PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 9, September 1996

THE LOCAL ZETA FUNCTION FOR THE NON-TRIVIAL CHARACTERS ASSOCIATED WITH THE SINGULAR JORDAN ALGEBRAS

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(Communicated by William W. Adams)

Abstract. This paper investigates the local integrals

$$Z_m(t,\chi) = \int_{H_m(O_C)} \chi(\det(x))|\det(x)|^s dx$$

where O_C represents the integers of a composition algebra over a non-archimedean local field K and χ is a non-trivial character on the units in the ring of integers of K extended to K^* by setting $\chi(\pi) = 1$. The local zeta function for the trivial character is known for all composition algebras C. In this paper, we show in the quaternion case that $Z(t,\chi) = 0$ for all non-trivial characters and then compute the local zeta function in the ramified quadratic extension case for χ equal to the quadratic character. In this latter case, $Z(t,\chi) = 0$ for any character of order greater than 2.

1. Introduction

Let K be a finite, algebraic extension of $\mathbf{Q}_p$, O_K the ring of integers in K, π the uniformizing element in O_K, πO_K the maximal ideal of O_K, $U_K = O_K - \pi O_K$ the group of units in O_K, and $O_K/\pi O_K$ the residue field of K with cardinality q.

With $f(x) \in K[x_1, x_2, ..., x_n] - \{0\}$ a non-constant polynomial over K and $\chi : U_K \to \mathbf{C}^\times$ a character of U_K which has been extended to K^* by setting $\chi(\pi) = 1$, one associates the Igusa local zeta function

$$Z(s,\chi) = \int_{O_K^n} \chi(f(x))|f(x)|^s dx,$$

for $\mathrm{Re}(s) > 0$, where $|\cdot|$ represents the p-adic absolute value on K, $\chi(0)$ is set equal to 0, and dx denotes the Haar measure on K^n normalized so that the measure of O_K^n is 1. Igusa [6] showed that this local zeta function has a meromorphic continuation to the whole complex plane and is, in fact, a rational function of $t = q^{-s}$. We write $Z(s,\chi) = Z(t,\chi)$.

Let $K' = K(\sqrt{\epsilon})$ be a quadratic unramified extension of K, where ϵ is a non-square unit in O_K, let $L = K(\sqrt{-\pi})$ be a ramified quadratic extension of K and let $D = K' \oplus \sqrt{\pi}K'$ be a quaternion division algebra. These vector spaces over K have natural involutions and form composition algebras C over K [2] and are called the singular composition algebras, as they arise for only finitely many primes.

Received by the editors July 5, 1994 and, in revised form, March 27, 1995.
1991 *Mathematics Subject Classification.* Primary 11R52, 11F85.

In addition, we can use the composition algebras and their involutions to form the vector space of $m \times m$ hermitian matrices over K, $H_m(C)$. These hermitian matrices are Jordan algebras with norm form equal to the formal determinant of the matrix [10, 5, 9].

This paper investigates the following integrals:

$$Z_m(t, \chi) = \int_{H_m(O_C)} \chi(\det(x)) |\det(x)|^s dx$$

where O_C represents the integers of the composition algebra. In [8, 11] the local zeta functions for the trivial character are given for all composition algebras. In this paper, we quickly show in the quaternion case that $Z(t, \chi) = 0$ for all non-trivial characters and then compute the local zeta function in the ramified quadratic extension case for χ equal to the quadratic character. In this case, $Z(t, \chi) = 0$ for any character of order greater than 2.

2. Quaternion case

Before we consider the quaternion case, we need the following simple lemma.

Lemma 1 ([7, page 82])**.** *Suppose that U_K acts on K^n as a group of measure-preserving homomorphisms in such a way that $f(u \cdot x) = u^\lambda f(x)$ for a fixed λ and for all $u \in U_K$ and $x \in K^n$. Then $Z(t, \chi) = 0$ for all χ of order not dividing λ.*

Proof. Since $x \to u \cdot x$ is a measure-preserving transformation and χ is multiplicative, we see immediately that $Z(t, \chi) = \chi(u)^\lambda Z(t, \chi)$ for all $u \in U_K$. Hence, if $\chi^\lambda \neq 1$ then there exists a $u \in U_K$ such that $\chi(u)^\lambda \neq 1$, which implies that $Z(t, \chi) = 0$. □

The division algebra D is not isomorphic to a full matrix algebra. For this reason, it is called the twisted case. The quaternion division algebra is isomorphic to a matrix subalgebra with coefficients in K' where the isomorphism ϕ is defined as follows for $\alpha \in D$ (and $a_i \in K$):

$$\phi : \alpha = a_1 + \sqrt{\epsilon} a_2 + (a_3 + \sqrt{\epsilon} a_4)\sqrt{\pi} \to \begin{pmatrix} a_1 + \sqrt{\epsilon} a_2 & \pi(a_3 + \sqrt{\epsilon} a_4) \\ a_3 - \sqrt{\epsilon} a_4 & a_1 - \sqrt{\epsilon} a_2 \end{pmatrix}.$$

Using this isomorphism, the $m \times m$ hermitian matrices with coefficients in D are isomorphic to a subalgebra of the $2m \times 2m$ alternating matrices with coefficients in L by the following prescription. For each entry $x_{i,j}$ in the hermitian matrix $H_m(D)$ substitute the 2×2 matrix $\phi(x_{i,j}) \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. In this way, we get a linear isomorphism ψ of $H_m(D)$ to a subalgebra of $\mathrm{Alt}_{2m}(K')$.

For example, the isomorphism ψ is defined as follows for $m = 2$:

$$\begin{pmatrix} a_1 & b_1 - \sqrt{\epsilon} b_2 - (b_3 + \sqrt{\epsilon} b_4)\sqrt{\pi} \\ b_1 + \sqrt{\epsilon} b_2 + (b_3 + \sqrt{\epsilon} b_4)\sqrt{\pi} & c_1 \end{pmatrix}$$

$$\to \begin{pmatrix} 0 & a_1 & b_3 - \sqrt{\epsilon} b_4 & b_1 - \sqrt{\epsilon} b_2) \\ -a_1 & 0 & -b_1 - \sqrt{\epsilon} b_2 & -\pi(b_3 + \sqrt{\epsilon} b_4) \\ -b_3 + \sqrt{\epsilon} b_4 & b_1 + \sqrt{\epsilon} b_2 & 0 & c_1 \\ -b_1 + \sqrt{\epsilon} b_2 & \pi(b_3 + \sqrt{\epsilon} b_4) & -c_1 & 0 \end{pmatrix}.$$

The Pfaffian of an alternating matrix is the square root of its determinant. If we denote the alternating image of $x \in H_m(D)$ by $\psi(x) = A$ then $\det(x) = \mathrm{Pf}(A)$.

In this case, we have a measure-preserving homomorphism for all $u' \in U_{K'}$ on the vector space of Hermitian matrices over D. For any $x \in H_m(D)$ and $u' \in U_{K'}$, we define the action $u'\cdot x = \begin{pmatrix} u' & 0 \\ 0 & 1_{m-1}\end{pmatrix} x \begin{pmatrix} \bar{u}' & 0 \\ 0 & 1_{m-1}\end{pmatrix}$, where $\bar{u}'$ is the conjugate of u' in $U_{K'}$.

Now, $f(u'\cdot x) = \det(u'\cdot x) = N_{K'/K}(u')\det(x) = uf(x)$ for any $u \in U_K$ since the norm map is surjective. Finally, we can apply the lemma above with $\lambda = 1$ and $f(x) = \det(x)$ to show that $Z_m(t,\chi) = 0$ for all non-trivial characters χ and for all $m \geq 1$.

3. Ramified quadratic case

In this case, we would like to compute the following integral:

$$Z_m(t,\chi) = \int_{H_m(O_L)} \chi(\det(x))|\det(x)|^s dx$$

for χ a non-trivial character on U_K and for $(2,q)=1$. For $x \in H_m(L)$, we have a measure-preserving homomorphism for all u by defining $u\cdot x = \begin{pmatrix} u & 0 \\ 0 & 1_{m-1}\end{pmatrix} x \begin{pmatrix} u & 0 \\ 0 & 1_{m-1}\end{pmatrix}$. This action implies that $\det(u\cdot x) = u^2\det(x)$. We can now apply Lemma 1 above with $\lambda = 2$ and $f(x) = \det(x)$ to show that $Z_m(t,\chi) = 0$ for all χ of order greater than 2. As mentioned above, the local zeta function for the trivial character χ_0 is known [11]. Adopting the notation $(a) = (1-q^{-a})$, $(a)_+ = (1+q^{-a})$, and $(a,b) = (1-q^{-a}t^b)$, we prove the following theorem.

Theorem 1. *For χ the unique quadratic character on U_K,*

$$Z_m(t,\chi) = \begin{cases} \chi(-1)^{m/2} q^{-m/2} \prod_{i=1}^{m/2} \frac{(2i-1)}{(2i,1)}, & m \text{ even}, \\ 0, & m \text{ odd}. \end{cases}$$

Proof. We will outline the proof of this theorem but note that it follows closely the procedure used in [11] to compute the integral in the trivial character case. We need to find the orbital decomposition of $H_m(O_L)$, to compute two partial integrals $I_{m,k}(t,\chi)$ and $J_{m,2r}(t,\chi)$, to use them and the orbital decompositions of $H_m(\mathbf{F}_q)$ and $\mathrm{Alt}_m(\mathbf{F}_q)$ to get a recursion relation for $Z_m(t,\chi)$, and finally to show that the expression above is the correct closed form solution.

Consider each entry in $H_m(O_L)$ modulo $\sqrt{-\pi}$ and then modulo π and see that

$$H_m(O_L) = H_m(\mathbf{F}_q) \oplus \sqrt{-\pi}\,\mathrm{Alt}_m(\mathbf{F}_q) \bmod \pi$$

where $H_m(\mathbf{F}_q)$ is the vector space of symmetric matrices over $\mathbf{F}_q$ and $\mathrm{Alt}_m(\mathbf{F}_q)$ are the skew-symmetric matrices over $\mathbf{F}_q$. The orbital structure of $H_m(\mathbf{F}_q)$ for $(q,2)=1$ under the action of $G = GL_m(\mathbf{F}_q)$ defined by $g\cdot a = ga\,{}^tg$ for $a \in H_m(\mathbf{F}_q)$ is as follows:

$$H_m(\mathbf{F}_q) = \{0\} \cup \left[\bigcup_{k=1}^{m} G\cdot\xi_k^1 \cup G\cdot\xi_k^2\right]$$

where $\xi_k^1 = \begin{pmatrix} 1_k & 0 \\ 0 & 0\end{pmatrix}$, $\xi_k^2 = \begin{pmatrix} 1_{k-1} & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 0\end{pmatrix}$, and $\varepsilon \in \mathbf{F}_q - (\mathbf{F}_q)^2$. This follows from the fact that $\alpha_1x_1^2 + \alpha_2x_2^2 = 1$ has solutions in $\mathbf{F}_q$ for all $\alpha_i \in \mathbf{F}_q^\times$ and that we can diagonalize any quadratic form by an $\mathbf{F}_q$-linear change of variables such that $\alpha_1x_1^2 + \alpha_2x_2^2 + \cdots + \alpha_kx_k^2 = y_1^2 + y_2^2 + \cdots + \alpha_1\alpha_2\cdots\alpha_ky_k^2$ for $\alpha_i \in \mathbf{F}_q^\times$. Note that any square α_i can be absorbed into y_k and that the decomposition above is disjoint.

The cardinality of these orbits can be computed using Dickson's formulae for the following groups [1, pages 78,160,94]:

$|GL_m(\mathbf{F}_q)| = q^{m^2}\prod_{i=1}^{m}(i)$, $\quad |Sp_{2r}(\mathbf{F}_q)| = q^{r(2r+1)}\prod_{i=1}^{r}(2i)$,

$$|SO_m(\xi_m^i)(\mathbf{F}_q)| = q^{m(m-1)/2} \begin{cases} \prod_{j=1}^{(m-1)/2}(2i), & m \text{ odd}, \\ (1-\chi(d)q^{-m/2}) \ \prod_{j=1}^{m/2-1}(2i), & m \text{ even}, \end{cases}$$
$$|O_m(\xi_m^i)(\mathbf{F}_q)| = 2|SO_m(\xi_m^i)(\mathbf{F}_q)|$$

where $d = (-1)^{m(m-1)/2}\det(\xi_m^i)$ for $i = 1, 2$ and χ is the quadratic character on $\mathbf{F}_q$. Thus, $|SO_{2n}(\xi_{2n}^i)(\mathbf{F}_q)|$ depends on

$$\chi((-1)^n\varepsilon) = \begin{cases} 1, & n \text{ even}, \varepsilon \in (\mathbf{F}_q^\times)^2, \\ -1, & n \text{ even}, \varepsilon \in (\mathbf{F}_q^\times) - (\mathbf{F}_q^\times)^2, \\ \chi(-1), & n \text{ odd}, \varepsilon \in (\mathbf{F}_q^\times)^2, \\ -\chi(-1), & n \text{ odd}, \varepsilon \in (\mathbf{F}_q^\times) - (\mathbf{F}_q^\times)^2. \end{cases}$$

Note that the stabilizer of ξ_k^i is the set of all $\left(\begin{smallmatrix} g_{11} & g_{12} \\ 0 & g_{22}\end{smallmatrix}\right) \in GL_m(\mathbf{F}_q)$ such that $g_{11} \in O_k(\xi_m^i)(\mathbf{F}_q)$, $g_{12} \in \mathrm{Mat}_{k,m-k}(\mathbf{F}_q)$, and $g_{22} \in GL_{m-k}(\mathbf{F}_q)$. Thus, we see that

$$|G\cdot\xi_k^i| = \frac{|GL_m(\mathbf{F}_q)|}{|GL_{m-k}(\mathbf{F}_q)|q^{k(m-k)}2|SO_k(\xi_m^i)(\mathbf{F}_q)|}.$$

In our calculation, we will need the cardinality of $|G\cdot\xi_k^1| - |G\cdot\xi_k^2|$. We make the convention that if $k = 0$ we simply mean the orbit of the 0 matrix, $|G\cdot 0| = 1$, and if a product is taken from a larger index to a smaller, we set it equal to 1. Using the formulae of Dickson with special attention to the case when k is even, we see that

$$|G\cdot\xi_k^1| - |G\cdot\xi_k^2| = \begin{cases} 1, & k = 0, \\ 0, & k \text{ odd}, \\ \chi(-1)^{k/2} \ (A - B), & k \text{ even}, \end{cases}$$

where

$$\begin{aligned} &A - B \\ &\quad = \frac{q^{m^2}\prod_{i=1}^m(i)}{2q^{(m-k)^2}\prod_{j=1}^{m-k}(j)q^{k(m-k)}q^{k(k-1)/2}\prod_{i=0}^{k/2-1}(2i)}\left(\frac{1}{1-q^{-k/2}} - \frac{1}{1+q^{-k/2}}\right) \\ &\quad = q^{k(2m-k)/2}\frac{\prod_{i=m-k+1}^m(i)}{\prod_{i=1}^{k/2}(2i)}. \end{aligned}$$

The orbital decomposition of $\mathrm{Alt}(\mathbf{F}_q)$ into disjoint orbits is known [4] to be

$$\mathrm{Alt}(\mathbf{F}_q) = \{0\} \cup \left\{ \bigcup_{k=1}^{[\frac{m}{2}]} G\cdot \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} \right\},$$

where $[\cdot]$ is the greatest integer function and E_r is the $2r \times 2r$ block matrix with r copies of $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)$ down the main diagonal and zeros elsewhere. The stabilizer of $\left(\begin{smallmatrix} E_r & 0 \\ 0 & 0\end{smallmatrix}\right)$ is the set of all $\left(\begin{smallmatrix} g_{11} & g_{12} \\ 0 & g_{22}\end{smallmatrix}\right) \in GL_m(\mathbf{F}_q)$ such that $g_{11} \in Sp_{2r}(\mathbf{F}_q)$, $g_{12} \in \mathrm{Mat}_{2r,m-2r}(\mathbf{F}_q)$, and $g_{22} \in GL_{m-2r}(\mathbf{F}_q)$. Thus, we see that

$$\left| G\cdot \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} \right| = q^{r(2m-2r-1)} \ \frac{\prod_{i=1}^{2r}(m-2r+i)}{\prod_{l=1}^{r}(2l)}.$$

Following the computation of the partial integrals in [8], we see that

$$\begin{aligned} I_{m,k}(t,\chi) &= \int_{\mathrm{diag}(\alpha_1,\dots,\alpha_k,0)+\sqrt{-\pi}\,\mathrm{Alt}_m(\mathbf{F}_q)+\pi H_m(O_L)} \chi(\det(x))|\det(x)|^s dx \\ &= \chi(\alpha_1)\dots\chi(\alpha_k) q^{-k(2m-k+1)/2} \\ &\quad \times \int_{\sqrt{-\pi}\,\mathrm{Alt}_{m-k}(\mathbf{F}_q)+\pi H_{m-k}(O_L)} \chi(\det(x))|\det(x)|^s dx \end{aligned}$$

and that

$$\begin{aligned} J_{m,2r}(t,\chi) &= \int_{\sqrt{-\pi}\left(\begin{smallmatrix} E_r & 0 \\ 0 & 0 \end{smallmatrix}\right)+\pi H_m(O_L)} \chi(\det(x))|\det(x)|^s dx \\ &= \chi(-1)^r q^{-m^2} t^{m-r} Z_{m-2r}(t,\chi) \end{aligned}$$

for $\alpha_1,\alpha_2,\dots,\alpha_k \in U_K$, $0 \le k \le m$, and $0 \le r \le [m/2]$.

Applying the key lemma for orbital decomposition [8] and using the formulae for the cardinality of the orbits of $H_m(\mathbf{F}_q)$ and the first partial integral, we find that

$$\begin{aligned} Z_m(t,\chi) = \sum_{k=0}^{m} (|G\cdot\xi_k^1| - |G\cdot\xi_k^2|) q^{-k(2m-k+1)/2} \\ \times \int_{\sqrt{-\pi}\,\mathrm{Alt}_{m-k}(\mathbf{F}_q)+\pi H_{m-k}(O_L)} \chi(\det(x))|\det(x)|^s dx. \end{aligned}$$

By our formula for the difference between these orbits, we know that the difference is non-zero only for even values of k, and our formula becomes

$$\begin{aligned} Z_m(t,\chi) = \sum_{k=0}^{[m/2]} \chi(-1)^k q^{-k} \frac{\prod_{i=m-2k+1}^{m}(i)}{\prod_{i=1}^{k}(2i)} \\ \times \int_{\sqrt{-\pi}\,\mathrm{Alt}_{m-2k}(\mathbf{F}_q)+\pi H_{m-2k}(O_L)} \chi(\det(x))|\det(x)|^s dx. \end{aligned}$$

Applying the key lemma again and using the formulae for the cardinality of the orbits of $\mathrm{Alt}_{m-k}(\mathbf{F}_q)$ and the second partial integral, we have that

$$\begin{aligned} Z_m(t,\chi) &= \sum_{k=0}^{[m/2]} \chi(-1)^k q^{-k} \frac{\prod_{i=m-2k+1}^{m}(i)}{\prod_{i=1}^{k}(2i)} \sum_{r=0}^{[(m-2k)/2]} \left| G\cdot\left(\begin{matrix} E_r & 0 \\ 0 & 0 \end{matrix}\right) \right| \\ &\quad \times \int_{\sqrt{-\pi}\left(\begin{smallmatrix} E_r & 0 \\ 0 & 0 \end{smallmatrix}\right)+\pi H_{m-2k}(O_L)} \chi(\det(x))|\det(x)|^s dx \\ &= \sum_{k=0}^{[m/2]} \chi(-1)^k q^{-k} \frac{\prod_{i=m-2k+1}^{m}(i)}{\prod_{i=1}^{k}(2i)} \sum_{r=0}^{[(m-2k)/2]} q^{r(2m-4k-2r-1)} \\ &\quad \times \frac{\prod_{i=1}^{2r}(m-2k-2r+i)}{\prod_{i=1}^{r}(2i)} \chi(-1)^r q^{-(m-2k)^2} t^{m-2k-r} \\ &\quad \times Z_{m-2k-2r}(t,\chi). \end{aligned}$$

We see immediately from this recursion formula that $Z_1(t,\chi) = 0$. This then implies, by induction, that $Z_m(t,\chi) = 0$ for all odd m. To complete the proof of our theorem, we let $m = 2n$, substitute our closed form expression for $Z_m(t,\chi)$ into the recursion formula above, divide both sides by $Z_m(t,\chi)$, and change the order of summation by letting $k \to n-k$. After these steps, the following identity remains to be proven:

$$1 = \sum_{k=0}^{n} \frac{\prod_{i=1}^{2(n-k)}(2k+i)}{\prod_{i=1}^{n-k}(2i)} q^{-4k^2} \frac{\prod_{i=k+1}^{n}(2i,1)}{\prod_{i=k+1}^{n}(2i-1)} \times \sum_{r=0}^{k} q^{2r(2k-r)} t^{2k-r} \frac{\prod_{i=1}^{2r}(2k-2r+i)\prod_{i=k-r+1}^{k}(2i,1)}{\prod_{i=1}^{r}(2i)\prod_{i=k-r+1}^{k}(2i-1)}.$$

Using the Gauss identity [3] and Lemma 3 in [11] with $x = q^{-2}$, we can show that the inner sum above is precisely $q^{2k^2}t^k$. For completeness, we state Lemma 3 in [11] without proof.

Lemma 3 ([11])**.** *For any non-negative integer k, the following identity holds:*

$$1 = \sum_{j=0}^{k} x^{j^2}\ t^j \prod_{i=1}^{k-j} \frac{(1-x^{j+i})}{(1-x^i)} \prod_{i=1}^{k-j}(1-x^{j+i}t).$$

Simplifying, our identity becomes:

$$1 = \sum_{k=0}^{n} q^{-2k^2} t^k \prod_{i=1}^{n-k} \frac{(2(k+i))}{(2i)} \prod_{i=k+1}^{n}(2i,1)$$

and another application of Lemma 3 in [11] proves the theorem. ☐

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