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Objective Knowledge and Rational Belief**

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Logic Group
Preprint Series
No. 55
September 1990



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1980 Mathematical Subject Classification: 03A05, 03B45

1987 Computing Review Classification: C.2.4, I.2.3

Key words and phrases: Logics of knowledge & belief, distributed systems

September, 1990

*The investigations were supported by the Foundation for Philosophical Research (SWON) which is subsidized by the Netherlands Organization for Scientific Research (NWO).

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Abstract In this paper, we study the logical relations between different notions of knowledge and belief by means of generalizations of the usual Kripke models for epistemic logic. We argue that the obtained generalized Kripke models might be useful for carefully distinguishing the many different notions of knowledge and belief. We identify the (highly idealized) notions of objective knowledge and rational (introspective) belief, which correspond with fairly standard notions of knowledge and belief, and propose a system **OKRIB** for combining both notions which differs essentially from some other such proposals found in the literature. We also consider some other notions of knowledge and belief, and study how they relate to objective knowledge and rational belief.

1. Introduction

Recently, there has been a considerable increase in the amount of theory about the logic of knowledge and belief (or epistemic logic), mainly through the impulses of researchers in the field of Artificial Intelligence and Computer Science. One of the main topics of discussion is the problem of *logical omniscience*, which consists, roughly speaking, of the fact that according to the traditional (possible world) semantics an agent knows/believes the logical consequences (including all tautologies) of anything he knows/believes. This property does not seem to be appropriate for resource-bounded agents, which have only limited reasoning powers.

On account of this problem of logical omniscience various authors have made a distinction between (logically closed) *implicit* and (not necessarily logically closed) *explicit* knowledge/belief. In the recent work on epistemic logic much attention is paid to the study of

different notions of explicit knowledge/belief and their relationship to implicit knowledge/belief. (See Halpern (1986) and Fagin & Halpern (1988).)

In this paper, the central topic is not the problem of logical omniscience but rather the relation between knowledge and belief. (The problem of logical omniscience will be treated in a sequel of this paper.) Although this relation has already been studied by philosophical logicians and by researchers in AI, we feel there are still a number of unclarified issues here. Since there are a number of different notions of knowledge and belief, an adequate study of the relation between knowledge and belief should first make clear of which notions the relation is studied. Therefore the notions whose interactions are studied in this paper are first treated separately. We concentrate on rather idealized notions of knowledge and belief which we call *objective knowledge* and *rational belief*, respectively. However, some other notions of knowledge and belief will also be considered. (All of these notions will be closed under logical consequence, though.)

We essentially use a possible worlds semantics, where the possible worlds are assumed to be complete and consistent. (In the already mentioned sequel of this paper, we allow partial and inconsistent worlds in order to capture beliefs which are not necessarily consistent or logically closed.) Similar approaches have already been discussed by several authors. However, our approach not only differs from these earlier ones in the details of the analysis, but also in the way the semantics is given an intuitive justification. Aspects of the agent's information and belief states are explicitly represented in models intimately related to the traditional Kripke models.

The remainder of this paper is organized as follows: In section 2 we define (generalized) functional Kripke models, sketch the relation between these models and traditional Kripke models, and give some arguments for preferring (generalized) functional Kripke models. The semantics of objective knowledge and rational belief are given in sections 3 and 4, respectively. In section 5 we combine both notions. Section 6 contains a comparison with some other opinions about the interaction of knowledge and belief and a discussion of some alternative notions of knowledge, and in section 7 we discuss alternative (more or less probabilistic) notions of belief. We end with a conclusion.

2. Functional Kripke models

Our basic language \mathcal{L} is built up from a set PL of proposition letters (p, q, r,...) and the logical constants \neg and \wedge . We will further use the constants \vee , \supset and \equiv , which are defined as usual.

Let for all $i \in \{1, 2, \dots, n\}$ $[i]$ be a modal operator. Then $\mathcal{L}_{[1][2]..[n]}$ is the smallest extension of \mathcal{L} such that if $1 \leq i \leq n$ and $\phi, \psi \in \mathcal{L}_{[1][2]..[n]}$, then $\phi \wedge \psi$, $\neg\phi$, and $[i]\phi \in \mathcal{L}_{[1][2]..[n]}$.

DEFINITION 2.1

- (i) A *frame* for $\mathcal{L}_{[1][2]..[n]}$ is a tuple $F = \langle W, R_{[1]}, R_{[2]}, \dots, R_{[n]} \rangle$, where $R_{[i]} \subseteq W \times W$.
- (ii) A *model* for $\mathcal{L}_{[1][2]..[n]}$ is a tuple $M = \langle w_0, W, R_{[1]}, R_{[2]}, \dots, R_{[n]}, \models \rangle$, where $w_0 \in W$, $F = \langle W, R_{[1]}, R_{[2]}, \dots, R_{[n]} \rangle$ is a frame for $\mathcal{L}_{[1][2]..[n]}$ and $\models : W \times \mathcal{L}_{[1][2]..[n]} \rightarrow \mathbf{2}$ such that $\models(w, \cdot)$ satisfies the standard propositional clauses and $\models(w, [i]\phi) = 1 \Leftrightarrow \forall w' (wR_{[i]}w' \Rightarrow \models(w', \phi) = 1)$. We say that M is *based on* F and w_0 is called M 's *designated world*.
- (iii) A *functional frame* for $\mathcal{L}_{[1][2]..[n]}$ is a tuple $F = \langle W, \sigma_{[1]}, \sigma_{[2]}, \dots, \sigma_{[n]} \rangle$, where $\sigma_{[i]} : W \rightarrow \wp W$.
- (iv) A *functional model* for $\mathcal{L}_{[1][2]..[n]}$ is a tuple $M = \langle w_0, W, \sigma_{[1]}, \sigma_{[2]}, \dots, \sigma_{[n]}, \models \rangle$, where $w_0 \in W$, $F = \langle W, \sigma_{[1]}, \sigma_{[2]}, \dots, \sigma_{[n]} \rangle$ is a functional frame for $\mathcal{L}_{[1][2]..[n]}$ and $\models : W \times \mathcal{L}_{[1][2]..[n]} \rightarrow \mathbf{2}$ such that $\models(w, \cdot)$ satisfies the standard propositional clauses and $\models(w, [i]\phi) = 1 \Leftrightarrow \forall w' \in \sigma_{[i]}(w) \models(w', \phi) = 1$. We say that M is *based on* F and w_0 is called M 's *designated world*.

We often write $w \models \phi$ for $\models(w, \phi) = 1$. If M is a (functional) model with valuation function \models and designated world w_0 , then $M \models \phi$ iff $w_0 \models \phi$. If F is a (functional) frame, then $F \models \phi$ iff for every (functional) Kripke model M based on F $M \models \phi$. Notice that with every frame we may associate an equivalent functional frame by defining $\sigma_{[i]}(w) = \{w' \mid wR_{[i]}w'\}$ and with every functional frame we may associate an equivalent frame by defining $wR_{[i]}w'$ iff $w' \in \sigma_{[i]}(w)$. In fact, we have a 1-1 correspondence between frames/models and functional frames/models. Furthermore, the best known simple properties of the accessibility relations of a traditional Kripke frame correspond with simple properties of the functions in the functional frame associated with the traditional Kripke frame:

PROPOSITION 2.2 Let $R \subseteq W \times W$ and let $\sigma : W \rightarrow \wp W$ be given by $\sigma(w) = \{w' \mid wRw'\}$.

- (i) R is reflexive iff $\forall w \in W w \in \sigma(w)$.
- (ii) R is transitive iff $\forall w, w' \in W (w \in \sigma(w') \Rightarrow \sigma(w) \subseteq \sigma(w'))$.
- (iii) R is euclidean iff $\forall w, w', w'' \in W (w', w'' \in \sigma(w) \Rightarrow w'' \in \sigma(w'))$.
- (iv) R is transitive and euclidean iff $\forall w, w' \in W (w \in \sigma(w') \Rightarrow \sigma(w) = \sigma(w'))$.

Proof. Trivial.

The reason for introducing functional Kripke models is that they form a bridge between the traditional Kripke models and the general kind of models we propose for epistemic logic:

DEFINITION 2.3 A *generalized functional model* for $\mathcal{L}_{[1][2]..[n]}$ is a tuple $M = \langle w_0, W, \Sigma_{[1]}, \Sigma_{[2]}, \dots, \Sigma_{[n]}, \sigma_{[1]}, \sigma_{[2]}, \dots, \sigma_{[n]}, F_{[1]}, F_{[2]}, \dots, F_{[n]}, \models \rangle$, where $w_0 \in W$, $\Sigma_{[i]}$ is a non-empty set, $\sigma_{[i]} : W \rightarrow \Sigma_{[i]}$, $F_{[i]}$ is a set of functions with domain $\Sigma_{[i]}$, and $\models : W \times \mathcal{L}_{[1][2]..[n]} \rightarrow 2$ such that $\models(w, \cdot)$ satisfies the standard propositional clauses and $\models(w, [i]\varphi)$ functionally depends on $\models(w, \{\varphi\})$, $\sigma_{[i]}$, w , and (a subset F of) $F_{[i]}$.

$\Sigma_{[i]}$ denotes a set of epistemic states, $\sigma_{[i]}$ associates with every world an element of $\Sigma_{[i]}$, and for every epistemic state $S \in \Sigma_{[i]}$ and $f \in F_{[i]}$ $f(S)$ denotes an aspect of the epistemic state S . Elements of $F_{[i]}$ are called *information extracting functions* or *projection functions*, since they extract information out of an epistemic state by "projecting" the epistemic state onto some aspect of the state. An example of a projection function is the function $\|\cdot\| : \Sigma_{[i]} \rightarrow \wp W$ given by $\|S\| = \{w \in W \mid w \text{ is possible according to } S\}$. Another example is the function Π such that $\Pi(S)$ yields a probability measure on $\wp\|S\|$ giving the probabilities which according to S have to be assigned to the (sets of) possible worlds. In general, we will only exhibit those projection functions which are mentioned in the valuation clauses of the modalities or which are used to single out a particular class of generalized functional models.

Generalized functional models generalize functional models in two respects: (1) epistemic states occur as atomic entities in the generalized models; they are no longer (indirectly) represented by means of sets of possible worlds and (2) in the generalized models modal operators may have nonstandard valuation clauses. Nevertheless, a class of generalized functional models will often turn out to be equivalent to a class of functional models which correspond with traditional Kripke models, so one can often invoke well-known results and methods to obtain soundness and completeness results.

In the following section, an example of the use of non-standard valuation clauses will be given. Our justification for the explicit representation of epistemic states in the generalized functional models is that it helps to systematize the study of the relations between various epistemic modalities. For example, the two central notions in this paper - objective knowledge and rational belief - refer to different epistemic states, and are therefore not as strongly related as some authors seem to think, whereas some (strongly related) notions of knowledge and belief might be considered to refer to different aspects of the same epistemic state. (Cf. sections 5 and 6.) In general, we believe the generalized functional models to be useful in bringing the study of epistemic logic to a higher level of 'informal rigour' (in the sense of Kreisel).

3. Objective knowledge

Let K_α be the operator with the following intuitive meaning:

$K_\alpha\phi$: agent α objectively knows that ϕ , i.e. ϕ is the case (in every world that is possible) given the information available to α .

Since we will restrict ourselves to the one agent case, we will usually omit the subscript. The notion of objective knowledge applies to any agent which is capable of processing information; we do not require the agent to consider the (im)possibility of some worlds. The logic of this notion of objective knowledge will be denoted by **OK**. Closely following the above intuitive explanation of the meaning of the operator K , we propose the following semantics for K :

DEFINITION 3.1 An *information state model* is a generalized functional model $M = \langle w_0, W, \Sigma_K, \kappa, \models \rangle$, where $w \models K\phi \Leftrightarrow \forall w' \in W (\kappa(w') = \kappa(w) \Rightarrow w' \models \phi)$. Elements of Σ_K are called *information states*.

The class of information state models is easily seen to be equivalent to a class of functional models which we call **OK**-models and which correspond with Kripke models for **S5**.

DEFINITION 3.2

- (i) An *OK-frame* is a functional frame $F = \langle W, \kappa \rangle$ for \mathcal{L}_K such that
 1. $\forall w \in W \ w \in \kappa(w)$
 2. $\forall w, w' \in W \ (w \in \kappa(w') \Rightarrow \kappa(w) = \kappa(w'))$
- (ii) An *OK-model* is a functional model based on an **OK**-frame.

PROPOSITION 3.3 Let $\phi \in \mathcal{L}_K$. The following are equivalent:

- (i) For every information state model $M \ M \models \phi$.
- (ii) $\models_{\mathbf{OK}} \phi$.
- (iii) $\models_{\mathbf{S5}} \phi$.

Proof. Let $M = \langle w_0, W, \Sigma_k, \kappa, \models \rangle$ be an information state model. Define $R \subseteq W \times W$ by wRw' iff $\kappa(w) = \kappa(w')$. Then $M' = \langle w_0, W, R, \models \rangle$ is an **S5** Kripke model equivalent to M . On the other hand, with every **S5** Kripke model $M = \langle w_0, W, R, \models \rangle$ one may associate an equivalent **OK**-model $M' = \langle w_0, W, \kappa, \models \rangle$ by $\kappa(w) = \{w' \in W \mid wRw'\}$. Finally, if $M = \langle w_0, W, \kappa, \models \rangle$ is an **OK**-model, then $M' = \langle w_0, W, \emptyset, \kappa, \models \rangle$ is an equivalent information state model.

Notice that the notion of information state does not have to be specified in order to determine the logic of K . The equivalence of information state models and OK -models of course does not show that information states can be equated with sets of possible worlds, since the equivalence depends for example on the fact that the language \mathcal{L}_K does not admit one to exploit probabilistic information which may be contained in the information states.

The above method of identifying the logic of objective knowledge as $S5$ is a generalization of the method by which the logic of the knowledge of processors in a distributed system is identified to be $S5$. In the latter case, one takes the information state of a processor to be identical to the internal state of the processor. Notice that although the introspection axioms ($K\phi \supset KK\phi$ and $\neg K\phi \supset K\neg K\phi$) are valid in OK , we did not assume our agents to have some kind of privileged access to or a perfect awareness of their epistemic attitudes.

4. Rational (introspective) belief

In this section we consider the operator B_α with the following intuitive meaning:

$B_\alpha\phi$: agent α rationally believes (is rationally convinced) that ϕ , i.e. ϕ is valid in every world that is considered possible by α .

Again we usually omit the subscript. Notice that the above notion of belief is a very strong one. In fact, if $B_\alpha\phi$, then α will be inclined to express his belief by saying that he *knows* that ϕ . If an agent says that he (only) *believes* that ϕ , then most likely a weaker notion of belief is used. Some possible weaker notions are treated in section 7.

Let $\beta(w)$ be the rational belief state (of an agent α) in w and remember that for any epistemic state S the set of worlds which are possible according to S is denoted by $\|S\|$. Hence $\|\beta(w)\|$ denotes the set of worlds which are considered possible (by α) in w . Rational belief states are assumed to satisfy the following two conditions:

CONSISTENCY: For every rational belief state S $\|S\| \neq \emptyset$.

INTROSPECTION: For every rational belief state S ($w \in \|S\| \Rightarrow \beta(w) = S$).

The consistency-condition derives from the assumed rationality of the agent: a rational agent will not adhere to an inconsistent set of beliefs, but will revise this set to a consistent one. The introspection-condition derives from the assumption that a rational agent is fully aware of his own belief state. Hence in every world he considers to be a possible candidate of the actual world he must have the same belief state. A rational agent is not infallible, hence we do *not*

require $w \in \|\beta(w)\|$. Some other conditions on rational belief states (concerning the relation between rational belief states and information states) are treated in the following section.

Notice that we do not equate a rational agent's belief state S with $\|S\|$, since we do not want to exclude the possibility that the agent has e.g. beliefs concerning the relative likelihood of subclasses of $\|S\|$. **RIB**, the logic of rational introspective belief has the following semantics:

DEFINITION 4.1 An *belief state model* is a generalized functional model $M = \langle w_0, W, \Sigma_B, \beta, \models \rangle$, which satisfies

1. $\forall w \in W \ \|\beta(w)\| \neq \emptyset$
2. $\forall w, w' \in W \ (w \in \|\beta(w')\| \Rightarrow \beta(w) = \beta(w'))$
3. $w \models B\phi \Leftrightarrow \forall w' \in \|\beta(w)\| \ w' \models \phi$.

Elements of Σ_B are called *belief states*.

Just as in the case of the information state models, the class of belief state models is equivalent to a class of functional models which correspond with very familiar Kripke models. These functional models are called **RIB**-models and they correspond with Kripke models for **KD45**.

DEFINITION 4.2

- (i) A **RIB**-frame is a functional frame $F = \langle W, \beta \rangle$ for \mathcal{L}_B such that
 1. $\forall w \in W \ \beta(w) \neq \emptyset$
 2. $\forall w, w' \in W \ (w \in \beta(w') \Rightarrow \beta(w) = \beta(w'))$.
- (ii) A **RIB**-model is a functional model based on a **RIB**-frame.

PROPOSITION 4.3 Let $\phi \in \mathcal{L}_B$. The following are equivalent:

- (i) For every belief state model $M \ M \models \phi$.
- (ii) $\models_{\mathbf{RIB}} \phi$.
- (iii) $\models_{\mathbf{KD45}} \phi$.

Proof. Similar to 3.3.

5. Combining knowledge and belief

The information objectively available to a rational introspective agent surely contains all information about his own rational belief state. Hence if a world w' is possible given the information available to an agent α in world w , then α 's rational belief state in w' is identical to that in w . Further, if a world w is a member of $\|S\|$, for some rational belief state of agent α ,

then it is reasonable to assume that the set of worlds which are possible according to α 's information state *in* w is a subset of $\|\Sigma\|$. (A world which is considered(!) to be objectively possible from a world believed to be a possible candidate for the actual world should itself be believed to be a possible candidate for the actual world.) Hence we arrive at the following additional assumptions on the relation between information classes and rational belief classes:

OBJECTIVITY: $w \in \|\kappa(w')\| \Rightarrow \beta(w) = \beta(w')$.

BELIEVED SOUNDNESS: $w \in \|\beta(w')\| \Rightarrow \|\kappa(w)\| \subseteq \|\beta(w')\|$.

Another interesting assumption is the following:

BELIEVED COMPLETENESS: $w \in \|\beta(w')\| \Rightarrow \|\kappa(w)\| \supseteq \|\beta(w')\|$.

This assumption might be appropriate for an agent who is very confident in the sense that he believes to have fully exploited the information available to him. However, in general we will not assume believed completeness. The semantics for **OK&(O)RIB**, the system in which objective knowledge and (overconfident) rational introspective belief are combined, becomes:

DEFINITION 5.1 An *information & belief state model* is a generalized functional model $M = \langle w_0, W, \Sigma_K, \Sigma_B, \kappa, \beta, \|\cdot\|_K, \|\cdot\|_B, \models \rangle$ such that $\langle w_0, W, \Sigma_K, \kappa, \models \rangle$ is an information state model, $\langle w_0, W, \Sigma_B, \beta, \models \rangle$ is a belief state model, $\|\cdot\|_K : \Sigma_K \rightarrow \wp W$, $\|\cdot\|_B : \Sigma_B \rightarrow \wp W$, and for all $w, w' \in W$: $w \in \|\kappa(w')\| \Rightarrow \beta(w) = \beta(w')$ and $w \in \|\beta(w')\| \Rightarrow \|\kappa(w)\| \subseteq \|\beta(w')\|$. An *information & overconfident belief state model* is an information & belief model which additionally satisfies $w \in \|\beta(w')\| \Rightarrow \|\kappa(w)\| \supseteq \|\beta(w')\|$.

DEFINITION 5.2

- (i) An **OK&RIB-frame** is a tuple $F = \langle W, \kappa, \beta \rangle$, where
 1. $\langle W, \kappa \rangle$ is an **OK-frame**
 2. $\langle W, \beta \rangle$ is a **RIB-frame**
 3. $\forall w, w' \in W (w \in \kappa(w') \Rightarrow \beta(w) = \beta(w'))$
 4. $\forall w, w' \in W (w \in \beta(w') \Rightarrow \kappa(w) \subseteq \beta(w'))$.
- (ii) An **OK&RIB-model** is a functional model based on an **OK&RIB-frame**.
- (iii) An **OK&ORIB-frame (model)** is an **OK&RIB-frame (model)** which additionally satisfies $\forall w, w' \in W (w \in \beta(w') \Rightarrow \kappa(w) \supseteq \beta(w'))$.

OK&(O)RIB-models are equivalent to information & (overconfident) belief state models and just as in the case of the **OK-** and the **RIB-models**, the **OK&(O)RIB-models** correspond to a relatively easy characterizable class of Kripke models:

PROPOSITION 5.3 Let $\varphi \in \mathcal{L}_{KB}$. The following are equivalent:

- (i) For every information & belief state model M $M \models \varphi$.
- (ii) $\models_{\mathbf{OK\&RIB}} \varphi$.
- (iii) φ is valid in every model for \mathcal{L}_{KB} such that R_K is an equivalence relation, R_B is serial, transitive, and euclidean, and $R_B R_K \subseteq R_B \supseteq R_K R_B$. (Since R_K is reflexive, the inclusions may be replaced by identities.)

PROPOSITION 5.4 Let $\varphi \in \mathcal{L}_{KB}$. The following are equivalent:

- (i) For every information & overconfident belief state model M $M \models \varphi$.
- (ii) $\models_{\mathbf{OK\&ORIB}} \varphi$.
- (iii) φ is valid in every model for \mathcal{L}_{KB} such that R_K is an equivalence relation, R_B is serial, transitive, and euclidean, $R_B R_K \subseteq R_B \supseteq R_K R_B$, and $\forall xyz (xR_B y \ \& \ xR_B z \Rightarrow yR_K z)$.

(The proofs of 5.3 and 5.4 are similar to that of 3.3)

LEMMA 5.5

- (i) $B\varphi \supset BK\varphi$ corresponds to $R_B R_K \subseteq R_B$.
- (ii) $B\varphi \supset KB\varphi$ corresponds to $R_K R_B \subseteq R_B$.
- (iii) $\neg B\varphi \supset B\neg K\varphi$ corresponds to $\forall xyz (xR_B y \ \& \ xR_B z \Rightarrow yR_K z)$.

Proof. (i) and (ii) are well-known. (iii) is proved in the standard way:

Consider a model M for \mathcal{L}_{KB} which satisfies $\forall xyz (xR_B y \ \& \ xR_B z \Rightarrow xR_K z)$ and assume $w_0 \models \neg B\varphi$. Then there exists a w such that $w_0 R_B w$ and $w \models \neg\varphi$. Since $w_0 R_B w'$ implies $w' R_K w$, we have $w_0 \models B\neg K\varphi$.

On the other hand, consider a frame $F = \langle W, R_K, R_B \rangle$ for \mathcal{L}_{KB} which does not satisfy $\forall xyz (xR_B y \ \& \ xR_B z \Rightarrow yR_K z)$. Then for some $w, w', w'' \in W$ we have $w R_B w'$, $w R_B w''$ and $\neg w R_K w''$. Let M be a functional model based on F such that w is M 's designated world and $\models(x, p) = 1 \Leftrightarrow x \in W \setminus \{w''\}$. Then $M \models \neg B\varphi \supset B\neg K\varphi$.

DEFINITION 5.6

- (i) **OK&RIB** is the system in \mathcal{L}_{KB} given by the following rules and axioms:
Modus Ponens, Necessitation for K and B , all propositional tautologies in \mathcal{L}_{KB} ,
Distribution for K and B , 4 ($\Box\varphi \supset \Box\Box\varphi$) for K and B , 5 ($\neg\Box\varphi \supset \Box\neg\Box\varphi$) for K and B ,
 $K\varphi \supset \varphi$, $B\varphi \supset \neg B\neg\varphi$, $B\varphi \supset BK\varphi$, $B\varphi \supset KB\varphi$.
- (ii) **OK&ORIB** = **OK&RIB** + $\neg B\varphi \supset B\neg K\varphi$.

PROPOSITION 5.7 (i) $\forall \varphi \in \mathcal{L}_{KB} (\vdash_{\mathbf{OK\&RIB}} \varphi \Leftrightarrow \models_{\mathbf{OK\&RIB}} \varphi)$
(ii) $\forall \varphi \in \mathcal{L}_{KB} (\vdash_{\mathbf{OK\&ORIB}} \varphi \Leftrightarrow \models_{\mathbf{OK\&ORIB}} \varphi)$

Proof. Immediate from Lemma 5.5 and other well-known results.

LEMMA 5.8 The following are theorems of **OK&RIB**:

- (i) $KK\phi \equiv K\phi$ (v) $KB\phi \equiv B\phi$
- (ii) $K\neg K\phi \equiv \neg K\phi$ (vi) $K\neg B\phi \equiv \neg B\phi$
- (iii) $BB\phi \equiv B\phi$ (vii) $BK\phi \equiv B\phi$
- (iv) $B\neg B\phi \equiv \neg B\phi$ (viii) $(B\neg\phi \supset B\neg K\phi) \wedge (B\neg K\phi \supset \neg B\phi)$

Proof. Only two cases are not completely trivial:

- (vi) $\neg B\phi \supset K\neg B\phi$: follows from (v) and the **S5**-property $\psi \supset K\neg K\neg\psi$ applied to $\psi = \neg B\phi$.
- (viii) $(B\neg\phi \supset B\neg K\phi)$ follows from (vii) and $K\phi \supset \neg K\neg\phi$;
 $(B\neg K\phi \supset \neg B\phi)$ follows from (vii) and $B\phi \supset \neg B\neg\phi$.

PROPOSITION 5.8

- (i) **OK&RIB** has 12 distinct modalities, viz. \cdot , K , B , $K\neg$, $B\neg$, $B\neg K$, and their negations. The implications among them are pictured in figure 1 below.
- (ii) **OK&ORIB** has 10 distinct modalities, viz. \cdot , K , B , $K\neg$, $B\neg$, and their negations. The implications among them are pictured in figure 2 below.

Proof. Immediate from lemma 5.7.

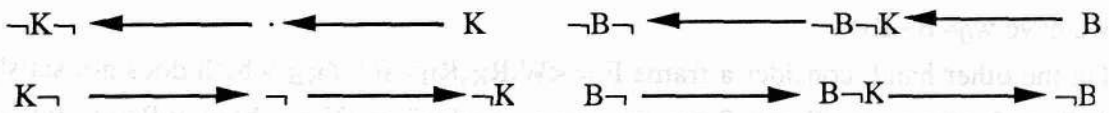


FIGURE 1. Modalities of **OK&RIB**.

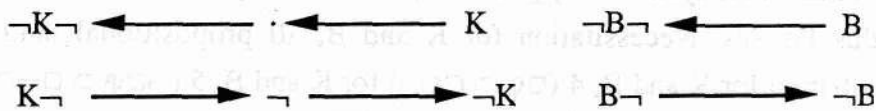


FIGURE 2. Modalities of **OK&ORIB**.

6. Knowledge which implies rational belief

It is no surprise that **OK&RIB** is not consistent with the ideas about the relation between knowledge and belief in Hintikka (1962) the notions of objective knowledge and rational belief are rather different from the originally studied epistemic notions. However, objective knowledge and rational belief do seem to correspond with the nowadays predominant notions of knowledge and belief. Nevertheless, **OK&RIB** differs in a number of respects from the system which was proposed in Kraus & Lehmann (1986) and thoroughly studied from a technical point of view in van der Hoek (1989). This system - let us call it **KL** - is more extensive than **OK&RIB**, since it also treats the notions of common knowledge and common belief. As far as pure knowledge propositions (i.e. propositions containing no other modal operators than K) and pure belief propositions are concerned, **KL** and **OK&RIB** agree. However, the **OK&RIB** theorem $B\phi \supset BK\phi$ is not derivable in **KL**, and adding it to **KL** would lead to the derivability of $B\phi \supset K\phi$.

As already observed by Wolfgang Lenzen (1978, p.80), $B\phi \supset BK\phi$ is only reasonable in case $B_\alpha\phi$ stands for " α is convinced that ϕ " and not in case $B_\alpha\phi$ means something like " α presumes that ϕ ". Hence at first sight the non-derivability of $B\phi \supset BK\phi$ in **KL** might be defended by arguing that **KL** does not intend to capture the notion of conviction. However, since an agent may be cautious and only presume something if he is convinced of it, the fact that $\mathbf{KL} + (B \supset BK) \vdash B\phi \supset K\phi$ indicates that there is something wrong with **KL**. Another indication is provided by the fact that $BK\phi \supset K\phi$ is a theorem of **KL**. This principle contradicts the intuition that you might believe you know that ϕ , while in fact ϕ is not true. In a sense, the principle is self-defeating: Let ψ be the sentence " $BK\phi \supset K\phi$ is not valid". The author of this paper believes to know that ψ . Hence $BK\psi$, where the author is the agent of K and B. But then the validity of $BK\phi \supset K\phi$ implies its invalidity.

KL has $K\phi \supset B\phi$ among its axioms. Deleting this axiom (which is not valid in **OK&RIB**) would block both the derivation of $BK\phi \supset K\phi$ in **KL** and the derivation of $B\phi \supset K\phi$ in $\mathbf{KL} + (B \supset BK)$. However, one frequently seems to use a notion of knowledge which *does* imply belief. For example, in the philosophical literature knowledge is often equated with (justified) true belief. Such a notion of knowledge K^+ is obviously different from our notion of objective knowledge and the justification given in section 3 for taking **S5** as the logic of K does not apply to K^+ . In fact, if one assumes $K^+\phi \supset B\phi$, then the negative introspection principle for K^+ ($\neg K^+\phi \supset K^+\neg K^+\phi$) no longer seems acceptable, since it would make it possible to derive $BK^+\phi \supset K^+\phi$ from the principle $B\neg\psi \supset \neg B\psi$.

Similar arguments are given in Lenzen (1978), where it is argued that the logic of justified true belief is at least as strong as **S4.2** ($= \mathbf{S4} + \neg\Box\neg\Box\phi \supset \Box\neg\Box\neg\phi$) and at most as strong as **S4.4** ($= \mathbf{S4} + \phi \supset (\neg\Box\neg\Box\phi \supset \Box\phi)$). The latter system is claimed to be the logic of

true belief. Indeed it is not difficult to check that if we define in \mathcal{L}_B $K^*\varphi = B\varphi \wedge \varphi$, then $\forall \varphi \in \mathcal{L}_{K^*}$ ($\models_{RIB}\varphi$ iff $\models_{S4.4}\varphi$).

In \mathcal{L}_{KB} we can define another notion of knowledge which implies belief: $K'\varphi = K\varphi \wedge B\varphi$. Below we show that the logic of this notion of rationally believed objective knowledge (**ROK**) lies between **S4.3** and **S4.4**:

DEFINITION 6.1 A *ROK-model* is a Kripke model for $\mathcal{L}_{K'}$ s.t. $R_{K'} = R_1 \cup R_2$, R_1 is an equivalence relation, R_2 is serial, transitive and euclidean and $R_1 R_2 = R_2 = R_2 R_1$.

DEFINITION 6.2 A relation R is called *semi-euclidean* iff $\forall xyz ((xRy \wedge xRz) \rightarrow (yRz \vee zRx))$.

LEMMA 6.3

- (i) If $R = R_1 \cup R_2$, where R_1 is an equivalence relation and R_2 is serial, transitive, euclidean, and $R_1 R_2 = R_2 = R_2 R_1$, then R is reflexive, transitive, and semi-euclidean.
- (ii) If R is reflexive, transitive, and semi-euclidean, then there exists R_1 and R_2 such that $R = R_1 \cup R_2$, R_1 is an equivalence relation and R_2 is serial, transitive and euclidean, and $R_1 R_2 = R_2 = R_2 R_1$.

Proof. (i) Trivial. (ii) Define xR_1y iff $x=y \vee (xRy \wedge \exists z(xRz \wedge \neg zRy))$ & xR_2y iff $\forall z(xRz \rightarrow zRy)$. Then R_1 and R_2 satisfy the requirements of (ii).

LEMMA 6.4 The transitive, semi-euclidean Kripke frames for \mathcal{L}_\square are characterized by the set $S = \{\square\varphi \supset \square\square\varphi, (\varphi \wedge \neg\square\varphi \wedge \square(\varphi \vee \square(\varphi \supset \square\varphi))) \supset \square\neg\square\varphi \mid \varphi \in \mathcal{L}_\square\}$.

Proof. It is easily checked that every formula of S is valid in every transitive, semi-euclidean frame. On the other hand, let $F = \langle W, R \rangle$ be a transitive frame for \mathcal{L}_\square . (It is well-known that there is a countermodel against $\square p \supset \square\square p$ based on any non-transitive frame.) Assume that R is not semi-euclidean. Then there exist $w, w', w'' \in W$ such that wRw' , wRw'' , $\neg w'Rw''$, and $\neg w''Rw$. Let $M = \langle w, W, R, \rangle$ be a functional model such that $\models(x, p) = 1$ iff $x = w$ or xRw or $\neg xRw''$ or $x \neq w''$. Then $M \not\models (p \wedge \neg\square p \wedge \square(p \vee \square(p \supset \square p))) \supset \square\neg\square p$.

PROPOSITION 6.5 Let $\varphi \in \mathcal{L}_{K'}$. The following are equivalent:

- (i) $\models_{ROK}\varphi$.
- (ii) $\models_{KT45'}\varphi$, where $(5') = (\varphi \wedge \neg K'\varphi \wedge K'(\varphi \vee K'(\varphi \supset K'\varphi))) \supset K'\neg K'\varphi$.
- (iii) For every reflexive, transitive, semi-euclidean Kripke model M for $\mathcal{L}_{K'}$ $M \models \varphi$.

Proof. Immediate from 6.3 and 6.4.

PROPOSITION 6.6 **KT45'** is (i) a proper extension of **S4.3** ($= \mathbf{S4} + \Box(\Box\phi \supset \Box\psi) \vee \Box(\Box\psi \supset \Box\phi)$) and (ii) a proper subsystem of **S4.4**.

Proof. (i) If R is semi-euclidean and transitive, then R is connected, i.e. $(xRy \ \& \ xRz) \Rightarrow (yRz$ or $zRy)$. **S4.3** is sound and complete with respect to reflexive transitive connected Kripke models. Hence **KT45'** is an extension of **S4.3**. The following **S4.3** countermodel against (5') shows that the inclusion is proper: $M = \langle w_0, W, R, I \rangle$, with $W = \{w_0, w_1, w_2\}$, $R = W \times W \setminus \{(w_1, w_0), (w_2, w_0), (w_2, w_1)\}$, $I : W \times PL \rightarrow \mathbf{2}$ is given by $I((w_0, p) = I(w_2, p) = 1$ and the value of I is 0 in all other cases.

(ii) **KT45'** is a subsystem of **S4.4**, since every **S4.4** Kripke model is a reflexive transitive semi-euclidean Kripke model. The following **KT45'** countermodel against the characteristic **S4.4** axiom $\phi \supset (\neg K' \neg K' \phi \supset K' \phi)$ shows that the inclusion is proper: $M' = \langle w_0, W, R', I \rangle$, with $R' = W \times W \setminus \{(w_2, w_0), (w_2, w_1)\}$, and W and I as in M of (i).

PROPOSITION 6.7 **ROK** has ten distinct modalities, viz $K', \neg K', K' \neg K', K' \neg K' \neg$, and their negations, with implications among them as pictured below.

Proof. That there are no more modalities follows from $K' \phi \equiv K' K' \phi$ and $\neg K' \neg K' \phi \supset K' \neg K' \neg \phi$. Most of the implications in figure 3 follow from T ($= \Box \phi \supset \phi$). $\neg K' \neg K' \phi \supset K' \neg K' \neg \phi$ follows from T and (5'). Finally, one can easily construct models showing that the implications pictured in figure 3 are proper.

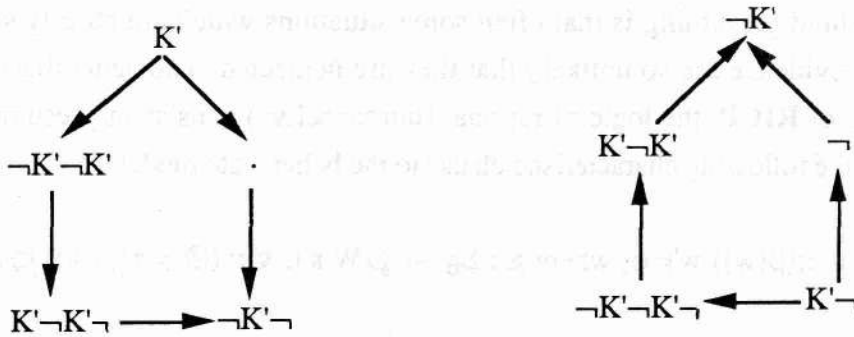


FIGURE 3. Modalities of **ROK**.

LEMMA 6.8

- (i) $\models_{\mathbf{OK\&RIB}} \neg K' \neg K' \phi \equiv B \phi$
- (ii) $\models_{\mathbf{RIB}} \neg K^* \neg K^* \phi \equiv B \phi$

Proof. (i) In **OK&RIB** we have $\neg B \phi \supset K' \neg B \phi$ (by 5.8(vi) and negative introspection for B) and $K' \neg B \phi \supset K' \neg K' \phi$. Hence $\neg B \phi \supset K' \neg K' \phi$. On the other hand, we have $B \phi \supset BK' \phi$ (by $(B \supset BK)$ and positive introspection for B) and $K' \neg K' \phi \supset \neg BK' \phi$ (by applying $K' \neg \psi \supset B \neg \psi$ and $B \neg \psi \supset \neg B \psi$ to $\psi = K' \phi$). Hence $K' \neg K' \phi \supset \neg B \phi$. (ii) Similar.

Wolfgang Lenzen suggested that for every reasonable notion of knowledge K and belief B the equivalence $B\phi \equiv \neg K\neg K\phi$ should hold. Additionally assuming $K\phi \supset \phi$ and $K\phi \supset KK\phi$ he arrived at his already mentioned conclusion that the logic of knowledge is at least as strong as **S4.2**.

7. Weak notions of belief

The notion of rational introspective belief of section 4 is a very strong notion of belief. In this section we will consider some weaker versions of belief which can still be considered rational and introspective. We take serious the remark in section 4 that in general a rational belief state S should not be equated with $\|S\|$ and we consider some modalities which refer to the (relative) likelihood of subclasses of $\|S\|$.

Consider P_α with the following intuitive meaning

$P_\alpha\phi$: agent α presumes ϕ , i.e. ϕ is true in every possible world which α considers to be not too unlikely.

The rationale behind presuming is that often some situations which are strictly speaking not excluded by the evidence are so unlikely that they are neglected. The generalized functional model semantics of **RICP**, the logic of rational (introspective) consistent presumed belief, is given by adding the following characteristic clause to the belief state models:

$$w \models P\phi \Leftrightarrow \forall w' \in \pi(\beta(w)) \ w' \models \phi, \text{ where } \pi : \Sigma_B \rightarrow \wp W \text{ s.t. } \forall w (\emptyset \neq \pi(\beta(w)) \subseteq \|\beta(w)\|)$$

Here $\pi(\beta(w))$ denotes the set of worlds which are not neglected. The requirement $\pi(\beta(w)) \neq \emptyset$ ensures that the presumed belief is consistent. If one drops this requirement, one obtains the semantics of **RIP**, the logic of (introspective) presumed belief. One easily obtains a functional model semantics for **RI(C)P**, which is equivalent to its generalized functional model semantics:

DEFINITION 7.1

(i) A **RIP-frame** is a functional frame $F = \langle W, \beta, \pi \rangle$ for \mathcal{L}_{BP} such that

1. $\forall w \ \pi(w) \subseteq \beta(w)$.
2. $\forall w, w' \in W (\beta(w) = \beta(w') \Rightarrow \pi(w) = \pi(w'))$

- (ii) A **RIP-model** is a functional model based on a **RIP-frame**.
- (iii) A **RICP-model/frame** is an **RIP-model/frame** with the additional assumption that $\pi(w) \neq \emptyset$.

PROPOSITION 7.2 $\forall \varphi \in \mathcal{L}_{\square} (\models_{\mathbf{RIB}} \varphi[\square := B] \Leftrightarrow \models_{\mathbf{RICP}} \varphi[\square := P])$

Proof. Consider a **RIB-frame** $F = \langle W, \beta \rangle$. Then $F' = \langle W, \beta, \beta \rangle$ is a **RICP-frame** such that the valid pure P-formulas in F' correspond with the valid pure B-formulas in F . On the other hand, if $F = \langle W, \beta, \pi \rangle$ is **RICP-frame**, then $F' = \langle W, \pi \rangle$ is a **RIB-frame** such that the valid pure B-formulas in F' correspond with the valid pure P-formulas in F .

Of course the above proposition does not imply that there is no difference between B and P. However, this difference will only show in contexts where these operators are combined with each other or other operators. For example, it is easy to see that (with respect to the obvious semantics) $B(P\varphi \supset \varphi)$ and $P\varphi \supset PK\varphi$ are invalid, whereas $B(B\varphi \supset \varphi)$ and $B\varphi \supset BK\varphi$ are valid.

Next consider the notion L_{α} with the following intuitive meaning:

$L_{\alpha}\varphi$: agent α finds it likely that φ , i.e. φ is true in every world of some subclass of $\|\beta(w)\|$ which is considered to be likely by α .

This notion of likely *belief* should not be confused with the notion of likelihood (also denoted by L) which is formalized in the system **LL** of Halpern & Rabin (1987). **LL** tries to capture the notion of *being* likely in a nonnumerical way. The characteristic clause of the semantics of **RIL**, the logic of rational introspective likely belief, is the following:

$w \models L\varphi \Leftrightarrow \exists A \in \lambda(\beta(w)) \forall w' \in A w' \models \varphi$, where $\lambda : \Sigma_B \rightarrow \wp^*(\wp^*(\|\beta(w)\|))$.
 ($\wp^*(X)$ denotes $\wp(X) \setminus \{\emptyset\}$.)

We have that $\models_{\mathbf{RIL}} \varphi \supset \psi$ implies that $\models_{\mathbf{RIL}} L\varphi \supset L\psi$, hence **RIL** is a classical modal logic (in the sense of Segerberg (1971)). However, **RIL** is not normal and not even regular, since neither $L(\varphi \supset \psi) \supset (L\varphi \supset L\psi)$ nor $(L\varphi \wedge L\psi) \supset L(\varphi \wedge \psi)$ is valid. Hence there is no standard Kripke model semantics or functional model semantics for **RIL**. (There does, of course, exist a neighbourhood model semantics for **RIL**.) From the assumption that only non-empty sets can be considered likely it follows that $L\neg\varphi \supset \neg L\varphi$ is valid. As a consequence of the introspection condition for rational belief states we further have $L(\neg)B\varphi \equiv (\neg)B\varphi$, $L(\neg)L\varphi \equiv (\neg)L\varphi$, and

$B(\neg)L\phi \equiv (\neg)L\phi$. (Hence different powers of L do not denote different levels of likelihood, as is the case in **LL**.)

A plausible reading of $L_\alpha\phi$ is " α 's subjective probability of ϕ is at least x ", for some particular $x \in (0,1]$. An alternative semantics for this notion can be obtained by using the projection function Π which assigns to every belief state $\beta(w)$ a (possibly generalized) probability measure P_w defined on (a σ -algebra/field \mathcal{F}_w of subsets of) $\|\beta(w)\|$. In that case one may define for some $x \in (0,1]$:

$$w \models L\phi \Leftrightarrow P_w(\{w' \in \|\beta(w)\| \mid w' \models \phi\}) \geq x \quad \text{or}$$

$$w \models L\phi \Leftrightarrow \exists A \in \mathcal{F}_w \forall w' \in A (w' \models \phi \text{ and } P_w(A) \geq x).$$

This semantics can easily be extended to interpret a richer formalism of rational subjective probability (**RSP**):

$$w \models P^{\geq x}\phi \Leftrightarrow P_w(\{w' \in \|\beta(w)\| \mid w' \models \phi\}) \geq x.$$

($P^{\leq x}\phi$, $P^x\phi$ and $P^{< x}\phi$ are defined analogously.)

In general, the relation between knowledge/belief and probability is a far from trivial matter. (Consider for example the problem mentioned in Fagin & Halpern (1988) of assigning the appropriate probability spaces.) Next to the notion of rational subjective belief, one has a notion of objective probability, and a 'subjectivized' notion of objective probability (the objective probability given the objective information available to a particular agent), and then some more. All the more reason to look for instruments which allow one to faithfully formalize the multitude of (different combinations of) epistemic notions. The generalized functional models might be considered to be such instruments, although they probably have to be sharpened in order to really cut through.

8 Conclusion

We introduced generalized functional models which generalize Kripke models by explicitly representing epistemic states. Those generalized functional models were used to identify the notions of objective knowledge and rational (introspective) belief and to determine the logical relation between the two notions. Although objective knowledge and rational belief seem to correspond to the predominant notions of (**S5**-)knowledge and (**KD45**-)belief, the system

OK&RIB combining the two notions turned out to be quite different from the well-known system (**KL**) of Kraus and Lehmann. We argued against **KL** by showing that negative introspection for knowledge is incompatible with the principle that knowledge implies belief (assuming beliefs to be consistent). Combining an **S5**-notion of knowledge with an **KD45**-notion of belief thus means dropping the knowledge-implies-belief-principle. Further, we briefly considered some notions of knowledge which imply belief and some weak notions of rational belief, partially as an illustration for the need of something like generalized functional models to keep track of the multitude of epistemic notions.

Acknowledgments

Albert Visser substantially influenced my ideas on the subject of this paper and members of the Amsterdam Working Group on Nonmonotonic Reasoning kindly commented on an early unpolished presentation of these ideas.

The investigations were supported by the Foundation for Philosophical Research (SWON), which is subsidized by the Netherlands Organization for Scientific Research (NWO).

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