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The Łojasiewicz Exponent at Infinity of Non-negative and Non-degenerate Polynomials

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Abstract Let f be a real polynomial, non-negative at infinity with non-compact zero-set. Suppose that f is non-degenerate in the Kushnirenko sense at infinity. In this paper we give a formula for the Łojasiewicz exponent at infinity of f and a formula for the exponent of growth of f in terms of its Newton polyhedron.

Keywords Łojasiewicz exponent · Newton polyhedron · Kushnirenko nondegeneracy

Mathematics Subject Classification 14P10

1 Introduction

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real polynomial, $f(0) = 0$ and $K \subset \mathbb{R}^n$ be a compact set. The well known classical inequality (see Łojasiewicz 1959) say that there exist positive constants C, α such that

$$|f(x)| \geq C \cdot \operatorname{dist}(x, f^{-1}(0))^{\alpha}, \tag{1}$$

for all $x \in K$.

If the set K is non-compact, it may happen that such C, α do not exist. One may check that it is impossible for polynomials

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$$f(x, y, z) = x^2(y^2 + z^4) \quad \text{and} \quad f(x, y) = x(y-1)[y^2 + (xy-1)^2]. \tag{2}$$

For this reason, some authors modify inequality (1) or its domain.

Hörmander (1958) considered a global version of inequality (1). Precisely, he proved the following

$$|f(x)| \cdot (1 + |x|)^{\beta} \geq C \cdot \operatorname{dist}(x, f^{-1}(0))^{\alpha},$$

for all $x \in \mathbb{R}^n$ and some positive constants C, α, β.

In some additional assumptions another global version of inequality (1) was given in Đinh (2014) i.e.

$$|f(x)|^{\alpha} + |f(x)|^{\beta} \geq C \cdot \operatorname{dist}(x, f^{-1}(0)),\ x \in \mathbb{R}^n,$$

for some positive constants C, α, β.

In turn, Hà and Duc (2010) Ha and Nguyen modified the zero set of polynomial $f: \mathbb{R}^2 \to \mathbb{R}$ in inequality (1):

$$|f(x)| \geq C \cdot \operatorname{dist}\left(x, f^{-1}(0)^{\mathbb{R}}\right)^{\alpha},$$

in some neighborhood at infinity, where $f^{-1}(0)^{\mathbb{R}}$ denotes real approximation at infinity of $\{x \in \mathbb{C}^2 \colon f(x) = 0\}$.

Another modification concerned both a zero set and a domain. Indeed, in Đinh (2013) Kurdyka and Le Gal established

$$|f(x)| \geq C \cdot \operatorname{dist}(x, Z)^{\alpha},\ x \in f^{-1}(-\delta, \delta)$$

for some positive constants δ, C, α, where $Z = \{x \in \mathbb{R}^n \colon f(x) \cdot \frac{\partial f}{\partial x_1}(x) = 0\}$, and f is a monic polynomial with respect to x_1. In this case constants C, α can be computed explicitly (see Hà et al. 2015).

If the set $f^{-1}(0)$ is compact, then

$$\operatorname{dist}(x, f^{-1}(0)) \approx |x|.$$

In this case for real polynomial $f: \mathbb{R}^n \to \mathbb{R}$, Gwoździewicz (1998) proved the following

$$|f(x)| \geq C \cdot |x|^{d-(d-1)^n},\ |x| > R,$$

where $d = \deg f > 2$ and $C, R > 0$.

Kollár (1988) gave similar result for complex polynomial mappings $F: \mathbb{C}^n \to \mathbb{C}^n$, $\# F^{-1}(0) < \infty$ i.e.

$$|F(x)| \geq C \cdot |x|^{d-d^n},\ |x| > R,$$

where $d = \deg F$ and $C, R > 0$.

In the paper we assume that $f^{-1}(0)$ is a non-compact set. We keep the form of inequality (1) , but we restrict the domain. Namely, we examine behavior of f:

(i) in the neighborhood of the level set $f^{-1}(0)$ at infinity i.e. in the set

$$\{x \in \mathbb{R}^n : \ \mathrm{dist}(x, f^{-1}(0)) < \varepsilon, \ |x| > R\},$$

(ii) or in the set

$$\{x \in \mathbb{R}^n : \ \mathrm{dist}(x, f^{-1}(0)) > R\}.$$

The lack of the distinction of these cases could lead to a situation that an exponent α in inequality (1) does not exist in neighborhood at infinity. See for example polynomials (2). In the case (i) and (ii) we give the following definitions.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial such that $f^{-1}(0)$ is a non-compact set. We define *the Łojasiewicz exponent of f at infinity* as the infimum of the exponents $l \in \mathbb{R}_+$ such that

$$|f(x)| \geq C \cdot \mathrm{dist}(x, f^{-1}(0))^l \quad \text{for all } x \text{ such that } \mathrm{dist}(x, f^{-1}(0)) < \varepsilon, \tag{3}$$

in some neighborhood of infinity for some $\varepsilon > 0$ and $C > 0$. We denote it by $\mathcal{L}_\infty(f)$. In cases where such l does not exist, we put

$$\mathcal{L}_\infty(f) = +\infty.$$

In Ðinh (2012) authors proved that there are no sequences of the first type if and only if there exist $C, \delta, \alpha > 0$ such that

$$|f(x)|^\alpha \geq C \cdot \mathrm{dist}(x, f^{-1}(0)) \quad \text{for all } x \in f^{-1}([-\delta, \delta]).$$

The sequence $(x_k)_{k=1}^\infty \subset \mathbb{R}^n$ is of the first type if $f(x_k) \to 0$ and $\mathrm{dist}(x, f^{-1}(0)) \nrightarrow 0$.

It is easy to observe that if the last inequality is true for some positive C, δ, α, then there exist $C, \varepsilon, l > 0$ such that inequality (3) is true. Hence if there are no sequences of the first type, then $\mathcal{L}_\infty(f)$ exists. However, in some cases $\mathcal{L}_\infty(f)$ exists but there is a sequence of the first type. For example $f(x, y) = x(y-1)[y^2 + (xy-1)^2]$.

In the paper we give an effective formula for the Łojasiewicz exponent at infinity in the class of non-negative and non-degenerate polynomials in terms of the Newton polyhedron (see Sect. 2). This result is a counterpart at infinity of the local result of the paper Bùi and Pham (2014).

2 Preliminaries

We denote by $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, we denote by x^α the monomial $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ and put $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $|x| = \max_{i=1}^n |x_i|$.

Let $f(x) = \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha x^\alpha$. Let us define the set $\mathrm{supp}(f) = \{\alpha \in \mathbb{Z}_+^n : c_\alpha \neq 0\}$ and call it *the support* of f. Define the set $\Gamma(f) = \mathrm{conv}\{\mathrm{supp}(f)\} \subset \mathbb{R}_+^n$ and call it *the Newton polyhedron at infinity of* f.

Let $q \in \mathbb{R}^n \backslash \{0\}$. Define

$$d(q, \Gamma(f)) = \min\{\langle q, \alpha\rangle : \alpha \in \Gamma(f)\},$$
$$\Delta(q, \Gamma(f)) = \{\alpha \in \Gamma(f) : \langle q, \alpha\rangle = d(q, \Gamma(f))\},$$

where $\langle \cdot, \cdot\rangle$ denotes the standard inner product in $\mathbb{R}^n \times \mathbb{R}^n$. We say that $\Delta \subset \Gamma(f)$ is *a face* of $\Gamma(f)$, if there exists $q \in \mathbb{R}^n \backslash \{0\}$ such that $\Delta = \Delta(q, \Gamma(f))$. By *a dimension* of a face Δ we mean the minimum of the dimensions of the affine subspace containing Δ. By *a vertice* of $\Gamma(f)$ we mean the 0-dimensional faces of $\Gamma(f)$. We define *Newton boundary at infinity* of f as the set of faces $\Delta \subset \Gamma(f)$ such that: if q is defining a vector for Δ then $q_i < 0$ for some $i \in \{1, \ldots, n\}$ and we denote it by $\Gamma_\infty(f)$. Denote by $\Gamma_\infty^k(f)$ the set of k-dimensional faces of $\Gamma_\infty(f)$, $k = 0, \ldots, n-1$. For $\Delta \in \Gamma_\infty(f)$ we define the polynomial

$$f_\Delta = \sum_{\alpha \in \Delta} c_\alpha x^\alpha.$$

and call it *the principal part of* f *at infinity with respect to face* Δ.

We say that f is *Kushnirenko non-degenerate at infinity on the face* $\Delta \in \Gamma_\infty(f)$ if the system of equations

$$\frac{\partial f_\Delta}{\partial x_1}(x) = \cdots = \frac{\partial f_\Delta}{\partial x_n}(x) = 0$$

has no solution in $(\mathbb{R}\backslash\{0\})^n \backslash K$, where $K \subset \mathbb{R}^n$ is a compact set. We say that f is *Kushnirenko non-degenerate at infinity* (shortly *non-degenerate*) if f is Kushnirenko non-degenerate at infinity on each face $\Delta \in \Gamma_\infty(f)$.

We say that f is *non-negative at infinity* (shortly *non-negative*) if there exists a compact set $K \subset \mathbb{R}^n$ such that $f(x) \geq 0$ for $x \in \mathbb{R}^n \backslash K$.

One of the main tool which we use in the paper is the following

Lemma 1 (Curve Selection Lemma at infinity, Đinh (2014), Lemma 1) *Let* $A \subset \mathbb{R}^n$ *be a semi-algebraic set, and let* $F := (f_1, \ldots, f_p) \colon \mathbb{R}^n \to \mathbb{R}^p$ *be a semi-algebraic map. Assume that there exists a sequence* $x^k \in A$ *such that* $\lim_{k\to\infty} |x^k| = \infty$ *and* $\lim_{k\to\infty} F(x^k) = y \in (\overline{\mathbb{R}})^p$*, where* $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$*. Then there exists an analytic curve* $\varphi \colon (0, \epsilon) \to A$ *of the form*

$$\varphi(t) = a^0 t^q + a^1 t^{q+1} + \ldots \quad ,$$

such that $a^0 \in \mathbb{R}^n \backslash \{0\}$*,* $q < 0$*,* $q \in \mathbb{Z}$*, and* $\lim_{t\to 0} F(\varphi(t)) = y$*.*

Let $A \subset \mathbb{N}^n$ be a finite set. Put

$$N_A(x) = \max_{\alpha \in A} |x^\alpha|.$$

Let V be the set of vertices of $\Gamma(f)$. Denote

$$N_\Gamma = N_V.$$

We recall two simple lemmas which will be used in the rest of the paper.

Lemma 2 (Đinh (2014), Lemma 11) *There exist some subset $J_1, \ldots, J_s$ of $\{1, \ldots, n\}$, with $J_i \not\subseteq J_j$ for $i \neq j$, such that*

$$N_\Gamma^{-1}(0) = \bigcup_{k=1}^{s} Z_k,$$

where $Z_k := \{x \in \mathbb{R}^n : x_j = 0,\ j \in J_k\}$.

For a given subset $J \subset \{1, \ldots, n\}$ we define

$$\mathbb{R}^J := \{\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n : \alpha_j = 0 \quad \text{for} \quad j \notin J\}.$$

Lemma 3 (Đinh (2014), Lemma 12) *Let $J_1, \ldots, J_s$ be as in Lemma 2. For every $(j_1, \ldots, j_s) \in J_1 \times \cdots \times J_s$, we have $V \cap \mathbb{R}^J \neq \emptyset$, where $J = \{j_1, \ldots, j_s\}$.*

3 The Main Theorem

Let $J_1, \ldots, J_s$ be as in Lemma 2 and let

$$\mathcal{P} = \{I \subset \{1, \ldots, n\} \colon I \neq \emptyset \wedge I \cap J_k = \emptyset \quad \text{for some} \quad k \in \{1, \ldots, s\}\}.$$

Observe that $\mathcal{P} \neq \emptyset$ i.e. $J_{k_0} \neq \{1, \ldots, n\}$ for some $k_0 \in \{1, \ldots, s\}$. Indeed, suppose to the contrary that $J_k = \{1, \ldots, n\}$ for any $k \in \{1, \ldots, s\}$. If $s > 1$, then by Lemma 2 it is not possible. Therefore $s = 1$. Hence $J_1 = \{1, \ldots, n\}$ and $N_\Gamma^{-1}(0) = \{0\}$. By Lemma 10

$$f^{-1}(0) \cap (\mathbb{R}^n \setminus K) = N_\Gamma^{-1}(0) \cap (\mathbb{R}^n \setminus K) = \emptyset,$$

for some compact set K. This gives a contradiction to the assumption that the set $f^{-1}(0)$ is not compact.

Let us fix $I \in \mathcal{P}$. We define $\varphi^I(x) = (\varphi_1^I(x), \ldots, \varphi_n^I(x))$, where

$$\varphi_i^I(x) = \begin{cases} 1 & \text{for } i \in I, \\ x_i & \text{for } i \notin I, \end{cases}$$

for $i = 1, \ldots, n$ and define $N_\Gamma^I = N_\Gamma \circ \varphi^I$.

Observe that

$$(N_\Gamma^I)^{-1}(0) = \bigcup_{k=1}^{s(I)} Z_k^I = \bigcup_{k=1}^{s(I)} \{x \in \mathbb{R}^{I'} : x_j = 0,\ j \in J_k^I\}, \quad J_l^I \not\subseteq J_m^I,\ l \neq m,$$

where $I' = \{1, \dots, n\} \setminus I$. Put

$$\alpha^I_{\max} = \max \left\{ \alpha^{\min}_J : J \in J^I_1 \times \cdots \times J^I_{s(I)} \right\},$$

where

$$\alpha^{\min}_J := \min \left\{ |\alpha| : \alpha \in V^{I'} \cap \mathbb{R}^J \right\},$$

and $V^{I'}$ denotes the projection of the set V onto $\mathbb{R}^{I'}$. Observe that $N^I_\Gamma = N_{V^{I'}}$.

Now, we give the main result of the paper.

Theorem 4 *Let $f : \mathbb{R}^n \to \mathbb{R}$, $f(0) = 0$, be a non-negative and non-degenerate polynomial. Then*

$$\mathcal{L}_\infty(f) = \max \left\{ \alpha^I_{\max} : I \in \mathcal{P} \right\}. \tag{4}$$

Remark 5 One can check that the assertions of the above theorems are also true if we assume Mikhailov–Gindikin non-degeneracy (see Đinh (2014), Section 5).

To illustrate the above theorems we give the following

Example 6 Let $f(x, y, z) = x^8(y^4 + z^6)$. It is easy to see that f is non-degenerate and non-negative. We have $V = \{(8, 4, 0), (8, 0, 6)\}$ and

$$N_\Gamma(x, y, z) = \max\{x^8 y^4, x^8 z^6\}, \quad (N_\Gamma)^{-1}(0) = \{x = 0\} \cup \{y = z = 0\}.$$

Hence $J_1 = \{1\}$, $J_2 = \{2, 3\}$.

We calculate $\mathcal{L}_\infty(f)$. We have $\mathcal{P} = \{\{1\}, \{2\}, \{3\}, \{2, 3\}\}$. For $I = \{1\}$ we obtain $I' = \{2, 3\}$ and

$$N^I_\Gamma(y, z) = \max\{y^4, z^6\}, \quad (N^I_\Gamma)^{-1}(0) = \{(0, 0)\}, \quad J^I_1 = \{2, 3\}.$$

Hence $J = \{2\}$ or $J = \{3\}$ and

$$\alpha^{\min}_2 = \min\{|\alpha| : \alpha \in \{4, 0\}\} = 4, \quad \alpha^{\min}_3 = \min\{|\alpha| : \alpha \in \{0, 6\}\} = 6.$$

Therefore

$$\alpha^{\{1\}}_{\max} = \max\{4, 6\} = 6.$$

Similarly we calculate

$$\alpha^{\{2\}}_{\max} = \alpha^{\{3\}}_{\max} = \alpha^{\{2,3\}}_{\max} = 8.$$

Finally we have

$$\mathcal{L}_\infty(f) = \max \left\{ \alpha^I_{\max} : I \in \mathcal{P} \right\} = \max\{6, 8\} = 8.$$

4 Auxiliary Results

The following lemmas will be used in the proof of Lemma 10. The proof of Lemma 7 is a simple transfer of its local counterpart [see Bùi and Pham (2014), Lemma 3.1]. We give it for a convenience of the reader.

Lemma 7 *Suppose that f is non-negative polynomial. Then for any face $\Delta \in \Gamma_\infty(f)$ we have $f_\Delta(x) \geq 0$ for $x \in (\mathbb{R}\setminus\{0\})^n \setminus K$, where K is a compact set.*

Proof Since f is non-negative there exists a compact set K such that $f(x) \geq 0$ for $x \in \mathbb{R}^n \setminus K$. Suppose to the contrary that there exists a face $\Delta \in \Gamma_\infty(f)$ and there exists a point $x^0 \in (\mathbb{R}\setminus\{0\})^n \setminus K$ such that $f_\Delta(x^0) < 0$. Let J be the smallest subset of $\{1, \ldots, n\}$ such that $\Delta \subset \mathbb{R}^J$. Hence, there exists a non-zero vector $a \in \mathbb{R}^n$, with $a_j < 0$ for some $j \in J$ and $a_j = 0$ for $j \notin J$ such that

$$\Delta = \{\nu \in \Gamma(f) \cap \mathbb{R}^J : \langle a, \nu \rangle = d(a, \Gamma(f))\}.$$

Define monomial curve $\varphi \colon (0, 1) \to \mathbb{R}^n$, $t \mapsto (\varphi_1(t), \ldots, \varphi_n(t))$, by

$$\varphi_j(t) = \begin{cases} x_j^0 t^{a_j} & \text{for } j \in J, \\ 0 & \text{for } j \notin J. \end{cases}$$

Put $d := d(a, \Gamma(f))$. Now, we may write f in the form:

$$f(\varphi(t)) = f_\Delta(x^0)t^d \quad + \quad \text{higher order terms in } t.$$

Since $f_\Delta(x^0) < 0$, we have

$$f(\varphi(t)) < 0 \quad \text{for all sufficiently small } t.$$

This gives a contradiction.

However counterpart of equivalence Bùi and Pham (2014, Lemma 3.2) is not true at infinity. The simple implication is the only one that holds.

Lemma 8 *If f is non-negative and non-degenerate, then for any face $\Delta \in \Gamma_\infty(f)$ we have $f_\Delta > 0$ on $(\mathbb{R}\setminus\{0\})^n \setminus K$.*

Proof Using Lemma 7 we obtain $f_\Delta(x) \geq 0$ for all $x \in (\mathbb{R}\setminus\{0\})^n \setminus K$, where K is a suitably chosen compact set. Suppose to the contrary that there exists a point $x^0 \in (\mathbb{R}\setminus\{0\})^n \setminus K$ such that $f_\Delta(x^0) = 0$. Therefore the function f_Δ attains a local minimum at the point x^0. Hence grad $f_\Delta(x^0) = 0$. This gives a contradiction to non-degeneracy of f.

The following lemma will be also applied in the proof of the Lemma 10.

Lemma 9 Gindikin (1974, Lemma 1) *Let* $\nu \in \mathbb{R}_+^n$, $\nu \in \operatorname{conv}\{\nu^1, \ldots, \nu^k\}$. *Then*

$$|x^\nu| \le \sum_{j=1}^{k} |x^{\nu_j}|.$$

The next lemma plays a crucial role in the proof of the main theorem. Its proof is a substantially analogous to the proof of Lemma 3.3 of the paper Bùi and Pham (2014). However we prove the second inequality in (5) without assumption of non-degeneracy and non-negativity, using Lemma 9.

Lemma 10 *If f is non-negative and non-degenerate then there exist some positive constants C_1 and C_2 such that*

$$C_1 N_\Gamma(x) \le f(x) \le C_2 N_\Gamma(x), \quad \textit{for all} \quad x \in \mathbb{R}^n \setminus K, \tag{5}$$

for some compact set $K \subset \mathbb{R}^n$.

Proof We will prove the first inequality. Suppose to the contrary that there exists a sequence $\{x^k\} \subset \mathbb{R}^n$ with $|x^k| > k$ and such that

$$f(x^k) < \frac{1}{k} N_\Gamma(x^k)$$

for all k. By Lemma 1, there exist an analytic curves $\varphi\colon (0,\epsilon) \to \mathbb{R}^n$, $t \mapsto (\varphi_1(t), \ldots, \varphi_n(t))$ and $\psi\colon (0,\epsilon) \to \mathbb{R}_+$ such that

$$|\varphi(t)| \to \infty, \quad |\psi(t)| \to 0 \quad \text{as} \quad t \to 0^+, \tag{6}$$

and

$$f(\varphi(t)) < \psi(t) N_\Gamma(\varphi(t)). \tag{7}$$

Let $J = \{j : \varphi_j \not\equiv 0\} \subset \{1, \ldots, n\}$. For $j \in J$ we can expand coordinate function φ_j, say

$$\varphi_j(t) = x_j^0 t^{a_j} \quad + \quad \text{higher order terms in } t,$$

where $x_j^0 \ne 0$ and $a_j \in \mathbb{N}$. From Condition (6), there exists $j \in J$ such that $a_j < 0$. If $\Gamma(f) \cap \mathbb{R}^J = \emptyset$, then for any vertex $\alpha \in V$, there exists $j \notin J$ such that $\alpha_j > 0$ ($V \subset \Gamma(f)$) and hence $(\varphi_j(t))^{\alpha_j} \equiv 0$. Then $(\varphi(t))^{\alpha_j} \equiv 0$. Hence

$$N_\Gamma(\varphi(t)) = \max_{\alpha \in V} |\varphi(t)^\alpha| \equiv 0.$$

This gives a contradiction to (7).

Therefore, $\Gamma(f) \cap \mathbb{R}^J \neq \emptyset$. Put

$$d = \min \left\{ \sum_{j \in J} a_j \alpha_j : \alpha \in \Gamma(f) \cap \mathbb{R}^J \right\},$$
$$\Delta = \left\{ \alpha \in \Gamma(f) \cap \mathbb{R}^J : \sum_{j \in J} a_j \alpha_j = d \right\}.$$

We can write

$$f(\varphi(t)) = f_\Delta(x^0) t^d \quad + \quad \text{higher order terms in } t,$$

where $x^0 = (x_1^0, \ldots, x_n^0)$ and $x_j^0 = 1$ for $j \notin J$. We will show that $f_\Delta(x^0) > 0$. Indeed, since f is non-negative and non-degenerate, it follows from Lemma 8 we have that $f_\Delta(x) > 0$ for $x \in (\mathbb{R} \backslash \{0\})^n \backslash K$, where K is a suitably chosen compact set. Therefore by quasi-homogeneity of f_Δ we have

$$f_\Delta(x^0) = \frac{f_\Delta((s^{a_j} x_j^0)_{j \in J})}{s^d} > 0,$$

where s is a positive number such that $s^{a_j} \cdot x_j^0$ is large enough for some $j \in J$. Hence

$$f(\varphi(t)) \quad \text{and} \quad t^d \tag{8}$$

are of the same order if $t \to 0^+$.

On the other hand, we have

$$N_\Gamma(\varphi(t)) = \max_{\alpha \in V} |\varphi(t)^\alpha| = \max_{\alpha \in \Delta} |(x^0)^\alpha| t^d \quad + \quad \text{higher order terms in } t.$$

Hence and by (8) we have a contradiction to (7).
Now we prove the second inequality in (b). Let $|x| \geq R \geq 1$, where R is sufficiently large. By Lemma 9 we have

$$\begin{aligned} f(x) = \sum_{\nu \in \operatorname{supp} f} c_\nu x^\nu \leq \max_{\nu \in \operatorname{supp} f} |c_\nu| \cdot \sum_{\nu \in \operatorname{supp} f} |x^\nu| \leq \\ \leq C_2 \cdot \max_{\nu \in V} |x^\nu| = C_2 \cdot N_\Gamma(x), \end{aligned}$$

where C_2 is a some positive constant. □

Let $A \subset \mathbb{N}^n$ be a finite set. Put

$$\mathcal{L}(N_A) = \inf\{l \in \mathbb{R}_+ : \exists_{C>0} |N_A(x)| \geq C \cdot \operatorname{dist}(x, N_A^{-1}(0))^l, \ \operatorname{dist}(x, N_A^{-1}(0)) < 1\}.$$

Now we give an effective formula to compute $\mathcal{L}(N_A)$.

Proposition 11 *We have*

$$\mathcal{L}(N_A) = \max\left\{\alpha_J^{\min} : J \in J_1 \times \cdots \times J_s\right\},$$

where

$$\alpha_J^{\min} := \min\left\{|\alpha| : \alpha \in A \cap \mathbb{R}^J\right\}.$$

Proof We first show that $\mathcal{L}(N_A) \leq \max\{\alpha_J^{\min} : J \in J_1 \times \cdots \times J_s\}$. Let us fix an arbitrary $x \in \mathbb{R}^n$ such that

$$\operatorname{dist}(x, N_A^{-1}(0)) = \delta < 1.$$

It is easy to check that

$$\operatorname{dist}(x, N_A^{-1}(0)) = \min_{k=1}^{s} \max_{j \in J_k} |x_j|.$$

Hence

$$\max_{j \in J_k} |x_j| \geq \delta \quad \text{for any} \quad k = 1, \ldots, s.$$

This means that for each $k = 1, \ldots, s$ there exists $j_k \in J_k$ such that

$$|x_{j_k}| \geq \delta.$$

Put $J = \{j_1, \ldots, j_s\}$. By Lemma 3 we have that $A \cap \mathbb{R}^J \neq \emptyset$. Let us choose $\alpha = (\alpha_1, \ldots, \alpha_s) \in A \cap \mathbb{R}^J$ such that

$$|\alpha| = \alpha_J^{\min}.$$

Hence

$$\begin{aligned} N_A(x) &= \max\{|x_{j_1}^{\alpha_j} \ldots x_{j_s}^{\alpha_s}|, \ldots\} \geq \delta^{\alpha_j} \ldots \delta^{\alpha_s} \\ &= \delta^{\alpha_J^{\min}} \geq \delta^{\max\{\alpha_J^{\min} : J \in J_1 \times \cdots \times J_s\}} \\ &= \operatorname{dist}(x, N_A^{-1}(0))^{\max\{\alpha_J^{\min} : J \in J_1 \times \cdots \times J_s\}}. \end{aligned}$$

This means that $\mathcal{L}(N_A) \leq \max\{\alpha_J^{\min} : J \in J_1 \times \cdots \times J_s\}$.

Now, we show that $\mathcal{L}(N_A) \geq \max\{\alpha_J^{\min} : J \in J_1 \times \cdots \times J_s\}$.

Let $(j_1, \ldots, j_s) \in J_1 \times \cdots \times J_s$ be such that realized the above maximum and let $J \subset \{1, \ldots, n\}$ be the minimal set such that $j_k \in J$, $k = 1, \ldots, s$. Put $A_J =$

$\mathbb{R}^J \cap A$. By Lemma 3 we have that $A_J \neq \emptyset$. Take the following parametrization $\varphi(t) = (\varphi_1(t), \ldots, \varphi_n(t))$, $|t| < 1$, where

$$\varphi_i(t) = \begin{cases} t & \text{for} \quad i \in J \\ 0 & \text{for} \quad i \notin J, \end{cases} \tag{9}$$

for $i = 1, \ldots, n$. We have

$$N_A(\varphi(t)) = \max_{\nu \in A_J} |\varphi(t)^\nu| = |t|^{\min\{|\nu|:\, \nu \in A_J\}} = |t|^{\alpha_J^{\min}} = \operatorname{dist}(\varphi(t), N_A^{-1}(0))^{\alpha_J^{\min}}.$$

Hence $\mathcal{L}(N_A) \geq \alpha_J^{\min}$. This ends the proof.

One can observe that the above proof in comparison with the proof of (Bùi and Pham 2014, Proposition 3.1) is more elementary.

5 Proof of the Main Theorem

Now, we are ready to give the proof of the main result.

Proof of Theorem 4. Since f is non-negative and non-degenerate polynomial, then by Lemma 10 there exist some positive constants C_1 and C_2 such that

$$C_1 N_\Gamma(x) \leq f(x) \leq C_2 N_\Gamma(x), \tag{10}$$

for all $x \in \mathbb{R}^n \backslash K$ and some compact set $K \subset \mathbb{R}^n$. Hence

$$f^{-1}(0) \cap (\mathbb{R}^n \backslash K) = N_\Gamma^{-1}(0) \cap (\mathbb{R}^n \backslash K). \tag{11}$$

We will show that there exist some positive constants D_1 and D_2 such that

$$D_1 \operatorname{dist}(x, N_\Gamma^{-1}(0)) \leq \operatorname{dist}(x, f^{-1}(0)) \leq D_2 \operatorname{dist}(x, N_\Gamma^{-1}(0)),$$

for all $x \in \mathbb{R}^n \backslash K_1$ and some compact set $K_1 \subset \mathbb{R}^n$, $K \subset K_1$. First, observe that

$$\operatorname{dist}(x, K) \leq |x| \leq 2 \operatorname{dist}(x, K), \quad x \in \mathbb{R}^n \backslash K_1, \tag{12}$$

for some compact set $K_1 \subset \mathbb{R}^n$, $K \subset K_1$. By (11), (12) and since $0 \in N_\Gamma^{-1}(0)$ we have

$$\begin{aligned} \operatorname{dist}(x, f^{-1}(0)) &= \min\{\operatorname{dist}(x, f^{-1}(0) \backslash K), \operatorname{dist}(x, f^{-1}(0) \cap K)\} \\ &\geq \min\{\operatorname{dist}(x, N_\Gamma^{-1}(0) \backslash K), \operatorname{dist}(x, K)\} \\ &\geq \min\{\operatorname{dist}(x, N_\Gamma^{-1}(0)), \frac{1}{2}|x|\} \end{aligned}$$

$$\geq \min\{\operatorname{dist}(x, N_\Gamma^{-1}(0)), \frac{1}{2}\operatorname{dist}(x, N_\Gamma^{-1}(0))\}$$
$$= \frac{1}{2}\operatorname{dist}(x, N_\Gamma^{-1}(0)),$$

for $x \in \mathbb{R}^n \setminus K_1$. Analogously, by (11), (12) and since $0 \in f^{-1}(0)$ we get

$$\operatorname{dist}(x, N_\Gamma^{-1}(0)) \geq \frac{1}{2}\operatorname{dist}(x, f^{-1}(0)),$$

for $x \in \mathbb{R}^n \setminus K_1$. Summing up we obtain

$$\frac{1}{2}\operatorname{dist}(x, N_\Gamma^{-1}(0)) \leq \operatorname{dist}(x, f^{-1}(0)) \leq 2\operatorname{dist}(x, N_\Gamma^{-1}(0)), \tag{13}$$

for $x \in \mathbb{R}^n \setminus K_1$. By (10) and (13) it follows that

$$\mathcal{L}_\infty(N_\Gamma) = \mathcal{L}_\infty(f). \tag{14}$$

By (14), it is enough to prove formula (4) for N_Γ. We first show that

$$\mathcal{L}_\infty(N_\Gamma) \leq \max\left\{\mathcal{L}(N_\Gamma^I) : I \in \mathcal{P}\right\}. \tag{15}$$

Let $x \in \mathbb{R}^n \setminus K$, where K is the same as in Lemma 10 and $\operatorname{dist}(x, N_\Gamma^{-1}(0)) < \varepsilon < 1$. It can be assumed that

$$\{x \in \mathbb{R}^n : |x| < 1\} \subset K.$$

Let $I \neq \emptyset$ be such that

$$|x_i| \geq 1, i \in I \quad \text{and} \quad |x_i| < 1, i \notin I.$$

It is easy to check that $I \in \mathcal{P}$. Since

$$(N_\Gamma^I)^{-1}(0) = N_\Gamma^{-1}(0) \cap \{x \in \mathbb{R}^n : x_i = 1 \quad \text{for} \quad i \in I\} \subset N_\Gamma^{-1}(0),$$

we have

$$\operatorname{dist}(x_I, (N_\Gamma^I)^{-1}(0)) \geq \operatorname{dist}(x_I, N_\Gamma^{-1}(0)). \tag{16}$$

It is easy to check that

$$\operatorname{dist}(x_I, N_\Gamma^{-1}(0)) = \operatorname{dist}(x, N_\Gamma^{-1}(0)). \tag{17}$$

By (16), (17) we obtain

$$|N_\Gamma(x)| \geq |N_\Gamma^I(x_I)| \geq C_I \operatorname{dist}(x_I, (N_\Gamma^I)^{-1}(0))^{\mathcal{L}(N_\Gamma^I)}$$

$$\geq C_I \operatorname{dist}(x_I, N_\Gamma^{-1}(0))^{\mathcal{L}(N_\Gamma^I)} = C_I \operatorname{dist}(x, N_\Gamma^{-1}(0))^{\mathcal{L}(N_\Gamma^I)}$$
$$\geq \min\{C_I \colon I \in \mathcal{P}\} \operatorname{dist}(x, N_\Gamma^{-1}(0))^{\max\{\mathcal{L}(N_\Gamma^I) \colon I \in \mathcal{P}\}}.$$

This gives (15).

Now we show that

$$\mathcal{L}_\infty(N_\Gamma) \geq \max\left\{\mathcal{L}(N_\Gamma^I) : I \in \mathcal{P}\right\}. \tag{18}$$

First we choose $I \in \mathcal{P}$ such that realizes the above maximum. Take the parametrization $\varphi \colon \mathbb{R}\setminus\{0\} \to \mathbb{R}^{\{1,\dots,n\}\setminus I}$ defined by formula (9) such that it realizes $\mathcal{L}(N_\Gamma^I)$. Let $\varepsilon > 0$. Let $(\varphi_\varepsilon)_i : \mathbb{R}\setminus\{0\} \to \mathbb{R}^n$ be defined in the following way

$$(\varphi_\varepsilon(t))_i = \begin{cases} \varphi_i(t) & \text{for } i \notin I, \\ t^{-\varepsilon} & \text{for } i \in I, \end{cases}$$

Observe that

$$\operatorname{dist}(\varphi_\varepsilon(t), N_\Gamma^{-1}(0)) = t.$$

Indeed, let $K = \{k \in \{1, \dots, s\} \colon J_k \cap I = \emptyset\} = \{k_1, \dots, k_r\}$. We have

$$\begin{aligned} \operatorname{dist}(\varphi_\varepsilon(t), N_\Gamma^{-1}(0)) &= \min_{k=1}^{s} \max_{j \in J_k} |\varphi_{\varepsilon,j}(t)| \\ &= \min_{k=1}^{s}\{\max\{\max_{j \in J_k \cap I} |\varphi_{\varepsilon,j}(t)|, \max_{j \in J_k \setminus I} |\varphi_{\varepsilon,j}(t)|\}\} \\ &= \min_{k=1}^{s}\{\max\{|t^{-\varepsilon}|, \max_{j \in J_k \setminus I} |\varphi_{\varepsilon,j}(t)|\}\} = \min_{k \in K}\{\max_{j \in J_k} |\varphi_j(t)|\}. \end{aligned}$$

Now, it is enough to show that

$$\min_{k \in K}\{\max_{j \in J_k} |\varphi_j(t)|\} = t. \tag{19}$$

Observe that

$$\begin{aligned} (N_\Gamma^I)^{-1}(0) &= N_\Gamma^{-1}(0) \cap \{x \in \mathbb{R}^n \colon x_i = 1,\ i \in I\} \\ &= \bigcup_{k=1}^{s} Z_k \cap \{x \in \mathbb{R}^n \colon x_i = 1,\ i \in I\} \\ &= \bigcup_{k=1}^{s}\{x \in \mathbb{R}^n \colon x_i = 0 \text{ for } i \in J_k,\ x_i = 1 \text{ for } i \in I\} \end{aligned}$$

$$= \bigcup_{k\in K} \{x \in \mathbb{R}^n : x_i = 0 \text{ for } i \in J_k,\ x_i = 1 \text{ for } i \in I\}$$
$$= \bigcup_{l=1}^{r} \{x \in \mathbb{R}^n : x_i = 0 \text{ for } i \in J_{k_l},\ x_i = 1 \text{ for } i \in I\}.$$

Let $(j_{k_1}, \ldots, j_{k_r}) \in J_{k_1} \times \cdots \times J_{k_r}$, $J = \{j_{k_1}, \ldots, j_{k_r}\}$ be the same as in definition of φ. It is obvious that $J_{k_l} \cap J \neq \emptyset$, $l = 1, \ldots, r$. Therefore

$$\max_{j\in J_{k_l}} |\varphi_j(t)| = t, \quad l = 1, \ldots, r.$$

This gives (19).

Let ν_I be the system of these coordinates of ν which are in I and $\nu_{I'}$ - system of the remaining ones. We have

$$\begin{aligned} |(N_\Gamma \circ \varphi_\varepsilon)(t)| &= \max_{\nu\in V} |\varphi_\varepsilon(t)^\nu| = \max_{\nu\in V}\{|t^{-\varepsilon}|^{\nu_I} \cdot |\varphi_{I'}(t)^{\nu_{I'}}|\} \\ &\leq |t^{-\varepsilon}|^{\max_{\nu\in V}|\nu_I|} \cdot \max_{\nu\in V} |\varphi_{I'}(t)^{\nu_{I'}}| = |t^{-\varepsilon}|^{\max_{\nu\in V}|\nu_I|} \cdot N_\Gamma(\varphi(t)) \\ &= |t^{-\varepsilon}|^{\max_{\nu\in V}|\nu_I|} \cdot |t|^{\mathcal{L}(N_\Gamma^I)} = |t|^{\mathcal{L}(N_\Gamma^I) - \varepsilon \max_{\nu\in V}|\nu_I|} \\ &= \operatorname{dist}(\varphi_\varepsilon(t), N_\Gamma^{-1}(0))^{\mathcal{L}(N_\Gamma^I) - \varepsilon \max_{\nu\in V}|\nu_I|}. \end{aligned}$$

It can be assumed that ε is such that $\mathcal{L}(N_\Gamma^I) - \varepsilon \max_{\nu\in V} |\nu_I| > 0$. Hence

$$\mathcal{L}_\infty(N_\Gamma) \geq \mathcal{L}(N_\Gamma^I) - \varepsilon \max_{\nu\in V} |\nu_I|.$$

By arbitrary choice of ε and I we obtain (18). Summing up we obtain

$$\mathcal{L}_\infty(N_\Gamma) = \max \left\{ \mathcal{L}(N_\Gamma^I) : I \in \mathcal{P} \right\}.$$

By Proposition 11 we have $\mathcal{L}(N_\Gamma^I) = \alpha_{\max}^I$ and hence we get the formula (4) for N_Γ. This ends the proof. □

6 Formula of Exponent of Growth

We also define *the exponent of growth of f at infinity* as the supremum of the exponents $l \in \mathbb{R}_+$ such that

$$|f(x)| \geq C \cdot \operatorname{dist}(x, f^{-1}(0))^l \quad \text{for all } x \text{ such that } \operatorname{dist}(x, f^{-1}(0)) > R,$$

in some neighborhood of infinity for some $R > 0$ and $C > 0$. We denote it by $\mathcal{E}_\infty(f)$. In the case that such l does not exist, we put

$$\mathcal{E}_\infty(f) = -\infty.$$

The second result is a formula of exponent of growth of polynomial f at infinity.

Theorem 12 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a non-negative and non-degenerate polynomial. Then*

$$\mathcal{E}_\infty(f) = \min\left\{\alpha_J^{\max} : J \in J_1 \times \cdots \times J_s\right\},$$

where

$$\alpha_J^{\max} := \max\left\{|\alpha| : \alpha \in V \cap \mathbb{R}^J\right\}.$$

Proof By Lemma 10 we have $\mathcal{E}_\infty(f) = \mathcal{E}_\infty(N_\Gamma)$. Therefore it is enough to prove this formula for N_Γ. We first show that $\mathcal{E}_\infty(N_\Gamma) \geq \min\left\{\alpha_J^{\max} : J \in J_1 \times \cdots \times J_s\right\}$. Let us fix arbitrary $x \in \mathbb{R}^n \setminus K$ such that

$$\operatorname{dist}(x, N_\Gamma^{-1}(0)) = \delta > 1.$$

Since

$$\operatorname{dist}(x, N_\Gamma^{-1}(0)) = \min_{k=1}^{s} \max_{j \in J_k} |x_j|,$$

hence we get

$$\max_{j \in J_k} |x_j| \geq \delta \quad \text{for any} \quad k = 1, \ldots, s.$$

This means that for each $k = 1, \ldots, s$ there exists $j_k \in J_k$ such that

$$|x_{j_k}| \geq \delta.$$

Put $J = \{j_1, \ldots, j_s\}$. By Lemma 3 we have $V \cap \mathbb{R}^J \neq \emptyset$. Let us choose $\alpha = (\alpha_1, \ldots, \alpha_s) \in V \cap \mathbb{R}^J$ such that

$$|\alpha| = \alpha_J^{\max}.$$

Hence

$$\begin{aligned} N_\Gamma(x) &= \max\{|x_{j_1}^{\alpha_j} \ldots x_{j_s}^{\alpha_s}|, \ldots\} \geq \delta^{\alpha_j} \ldots \delta^{\alpha_s} \\ &= \delta^{\alpha_J^{\max}} \geq \delta^{\min\{\alpha_J^{\max} : J \in J_1 \times \cdots \times J_s\}} \\ &= \operatorname{dist}(x, N_\Gamma^{-1}(0))^{\min\{\alpha_J^{\max} : J \in J_1 \times \cdots \times J_s\}}. \end{aligned}$$

This means that $\mathcal{E}_\infty(N_\Gamma) \geq \min\left\{\alpha_J^{\max} : J \in J_1 \times \cdots \times J_s\right\}$.

Now, we show that $\mathcal{E}_\infty(N_\Gamma) \leq \min\left\{\alpha_J^{\max} : J \in J_1 \times \cdots \times J_s\right\}$.

Let $(j_1, \dots, j_s) \in J_1 \times \cdots \times J_s$ and let $J \subset \{1, \dots, n\}$ be the minimal set such that $j_k \in J$, $k = 1, \dots, s$. Put $V_J = \mathbb{R}^J \cap V$. By Lemma 3 we have that $V_J \neq \emptyset$. Take the following parametrization $\varphi(t) = (\varphi_1(t), \dots, \varphi_n(t))$, $|t| > 1$, where

$$\varphi_i(t) = \begin{cases} t & \text{for} \quad i \in J \\ 0 & \text{for} \quad i \notin J, \end{cases}$$

for $i = 1, \dots, n$. We have

$$N_\Gamma(\varphi(t)) = \max_{\nu \in V_J} |\varphi(t)^\nu| = |t|^{\max\{|\nu| :\, \nu \in V_J\}} = |t|^{\alpha_J^{\max}} = \operatorname{dist}(\varphi(t), N_\Gamma^{-1}(0))^{\alpha_J^{\max}}.$$

Hence $\mathcal{E}_\infty(N_\Gamma) \leq \alpha_J^{\max}$ and by arbitrary choice of $(j_1, \dots, j_s) \in J_1 \times \cdots \times J_s$ we have

$$\mathcal{E}_\infty(N_\Gamma) \leq \min \left\{ \alpha_J^{\max} : J \in J_1 \times \cdots \times J_s \right\}.$$

This ends the proof. □

Example 13 Let again $f(x, y, z) = x^8(y^4 + z^6)$ and $V = \{(8, 4, 0), (8, 0, 6)\}$ and $J_1 = \{1\}$, $J_2 = \{2, 3\}$.

We calculate $\mathcal{E}_\infty(f)$. Take $J \in J_1 \times J_2$. Then $J = \{1, 2\}$ or $J = \{1, 3\}$. We calculate

$$\alpha_{\{1,2\}}^{\max} = \max\{|\alpha| : \alpha \in V \cap \mathbb{R}^{\{1,2\}}\} = \max\{|\alpha| : \alpha \in \{(8, 4)\}\} = 12$$

and

$$\alpha_{\{1,3\}}^{\max} = \max\{|\alpha| : \alpha \in V \cap \mathbb{R}^{\{1,3\}}\} = \max\{|\alpha| : \alpha \in \{(8, 6)\}\} = 14.$$

Finally

$$\mathcal{E}_\infty(f) = \min \left\{ \alpha_{\{1,2\}}^{\max}, \alpha_{\{1,3\}}^{\max} \right\} = \min\{12, 14\} = 12.$$

References

Bùi, N.T.N., Pham, T.S.: Computation of the Łojasiewicz exponent of nonnegative and nondegenerate analytic functions. Int. J. Math. **25**, 1450092 (2014)

Đinh, S.T., Hà, H.V., Tháo, N.T.: Łojasiewicz inequality for polynomial functions on non-compact domains. Int. J. Math. **23**, 1250033 (2012)

Đinh, S.T., Kurdyka, K., Le Gal, O.: Łojasiewicz inequality on non-compact domains and singularities at infinity. Int. J. Math. **24**, 1350079 (2013)

Đinh, S.T., Hà, H.V., Pham, T.S., Tháo, N.T.: Global Łojasiewicz-type inequality for non-degenerate polynomial maps. J. Math. Anal. Appl. **410**, 541–560 (2014)

Gindikin, S.G.: Energy estimates connected with the Newton polyhedron. (Russian) Collection of articles dedicated to the memory of Ivan Georgievič Petrovskiĭ. Trudy Moskov. Mat. Obšč. **31**, 189–236 (1974)

Gwoździewicz, J.: Growth at infinity of a polynomial with a compact zero set. Banach Cent. Publ. **44**, 123–128 (1998)

Hà, H.V., Duc, N.H.: Łojasiewicz inequality at infinity for polynomials in two real variables. Math. Z. **266**, 243–264 (2010)

Hà, H.V., Ngai, H.V., Pham, T.S.: A global smooth version of the classical Łojasiewicz inequality. J. Math. Anal. Appl. **421**, 1559–1572 (2015)

Hörmander, L.: On the division of distributions by polynomials. Ark. Mat. **3**, 555–568 (1958)

Kollár, J.: Sharp effective Nullstellensatz. J. Am. Math. Soc. **1**, 963–975 (1988)

Łojasiewicz, S.: Sur le problème de la division. (French). Stud. Math. **18**, 87–136 (1959)