THE LOGARITHMS OF DEHN TWISTS

NARIYA KAWAZUMI

This is a joint work with Yusuke Kuno (Hiroshima) [6].

Let Σ be an oriented connected compact surface of genus $g\,(\geq 1)$ with 1 boundary component. Choose a basepoint $* \in \partial\Sigma$. We denote $\pi := \pi_1(\Sigma, *)$ and $H := H_1(\Sigma; \mathbb{Q})$. The simple loop going around the boundary in the opposite direction defines an element $\zeta \in \pi$.

Any simple closed curve $C \subset \Sigma$ defines the right handed Dehn twist t_C along C as an element of the mapping class group of the surface Σ relative to the boundary $\partial\Sigma$. The classical formula says the action $|t_C|$ of the Dehn twist t_C on the homology group H is given by

$$|t_C| = 1_H - [C] \otimes [C] \in \mathrm{Hom}(H, H),$$

where $[C] \in H$ is the homology class of C with a fixed orientation, and we identify $H \otimes H = \mathrm{Hom}(H, H)$, $Y \otimes Z \mapsto (X \mapsto (X \cdot Y)Z)$, by the Poincaré duality. Our result generalizes this formula to the action of t_C on the completed group ring $\widehat{\mathbb{Q}\pi}$, where the completion is induced by the augmentation ideal $I\pi \subset \mathbb{Q}\pi$.

Massuyeau [10] introduced the notion of a symplectic expansion of the group π, which provides an isomorphism of pairs of complete Hopf algebras

$$\theta : (\widehat{\mathbb{Q}\pi}, \widehat{\mathbb{Q}\langle\zeta\rangle}) \cong (\widehat{T}, \mathbb{Q}[[\omega]]).$$

Here $\widehat{T} := \prod_{m=0}^{\infty} H^{\otimes m}$ is the completed tensor algebra generated by H, and $\omega \in H^{\otimes 2}$ is the symplectic form. Explicit symplectic expansions have been constructed by Kawazumi [5] (over $\mathbb{R}$), Massuyeau [10], Kuno [9] and Bene-Kawazumi-Kuno-Penner [1].

Any mapping class φ on the surface Σ relative to the boundary $\partial\Sigma$ defines an automorphism of the complete Hopf algebra $\widehat{\mathbb{Q}\pi}$. By using the isomorphism θ, we may regard it as an automorphism $T^\theta(\varphi)$ of the complete Hopf algebra $\widehat{T}$, which we call *the total Johnson map* [4]. Our result describes the map $T^\theta(t_C)$ in an explicit way.

We introduce a linear map $N : \widehat{T} \to H \otimes \widehat{T} \subset \widehat{T}$ by $N|_{H^{\otimes 0}} = 0$ and $N(X_1 X_2 \cdots X_m) = \sum_{i=1}^{m} X_i \cdots X_m X_1 \cdots X_{i-1}$, $X_j \in H$. Here and for the rest of this report, we often drop the symbol $\otimes$. Take the logarithm of the symplectic expansion θ, $\ell^\theta := \log\theta : \pi \to H \otimes \widehat{T}$. Then we define a map $L^\theta : \pi \to \mathrm{Hom}(H, \widehat{T})$ by

$$L^\theta(x) := \frac{1}{2}N(\ell^\theta(x)\ell^\theta(x)) = N\theta(\frac{1}{2}(\log x)^2) \in H \otimes \widehat{T} = \mathrm{Hom}(H, \widehat{T})$$

for $x \in \pi$. It is easy to show $L^\theta(x)$ is an invariant of unoriented loops on the surface Σ. Here we identify $H \otimes \widehat{T} = \mathrm{Hom}(H, \widehat{T})$ by the Poincaré duality. Further the space $\mathrm{Hom}(H, \widehat{T})$ is naturally identified with $\mathrm{Der}(\widehat{T})$, the Lie algebra of derivations of the algebra $\widehat{T}$. In particular, $L^\theta(x)$ is regarded as a derivation of $\widehat{T}$, so that we may define an algebra automorphism of $\widehat{T}$ by taking the exponential $e^{-L^\theta(x)}$.

Theorem 0.1. *For any symplectic expansion θ and a simple closed curve $C \subset \Sigma$, we have*

$$T^\theta(t_C) = e^{-L^\theta(C)}$$

as algebra automorphisms of the algebra $\widehat{T}$. In other words, the invariant $-L^\theta(C)$ is **the logarithm of the Dehn twist** t_C.

The degree 0 part of this formula is exactly the classical formula stated above. As a corollary of the theorem, the action of the Dehn twist t_C on N_k, the k-th nilpotent quotient of π, depends only on the conjugacy class of a based loop representing C in N_k. Moreover, using the exponential $e^{-L^\theta(C)}$, we can define the *Dehn twist along a not-necessarily-simple closed curve* C as an automorphism of the completed group ring $\widehat{\mathbb{Q}\pi}$. It would be intersting if one could realize this automorphism in a geometric context.

The key to the proof of the theorem is a geometric interpretation of symplectic derivations of the algebra $\widehat{T}$. Under the identification $\widehat{T}_1 = \mathrm{Der}(\widehat{T})$, the subspace $N(\widehat{T}_1)$ is exactly equal to the Lie algebra of symplectic derivations, $\mathrm{Der}_\omega(\widehat{T}) = \{D \in \mathrm{Der}(\widehat{T}); D\omega = 0\}$. Kontsevich's "associative" Lie algebra [8] is a Lie subalgebra of $\mathrm{Der}_\omega(\widehat{T})$. Let $\mathbb{Q}\hat{\pi}$ be the Goldman Lie algebra of the surface Σ [2]. We have a natural homomorphism of Lie algebras

$$\sigma : \mathbb{Q}\hat{\pi} \to \mathrm{Der}(\mathbb{Q}\pi).$$

In general, let M be a d-dimensional oriented C^∞ manifold, and $*$ a basepoint on M. Then we can construct a natural map $H_i(L(M \setminus \{*\})) \otimes H_j(\Omega(M, *)) \to H_{i+j+2-d}(\Omega(M, *))$ in a similar way to [3], where $\Omega(M, *) = \mathrm{Map}((S^1, 0), (M, *))$ and $L(M \setminus \{*\}) = \mathrm{Map}(S^1, M \setminus \{*\})$.

For any symplectic expansion θ, we define a map

$$-\lambda_\theta : \mathbb{Q}\hat{\pi} \to N(\widehat{T}_1) = \mathrm{Der}_\omega(\widehat{T}), \quad x \mapsto -N\theta(x).$$

Theorem 0.2. *The diagram*

$$\begin{array}{ccc} \mathbb{Q}\hat{\pi} \times \mathbb{Q}\pi & \xrightarrow{\sigma} & \mathbb{Q}\pi \\ {\scriptstyle -\lambda_\theta \times \theta}\downarrow & & \downarrow{\scriptstyle \theta} \\ \mathrm{Der}_\omega(\widehat{T}) \times \widehat{T} & \longrightarrow & \widehat{T}, \end{array}$$

where the bottom horizontal arrow means the derivation, commutes.

Let $\{\alpha_i, \beta_i\} \subset \pi$ be a symplectic generating system. The Dehn twist along α_1 satisfies $t_{\alpha_1}(\alpha_1) = \alpha_1$ and $t_{\alpha_1}(\beta_1) = \beta_1\alpha_1$. Hence the "logarithm" $\log(t_{\alpha_1})$ should satisfy $\log(t_{\alpha_1})(\beta_1) = \beta_1 \log \alpha_1$. On the other hand, we have $\sigma({\alpha_1}^n)(\beta_1) = n\beta_1{\alpha_1}^n$ for any $n \geq 0$, so that $\sigma(f(\alpha_1))(\beta_1) = \beta_1\alpha_1 f'(\alpha_1)$ for any formal power series $f(x)$ in $x - 1$. If $f(x)$ satisfies $xf'(x) = \log(x)$ and $f(1) = 0$, then it must be $\frac{1}{2}(\log x)^2$. This is the reason why the logarithm of $T^\theta(t_C)$ equals $-L^\theta(C) = -N\theta(\frac{1}{2}(\log x)^2)$, where C is represented by $x \in \pi$.

The map $-\lambda_\theta : \mathbb{Q}\hat{\pi} \to \mathrm{Der}_\omega(\widehat{T})$ is a Lie algebra homomorphism. Using this homomorphism, we can compute the center of the Goldman Lie algebra of an oriented surface of infinite genus [7].

References

1. A. J. Bene, N. Kawazumi, Y. Kuno and R. C. Penner, in preparation.
2. W. Goldman, *Invariant functions on Lie groups and Hamiltonian flows of surface groups representations*, Invent. math. 85 (1986) 263–302.
3. M. Chas and D. Sullivan, *String topology*, preprint, arXiv: math.GT/9911159.
4. N. Kawazumi, *Cohomological aspects of Magnus expansions*, preprint, arXiv: math.GT/0505497.
5. ———, *Harmonic Magnus Expansion on the Universal Family of Riemann Surfaces*, preprint, arXiv: math.GT/0603158.
6. N. Kawazumi and Y. Kuno, *The logarithms of Dehn twists*, preprint, arXiv: 1008.5017.
7. ———, *The Chas-Sullivan conjecture for a surface of infinite genus*, preprint, arXiv: 1009.4985.
8. M. Kontsevich, *Formal (non)-commutative symplectic geometry*, in: *The Gel'fand Mathematical Seminars, 1990-1992*, Birkhäuser, Boston, 1993, 173–187.
9. Y. Kuno, *A combinatorial construction of symplectic expansions*, preprint, arXiv: 1009.2219.
10. G. Massuyeau, *Infinitesimal Morita homomorphisms and the tree-level of the LMO invariant*, preprint, arXiv: 0809.4629.

Department of Mathematical Sciences, University of Tokyo, Komaba, Tokyo 153-8914, Japan

E-mail address: kawazumi@ms.u-tokyo.ac.jp