

THE LOG-TERM OF THE DISC BUNDLE OVER A HOMOGENEOUS HODGE MANIFOLD

ANDREA LOI, ROBERTO MOSSA, AND FABIO ZUDDAS

ABSTRACT. We show the vanishing of the log-term in the Fefferman expansion of the Bergman kernel of the disk bundle over a compact simply-connected homogeneous Kähler-Einstein manifold of classical type. Our results extends that in [13] for the case of Hermitian symmetric spaces of compact type.

CONTENTS

1.	Introduction	1
2.	Kähler structures and complex coordinates of flag manifolds	3
3.	Calabi’s diastasis function of classical flag manifolds	7
4.	Proof of Theorem 1.1	11
	Appendix A. Classical Lie Algebras	16
	References	19

1. INTRODUCTION

We recall the basic framework. Let (L, h) be a positive Hermitian line bundle over a compact Kähler manifold (M, g) of complex dimension n , such that $\text{Ric}(h) = \omega_g$, where ω_g denotes the Kähler form associated to g and $\text{Ric}(h)$ is the two-form on M whose local expression is given by

$$\text{Ric}(h) = -\frac{i}{2} \partial \bar{\partial} \log h(\sigma(x), \sigma(x)), \quad (1)$$

for a trivializing holomorphic section $\sigma : U \rightarrow L \setminus \{0\}$. In the quantum mechanics terminology the pair (L, h) is called a *geometric quantization* of (M, ω_g) and L the *quantum line bundle*. Notice that such an h exists if and only if $\frac{\omega_g}{\pi}$ is an integral Kähler form which represents the first Chern class $c_1(L)$ of L . Consider the negative

2000 *Mathematics Subject Classification.* 53D05; 53C55; 53D05; 53D45.

Key words and phrases. Bergman kernel; Ramadanov conjecture; homogeneous space; log-term.

The first two authors were supported by Prin 2010/11 – Varietà reali e complesse: geometria, topologia e analisi armonica – Italy and also by INdAM-GNSAGA - Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni; the third author was supported by the FIRB project 2012 “Geometria Differenziale e Teoria geometrica delle funzioni”.

Hermitian line bundle (L^*, h^*) over (M, g) dual to (L, h) and let $D \subset L^*$ be the unit disc bundle over M , i.e.

$$D = \{v \in L^* \mid \rho(v) := 1 - h^*(v, v) > 0\}. \quad (2)$$

It is not hard to see (and well-known) that the condition $\text{Ric}(h) = \omega_g$ implies that D is a strongly pseudoconvex domain in L^* with smooth boundary $X = \partial D = \{v \in L^* \mid \rho(v) = 0\}$. Consider the Bergman space \mathcal{B}_D consisting of holomorphic $(n+1, 0)$ -forms η on D such that $\frac{i^{n+1}}{2^{n+1}} \int_D \eta \wedge \bar{\eta} < \infty$ and the corresponding Bergman kernel Berg_D namely the $(n+1, n+1)$ -form given by:

$$\text{Berg}_D = \sum_j \eta_j \wedge \bar{\eta}_j,$$

where $\{\eta_j\}$ is an orthonormal basis of \mathcal{B}_D .

We say that the log-term of the Bergman kernel Berg_D vanishes if there exists a non-vanishing $(n+1, n+1)$ -form a on \bar{D} (the closure of D) such that

$$\text{Berg}_D(v) = a(v)\rho(v)^{-n-2}, \quad v \in D. \quad (3)$$

The main goal of this paper is to study the Bergman kernel Berg_D of the disk bundle $D \subset L^*$, when L is the anticanonical bundle K^* over a compact homogeneous Kähler-Einstein manifold (M, g) (and hence $L^* = K$ is the canonical bundle).

The following theorem represents our main result.

Theorem 1.1. *Let (M, g) be a compact and simply-connected homogeneous Kähler-Einstein manifold of classical type with $\frac{\omega_g}{\pi} \in c_1(L)$, $L = K^*$, and $D \subset L^*$ be the corresponding disk bundle. Then the log-term of the Bergman kernel Berg_D vanishes.*

Problems and results of this kind go back to the celebrated Fefferman's theorem ([14]) about the expansion of the Bergman kernel for domains in \mathbb{C}^n . Let us recall that, for a strongly pseudoconvex domain $D \subseteq \mathbb{C}^n$ given by a defining function ψ , the Bergman kernel $B(z) = \sum |f_j(z)|^2$ of D (where f_j is an orthonormal basis for the Bergman space of holomorphic square integrable functions $f : D \rightarrow \mathbb{C}$) admits the following decomposition

$$B(z) = \phi(z)\psi(z)^{-n-1} + \tilde{\phi}(z) \log \psi(z), \quad (4)$$

where ϕ does not vanish on ∂D and $\tilde{\phi}$ is said to be the *log-term*. In [25], Ramadanov conjectures that if the log-term of D vanishes, then D is biholomorphic to the unit ball in \mathbb{C}^n . The conjecture has been proved in some special cases, among which domains in \mathbb{C}^2 and domains with rotational symmetries (see, for example, [16], [23]). A decomposition analogous to (4) has been proved by Boutet de Monvel and Sjöstrand for the Szegő kernel ([10]), and there is a corresponding version of the

Ramadanov conjecture. It is then natural to consider the analogous definitions and questions for strictly pseudoconvex domains in complex manifolds. Our Theorem 1.1 extends that of M. Engliš and G. Zhang [13] when M is an Hermitian symmetric space of compact type. In that paper they ask (see Question 4 at page 911) if the vanishing of the log-term of the Bergman kernel of the disk bundle over a compact Kähler manifold (M, g) should imply that the manifold is symmetric. Theorem 1.1 provides a negative answer to this question: one can simply take a compact and simply-connected homogeneous Kähler-Einstein manifold of classical type which is not symmetric.¹

The proof of Theorem 1.1 is based on the following two facts satisfied by any homogeneous Kähler-Einstein manifold as in the theorem:

- the quantization bundle $L = K^* \rightarrow M$ is regular, namely its associated Kempf distortion function $T_{m,g}$ is a positive constant for all positive integer m (see Lemma 4.3 below);
- (M, g) admits a Calabi's diastasis function whose maximal domain of definition is an open and dense subset of M biholomorphically equivalent to \mathbb{C}^n , where n is the complex dimension of M (see Theorem 3.5).

We think that the result of Theorem 1.1 can be extended also for the disk bundle $D_m \subset L^{*m}$ associated to the Kähler-Einstein form $\frac{m\omega_g}{\pi} \in c_1(L^m)$, for any $m \geq 1$. Actually we believe that the result holds true for all homogeneous Kähler metrics not necessarily Einstein or of classical type.

The paper is organized as follows. In Section 2 we are going to give a self-contained exposition of the results proved in [2] about G -invariant Kähler metrics on homogeneous Hodge manifolds G/K of classical type and on the existence of an open dense subset F_{reg} of G/K biholomorphically equivalent to the Euclidean space. In Section 3, after recalling the construction of an explicit Kähler potential in F_{reg} due to [2], we will prove Theorem 3.5, namely that this potential coincides with Calabi's diastasis function having F_{reg} as its maximal domain of definition. Finally, Section 4 is dedicated to the proof of Theorem 1.1. The paper ends with an Appendix which contains the basic material on semisimple Lie algebras used in this paper.

2. KÄHLER STRUCTURES AND COMPLEX COORDINATES OF FLAG MANIFOLDS

The material of this section is based on [1] and [2]. Here and below, G will denote a compact semisimple group with Lie algebra \mathfrak{g} and $G^{\mathbb{C}}$, $\mathfrak{g}^{\mathbb{C}}$ the corresponding complexifications. Given $Z \in \mathfrak{g}$, let us consider the orbit $F = \text{Ad}_G Z$ of Z for the

¹ It is worth mentioning that in [6] (see also [13]) it is proven the analogous of Theorem 1.1 for the Szegő kernel, namely the vanishing of the log-term of the Szegő kernel of the disk bundle over a compact homogenous Hodge manifold (not necessarily of classical type).

adjoint action of G on \mathfrak{g} . Then F is diffeomorphic to the quotient manifold G/K , being K the stabilizer of Z with respect to the adjoint action, and is called a *flag manifold*.

Recall (see, for example, [8]) that each compact homogeneous Kähler manifold M is the Kähler product of a flat complex torus and a simply-connected compact homogeneous Kähler manifold, and admits a Kähler-Einstein structure if and only if is a torus or is simply-connected. In the simply-connected case, M is isomorphic, as a homogeneous complex manifold, to an orbit of the adjoint action of its connected group of isometries G (which, being compact and with no center, is semisimple). In this paper we restrict to the simply-connected case.

We are going to recall how one can describe combinatorially all the G -invariant complex structures on a flag manifold F via root systems and Dynkin diagrams. In what follows, \mathfrak{k} and $\mathfrak{k}^{\mathbb{C}}$ denote respectively the Lie algebra of K and its complexification. As it is known from the theory of complex semisimple Lie algebras, given a Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$ (i.e. a maximal abelian subalgebra such that ad_X is diagonalizable for each $X \in \mathfrak{h}^{\mathbb{C}}$), one has the decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} + \sum_{\alpha \in R} \mathbb{C}E_{\alpha}, \quad (5)$$

where a *root* $\alpha \in R$ is a functional $\mathfrak{h}^{\mathbb{C}} \rightarrow \mathbb{C}$ such that $[H, E_{\alpha}] = \alpha(H)E_{\alpha}$ for each $H \in \mathfrak{h}^{\mathbb{C}}$ and for some $E_{\alpha} \in \mathfrak{g}^{\mathbb{C}}$ (called *root vector* of α). The set R of roots is called the *root system* of $\mathfrak{g}^{\mathbb{C}}$. Recall also that any two Cartan subalgebras $\mathfrak{h}_1^{\mathbb{C}}$ and $\mathfrak{h}_2^{\mathbb{C}}$ are conjugate, i.e. there exists $g \in G^{\mathbb{C}}$ such that $Ad_g(\mathfrak{h}_1^{\mathbb{C}}) = \mathfrak{h}_2^{\mathbb{C}}$. Since by definition of semisimple algebra the Killing form B of $\mathfrak{g}^{\mathbb{C}}$ is nondegenerate, to every root $\alpha \in R$ is associated by duality a vector $H_{\alpha} \in \mathfrak{h}^{\mathbb{C}}$ which satisfies $B(H, H_{\alpha}) = \alpha(H)$ for every $H \in \mathfrak{h}^{\mathbb{C}}$. The real vector space \mathfrak{h} spanned by the H_{α} 's, $\alpha \in R$, is a real form of $\mathfrak{h}^{\mathbb{C}}$ on which B is real and positive definite (Theorem 4.4 in [15]). We can then define a scalar product between the roots by $\langle \alpha, \beta \rangle := B(H_{\alpha}, H_{\beta})$.

A *basis* Π of the root system is a subset $\Pi \subseteq R$ such that every root $\alpha \in R$ can be written as a linear combination of the elements of Π with the coefficients either all non-negative or all non-positive. In the first (resp. second) case, α is said to be positive (resp. negative). The set of positive roots with respect to a fixed basis will be denoted by R^+ . It can be shown that any subset $R^+ \subseteq R$ such that $R = R^+ \cup (-R^+)$, $R^+ \cap (-R^+) = \emptyset$ and $\alpha + \beta \in R^+$ provided $\alpha, \beta \in R^+$, $\alpha + \beta \in R$, is the set of the positive roots with respect to some basis.

The Lie algebra $\mathfrak{g}^{\mathbb{C}}$ can be combinatorially represented by a *Dynkin diagram*, which is constructed as follows: fixed a basis Π of R , the Dynkin diagram is the linear graph having $\#\Pi$ vertices, one for each element of Π , and such that the number of edges between two vertices depends on the value of the scalar product between the

corresponding roots (more precisely, if the angle between the roots is θ , the number of edges is equal to $4 \cos^2 \theta$, and one proves that this number can be only 0,1,2,3). Moreover, an edge between two vertices is oriented with an arrow if and only if the corresponding roots have different norms (and the arrow goes from the longer root to the shorter one). In fact, the diagram does not depend on the choice of the basis. The choice of a basis is called an *equipment* for the diagram. In the Appendix, we give for each of the classical groups $G = SU(d), Sp(d), SO(2d), SO(2d + 1)$ an equipment which will have a fundamental role in the proofs of our results (see Remark 2.3).

Now, the complexification $\mathfrak{k}^{\mathbb{C}}$ of the Lie algebra of the stabilizer subgroup K turns out to be reductive (i.e. it decomposes as the direct sum of its center and a semisimple part) so we have

$$\mathfrak{k}^{\mathbb{C}} = Z(\mathfrak{k}^{\mathbb{C}}) \oplus (\mathfrak{h}^{\mathbb{C}})' + \sum_{\alpha \in R_K} \mathbb{C}E_{\alpha} \quad (6)$$

where $Z(\mathfrak{k}^{\mathbb{C}})$ is the center of $\mathfrak{k}^{\mathbb{C}}$ and $(\mathfrak{h}^{\mathbb{C}})'$, $R_K \subseteq R$ are respectively a Cartan subalgebra and the root system of the semisimple part of $\mathfrak{k}^{\mathbb{C}}$ (with respect to $(\mathfrak{h}^{\mathbb{C}})'$).

The elements of R_K (resp. of the complementary subset R_M) are usually called *white roots* (resp. *black roots*). The reason is that one can represent the flag manifold G/K on the Dynkin diagram of G , equipped with a given basis Π , by painting black the vertices corresponding to roots belonging to R_M . One gets a decomposition $\Pi = \Pi_K \cup \Pi_M$ of the basis Π and the resulting diagram is called *painted Dynkin diagram*.² Looking at a painted diagram, one can easily recover the root decomposition and the flag manifold: indeed, a root α belongs to R_M if and only if $\alpha = \sum_{\beta \in \Pi} c_{\beta} \beta$ with $c_{\beta} \neq 0$ for some $\beta \in \Pi_M$; moreover, deleting the black nodes from the diagram one gets the Dynkin diagram of the semisimple part of K .

Definition 2.1. *A subset $Q \subseteq R_M$ is said to be maximal closed nonsymmetric if it satisfies the following conditions:*

- (i) $Q \cup (-Q) = R_M$;
- (ii) $Q \cap (-Q) = \emptyset$;
- (iii) for any $\alpha, \beta \in Q$ such that $\alpha + \beta \in R$ one has $\alpha + \beta \in Q$.

Then we have

Proposition 2.2. *(Corollary 3.1 in [2]) There exists a one-to-one correspondence between G -invariant complex structures on G/K and maximal closed nonsymmetric subsets Q in R_M . More precisely, given the decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} + \mathfrak{m}^{\mathbb{C}}$, $\mathfrak{m}^{\mathbb{C}} =$*

²Some authors (for example [9]) reverse the notation and paint black the roots in R_K .

$\sum_{\alpha \in R_M} \mathbb{C}E_\alpha$, the G -invariant complex structure J_Q associated to Q is determined (on the complexified tangent space at the base point $o = K$, naturally identified with $\mathfrak{m}^\mathbb{C}$) by $(J_Q)_o(E_{\pm\alpha}) = \pm iE_\alpha$ ($\alpha \in Q$).

The following remark is crucial for the results of the next section.

Remark 2.3. Let $F = G/K$ be a flag manifold endowed with an invariant complex structure. We claim that, fixed any basis Π for the root system R of G , one can find a painting of the Dynkin diagram of G equipped with Π so that the associated flag manifold, endowed with the complex structure determined by the maximal closed nonsymmetric subset R_M^+ of the black roots which are positive with respect to Π is G -diffeomorphic to F .

In order to see that, fix a basis Π_K for the set R_K of white roots of F , and let R_K^+ be the induced set of white positive roots. If Q is the maximal closed nonsymmetric subset of R_M corresponding to the complex structure of F , take $R^+ := Q \cup R_K^+$ as the set of positive roots in R (in fact, it is easily seen to satisfy the three properties needed to be the set of positive roots induced by a basis) and let Π' be the basis of R which induces R^+ . Clearly, Π' contains Π_K . Now, the Dynkin diagram of G equipped with Π' and with nodes in $\Pi' \setminus \Pi_K$ painted black represents F and Q is the set of positive black roots with respect to this equipment. Then, it is enough to apply to this painted diagram a Weyl group transformation w (see for example Section II.6 in [17]) such that $w(\Pi') = \Pi$ in order to get a repainted diagram equipped with Π and such that Q is sent to the set of black roots which are positive with respect to Π , as claimed. By this remark, in the results we are going to prove about the flag manifolds G/K of the classical groups $G = SU(d), SO(2d), SO(2d+1), Sp(d)$, it is not restrictive to assume that G/K is represented by a painted Dynkin diagram equipped with the canonical equipment Π_{can} (see the Appendix) and with the invariant complex structure of G/K given by the set R_M^+ of black positive roots (with respect to Π_{can}).

Let now Π be a basis of R and let $\Pi = \Pi_K \cup \Pi_M$ be the decomposition into white and black nodes, with $\Pi_K = \{\beta_1, \dots, \beta_k\}$, $\Pi_M = \{\alpha_1, \dots, \alpha_m\}$. As we have recalled at the beginning of this section, the Killing form induces by duality a scalar product \langle, \rangle on the real space \mathfrak{h}^* spanned by the roots. The *fundamental weight* $\bar{\alpha}_i$ associated with α_i , $i = 1, \dots, m$ is the element of \mathfrak{h}^* defined by

$$\frac{2\langle \bar{\alpha}_i, \alpha_j \rangle}{\|\alpha_j\|^2} = \delta_{ij}, \quad \langle \bar{\alpha}_i, \beta_j \rangle = 0 \quad (7)$$

If we denote by $\mathfrak{t} = Z(\mathfrak{k}) \cap \mathfrak{h}$ the intersection between the center $Z(\mathfrak{k})$ of \mathfrak{k} and \mathfrak{h} , then the fundamental weights form a basis of the real space \mathfrak{t}^* .

Proposition 2.4. ([2], Proposition 2.2) *There exists a natural isomorphism between \mathfrak{t}^* and the space of closed invariant 2-forms on F given by*

$$\xi \in \mathfrak{t}^* \rightarrow \omega_\xi = \frac{i}{2\pi} \sum_{\alpha \in R_M} \frac{2\langle \xi, \alpha \rangle}{\langle \alpha, \alpha \rangle} \omega^\alpha \wedge \omega^{-\alpha} \quad (8)$$

where the ω^α 's are 1-forms on $\mathfrak{m}^\mathbb{C}$, dual to the E_α 's.

If J_Q is the complex structure associated to $Q = R_M^+$ with the given equipment Π , then ω_ξ is Kähler with respect to J_Q if and only if all the coordinates of ξ with respect to $\{\bar{\alpha}_1, \dots, \bar{\alpha}_m\}$ are positive. Moreover, ω_ξ is integral if and only if the coordinates of ξ with respect to $\{\bar{\alpha}_1, \dots, \bar{\alpha}_m\}$ are integers.

By Proposition 2.2, invariant complex structures on G/K are in one-to-one correspondence with maximal closed nonsymmetric subsets $Q \subseteq R_M$. Indeed, the manifold G/K endowed with the complex structure J_Q is biholomorphic to the complex homogeneous manifold $G^\mathbb{C}/K^\mathbb{C}G^Q$, where $G^Q = \exp(\mathfrak{g}^Q)$ and $\mathfrak{g}^Q = \sum_{\alpha \in Q} \mathbb{C}E_\alpha$ (see [1], [2] and also [7] for more details). Using this, in [2] it is given an explicit system of complex coordinates on an open subset of G/K as follows.

Since the product $G_{reg}^\mathbb{C} = G^{-Q}K^\mathbb{C}G^Q$ (where $G^{-Q} = \exp(\mathfrak{g}^{-Q})$ and $\mathfrak{g}^{-Q} = \sum_{\alpha \in -Q} \mathbb{C}E_\alpha$) defines an open dense subset in $G^\mathbb{C}$, its image in $G^\mathbb{C}/K^\mathbb{C}G^Q$ via the natural projection $G^\mathbb{C} \rightarrow G^\mathbb{C}/K^\mathbb{C}G^Q$ defines an open dense subset in G/K , denoted $F_{reg} = G_{reg}^\mathbb{C}/K^\mathbb{C}G^Q$. Clearly, $F_{reg} \simeq G^{-Q}$. Then, by

$$z = (z_\alpha)_{\alpha \in -Q} \in \mathbb{C}^n \mapsto \exp\left(\sum_{\alpha \in -Q} z_\alpha E_\alpha\right) \in G^{-Q} \simeq F_{reg} \subseteq F \quad (9)$$

(where n is the cardinality of Q) one defines a system of complex coordinates on F_{reg} .

3. CALABI'S DIASTASIS FUNCTION OF CLASSICAL FLAG MANIFOLDS

We now assume that G is one of the classical groups $SU(d), Sp(d), SO(2d), SO(2d+1)$ (where the orthogonal groups are realized as groups of matrices as in the Appendix). In this section, after recalling the construction of an explicit Kähler potential for the invariant Kähler form ω_ξ (in the notation of Proposition 2.4) in the coordinates defined at the end of the previous section we prove, in Theorem 3.5, that this potential is indeed Calabi's diastasis function.

Definition 3.1. ([2], Definition 8.1) *Let $F = G/K$, $G \subseteq GL(N, \mathbb{C})$, be a flag manifold. A principal minor Δ_k , $k = 1, \dots, N-1$, (i.e. the function $GL(N, \mathbb{C}) \rightarrow \mathbb{C}$ associating to $A \in GL(N, \mathbb{C})$ the determinant $\Delta_k(A)$ of the submatrix of A given by the first k rows and columns of A) is said to be F -admissible if for every $A \in K^\mathbb{C}$ and every $v = (v_1, \dots, v_N) \in \mathbb{C}^N$, $v_{k+1} = \dots = v_N = 0$ implies $(vA)_{k+1} = \dots = (vA)_N = 0$.*

Example 3.2. For the flag manifolds of the classical groups (see, for example, [7])

$$\begin{aligned} G/K &= SU(d)/S(U(d_1) \times \cdots \times U(d_s)) \quad (d = d_1 + \cdots + d_s, s \geq 1): \\ G/K &= Sp(d)/U(d_1) \times \cdots \times U(d_s) \times Sp(l) \\ G/K &= SO(2d+1)/U(d_1) \times \cdots \times U(d_s) \times SO(2l+1) \\ G/K &= SO(2d)/U(d_1) \times \cdots \times U(d_s) \times SO(2l) \\ &\quad (d = d_1 + \cdots + d_s + l, s, l \geq 0, l \neq 1) \end{aligned}$$

it is easy to see that a minor Δ_k is admissible if and only if $k = d_1 + \cdots + d_j$, for some $j = 1, \dots, s-1$ (resp. $j = 1, \dots, s$) in the case $G = SU(d)$ (resp. in all the other cases).

We have the following:

Theorem 3.3. ([2], Proposition 8.2) Let $F = G/K$, $G = SU(d), Sp(d), SO(2d), SO(2d+1)$, be a flag manifold represented by a painted Dynkin diagram endowed with the canonical equipment Π_{can} given in the Appendix and let $\{\alpha_{k_1}, \dots, \alpha_{k_m}\}$, $k_1 < \cdots < k_m$, be the set of black nodes, with associated fundamental weights $\bar{\alpha}_{k_i}$, $i = 1, \dots, m$. Let F be endowed with the G -invariant complex structure determined by $Q = R_M^+$. Then, in the holomorphic coordinates $z = (z_\alpha)_{\alpha \in -Q}$ defined in (9) on the open dense subset F_{reg} , a Kähler potential for the invariant Kähler form ω_ξ , where $\xi = \sum_{j=1}^m c_j \bar{\alpha}_{k_j}$, $c_j > 0$, is

$$z = (z_\alpha)_{\alpha \in -Q} \mapsto \sum_{j=1}^m c_j \ln \Delta_{k_1 + \cdots + k_j} (\overline{^T \exp(Z(z))} \exp(Z(z))) \quad (10)$$

where

$$Z(z) = \sum_{\alpha \in -Q} z_\alpha E_\alpha \quad (11)$$

and $\Delta_{k_1 + \cdots + k_j}$ is the j -th F -admissible minor.

Notice that, by Remark 2.3, the assumptions on the equipment and on Q are not restrictive. In order to prove the main result of this section, Theorem 3.5 below, we need the following:

Lemma 3.4. Let $F = G/K$, $G = SU(d), Sp(d), SO(2d), SO(2d+1)$, be a flag manifold endowed with the complex structure associated to the maximal closed non-symmetric subset $Q = R_M^+$ with respect to the canonical equipment Π_{can} . Let $Z(z) \in \mathfrak{g}^{-Q}$ given by (11). Then, for every $i, j = 1, \dots, d$ we have:

- (a) If the entry $\exp(Z(z))_{ij}$, $i \neq j$, is non-identically zero then the same is true for $Z(z)_{ij}$;
- (b) $\exp(Z(z))_{ii} = 1$.

Proof. Let us first deal with the case $G = SU(d)$. If $\exp(Z(z))_{ij}$, $i \neq j$, is non-identically zero, then there exists $k > 0$ such that $(Z(z)^k)_{ij} \neq 0$. So part (a) will be proved if we show that if $(Z(z)^k)_{ij}$ is non-identically zero then the same is true for $Z(z)_{ij}$. Let us see this by induction on k . For $k = 1$ this is obvious. For $k > 1$, from $(Z(z)^k)_{ij} = \sum_{l=1}^d (Z(z)^{k-1})_{il} Z(z)_{lj} \neq 0$ it follows that there exists $l = 1, \dots, d$ such that $(Z(z)^{k-1})_{il} \neq 0$ and $Z(z)_{lj} \neq 0$. By the inductive assumption, we have $Z(z)_{il} \neq 0$ and $Z(z)_{lj} \neq 0$. But for $Z(z) \in \mathfrak{g}^{-Q}$ we have $Z(z)_{ij} \neq 0 \Leftrightarrow e_i - e_j \in -Q$ (see the Appendix). So we have $e_i - e_l \in -Q$ and $e_l - e_j \in -Q$, and by Definition 2.1 (iii) we have $e_i - e_j = (e_i - e_l) + (e_l - e_j) \in -Q$, which in turn implies $Z(z)_{ij} \neq 0$, as required.

To show (b), we prove that $(Z^k)_{ii} = 0$ for every $k > 0$. For $k = 1$, this follows from $Z(z) = \sum_{\alpha \in -Q} z_\alpha E_\alpha$ and from the definition of E_α given in the Appendix. Let $k > 1$: if in the sum $(Z(z)^k)_{ii} = \sum_{l=1}^d (Z(z)^{k-1})_{il} Z(z)_{li}$ we have $(Z(z)^{k-1})_{il} \neq 0$, $Z(z)_{li} \neq 0$ for some $l \neq i$, it follows by (a) that $Z(z)_{il} \neq 0$, so that $e_i - e_l, e_l - e_i \in -Q$, against Definition 2.1 (ii). So $(Z(z)^k)_{ii} = (Z(z)^{k-1})_{ii} Z(z)_{ii} = 0$ by the case $k = 1$. (Notice that the above arguments hold true for general maximal closed nonsymmetric subset Q).

Let now $G = Sp(d), SO(2d), SO(2d+1)$. By the explicit matrix representation of the root vectors E_α given in the Appendix we see that, under the assumption $Q = R_M^+$, in all these cases the matrices E_α , $\alpha \in -Q$ are block matrices of the kind $\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$, so that we have $\exp(Z(z)) = \begin{pmatrix} \exp(\tilde{Z}(z)) & 0 \\ * & * \end{pmatrix}$, where $\tilde{Z}(z) \in M_d(\mathbb{C})$ is the matrix having z_α in the place ij if $e_i - e_j = \alpha \in -Q$ and 0 elsewhere. Then, by the assumption $i, j = 1, \dots, d$, to conclude the proof of (a), (b) it is enough to show respectively that (a') If the entry $\exp(\tilde{Z}(z))_{ij}$, $i \neq j$, is non-identically zero then the same is true for $\tilde{Z}(z)_{ij}$; (b') $\exp(\tilde{Z}(z))_{ii} = 1$. But, since $\tilde{Z}(z)$ is constructed by using only the roots in $-Q$ of the kind $e_i - e_j$, these two claims are true by the same arguments used for the case $G = SU(d)$. \square

Now we are ready to prove the main result of this section.

Theorem 3.5. *Let F be a flag manifold of classical type. Then the potential given by formula (10) is Calabi's diastasis function. Moreover, $F_{reg} \subset F$ is the domain of maximal extension of Calabi's diastasis function.*

Proof. Without loss of generality we can assume $F = G/K$ is irreducible and hence $G = SU(d), Sp(d), SO(2d), SO(2d+1)$. Recall that Calabi's diastasis function is the (unique determined) potential around a point p such that, in any given system of complex coordinates $z = (z_1, \dots, z_n)$ centered at p , its power series development in z and \bar{z} does not contain the monomials z^J or \bar{z}^J , for any nonzero multiindex $J = (j_1, \dots, j_n) \in (\mathbb{Z}_{\geq 0})^n$ (see [11], [18] and [20]).

Let us denote by the brevity of notation $A = A(z, \bar{z}) = \overline{\exp(Z(z))} \exp(Z(z))$, $\Psi_l(z, \bar{z}) = \Delta_l(A)$ and $\psi_l = \ln \Psi_l$. We are going to prove that, when l is chosen so that Δ_l is an F -admissible minor, then the power series development of ψ_l around $z = 0$ does not contain the monomials z^J , i.e. $\frac{\partial^{|J|} \psi_l}{\partial z^J}(0) = 0$ (the case of the monomials \bar{z}^J is similar and we leave it to the reader).

Since $Z(0)$ is the null matrix, notice that we have $\exp(Z(0)) = I$ and $\Psi_l(0) = 1$. From this and from the fact (easily seen by induction) that

$$\frac{\partial^{|J|} \psi_l}{\partial z^J} = \frac{1}{\Psi_l} \frac{\partial^{|J|} \Psi_l}{\partial z^J} + (\text{terms containing derivatives of } \Psi_l \text{ order } < |J|)$$

one sees that if the power series development of Ψ_l at $z = 0$ does not contain the terms z^J then the same is true for the power series development of ψ_l . So, assume by contradiction that $\frac{\partial^{|J|} \Psi_l}{\partial z^J}(0) \neq 0$ for some $J \neq 0$. Then, by the very definition of determinant $\Psi_l = \Delta_l(A) = \sum_{\sigma \in S_l} \epsilon(\sigma) A_{1\sigma(1)} \cdots A_{l\sigma(l)}$, it is clear that there exist $\sigma \in S_l$ and $K_1, \dots, K_l \in (\mathbb{Z}_{\geq 0})^n$ ($K_i = 0$ is allowed), $K_1 + \dots + K_l = J$, such that $\frac{\partial^{|K_i|} A_{i\sigma(i)}}{\partial z^{K_i}}(0) \neq 0$ for every $i = 1, \dots, l$.

But, since $A_{ij} = \sum_s \overline{\exp(Z(z))}_{si} \exp(Z(z))_{sj}$ and for every $K \in (\mathbb{Z}_{\geq 0})^n$ one has

$$\frac{\partial^{|K|} A_{ij}}{\partial z^K}(0) = \sum_s \overline{\exp(Z(0))}_{si} \frac{\partial^{|K|} \exp(Z(z))_{sj}}{\partial z^K}(0) = \frac{\partial^{|K|} \exp(Z(z))_{ij}}{\partial z^K}(0),$$

we conclude that there exist $\sigma \in S_l$ and $K_1, \dots, K_l \in (\mathbb{Z}_{\geq 0})^n$, $K_1 + \dots + K_l \neq 0$, such that

$$\frac{\partial^{|K_i|} \exp(Z(z))_{i\sigma(i)}}{\partial z^{K_i}}(0) \neq 0 \tag{12}$$

for every $i = 1, \dots, l$.

Now, by Lemma 3.4 (b), if $\sigma(i) = i$, then $\exp(Z(z))_{i\sigma(i)} = 1$, so that $K_i = 0$ in this case. Since $K_1 + \dots + K_l \neq 0$, it must exist $i_1 = 1, \dots, l$ such that $\sigma(i_1) \neq i_1$. Take the cyclic permutation $i_1, i_2 = \sigma(i_1), \dots, i_s = \sigma(i_{s-1}), i_1 = \sigma(i_s)$ starting at i_1 . By (12) we have

$$\exp(Z(z))_{i_1 i_2}, \exp(Z(z))_{i_2 i_3}, \dots, \exp(Z(z))_{i_s i_1} \neq 0. \tag{13}$$

Since $\{i_1, \dots, i_s\} \subseteq \{1, \dots, l\}$ and Δ_l is an admissible minor, as we have noticed in Examples (3.2) we have $l \leq d$ for all the classical groups $G = SU(d), Sp(d), SO(2d), SO(2d+1)$ and then, by Lemma 3.4 (a), conditions (12) imply

$$(Z(z))_{i_1 i_2}, (Z(z))_{i_2 i_3}, \dots, (Z(z))_{i_s i_1} \neq 0.$$

By definition of $Z(z)$ and by the description of the root vectors given in the Appendix, it means that $e_{i_1} - e_{i_2}, e_{i_2} - e_{i_3}, \dots, e_{i_s} - e_{i_1} \in -Q$. By Definition 2.1

(iii) we have $e_{i_2} - e_{i_3} + \cdots + e_{i_s} - e_{i_1} = e_{i_2} - e_{i_1} \in -Q$ which, together with $e_{i_1} - e_{i_2} \in -Q$, contradicts Definition 2.1 (ii).

Finally, in order to prove the last assertion in the statement of the theorem we will show that $D(z, \bar{z}) = \sum_{j=1}^m c_j \ln \Delta_{k_1+\cdots+k_j}(\overline{^T \exp(Z(z))} \exp(Z(z)))$ on \mathbb{C}^n goes to infinity for $z = a\xi = (a_1\xi, \dots, a_n\xi)$, when $\xi \in \mathbb{C}$, $|\xi| \rightarrow \infty$, for every $a \neq 0$.

First, we claim that, for each admissible minor Δ_I , the function $\Delta_I(\overline{^T \exp(Z(z))} \exp(Z(z)))$ is a polynomial in z, \bar{z} . Indeed, under the assumption $Q = R_M^+$ with respect to the canonical equipment, the matrix $Z(z)$ is clearly nilpotent lower triangular in the case $G = SU(d)$, while in the cases $G = Sp(d), SO(2d)$ (resp. $G = SO(2d+1)$) is a block matrix of the kind

$$\begin{pmatrix} \tilde{Z}(z) & 0 \\ * & -{}^T \tilde{Z}(z) \end{pmatrix} \text{ (resp. of the kind } \begin{pmatrix} \tilde{Z}(z) & 0 & 0 \\ * & -{}^T \tilde{Z}(z) & * \\ * & 0 & 0 \end{pmatrix}), \text{ where } \tilde{Z}(z) \text{ is}$$

nilpotent lower triangular, which is easily seen to imply again that $Z(z)$ is nilpotent. So, in all the cases, the entries of $\exp(Z(z))$ are polynomials in z , which clearly implies the claim. Set $P_j(z, \bar{z}) = \Delta_{k_1+\cdots+k_j}(\overline{^T \exp(Z(z))} \exp(Z(z)))$. The diastasis $D = \sum_{j=1}^m c_j \ln P_j = \ln(P_1^{c_1} \cdots P_m^{c_m})$, with $c_j > 0$, does not tend to infinity in the direction $z = a\xi$ if and only if $P_j(a\xi, \bar{a}\bar{\xi})$, $j = 1, \dots, m$ stays bounded, which for a polynomial is possible only when it is constant. This implies that the potential itself is constant along the same direction, i.e. $\phi(\xi) := D(a\xi, \bar{a}\bar{\xi})$ is constant. But this contradicts the fact that D is a potential for a Kähler metric, since we have

$$0 = \frac{\partial^2 \phi}{\partial \xi \partial \bar{\xi}} = \sum_{i,j=1}^n \frac{\partial^2 D}{\partial z_i \partial \bar{z}_j}(a\xi, \bar{a}\bar{\xi}) a_i \bar{a}_j$$

which is not possible if $a \neq 0$. \square

Remark 3.6. We believe that our results generalize to the homogeneous manifolds of the exceptional groups E_i ($i = 6, 7, 8$), F_4 , G_2 (see, for example, Table 4 in [9] for a complete list in terms of painted Dynkin diagrams). In particular, in [12], by using the theory of Jordan triple systems, it was proved that the symmetric spaces of both classical and exceptional groups admit a system of complex coordinates defined on an open dense subset biholomorphic to \mathbb{C}^n .

4. PROOF OF THEOREM 1.1

In order to prove Theorem 1.1 we need the following two lemmata which show how to extend L^2 -holomorphic functions to holomorphic sections of holomorphic line bundles.

Lemma 4.1. *Let M be a compact Hodge manifold of complex dimension n with Kähler form ω and let (L, h) be a Hermitian line bundle such that $\text{Ric}(h) = \omega$.*

Assume that there exists a divisor $Y \subset M$ such that the restriction of L to $M \setminus Y$ is the trivial holomorphic line bundle and let $\sigma : M \setminus Y \rightarrow L$ be a trivializing holomorphic section. Let f be a holomorphic function on $M \setminus Y$ such that

$$\int_{M \setminus Y} |f(x)|^2 h(\sigma(x), \sigma(x)) \frac{\omega^n}{n!} < \infty.$$

Then f extends to a (unique) global holomorphic section, namely there exists $s \in H^0(L)$ such that $s(x) = f(x)\sigma(x)$ for all $x \in M \setminus Y$.

Proof. Choose trivializing holomorphic sections $\tau_\alpha : U_\alpha \rightarrow L$ defined on open and connected open subsets $U_\alpha \subset M$ such that $Y \subset \cup_\alpha U_\alpha$. Let $g_\alpha : U_\alpha \setminus Y \rightarrow \mathbb{C}$ be the holomorphic function such that $\tau_\alpha(x) = g_\alpha(x)\sigma(x)$ for $x \in U_\alpha \setminus Y$. It follows by the assumption that, for each α ,

$$\int_{U_\alpha \setminus Y} \frac{|f(x)|^2}{|g_\alpha(x)|^2} h(\tau_\alpha(x), \tau_\alpha(x)) \frac{\omega^n}{n!} = \int_{U_\alpha \setminus Y} |f(x)|^2 h(\sigma(x), \sigma(x)) \frac{\omega^n}{n!} < \infty.$$

Fix such an α . Thus, the function $\frac{f}{g_\alpha} : U_\alpha \setminus Y$ is a L^2 -bounded holomorphic function on $U_\alpha \setminus Y$. Since the set $Y \cap U_\alpha$ is an analytic subset of U_α it follows by [24, Thm. 5.17, p. 101] that $\frac{f}{g_\alpha}$ extends to a holomorphic function to all of U_α . The desired global holomorphic section $s \in H^0(L)$ is then given by:

$$s(x) = \begin{cases} f(x)\sigma(x) & \text{if } x \in M \setminus Y, \\ \frac{f(x)}{g_\alpha(x)}\tau_\alpha(x) & \text{if } x \in Y \cap U_\alpha. \end{cases}$$

□

Lemma 4.2. *Let N be a (not necessarily compact) complex manifold of dimension m and let Z be an analytic subset of N such that $V = N \setminus Z$ can be equipped with complex coordinates w_1, \dots, w_m . Let f be a holomorphic function on V such that*

$$\int_V |f(x)|^2 d\nu < \infty,$$

where $d\nu := \frac{i^m}{2^m} dw_1 \wedge d\bar{w}_1 \wedge \dots \wedge dw_m \wedge d\bar{w}_m$. Then there exists a $(m, 0)$ -form η on N such that $\eta|_V = f dw_1 \wedge \dots \wedge dw_m$.

Proof. The proof can be obtained by an argument similar to that of the previous lemma using the line bundle of $\Lambda^{m,0}K$ whose sections are holomorphic m -forms on N . □

Another important ingredient in the proof of Theorem 1.1 is the concept of regular quantization. Let (L, h) be a positive Hermitian line bundle over a compact Kähler manifold (M, g) of complex dimension n , such that $\text{Ric}(h) = \omega_g$ as in the introduction. Let $m \geq 1$ be an integer and consider the Kempf distortion function

associated to mg , i.e.

$$T_{mg}(x) = \sum_{j=0}^{d_m} h_m(s_j(x), s_j(x)) \quad (14)$$

where h_m is an hermitian metric on L^m such that $\text{Ric}(h_m) = m\omega_g$ and s_0, \dots, s_{d_m} , $d_m + 1 = \dim H^0(L^m)$ is an orthonormal basis of $H^0(L^m)$ (the space of holomorphic sections of L^m) with respect to the L^2 -scalar product

$$\langle s, t \rangle_m = \int_M h_m(s(x), t(x)) \frac{\omega_g^n(x)}{n!}, \quad s, t \in H^0(L^m). \quad (15)$$

(In the quantum geometric context m^{-1} plays the role of Planck's constant, see e.g. [3]). As suggested by the notation this function depends only on the Kähler metric g and on m and not on the orthonormal basis chosen.

The function $T_{mg}(x)$ admits the following asymptotic expansion (called Tian-Yau-Zelditch expansion or, for short, TYZ expansion)

$$T_{mg}(x) \sim \sum_{j=0}^{\infty} a_j(x) m^{n-j}, \quad (16)$$

where $a_j(x)$, $j = 0, 1, \dots$, are smooth coefficients with $a_0(x) = 1$. More precisely, for any nonnegative integers r, k the following estimate holds:

$$\|T_{mg}(x) - \sum_{j=0}^k a_j(x) m^{n-j}\|_{C^r} \leq C_{k,r} m^{n-k-1}, \quad (17)$$

where $C_{k,r}$ are constants depending on k, r and on the Kähler form ω_g and $\|\cdot\|_{C^r}$ denotes the C^r norm in local coordinates. In [21] Z. Lu, by means of Tian's peak section method, proved that each of the coefficients $a_j(x)$ in (16) is a polynomial of the curvature and its covariant derivatives at x of the metric g which can be found by finitely many algebraic operations. Furthermore, he explicitly computes a_j with $j \leq 3$. The reader is referred to [6] and references therein for details on Kempf distortion function and its link with the TYZ expansion.

Prescribing the values of the coefficients of the TYZ expansion gives rise to interesting elliptic PDEs as shown by Z. Lu and G. Tian [22]. The main result obtained there, is that if the log-term of the Bergman kernel of the unit disk bundle over M defined in the introduction vanishes then $a_k = 0$, for $k > n$ (n being the complex dimension of M). Moreover Z. Lu has conjectured (private communication) that the converse is true, namely if $a_k = 0$, for $k > n$ then the log-term vanishes.

A Kähler manifold (M, g) admits a *regular quantization* if there exists a positive Hermitian line bundle (L, h) as above such that the Kempf distortion $T_{mg}(x)$ is a constant T_{mg} (depending on m) for all non-negative integer $m \geq 1$. One also refers to (L, h) as a regular quantization of (M, g) .

Regular quantizations play a prominent role in the study of Berezin quantization of Kähler manifolds (see [3], [4], [5] and [19] and references therein). From our point of view we are interested in the following result.

Lemma 4.3. *Any geometric quantization (L, h) of a compact, simply-connected and homogeneous Kähler manifold (M, g) is regular.*

Proof. See [3, Theorem 5.1]. □

We are now in the position to prove Theorem 1.1.

Proof of Theorem 1.1. A compact homogeneous Hodge manifold (M, g) is a flag manifold (F, ω) , where $\omega = \omega_\xi$ is an invariant Kähler form given by (8) and ω/π is integral. It follows by Theorem 3.3 that there exists a dense and open subset $U = F_{reg} \subset M$ biholomorphic to \mathbb{C}^n (with n the complex dimension of M) and a global Kähler potential $\Phi : U \rightarrow \mathbb{R}$ for the metric g , i.e. $\omega_g = \frac{i}{2}\partial\bar{\partial}\Phi$. Moreover, since we are assuming that (M, g) is of classical type, it follows by Theorem 3.5 that Φ is Calabi's diastasis function for the metric g and Φ blows up on the points of $Y = M \setminus U$. On the other hand, (M, g) can be Kähler embedded into some complex projective space (see Section 1.2. in [12]), namely there exists a positive integer N and a holomorphic embedding $f : M \rightarrow \mathbb{C}P^N$ such that $f^*g_{FS} = g$, where g_{FS} denotes the Fubini–Study metric of $\mathbb{C}P^N$. Hence, by a theorem of Calabi [11], Y turns out to be the divisor on M given by the pull-back via the embedding f of the hyperplane divisor of $\mathbb{C}P^N$. As U is contractible the restriction of L^* to U is holomorphically trivial, $L^*|_U \cong U \times \mathbb{C}$. Let $\sigma : U \rightarrow L$ be a trivializing holomorphic section of L and let $\tau : U \rightarrow L^*$ be the dual section trivializing L^* , namely $\tau(x)(\sigma(x)) = 1$. Then $h^*(\tau(x), \tau(x)) = (h(\sigma(x), \sigma(x)))^{-1}$ and it follows by (2) that the restriction of the disc bundle D to U is biholomorphic to the Hartogs domain $H \subset \mathbb{C}^{n+1}$ given by:

$$H = \{(z, \lambda) \in \mathbb{C}^{n+1} \mid |\lambda|^2 < h(z), z \in \mathbb{C}^n\}, \quad (18)$$

where $h(z) := h(\sigma(x), \sigma(x))$ and we identify a point $x \in U$ with its coordinates $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. Moreover, the restriction of ∂D to U is identified with

$$\partial H = \{(z, \lambda) \in \mathbb{C}^{n+1} \mid |\lambda|^2 = h(z)\}.$$

By Lemma 4.2 (with $N = D$, $m = n + 1$, $Z = \pi^{-1}(Y)$, $\pi : D \rightarrow M$ the bundle projection and hence $V = H$) the Bergman space \mathcal{B}_D can be identified with the (usual) Bergman space

$$\mathcal{L}^2(H) = \{f \in \text{Hol}(H) \mid \|f\|^2 = \frac{i}{2} \int_H |f(z, \lambda)|^2 d\lambda \wedge d\bar{\lambda} \wedge d\mu < \infty\},$$

where $d\mu := \frac{i^n}{2^n} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$. Let $K(z, \lambda)$ be the reproducing kernel of $\mathcal{L}^2(H)$. In view of (3), we need to show that there exists a non vanishing smooth function $a \in C^\infty(\overline{H})$ such that

$$K(z, \lambda) = a(z, \lambda) \rho(z, \lambda)^{-n-2}, \quad \rho(z, \lambda) = 1 - \frac{|\lambda|^2}{h(z)}. \quad (19)$$

The Hilbert space $\mathcal{L}^2(H)$ admits the Fourier decomposition into irreducible factors with respect to the natural S^1 -action $s(z, \lambda) \mapsto (z, s\lambda)$, $s \in S^1$ and hence we can write

$$\mathcal{L}^2(H) = \bigoplus_{m=0}^{+\infty} \mathcal{L}_m^2(H),$$

where

$$\mathcal{L}_m^2(H) = \{f \in \mathcal{L}^2(H) \mid f(z, s\lambda) = s^m f(z, \lambda), s \in S^1\}.$$

Now, the assumption that $\frac{\omega_g}{\pi} \in c_1(K^*)$ implies that the Einstein constant of g equals 2, i.e.

$$\rho_{\omega_g} = -i\partial\bar{\partial} \log(\det g) = 2\omega_g.$$

By (1), $\frac{\omega_g^n}{n!} = \det(g)d\mu$ and the $\partial\bar{\partial}$ -lemma there exists a holomorphic function φ on U such that $\frac{\omega_g^n}{n!} = e^{\varphi+\bar{\varphi}} h d\mu$.

If $\{s_j^m\}$ is an orthonormal basis of $(H^0(L^m), \langle \cdot, \cdot \rangle_m)$ (given by (15)), let us define functions $f_j(z, \lambda) \in \mathcal{L}_m^2(H)$ by $f_j(z, \lambda) = \lambda^m f_j(z)$ where

$$f_j(z)\sigma_m(z) = \sqrt{\frac{(m+1)}{\pi}} e^{\varphi} s_j^m(z),$$

and σ_m is the trivializing section of L^m given by $\sigma_m = \sigma^{\otimes m}$. Notice that

$$\begin{aligned} \frac{i}{2} \int_H f_j(z, \lambda) \bar{f}_k(z, \lambda) d\lambda \wedge d\bar{\lambda} \wedge d\mu &= \frac{i}{2} \int_{\mathbb{C}^n} \int_{|\lambda|^2 < h(z)} |\lambda|^{2m} f_j(z) \bar{f}_k(z) d\lambda \wedge d\bar{\lambda} \wedge d\mu = \\ &= \frac{\pi}{m+1} \int_{\mathbb{C}^n} h(z)^{m+1} f_j(z) \bar{f}_k(z) d\mu = \frac{\pi}{m+1} \int_{\mathbb{C}^n} e^{-\varphi-\bar{\varphi}} h(z)^m f_j(z) \bar{f}_k(z) \frac{\omega_g^n}{n!} = \\ &= \int_{\mathbb{C}^n} h_m(s_j^m(z), s_k^m(z)) \frac{\omega_g^n}{n!} = \delta_{jk}. \end{aligned} \quad (20)$$

Combining this computation with Lemma 4.1 we see that $\{f_j(z, \lambda)\}_{j=0, \dots, N_m}$ is an orthonormal basis of $(\mathcal{L}_m^2(H), \|\cdot\|)$. Thus

$$\begin{aligned} K_m(z, \lambda) &= \sum_{j=0}^{N_m} |f_j(z, \lambda)|^2 = \left(\frac{|\lambda|^2}{h(z)} \right)^m \sum_{j=0}^{N_m} h(z)^m |f_j(z)|^2 \\ &= \left(\frac{|\lambda|^2}{h(z)} \right)^m \frac{(m+1) e^{\varphi+\bar{\varphi}}}{\pi} T_{mg}(z), \end{aligned}$$

where T_{mg} is Kempf distortion function given by (14). It follows by Lemma 4.3 and by the very definition of Kempf distortion function that

$$T_{mg}(z) = T_{mg} = \frac{h^0(L^m)}{V(M)}, \quad (21)$$

where $V(M) = \int_M \frac{\omega^n}{n!}$. By Riemann–Roch theorem $h^0(L^m)$ is a monic polynomial in m of degree n so it can be written as linear combination of the binomial coefficients $C_k^{m+k} = \frac{(m+k)!}{m!k!}$, namely

$$h^0(L^m) = \sum_{k=0}^n d_k C_k^{m+k}, \quad d_n = n!.$$

Observe that

$$m C_k^{m+k} = (k+1) C_{k+1}^{(m-1)+(k+1)}$$

and

$$\sum_{m=0}^{\infty} C_k^{m+k} x^m = \frac{1}{(1-x)^{k+1}}, \quad 0 < x < 1.$$

Hence

$$\begin{aligned} K(z, \lambda) &= \sum_{m=0}^{\infty} K_m(z, \lambda) = \frac{e^{\varphi+\bar{\varphi}}}{\pi V(M)} \sum_{k=0}^n d_k \sum_{m=0}^{\infty} (m+1) C_k^{m+k} \left(\frac{|\lambda|^2}{h(z)} \right)^m \\ &= \frac{e^{\varphi+\bar{\varphi}}}{\pi V(M)} \sum_{k=0}^n d_k \sum_{m=0}^{\infty} \left((k+1) C_{k+1}^{(m-1)+(k+1)} + C_k^{m+k} \right) \left(\frac{|\lambda|^2}{h(z)} \right)^m \\ &= \frac{e^{\varphi+\bar{\varphi}}}{\pi V(M)} \sum_{k=0}^n d_k \left((k+1) \rho(z, \lambda)^{-k-2} + \rho(z, \lambda)^{-k-1} \right) = a(z, \lambda) \rho(z, \lambda)^{-n-2}, \end{aligned}$$

where

$$a(z, \lambda) = \frac{e^{\varphi+\bar{\varphi}}}{\pi V(M)} \sum_{k=0}^n d_k \left((k+1) \rho(z, \lambda)^{n-k} + \rho(z, \lambda)^{n-k+1} \right).$$

This proves (19) and concludes the proof of the theorem (notice that when $\rho(z, \lambda) \rightarrow 0$ then $a(z, \lambda) \rightarrow \frac{e^{\varphi+\bar{\varphi}}}{\pi V(M)} d_n(n+1) \neq 0$, so a does not vanish on ∂D). \square

APPENDIX A. CLASSICAL LIE ALGEBRAS

We now describe root systems, root vectors and Dynkin diagram for the classical groups $G = SU(d), Sp(d), SO(2d), SO(2d+1)$ following [15, Section III.8]. The diagrams are endowed with the equipment Π_{can} we have used throughout the paper and which we have referred to as the canonical equipment.

Example A.1. If $G = SU(d)$, $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(d, \mathbb{C})$ is the set of matrices with null trace, and let $\mathfrak{h}^{\mathbb{C}}$ be given by the diagonal matrices in $\mathfrak{sl}(d, \mathbb{C})$; for any $H = \text{diag}(h_1, \dots, h_d)$ let $e_i(H) = h_i$: then the root system is $R = \{e_i - e_j \mid i \neq j\}$ and E_{α} , $\alpha = e_i - e_j$, is the matrix E_{ij} having 1 in the ij place and 0 anywhere else.

The Killing form B satisfies $B(X, Y) = 2d \operatorname{tr}(XY)$ for all $X, Y \in \mathfrak{sl}(d, \mathbb{C})$. The canonical basis is

$$\Pi_{can} = \{\alpha_1 = e_1 - e_2, \dots, \alpha_{d-1} = e_{d-1} - e_d\}.$$

The Dynkin diagram, with this equipment, is

$$\overset{\circ}{\alpha_1} - \overset{\circ}{\alpha_2} - \cdots - \overset{\circ}{\alpha_{d-2}} - \overset{\circ}{\alpha_{d-1}}$$

Example A.2. For $G = Sp(d)$, $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sp}(d, \mathbb{C})$ is the set of $2d \times 2d$ block matrices of the kind $\begin{pmatrix} Z_1 & Z_2 \\ Z_3 & -{}^T Z_1 \end{pmatrix}$, where Z_2, Z_3 are symmetric. Let the Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ be given by diagonal matrices $H = \operatorname{diag}(h_1, \dots, h_d, -h_1, \dots, -h_d)$ in $\mathfrak{sp}(d, \mathbb{C})$, and if for any such H we have $e_i(H) = h_i$, $i = 1, \dots, d$, then the root system is $R = \{\pm e_i \pm e_j\}$ (the case $i = j$ is allowed when the signs are equal). The root vector E_{α} is given by

$\begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix}$ if $\alpha = e_i - e_j$, $\begin{pmatrix} 0 & E_{ij} + E_{ji} \\ 0 & 0 \end{pmatrix}$ if $\alpha = e_i + e_j$ and $\begin{pmatrix} 0 & 0 \\ E_{ij} + E_{ji} & 0 \end{pmatrix}$ if $\alpha = -e_i - e_j$ and the Killing form B is given by $B(X, Y) = 2(d+1) \operatorname{tr}(XY)$ for all $X, Y \in \mathfrak{sp}(d, \mathbb{C})$.

The canonical basis is given by

$$\Pi_{can} = \{\alpha_1 = e_1 - e_2, \dots, \alpha_{d-1} = e_{d-1} - e_d, \alpha_d = 2e_d\}.$$

The Dynkin diagram, with this equipment, is

$$\overset{\circ}{\alpha_1} - \overset{\circ}{\alpha_2} - \cdots - \overset{\circ}{\alpha_{d-1}} \Leftarrow \overset{\circ}{\alpha_d}$$

Example A.3. Let $G = SO(2d)$. Here and throughout the paper we identify the complexification $SO(2d, \mathbb{C})$ with the subgroup of $GL(2d, \mathbb{C})$ leaving invariant the quadratic form $z_1 z_{d+1} + \cdots + z_d z_{2d}$. Then $\mathfrak{g}^{\mathbb{C}} = \mathfrak{so}(2d, \mathbb{C})$ is the set of $2d \times 2d$ block matrices of the kind $\begin{pmatrix} Z_1 & Z_2 \\ Z_3 & -{}^T Z_1 \end{pmatrix}$, where Z_2, Z_3 are skew-symmetric. Let the Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ be given by diagonal matrices $H = \operatorname{diag}(h_1, \dots, h_d, -h_1, \dots, -h_d)$ in $\mathfrak{so}(2d, \mathbb{C})$, and if for any such H we have $e_i(H) = h_i$, $i = 1, \dots, d$, then the root system is $R = \{\pm e_i \pm e_j \ (i \neq j)\}$. The root vector E_{α} is given by

$\begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix}$ if $\alpha = e_i - e_j$, $\begin{pmatrix} 0 & E_{ij} - E_{ji} \\ 0 & 0 \end{pmatrix}$ if $\alpha = e_i + e_j$ ($i < j$) and $\begin{pmatrix} 0 & 0 \\ E_{ij} - E_{ji} & 0 \end{pmatrix}$ if $\alpha = -e_i - e_j$ ($i < j$) and the Killing form B is given by $B(X, Y) = 2(d-1) \operatorname{tr}(XY)$ for all $X, Y \in \mathfrak{so}(2d, \mathbb{C})$.

The canonical basis is given by

$$\Pi_{can} = \{\alpha_1 = e_1 - e_2, \dots, \alpha_{d-1} = e_{d-1} - e_d, \alpha_d = e_{d-1} + e_d\}.$$

The Dynkin diagram, with this equipment, is

$$\begin{array}{ccccccc} & & & & \circ_{\alpha_d} & & \\ & & & & | & & \\ \circ_{\alpha_1} & - & \circ_{\alpha_2} & - & \dots & - & \circ_{\alpha_{d-2}} & - & \circ_{\alpha_{d-1}} \end{array}$$

Example A.4. Let $G = SO(2d+1)$. Here and throughout the paper we identify the complexification $SO(2d+1, \mathbb{C})$ with the subgroup of $GL(2d+1, \mathbb{C})$ leaving invariant the quadratic form $2(z_1 z_{d+1} + \dots + z_d z_{2d}) + z_{2d+1}$. Then $\mathfrak{g}^{\mathbb{C}} = \mathfrak{so}(2d+1, \mathbb{C})$ is the set

of $(2d+1) \times (2d+1)$ block matrices of the kind $\begin{pmatrix} Z_1 & Z_2 & u \\ Z_3 & -^T Z_1 & v \\ -^T v & -^T u & 0 \end{pmatrix}$, where Z_2, Z_3

are skew-symmetric and $u, v \in \mathbb{C}^d$. Let the Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ be given by diagonal matrices

$H = \operatorname{diag}(h_1, \dots, h_d, -h_1, \dots, -h_d, 0)$ in $\mathfrak{so}(2d+1, \mathbb{C})$, and if for any such H we have $e_i(H) = h_i$, $i = 1, \dots, d$, then the root system is $R = \{\pm e_i \pm e_j$ ($i \neq j$), $\pm e_i\}$.

The root vector E_{α} is given by

$$\begin{aligned} & \begin{pmatrix} E_{ij} & 0 & 0 \\ 0 & -E_{ji} & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ if } \alpha = e_i - e_j, \quad \begin{pmatrix} 0 & E_{ij} - E_{ji} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ if } \alpha = e_i + e_j \text{ (} i < j \text{),} \\ & \begin{pmatrix} 0 & 0 & 0 \\ E_{ij} - E_{ji} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ if } \alpha = -e_i - e_j \text{ (} i < j \text{),} \quad \begin{pmatrix} 0 & 0 & E_i \\ 0 & 0 & 0 \\ 0 & -^T E_i & 0 \end{pmatrix} \text{ if } \alpha = e_i \text{ and} \\ & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & E_i \\ -^T E_i & 0 & 0 \end{pmatrix} \text{ if } \alpha = -e_i \text{ (being } E_i \text{ the } i\text{-th vector of the canonical basis} \end{aligned}$$

of \mathbb{C}^d) and the Killing form B is given by $B(X, Y) = (2d-1) \operatorname{tr}(XY)$ for all $X, Y \in \mathfrak{so}(2d+1, \mathbb{C})$.

The canonical basis is given by

$$\Pi_{can} = \{\alpha_1 = e_1 - e_2, \dots, \alpha_{d-1} = e_{d-1} - e_d, \alpha_d = e_d\}.$$

The Dynkin diagram, with this equipment, is

$$\alpha_1 \circ - \alpha_2 \circ - \cdots - \alpha_{d-1} \circ \Rightarrow \alpha_d \circ$$

Remark A.5. In each of the above examples, the given Cartan subalgebra is the complexification of the Lie algebra of a maximal torus T in the compact group G (more precisely, for $G = SU(d), Sp(d), SO(2d), SO(2d + 1)$ we have respectively $T = \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_d}), \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_d}, e^{-i\theta_1}, \dots, e^{-i\theta_d}), \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_d}, e^{-i\theta_1}, \dots, e^{-i\theta_d}, 1)$). In general, given a maximal torus T in a compact connected Lie group G , its Lie algebra \mathfrak{t} is a maximal abelian subalgebra of \mathfrak{g} and its complexification $\mathfrak{t}^{\mathbb{C}}$ is a Cartan subalgebra of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$.

REFERENCES

- [1] D. V. Alekseevsky, *Flag manifolds*, Preprint ESI 415 (1997).
- [2] D. V. Alekseevsky, A. M. Perelomov, *Invariant Kaehler-Einstein metrics on compact homogeneous spaces*, *Funct. Anal. Appl.*, 20 (3) (1986) 171-182.
- [3] C. Arezzo, A. Loi, *Quantization of Kähler manifolds and the asymptotic expansion of Tian-Yau-Zelditch*, *J. Geom. Phys.* 47 (2003), 87-99.
- [4] C. Arezzo and A. Loi, *Moment maps, scalar curvature and quantization of Kähler manifolds*, *Comm. Math. Phys.* 246 (2004), 543-549.
- [5] C. Arezzo, A. Loi, F. Zuddas, *On homothetic balanced metrics*, *Ann. Global Anal. Geom.* 41, n. 4 (2012), 473-491.
- [6] C. Arezzo, A. Loi, F. Zuddas, *Szego Kernel, regular quantizations and spherical CR-structures*. *Math. Z.* (2013) 275, 1207-1216.
- [7] A. Arvanitoyeorgos, *Geometry of flag manifolds*, *International Journal of Geometric Methods in Modern Physics* Vol.3, Nos. 5, 6 (2006), 957-974.
- [8] A. L. Besse, *Einstein manifolds*, Springer-Verag (1987).
- [9] M. Bordemann, M. Forger, H. Roemer, *Homogeneous Kaehler manifolds: paving the way towards new supersymmetric sigma models*, *Commun. Math. Phys.* 102, 605-647 (1986).
- [10] L. Boutet de Monvel and J. Sjöstrand, *Sur la singularité des noyaux de Bergman et de Szegő*, *Journées: Equations aux Dérivées Partielles de Rennes (1975)*, Soc. Math. France, Paris, 1976, pp. 123-164. Astérisque, No. 3435.
- [11] E. Calabi, *Isometric Imbedding of Complex Manifolds*, *Ann. of Math.* Vol. 58 No. 1, 1953.
- [12] A. J. Di Scala, H. Hideyuki, A. Loi, *Kähler immersions of homogeneous Kähler manifolds into complex space forms*, *Asian Journal of Mathematics* Vol. 16 No. 3 (2012), 479-488.
- [13] M. Engliš, G. Zhang, *Ramadanov conjecture and line bundles over compact Hermitian symmetric spaces*. *Math. Z.* 264 (2010), no. 4, 901-912.
- [14] C. Fefferman, *The Bergman kernel and biholomorphic mappings of pseudoconvex domains*, *Invent. Math.* 26 (1974), 1-65.
- [15] S. Helgason, *Differential geometry, Lie groups and symmetric spaces*, *Pure and Applied Mathematics*, vol. 80, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1978.

- [16] K. Hirachi: *The second variation of the Bergman kernel of ellipsoids*, Osaka J. Math. 30 (1993), 457-473.
- [17] A. W. Knap, *Lie groups beyond an introduction*, Progress in Mathematics, 140, Birkhaeuser 1996.
- [18] A. Loi, *Calabi's diastasis function for Hermitian symmetric spaces*, Differential Geom. Appl. 24 (2006), 311-319.
- [19] A. Loi, R. Mossa, *Berezin quantization of homogeneous bounded domains*, Geom. Dedicata 161 (2012), 119-128.
- [20] A. Loi, R. Mossa, *The diastatic exponential of a symmetric space*, Math. Z. 268 (2011), no. 3-4, 1057-1068.
- [21] Z. Lu, *On the lower order terms of the asymptotic expansion of Tian-Yau-Zelditch*, Amer. J. Math. **122** (2000), no. 2, 235-273.
- [22] Z. Lu and G. Tian, *The log term of Szegő Kernel*, Duke Math. J. 125 (2004), 351-387.
- [23] N. Nakazawa, *Asymptotic expansion of the Bergman kernel for strictly pseudoconvex complete Reinhardt domains in \mathbb{C}^2* , Proc. Japan Acad. Ser. A Math. Sci. 66 (1990), 39-41.
- [24] T. Ohsawa, *Analysis of several complex variables*, Translations of Mathematical Monographs, Vol. 211 American Mathematical Society, 2002.
- [25] I. P. Ramadanov: *A characterization of the balls in \mathbb{C}^n by means of the Bergman kernel*, C. R. Acad. Bulgare Sci. 34 (1981), 927-929.

(ANDREA LOI) DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI CAGLIARI (ITALY)

E-mail address: `loi@unica.it`

(ROBERTO MOSSA) DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI CAGLIARI (ITALY)

E-mail address: `roberto.mossa@gmail.com`

(FABIO ZUDDAS) DIPARTIMENTO DI MATEMATICA E INFORMATICA, VIA DELLE SCIENZE 206,
UDINE (ITALY)

E-mail address: `fabio.zuddas@uniud.it`