I} - \frac{A_n^{(2)}(0)}{n^6 z} \right)^{-1} \left( \mathbb{I} - \frac{A_n^{(1)}(0)}{n^3 z} \right)^{-1} N(z), \quad z \in \partial D(0, R). \quad (5.96)$$

In (5.65) we already noted that  $A_n^{(1)}(0) = \mathcal{O}(n^2)$  and therefore

$$\left( \mathbb{I} - \frac{A_n^{(1)}(0)}{n^3 z} \right)^{-1} = \mathbb{I} + \mathcal{O}(n^{-1}) \quad \text{as } n \rightarrow \infty,$$

uniformly for  $|z| = R$ . From the definition (5.75) we see that  $A_n^{(2)}$  contains terms with products of two  $A_n^{(1)}$  factors which by (5.66) gives the estimate  $A_n^{(2)}(z) = \mathcal{O}(n^{9/2})$  for  $|z| = r_n$ . Since  $A_n^{(2)}$  is analytic, we may then use the maximum modulus principle to conclude that also  $A_n^{(2)}(0) = \mathcal{O}(n^{9/2})$  as  $n \rightarrow \infty$ . As a result

$$\left( \mathbb{I} - \frac{A_n^{(2)}(0)}{n^6 z} \right)^{-1} = \mathbb{I} + \mathcal{O}(n^{-\frac{3}{2}})$$

uniformly for  $|z| = R$ . From (5.84) we get that  $A_n^{(3)}$  has terms with three  $A_n^{(1)}$  factors. We again use the estimate (5.66) (now with  $k = 3$ ). Then reasoning as before we obtain  $A_n^{(3)}(0) = \mathcal{O}(n^7)$  as  $n \rightarrow \infty$ . This leads to

$$\left( \mathbb{I} - \frac{A_n^{(3)}(0)}{n^9 z} \right)^{-1} = \mathbb{I} + \mathcal{O}(n^{-2})$$

uniformly for  $|z| = R$ . Inserting all the  $\mathcal{O}$  estimates in (5.96) is, we obtain (5.95) and we are done.  $\square$

**Remark 5.24.** As already noted at the end of the introduction, the matching condition can be an issue in larger size Riemann-Hilbert problems, and other constructions have been proposed before in the literature. In [19, 20, 29] the matching condition was established by means of a modification of the global parametrix. It also relied on the nilpotency of a term that appears in the jump matrix which is analogous to our matrix  $A_n^{(1)}$ .

An iterative method was designed by Bertola and Bothner [6] in a situation that is similar to ours. At the initial step of the construction of the local parametrix also a ‘‘matching’’ of the form  $\mathbb{I} + \mathcal{O}(n^{1/2})$  is obtained. Bertola and Bothner also modify the analytic prefactor step by step to improve the matching. They use very detailed information on the  $\mathcal{O}(n^{1/2})$  term which is also nilpotent.

It differs from our method since we do not use the nilpotency, but instead we have control on the behavior of the powers of  $A_n^{(1)}$  as  $n \rightarrow \infty$ , see (5.62). We also need technical estimates on the analytic prefactor  $E_n$  as in Lemma 5.13. Having these estimates our transformations proceed in a systematic fashion where the estimates come from a rather straightforward counting procedure, namely we count the number of  $A_n^{(1)}$  factors as we go along. We also introduced the novel trick to construct a matching condition on two circles.

We expect that the ideas behind our method may be more wider applicable. The first test case will be the generalization of this work to the case  $\theta = \frac{1}{r}$  with  $r \geq 3$ .

## 6 Final transformation and proof of Theorem 1.2

### 6.1 Final transformation $S \mapsto R$

Having the global parametrix  $N$ , and the local parametrices  $P$  (in the neighborhood  $D(0, R)$  of 0) and  $Q$  (in the neighborhood  $D(q, r_q)$  of  $q$ ) we define the final transformation as follows.

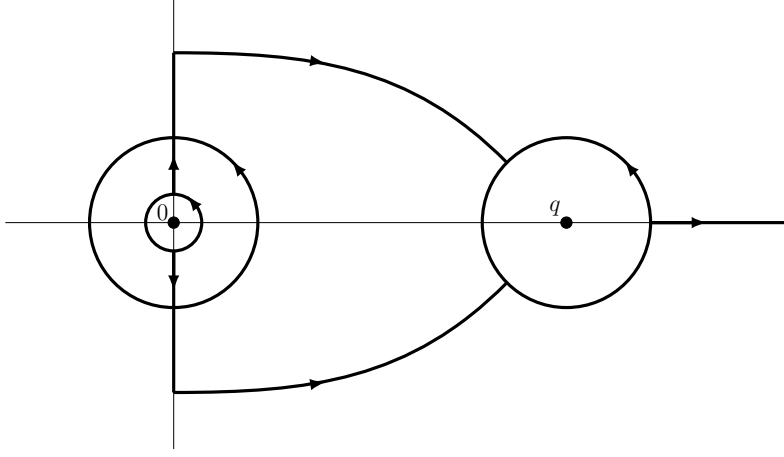


Figure 6: The contour  $\Sigma_R$  for  $R$ .

**Definition 6.1.** We define  $R$  as

$$R(z) = \begin{cases} S(z)N^{-1}(z), & z \in \mathbb{C} \setminus (\Sigma_S \cup \overline{D(0, R)} \cup \overline{D(q, r_q)}), \\ S(z)P^{-1}(z), & z \in D(0, r_n) \setminus (\Sigma_S \cup \partial D(0, r_n)), \\ S(z)Q^{-1}(z), & z \in D(q, r_q) \setminus \Sigma_S. \end{cases} \quad (6.1)$$

Since the jumps of  $S$  and  $N$  agree on  $(-\infty, 0) \cup (0, q)$  we find that  $R$  is analytic across  $(-\infty, -R)$  and  $(R, q - r_q)$ . Also the jumps of  $S$  and  $Q$  agree inside  $D(q, r_q)$  and so  $R$  is analytic in  $D(q, r_q)$ . Finally, the jumps of  $S$  and  $P$  agree inside  $D(0, r_n)$  and on  $(-R, 0)$  and  $(0, R)$ . Therefore  $R$  has analytic continuation to  $\mathbb{C} \setminus (\Sigma_R \cup \{0\})$  where

$$\Sigma_R = \partial D(0, r_n) \cup \partial D(0, R) \cup \partial D(q, r_q) \cup (\Delta_1^\pm \setminus (D(0, r_n) \cup D(q, r_q))) \cup [q + r_q, \infty) \quad (6.2)$$

see Figure 6. We show that the isolated singularity at  $0$  is removable.

**Lemma 6.2.** *The singularity of  $R$  at  $0$  is removable, and thus  $R$  has analytic continuation to  $\mathbb{C} \setminus \Sigma_R$ .*

*Proof.* We analyse the behavior of  $R(z)$  as  $z \rightarrow 0$  in the left half plane. We have from (3.28) and (5.16)

$$R(z) = S(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & z^{-\beta} & 0 \\ 0 & 0 & z^{-\beta} \end{pmatrix} D_0^{-n}(z) \Psi_\alpha^T(n^3 f(z)) \frac{-1}{4\pi^2} \begin{pmatrix} 1 & 0 & 0 \\ -2\alpha - \frac{1}{2} & -1 & 0 \\ \alpha(\alpha + \frac{1}{2}) & 2\alpha + \frac{1}{2} & 1 \end{pmatrix} E_{in}^{-1}(z) \quad (6.3)$$

We have from (4.44) that, as  $z \rightarrow 0$  with  $\text{Re } z < 0$ ,

$$S(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & z^{-\beta} & 0 \\ 0 & 0 & z^{-\beta} \end{pmatrix} = \mathcal{O} \begin{pmatrix} 1 & h_{-\alpha-\frac{1}{2}}(z) & h_{-\alpha-\frac{1}{2}}(z) \\ 1 & h_{-\alpha-\frac{1}{2}}(z) & h_{-\alpha-\frac{1}{2}}(z) \\ 1 & h_{-\alpha-\frac{1}{2}}(z) & h_{-\alpha-\frac{1}{2}}(z) \end{pmatrix},$$

since  $z^{-\beta-\frac{1}{4}} h_{\alpha+\frac{1}{2}}(z) = h_{-\alpha-\frac{1}{2}}(z)$ . Since  $D_0^{-n}(z)$  is a diagonal matrix which remains bounded as  $z \rightarrow 0$ , we also find

$$S(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & z^{-\beta} & 0 \\ 0 & 0 & z^{-\beta} \end{pmatrix} D_0^{-n}(z) = \mathcal{O} \begin{pmatrix} 1 & h_{-\alpha-\frac{1}{2}}(z) & h_{-\alpha-\frac{1}{2}}(z) \\ 1 & h_{-\alpha-\frac{1}{2}}(z) & h_{-\alpha-\frac{1}{2}}(z) \\ 1 & h_{-\alpha-\frac{1}{2}}(z) & h_{-\alpha-\frac{1}{2}}(z) \end{pmatrix}, \quad (6.4)$$

as  $z \rightarrow 0$  with  $\text{Re } z < 0$ . For  $\Psi_\alpha$  we recall the behavior (3.31) which for general  $\alpha > -1$  is

$$\Psi_\alpha^T(z) = \mathcal{O} \begin{pmatrix} h_\alpha(z) & h_\alpha(z) & h_\alpha(z) \\ z^\alpha & z^\alpha & z^\alpha \\ z^\alpha & z^\alpha & z^\alpha \end{pmatrix} \quad (6.5)$$

as  $z \rightarrow 0$  with  $\operatorname{Re} z < 0$ .

The remaining factors in (6.3) are either constant or analytic at  $z = 0$  and therefore remain bounded as  $z \rightarrow 0$ . Then using (6.4) and (6.5) in (6.3) we get

$$R(z) = \mathcal{O}(h_\alpha(z)) + \mathcal{O}\left(z^\alpha h_{-\alpha-\frac{1}{2}}(z)\right) = \begin{cases} z^\alpha, & -1 < \alpha < -\frac{1}{2}, \\ z^{-\frac{1}{2}} \log(z), & \alpha = -\frac{1}{2}, \\ z^{-\frac{1}{2}}, & \alpha > -\frac{1}{2}, \end{cases}$$

as  $z \rightarrow 0$  with  $\operatorname{Re} z < 0$ .

In all cases we have  $zR(z) \rightarrow 0$  as  $z \rightarrow 0$  with  $\operatorname{Re} z < 0$ . This implies that  $R$  cannot have a pole at  $z = 0$ . There cannot be an essential singularity either, and therefore the singularity at  $z = 0$  is indeed removable.  $\square$

As a result of the steepest descent analysis, and in particular the matching condition in Theorem 5.23 we now have.

**Proposition 6.3.** (a) *As  $n \rightarrow \infty$  we have*

$$R_+(z) = R_-(z) \left( \mathbb{I} + \mathcal{O}\left(\frac{1}{n}\right) \right), \quad \text{for } z \in \partial D(0, R) \cup \partial D(0, r_n) \cup \partial D(q, r_q), \quad (6.6)$$

$$R_+(z) = R_-(z) \left( \mathbb{I} + \mathcal{O}\left(e^{-c_1\sqrt{n}}\right) \right), \quad \text{for } z \in \Delta_1^\pm \cap A(0; r_n, R), \quad (6.7)$$

$$R_+(z) = R_-(z) \left( \mathbb{I} + \mathcal{O}\left(e^{-c_2n}\right) \right), \quad \text{on remaining parts of } \Sigma_R. \quad (6.8)$$

where  $c_1, c_2$  are certain positive constants. All  $\mathcal{O}$  terms in (6.6)-(6.8) are uniform for  $z$  on the indicated contours.

(b) *We have*

$$R(z) = \mathbb{I} + \mathcal{O}\left(\frac{1}{n}\right) \quad \text{as } n \rightarrow \infty \quad (6.9)$$

uniformly for  $z \in \mathbb{C} \setminus \Sigma_R$ .

*Proof.* For  $z \in \partial D(0, r_n)$  we have by the definition (6.1) that

$$R_-^{-1}(z)R_+(z) = (S(z)P_-(z))^{-1}S(z)P_+(z) = P_-^{-1}(z)P_+(z)$$

since  $S$  has no jump on  $\partial D(0, r_n)$ . Then (6.6) for  $z \in \partial D(0, r_n)$  follows from (5.94). Similarly, (6.6) for  $z \in \partial D(0, R)$  follows from (5.95) and for  $z \in \partial D(q, r_q)$  it follows from the matching condition (4.52) for the local parametrix at  $q$ .

For  $z \in \Sigma_R$  outside the disks  $D(0, R)$  and  $D(q, r_q)$  we have by (6.1)

$$R_-^{-1}(z)R_+(z) = N(z)S_-^{-1}(z)S_+(z)N^{-1}(z)$$

and  $S_-^{-1}(z)S_+(z) = \mathbb{I} + \mathcal{O}(e^{-cn})$ , which follows from the jumps in the RH problem for  $S$ , since  $\operatorname{Re} \varphi(z) \leq -c < 0$  on  $(\Delta_1^+ \cup \Delta_1^-) \setminus (D(0, R) \cup D(q, r_q))$ , while  $\operatorname{Re} \varphi(x) \geq c > 0$  on  $(q + r_q, \infty)$ . Then the estimate (6.8) also follows.

For (6.7) we have to be a little bit more careful. For  $z \in \Delta_1^\pm \cap A(0; r_n, R_n)$  we have  $R(z) = S(z)P^{-1}(z)$  and  $P_+(z) = P_-(z)$  by (5.8). By the jump of  $S$  in (4.39), we get

$$\begin{aligned} R_-^{-1}(z)R_+(z) &= P(z)S_-^{-1}(z)S_+(z)P^{-1}(z) \\ &= \mathbb{I} + z^{-\beta} e^{2n\varphi(z)} P(z) \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} P^{-1}(z). \end{aligned}$$

On  $\Delta_1^\pm$  we have by (4.46) that  $\operatorname{Re} \varphi(z) \leq -c|z|^{1/3}$  for some constant  $c > 0$ . Since  $|z| \geq n^{-\frac{3}{2}}$  on the annulus  $A(0; r_n, R)$  we have  $|e^{2n\varphi(z)}| \leq e^{-2c\sqrt{n}}$  for  $z \in \Delta_1^\pm \cap A(0; r_n, R)$ . Then (6.7) follows since the entries of  $P(z)$  and  $P^{-1}(z)$  have at most a power law singularity as  $z \rightarrow 0$ .

Part (b) follows from part (a) by standard arguments from Riemann-Hilbert theory. Compare for example with the arguments in Appendix A from [8] that deal explicitly with a RH problem containing a varying shrinking contour.  $\square$

## 6.2 Rewriting of the correlation kernel

Before we take the scaling limit we follow the transformations  $Y \mapsto X \mapsto T \mapsto S \mapsto R$  to see their effect on the correlation kernel for  $x, y$  near 0.

**Lemma 6.4.** *For  $x, y \in (0, r_n)$ , we have*

$$K_{V,n}^{\alpha, \frac{1}{2}}(x, y) = e^{\frac{2}{3}n(V(x)-V(y))} \frac{1}{2\pi i(x-y)} \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} \Phi_{\alpha,+}^{-1}(n^3 f(y)) E_{in}^{-1}(y) R^{-1}(y) R(x) E_{in}(x) \Phi_{\alpha,+}(n^3 f(x)) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}. \quad (6.10)$$

*Proof.* We start with the formula (2.6) for the correlation kernel in terms of  $Y$ . The first two transformations  $Y \mapsto X \mapsto T$  from Definitions 4.1 and 4.4 lead to

$$K_{V,n}^{\alpha, \frac{1}{2}}(x, y) = \frac{\sqrt{2}y^\beta e^{-nV(y)}}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} X_+^{-1}(y) X_+(x) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ = \frac{y^\beta e^{-n(V(y)-g_{1,+}(y)+g_2(y)+\ell)} e^{ng_{1,+}(x)}}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} T_+^{-1}(y) T_+(x) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The next transformation  $T \mapsto S$  is the opening of lenses. For  $x, y \in (0, q)$  we use (4.35) to obtain

$$K_{V,n}^{\alpha, \frac{1}{2}}(x, y) = e^{-n(V(y)-g_{1,+}(y)+g_2(y)-\varphi_+(y)+\ell)} e^{n(g_{1,+}(x)+\varphi_+(x))} \frac{1}{2\pi i(x-y)} \begin{pmatrix} -e^{n\varphi_+(y)} & y^\beta e^{-n\varphi_+(y)} & 0 \end{pmatrix} S_+^{-1}(y) S_+(x) \begin{pmatrix} e^{-n\varphi_+(x)} \\ x^{-\beta} e^{n\varphi_+(x)} \\ 0 \end{pmatrix}. \quad (6.11)$$

The scalar prefactor simplifies because of (4.26) and we find

$$K_{V,n}^{\alpha, \frac{1}{2}}(x, y) = e^{\frac{1}{2}n(V(x)+g_2(x)-V(y)-g_2(y))} \frac{1}{2\pi i(x-y)} \begin{pmatrix} -e^{n\varphi_+(y)} & y^\beta e^{-n\varphi_+(y)} & 0 \end{pmatrix} S_+^{-1}(y) S_+(x) \begin{pmatrix} e^{-n\varphi_+(x)} \\ x^{-\beta} e^{n\varphi_+(x)} \\ 0 \end{pmatrix}. \quad (6.12)$$

Now we take  $x, y \in (0, r_n)$ . We write  $S = RP$  as in (6.1) and  $P$  is given by (5.16). Then

$$S_+(x) \begin{pmatrix} e^{-n\varphi_+(x)} \\ x^{-\beta} e^{n\varphi_+(x)} \\ 0 \end{pmatrix} = R(x) E_{in}(x) \Phi_{\alpha,+}(n^3 f(x)) \begin{pmatrix} e^{-n(\varphi_+(x)-\frac{4}{3}\varphi_{1,+}(x)-\frac{2}{3}\varphi_{2,+}(x))} \\ e^{n(\varphi_+(x)-\frac{2}{3}\varphi_{1,+}(x)+\frac{2}{3}\varphi_{2,+}(x))} \\ 0 \end{pmatrix} \\ = (-1)^n e^{\frac{1}{3}n(\varphi_{1,+}(x)+2\varphi_{2,+}(x))} R(x) E_{in}(x) \Phi_{\alpha,+}(n^3 f(x)) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad (6.13)$$

where we used (5.16) and the formula (5.15) for  $D_0$  for the first identity, and the fact that  $\varphi_+(x) = \varphi_{1,+}(x) - \pi i$  by (5.13) for the second identity in (6.13). In a similar way,

$$\begin{pmatrix} -e^{n\varphi_+(y)} & y^\beta e^{-n\varphi_+(y)} & 0 \end{pmatrix} S_+^{-1}(y) \\ = (-1)^n e^{-\frac{1}{3}n(\varphi_{1,+}(y)+2\varphi_{2,+}(y))} \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} \Phi_{\alpha,+}^{-1}(n^3 f(y)) E_{in}^{-1}(y) R^{-1}(y). \quad (6.14)$$

We insert (6.13) and (6.14) into (6.11). Then (6.10) follows, since we also have

$$\frac{1}{2}V(x) + g_2(x) + \frac{1}{3}\varphi_{1,+}(x) + \frac{2}{3}\varphi_{2,+}(x) = \frac{2}{3}V(x) + \frac{1}{6}\ell$$

because of (2.17), (5.13), and (5.14). □

### 6.3 Proof of Theorem 1.2

In view of (6.4) and (2.28) the strategy for the proof of Theorem 1.2 is now clear.

*Proof of Theorem 1.2.* We put  $x_n = \frac{x}{(c_V n)^3}$  and  $y_n = \frac{y}{(c_V n)^3}$  with  $x, y > 0$  fixed and  $c_V$  is the constant defined in Theorem 1.2. It follows from Proposition 5.4 that  $c_V^3 = f'(0)$  and therefore  $\Phi_{\alpha,+}(n^3 f(x_n)) \rightarrow \Phi_{\alpha,+}(x)$  and  $\Phi_{\alpha,+}(n^3 f(y_n)) \rightarrow \Phi_{\alpha,+}(y)$  as  $n \rightarrow \infty$ , uniformly for  $x, y$  in compact subsets of  $(0, \infty)$ .

We need to show that

$$E_{in}^{-1}(y_n)R^{-1}(y_n)R(x_n)E_{in}(x_n) \rightarrow \mathbb{I} \quad (6.15)$$

as  $n \rightarrow \infty$ , and then (1.10) follows indeed from (6.10) and (2.28). The proof of (6.15) (in a more precise form) is in Lemma 6.7 below. To establish this lemma, we first need two other lemmas that we state and prove separately.  $\square$

We assume  $x_n = \frac{x}{(c_V n)^3}$  and  $y_n = \frac{y}{(c_V n)^3}$ .

**Lemma 6.5.** *We have uniformly for  $x, y$  in compact subsets of  $[0, \infty)$ ,*

$$R^{-1}(y_n)R(x_n) = \mathbb{I} + \mathcal{O}\left((x-y)n^{-\frac{5}{2}}\right) \quad (6.16)$$

as  $n \rightarrow \infty$ .

*Proof.* Cauchy's integral formula yields that

$$\begin{aligned} R(x_n) - R(y_n) &= \frac{1}{2\pi i} \oint_{|z|=r_n} \left( \frac{R(z) - \mathbb{I}}{z - x_n} - \frac{R(z) - \mathbb{I}}{z - y_n} \right) dz \\ &= \frac{x_n - y_n}{2\pi i} \oint_{|z|=r_n} \frac{R(z) - \mathbb{I}}{(z - x_n)(z - y_n)} dz \end{aligned}$$

By (6.9) we have  $R(z) - \mathbb{I} = \mathcal{O}(n^{-1})$  and it follows by easy estimation that  $R(x_n) - R(y_n) = \mathcal{O}\left((x-y)n^{-\frac{5}{2}}\right)$  as  $n \rightarrow \infty$ . As a result, since  $R^{-1}(y_n)$  remains bounded,

$$R^{-1}(y_n)R(x_n) = \mathbb{I} + R^{-1}(y_n)(R(x_n) - R(y_n)) = \mathbb{I} + \mathcal{O}\left((x-y)n^{-\frac{5}{2}}\right).$$

and the constant implied in the  $\mathcal{O}$  term is uniform for  $x$  and  $y$  in compact subsets of  $[0, \infty)$ .  $\square$

**Lemma 6.6.** *We have uniformly for  $x, y$  in compact subsets of  $[0, \infty)$ ,*

$$E_{in}^{-1}(y_n)E_{in}(x_n) = \mathbb{I} + \mathcal{O}\left((x-y)n^{-\frac{1}{2}}\right) \quad (6.17)$$

as  $n \rightarrow \infty$ .

*Proof.* We note that by Definition 5.22 and (5.45), (5.71), (5.79), (5.89)

$$E_{in}(z) = \frac{\sqrt{3}}{2\pi} n^{2\beta} \left( \frac{f(z)}{z} \right)^{\frac{2\beta}{3}} \left( \mathbb{I} - B_n^{(3)}(z) \right) \left( \mathbb{I} - B_n^{(2)}(z) \right) \left( \mathbb{I} - B_n^{(1)}(z) \right) E_n(z) \quad (6.18)$$

with

$$B_n^{(k)}(z) = \frac{A_n^{(k)}(z) - A_n^{(k)}(0)}{n^{3k}z}, \quad \text{for } k = 1, 2, 3. \quad (6.19)$$

We are going to show that for every  $C > 0$  and for  $|z|, |s| \leq Cn^{-\frac{5}{2}}$  we have as  $n \rightarrow \infty$ ,

$$E_n^{-1}(z)A_n^{(k)}(s)E_n(z) = \mathcal{O}\left(n^{\frac{5(k-1)}{2}}\right), \quad (6.20)$$

$$E_n^{-1}(z)B_n^{(k)}(s)E_n(z) = \mathcal{O}\left(n^{-\frac{k}{2}}\right), \quad (6.21)$$

for  $k = 1, 2, 3$ .

We start by noting that for  $z, s = \mathcal{O}(n^{-\frac{5}{2}})$  one has by Lemma 5.13 that  $E_n^{-1}(z)E_n(s) = \mathcal{O}(1)$ , and then (6.20) with  $k = 1$  follows immediately from this and the definition (5.55) of  $A_n^{(1)}$ .

Note that by Cauchy's formula

$$E_n^{-1}(z) \frac{A_n^{(1)}(s) - A_n^{(1)}(0)}{s} E_n(z) = \frac{1}{2\pi i} \oint_{|t|=2Cn^{-\frac{5}{2}}} \frac{E_n^{-1}(z)A_n^{(1)}(t)E_n(z)}{t(t-s)} dt \quad (6.22)$$

which we can estimate because of (6.20) with  $k = 1$  to give

$$E_n^{-1}(z) \frac{A_n^{(1)}(s) - A_n^{(1)}(0)}{s} E_n(z) = \frac{\mathcal{O}(1)}{n^{-\frac{5}{2}}} = \mathcal{O}\left(n^{\frac{5}{2}}\right) \quad (6.23)$$

whenever  $|z|, |s| \leq Cn^{-\frac{5}{2}}$ . Then (6.21) with  $k = 1$  follows because of (6.19).

Then by (5.75), (6.19), and (6.20) with  $k = 1$

$$\begin{aligned} E_n^{-1}(z)A_n^{(2)}(s)E_n(z) &= n^3 \left( E_n^{-1}(z)A_n^{(1)}(0)E_n(z) \right) \left( E_n^{-1}(z)B_n^{(1)}(s)E_n(z) \right) \\ &= n^3 \cdot \mathcal{O}(1) \cdot \mathcal{O}\left(n^{-\frac{1}{2}}\right) = \mathcal{O}\left(n^{\frac{5}{2}}\right) \end{aligned}$$

for  $|z|, |s| \leq Cn^{-\frac{5}{2}}$ , which proves (6.20) with  $k = 2$ .

From formula (5.84) for  $A_n^{(3)}$  we find in a similar way

$$\begin{aligned} E_n^{-1}(z)A_n^{(3)}(s)E_n(z) &= -n^6 \left( E_n^{-1}(z)B_n^{(1)}(s)E_n(z) \right) \left( E_n^{-1}(z)A_n^{(1)}(s)E_n(z) \right) \left( E_n^{-1}(z)B_n^{(1)}(s)E_n(z) \right) \\ &= n^6 \cdot \mathcal{O}\left(n^{-\frac{1}{2}}\right) \cdot \mathcal{O}(1) \cdot \mathcal{O}\left(n^{-\frac{1}{2}}\right) = \mathcal{O}\left(n^5\right) \end{aligned}$$

for  $|z|, |s| \leq Cn^{-\frac{5}{2}}$ . This proves (6.20) with  $k = 3$ .

Finally, (6.21) for  $k = 2, 3$  follows from (6.20) with  $k = 2, 3$  in the same way as we obtained (6.21) for  $k = 1$  with the identity (6.22) and the estimate (6.23).

Having (6.21) we go back to (6.18) which we rewrite and estimate for  $z = \mathcal{O}(n^{-3})$  as

$$\begin{aligned} E_{in}(z) &= \frac{\sqrt{3}}{2\pi} n^{2\beta} \left( \frac{f(z)}{z} \right)^{\frac{2\beta}{3}} E_n(z) \left( \mathbb{I} - E_n^{-1}(z)B_n^{(3)}(z)E_n(z) \right) \\ &\quad \times \left( \mathbb{I} - E_n^{-1}(z)B_n^{(2)}(z)E_n(z) \right) \left( \mathbb{I} - E_n^{-1}(z)B_n^{(1)}(z)E_n(z) \right) \\ &= \frac{\sqrt{3}}{2\pi} n^{2\beta} \left( \frac{f(z)}{z} \right)^{\frac{2\beta}{3}} E_n(z) \left( \mathbb{I} + \mathcal{O}(n^{-\frac{3}{2}}) \right) \left( \mathbb{I} + \mathcal{O}(n^{-1}) \right) \left( \mathbb{I} + \mathcal{O}(n^{-\frac{1}{2}}) \right) \\ &= \frac{\sqrt{3}}{2\pi} n^{2\beta} \left( \frac{f(z)}{z} \right)^{\frac{2\beta}{3}} E_n(z) \left( \mathbb{I} + \mathcal{O}(n^{-\frac{1}{2}}) \right). \end{aligned}$$

Since all factors are analytic and invertible we in fact have

$$E_{in}(z) = \frac{\sqrt{3}}{2\pi} n^{2\beta} \left( \frac{f(z)}{z} \right)^{\frac{2\beta}{3}} E_n(z) (\mathbb{I} + C_n(z)), \quad (6.24)$$

where  $C_n$  is analytic and

$$C_n(z) = \mathcal{O}(n^{-\frac{1}{2}}), \quad \text{for } z = \mathcal{O}(n^{-3}). \quad (6.25)$$

We use (6.24) for  $z = x_n$  and  $z = y_n$ . Since  $f$  is analytic with  $f(0) = 0$ ,  $f'(0) > 0$ , it is easy to check that

$$\left( \frac{f(x_n)}{x_n} \right)^{\frac{2\beta}{3}} \left( \frac{f(y_n)}{y_n} \right)^{-\frac{2\beta}{3}} = 1 + \mathcal{O}\left((x-y)n^{-3}\right) \quad (6.26)$$



as  $n \rightarrow \infty$ . In view of (5.57) we have  $E_n^{-1}(y_n)E_n(x_n) = \mathbb{I} + \mathcal{O}\left((x-y)n^{-\frac{1}{2}}\right)$  and we obtain from this, (6.24), (6.25), and (6.26)

$$\begin{aligned} E_{in}^{-1}(y_n)E_{in}(x_n) &= (\mathbb{I} + C_n(y_n))^{-1} \left( \mathbb{I} + \mathcal{O}\left((x-y)n^{-\frac{1}{2}}\right) \right) (\mathbb{I} + C_n(x_n)) \\ &= (\mathbb{I} + C_n(y_n))^{-1} (\mathbb{I} + C_n(x_n)) + \mathcal{O}\left((x-y)n^{-\frac{1}{2}}\right) \\ &= \mathbb{I} + (\mathbb{I} + C_n(y_n))^{-1} (C_n(x_n) - C_n(y_n)) + \mathcal{O}\left((x-y)n^{-\frac{1}{2}}\right) \end{aligned}$$

From the fact that  $C_n(z)$  is analytic with  $C_n(z) = \mathcal{O}\left(n^{-\frac{1}{2}}\right)$  for  $z = \mathcal{O}(n^{-3})$  it follows (with an argument based on Cauchy's formula), that

$$C_n(x_n) - C_n(y_n) = \mathcal{O}\left((x-y)n^{-\frac{1}{2}}\right)$$

and (6.17) follows. □

**Lemma 6.7.** *As  $n \rightarrow \infty$*

$$E_{in}^{-1}(y_n)R^{-1}(y_n)R(x_n)E_{in}(x_n) = \mathbb{I} + \mathcal{O}\left((x-y)n^{-\frac{1}{2}}\right) \quad (6.27)$$

*uniformly for  $x, y$  in compact subsets of  $[0, \infty)$ .*

*Proof.* By (6.16) and (6.17) we have

$$\begin{aligned} E_{in}^{-1}(y_n)R^{-1}(y_n)R(x_n)E_{in}(x_n) &= E_{in}^{-1}(y_n) \left( \mathbb{I} + \mathcal{O}\left((x-y)n^{-5/2}\right) \right) E_{in}(x_n) \\ &= \mathbb{I} + \mathcal{O}\left((x-y)n^{-\frac{1}{2}}\right) + E_{in}^{-1}(y_n) \mathcal{O}\left((x-y)n^{-\frac{5}{2}}\right) E_{in}(x_n). \quad (6.28) \end{aligned}$$

From (6.24) and (6.26) we obtain

$$\begin{aligned} E_{in}^{-1}(y_n) \mathcal{O}\left((x-y)n^{-\frac{5}{2}}\right) E_{in}(x_n) &= (\mathbb{I} + C_n(y_n))^{-1} E_n^{-1}(y_n) \mathcal{O}\left((x-y)n^{-\frac{5}{2}}\right) E_n(x_n) (\mathbb{I} + C_n(x_n)) \\ &= E_n^{-1}(y_n) \mathcal{O}\left((x-y)n^{-\frac{5}{2}}\right) E_n(x_n) \quad (6.29) \end{aligned}$$

where in the second step we used (6.25). Finally recall that  $E_n(x_n) = \mathcal{O}(n)$  and  $E_n^{-1}(y_n) = \mathcal{O}(n)$  by Lemma 5.13. Combining this with (6.28), (6.29), we find (6.27), and the lemma follows. □

## Acknowledgements

ABJK is supported by long term structural funding - Methusalem grant of the Flemish Government, and by FWO Flanders projects G.0864.16 and EOS G0G9118N. LDM is supported by a PhD fellowship of the Research Foundation Flanders (FWO).

## References

- [1] G. Akemann and J.R. Ipsen, “Recent exact and asymptotic results for products of independent random matrices”, Acta Phys. Pol. B 46 (2015), 1747–1784.
- [2] G. Akemann, J.R. Ipsen, and M. Kieburg, “Products of rectangular random matrices: singular values and progressive scattering”, Phys. Rev. E 88 (2013), 052118.
- [3] G. Akemann, M. Kieburg, and L. Wei, “Singular value correlation functions for products of Wishart random matrices”, J. Phys. A: Math. Theor. 46 (2013), 275205.
- [4] G. Akemann and E. Strahov, “Dropping the independence: singular values for products of two coupled random matrices”, Comm. Math. Phys. 345 (2016), 101–140.

- [5] G. Akemann and E. Strahov, “*Hard edge limit of the product of two strongly coupled random matrices*”, *Nonlinearity* 29 (2016), 3743–3776.
- [6] M. Bertola and T. Bothner, “*Universality conjecture and results for a model of several coupled positive-definite matrices*”, *Comm. Math. Phys.* 337 (2015) 1077–1141.
- [7] P.M. Bleher and A.B.J. Kuijlaars, “*Random matrices with external source and multiple orthogonal polynomials*”, *Internat. Math. Research Notices* 2004:3 (2004), 109–129.
- [8] P.M. Bleher and A.B.J. Kuijlaars, “*Large  $n$  limit of Gaussian random matrices with external source, part III: double scaling limit*”, *Comm. Math. Phys.* 270 (2007), 481–517.
- [9] T. Bloom, N. Levenberg, V. Totik, and F. Wielonsky, “*Modified logarithmic potential theory and applications*”, *Int. Math. Res. Not. IMRN* 2017, no. 4, 1116–1154.
- [10] A. Borodin, “*Biorthogonal ensembles*”, *Nucl. Phys. B* 536 (1999) 704–732.
- [11] R. Butez, “*Large deviations for biorthogonal ensembles and variational formulation for the Dykema-Haagerup distribution*”, *Electron. Commun. Probab.* 22 (2017), Paper No. 37, 11 pp.
- [12] D. Cheliotis, “*Triangular random matrices and biorthogonal ensembles*”, *Statist. Probab. Lett.* 134 (2018), 36–44.
- [13] T. Claeys and S. Romano, “*Biorthogonal ensembles with two-particle interactions*”, *Nonlinearity* 27 (2014), 2419–2444.
- [14] E. Daems and A.B.J. Kuijlaars, “*A. Christoffel–Darboux formula for multiple orthogonal polynomials*”, *J. Approx. Theory* 130 (2004), 188–200.
- [15] P. Deift, “*Orthogonal Polynomials and Random Matrices: a Riemann–Hilbert approach.*”, *Courant Lecture Notes in Mathematics* Vol. 3, Amer. Math. Soc., Providence R.I. 1999.
- [16] P. Deift, T. Kriecherbauer, K.T.-R. McLaughlin, S. Venakides, and X. Zhou, “*Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory*”, *Comm. Pure Appl. Math.* 52 (1999), 1335–1425.
- [17] P. Deift, T. Kriecherbauer, K.T.-R. McLaughlin, S. Venakides, and X. Zhou, “*Strong asymptotics of orthogonal polynomials with respect to exponential weights*”, *Comm. Pure Appl. Math.* 52 (1999), 1491–1552.
- [18] P. Deift and X. Zhou, “*A steepest descent method for oscillatory Riemann–Hilbert problems. Asymptotics for the MKdV equation*”, *Ann. Math.* 137 (1993), 295–368.
- [19] S. Delvaux, A.B.J. Kuijlaars, and L. Zhang, “*Critical behavior of nonintersecting Brownian motions at a tacnode*”, *Comm. Pure Appl. Math.* 64 (2011), 1305–1383.
- [20] K. Deschout and A.B.J. Kuijlaars, “*Double scaling limit for modified Jacobi–Anglesco polynomials*”, in: *Notions of Positivity and the Geometry of Polynomials*, Trends in Mathematics, (P. Brändén, M. Passare, and M. Putinar, eds.), Springer Basel (2011), pp. 115–161.
- [21] P. Eichelsbacher, J. Sommerauer, and M. Stolz, “*Large deviations for disordered bosons and multiple orthogonal polynomial ensembles*”, *J. Math. Phys.* 52 (2011), no. 7, 073510, 16 pp.
- [22] A.S. Fokas, A.R. Its, and A.V. Kitaev, “*The isomonodromy approach to matrix models in 2D quantum gravity*”, *Comm. Math. Phys.* 147 (1992), 395–430.
- [23] P.J. Forrester and D-Z. Liu, “*Singular values for products of complex Ginibre matrices with a source: hard edge limit and phase transition*”, *Comm. Math. Phys.* 344 (2016), 333–368.
- [24] P.J. Forrester, D-Z. Liu, and P. Zinn-Justin, “*Equilibrium problems for Raney densities*”, *Nonlinearity* 28 (2015), 2265–2277.

- [25] P.J. Forrester and D. Wang, “*Muttalib-Borodin ensembles in random matrix theory – realisations and correlation functions*”, *Electron. J. Probab.* 22 (2017), paper no. 54, 43 pp.
- [26] M. Kieburg, A.B.J. Kuijlaars, and D. Stivigny, “*Singular value statistics of matrix products with truncated unitary matrices*”, *Int. Math. Res. Not. IMRN* 2016, 3392-3424.
- [27] A.B.J. Kuijlaars, “*A vector equilibrium problem for Muttalib-Borodin biorthogonal ensembles*”, *SIGMA* 12 (2016), 065, 15 pp.
- [28] A.B.J. Kuijlaars, “*Riemann-Hilbert analysis for orthogonal polynomials. In: “Orthogonal Polynomials and Special Functions” (E. Koelink and W. Van Assche, eds)*”, *Lecture Notes in Math.* 1817, Springer, Berlin, 2003, pp. 167-210.
- [29] A.B.J. Kuijlaars, A. Martínez-Finkelshtein, and F. Wielonsky, “*Non-intersecting squared Bessel paths: critical time and double scaling limit*”, *Comm. Math. Phys.* 308 (2011), 227-279.
- [30] A.B.J. Kuijlaars, K.T-R. McLaughlin, W. Van Assche, and M. Vanlessen, “*The Riemann-Hilbert approach to strong asymptotics for orthogonal polynomials on  $[-1, 1]$* ”, *Adv. Math.* 188 (2004), 337-398.
- [31] A.B.J. Kuijlaars and D. Stivigny, “*Singular values of products of random matrices and polynomial ensembles*”, *Random Matrices Theory Appl.* 3 (2014), 1450011, 22 pp.
- [32] A.B.J. Kuijlaars and L. Zhang, “*Singular values of products of Ginibre random matrices, multiple orthogonal polynomials and hard edge scaling limits*”, *Comm. Math. Phys.* 332 (2014), 759-781.
- [33] D.-Z. Liu, D. Wang, and L. Zhang, “*Bulk and soft-edge universality for singular values of products of Ginibre random matrices*”, *Ann. Inst. Henri Poincaré Probab. Stat.* 52 (2016), 1734-1762.
- [34] T. Lueck, H.-J. Sommers and M. R. Zirnbauer, “*Energy correlations for a random matrix model of disordered bosons*”, *J. Math. Phys.* 47 (2006), 103304.
- [35] Y. L. Luke, “*The special functions and their approximations*”, Vol. 1, Academic Press, New York, 1969.
- [36] K.A. Muttalib, “*Random matrix models with additional interactions*”, *J. Phys. A: Math. Gen.* 28 (1995), L159-164.
- [37] F.W.J. Olver, D.W. Lozier, R.F. Boisvert, and C.W. Clark, eds., “*NIST Handbook of Mathematical Functions*”, Cambridge University Press, Cambridge, U.K., 2010.
- [38] E.B. Saff and V. Totik, “*Logarithmic Potentials with External Fields*”, Springer-Verlag, Berlin, 1997.
- [39] W. Van Assche, “*Mehler-Heine asymptotics for multiple orthogonal polynomials*”, *Proc. Amer. Math. Soc.* 145 (2017), 303-314.
- [40] W. Van Assche, J.S. Geronimo, and A.B.J. Kuijlaars, “*Riemann-Hilbert problems for multiple orthogonal polynomials*”, pp. 23-59 in: “*NATO ASI Special Functions 2000. Current Perspective and Future Directions*”, (J. Bustoz, M.E.H. Ismail, S.K. Suslov eds.), *Nato Science Series II Vol.30*, Kluwer Academic Publishers, Dordrecht, 2001.
- [41] L. Zhang, “*Local universality in biorthogonal Laguerre ensembles*”, *J. Stat. Phys.* 161 (2015), 688-711.
- [42] L. Zhang, “*On Wright’s generalized Bessel kernel*”, *Physica D* 340 (2017), 27-39.