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THE ŁOJASIEWICZ EXPONENT OF AN ISOLATED WEIGHTED HOMOGENEOUS SURFACE SINGULARITY

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Abstract. We give an explicit formula for the Łojasiewicz exponent of an isolated weighted homogeneous surface singularity in terms of its weights. From the formula we get that the Łojasiewicz exponent is a topological invariant of these singularities.

1. Introduction

Let $f = f(z_1, \dots, z_n) \in \mathbb{C}\{z_1, \dots, z_n\}$ be a convergent power series defining an *isolated singularity* at the origin $\mathbf{0} \in \mathbb{C}^n$; i.e. $f(\mathbf{0}) = 0$ and the gradient of f,

$$\nabla f := \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n} \right) : (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}^n, \mathbf{0}),$$

has an isolated zero at $\mathbf{0} \in \mathbb{C}^n$. The *Lojasiewicz exponent* $\mathcal{L}_0(f)$ *of* f is by definition the smallest $\theta > 0$ such that there exists a neighbourhood U of $\mathbf{0} \in \mathbb{C}^n$ and a constant $c > 0$ such that

$$|\nabla f(z)| \geqslant c\,|z|^{\theta} \quad \text{for all} \quad z \in U.$$

B. Teissier proved that $\mathcal{L}_0(f) + 1$ is equal to the maximal polar invariant of the singularity f ([T], Corollary 2). In particular $\mathcal{L}_0(f)$ depends only on the analytical type of the germ $\{f = 0\}$ (even more: $\mathcal{L}_0(f)$ is an invariant of the "c-cosécance" introduced in [T]). It is an open question whether $\mathcal{L}_0(f)$ is a topological invariant of an isolated singularity f. Let $\mathrm{Suff}_0(f)$ be the C^0-degree of sufficency of f, i.e. the smallest integer r such that f is topologically equivalent to $f + g$ for all g with $\operatorname{ord} g \geq r + 1$. Then $\mathrm{Suff}_0(f) = [\mathcal{L}_0(f)] + 1$ ([T], Theorem 8), where $[a]$ is the integral part of $a \in \mathbb{R}$. The Łojasiewicz exponent can be calculated by means of analytic paths $\varphi(t) = (\varphi_1(t), \dots, \varphi_n(t)) \in \mathbb{C}\{t\}^n$, $\varphi(0) = \mathbf{0}$, $\varphi(t) \neq 0$ in $\mathbb{C}\{t\}^n$. If $\operatorname{ord} \varphi := \inf_{i=1}^n \operatorname{ord} \varphi_i$, then

$$\mathcal{L}_0(f) = \sup_{\varphi} \frac{\operatorname{ord}((\nabla f) \circ \varphi)}{\operatorname{ord} \varphi}$$

(by the Curve Selection Lemma; see also [L-JT]). In the two-dimensional case there are many explicit formulas for $\mathcal{L}_0(f)$ in various terms (see [KL], [CK1], [CK2], [L]). In this paper we investigate the problem of determining the Łojasiewicz exponent

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for weighted homogeneous isolated singularities. Let us recall that if $(w_1, \ldots, w_n)$ is a sequence of n rational numbers (*weights*) such that $w_i \geq 2$ for $i = 1, \ldots, n$, then a polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$ is called *weighted homogeneous of type* $(w_1, \ldots, w_n)$ if f may be written as a sum of monomials $z_1^{\alpha_1} \ldots z_n^{\alpha_n}$ with

$$\frac{\alpha_1}{w_1} + \ldots + \frac{\alpha_n}{w_n} = 1.$$

For another definition of weighted homogeneous polynomials see the Appendix.

The set of weights $\{w_1, \ldots, w_n\}$ of a weighted homogeneous polynomial f defining an isolated singularity is an analytic invariant of the germ $\{f = 0\}$ [S]. Many topological invariants of weighted homogeneous isolated singularities are expressed in terms of weights: for instance, the Milnor number $\mu_0(f)$ of f and the characteristic monodromy polynomial $\Delta_f(t)$ [MO], and in the case of weighted homogeneous isolated surface singularities, the multiplicity of f [Y], the fundamental group $\pi(K_f)$ of the link of f and the minimal resolution of f [OW].

In this note we will give a formula for the Łojasiewicz exponent of weighted homogeneous isolated surface singularities in terms of its weights. Precisely, the Łojasiewicz exponent is equal to the maximum of its weights minus one. As a corollary we obtain that in this class of singularities $\mathcal{L}_0(f)$ is a topological invariant.

Estimations of the Łojasiewicz exponent for quasi-homogeneous isolated singularities in the real and complex cases are in a recent preprint by Haraux and Pham [HP]. Estimations in the general case can be found in [Lt], [F], [P1], [A].

2. Results

The main result of this paper is the following:

Theorem 1. *Let $f = f(z_1, z_2, z_3)$ be a weighted homogeneous polynomial of type (w_1, w_2, w_3) defining an isolated singularity at the origin $\mathbf{0} \in \mathbb{C}^3$. Then*

$$\mathcal{L}_0(f) = \max_{i=1}^{3}(w_i - 1). \tag{2.1}$$

An analogous formula also holds in the case $n = 2$ (Corollary 4). In the general case we have only the inequality "$\leq$"in (2.1); the equality holds under additional assumptions (Propositions 1 and 2 in Section 3).

The proof of the above theorem is given in Section 5.

Corollary 1. $\mathrm{Suff}_0(f) = \left[\max_{i=1}^{3}(w_i)\right]$.

Since weights are a topological invariant of weighted homogeneous surface singularities [Y], Theorem B, we obtain

Corollary 2. *The Łojasiewicz exponent $\mathcal{L}_0(f)$ of weighted homogeneous isolated surface singularities f is a topological invariant.*

It means that if f, f' are weighted homogeneous isolated surface singularities and $(\mathbb{C}^3, V(f), \mathbf{0})$ is homeomorphic to $(\mathbb{C}^3, V(f'), \mathbf{0})$, then $\mathcal{L}_0(f) = \mathcal{L}_0(f')$.

From Corollary 1 we easily get

Corollary 3. $\deg f \leq \mathrm{Suff}_0(f)$.

The above inequality may be strict.

Example 1. Let a, b be integers such that $b \geq 2$ and $\frac{a}{2} > b-1$. The polynomial $f = z_1^a z_2 + z_2^b + z_3^2$ is of type $(\frac{ab}{b-1}, b, 2)$ and defines an isolated singularity at $\mathbf{0} \in \mathbb{C}^3$. Then $\deg f = a+1$ and $\mathrm{Suff}_0(f) = \left[\frac{ab}{b-1}\right] > \deg f$.

The crucial role in the proof of the main theorem is played by the following result concerning arbitrary isolated surface singularities.

Theorem 2. *Let $f = f(z_1, z_2, z_3)$ be an isolated surface singularity and*

$$V\left(\frac{\partial f}{\partial z_2},\ \frac{\partial f}{\partial z_3}\right) \subset V(z_1).$$

Then

$$z_1 \in \left(\frac{\partial f}{\partial z_2}, \frac{\partial f}{\partial z_3}\right) \quad \textit{in} \quad \mathbb{C}\{z_1, z_2, z_3\}.$$

The proof of the above theorem is given in Section 4.

To generalize Theorem 1 to the n-dimensional case it is enough to prove the last theorem in the n-dimensional case in the following formulation.

Problem 1. Let $f = f(z_1, \dots, z_n)$ be an isolated singularity and

$$V\left(\frac{\partial f}{\partial z_2}, \dots, \frac{\partial f}{\partial z_n}\right) \subset V(z_1).$$

Then

$$z_1 \in \left(\frac{\partial f}{\partial z_2}, \dots, \frac{\partial f}{\partial z_n}\right) \quad \text{in} \quad \mathbb{C}\{z_1, \dots, z_n\}.$$

Remark 1. Theorem 1 implies that the maximal polar invariant of a weighted homogeneous isolated surface singularity is equal to its maximal weight.

3. Upper bound for the Łojasiewicz exponent of weighted homogeneous isolated singularities

In this section we will prove

Proposition 1. *Let $f \in \mathbb{C}\{z_1, \dots, z_n\}$ be a weighted homogeneous isolated singularity of type $(w_1, \dots, w_n)$ at $\mathbf{0} \in \mathbb{C}^n$. Then*

$$\mathcal{L}_0(f) \leq \max_{i=1}^{n}(w_i - 1).$$

Remark 2. If f is a homogeneous isolated singularity of degree $d > 1$, then $\mathcal{L}_0(f) = d-1$ ([P2], Lemma 2.4). In this case we have $w_i = d$ for $i = 1, \dots, n$.

We will get Proposition 1 from an estimation of the Łojasiewicz exponent for semi-weighted homogeneous mappings given in [P2] (see also [F], Theorem 3.2). First we recall the notion of the Łojasiewicz exponent for holomorphic mappings with an isolated zero.

Let $\boldsymbol{f} = (f_1, \dots, f_n) \in \mathbb{C}\{z_1, \dots, z_n\}^n$ define a germ of the holomorphic mapping $\boldsymbol{f} : (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}^n, \mathbf{0})$ with an isolated zero at $\mathbf{0} \in \mathbb{C}^n$. The *Łojasiewicz exponent* $l_0(\boldsymbol{f})$ *of* $\boldsymbol{f}$ is by definition the smallest $\theta > 0$ such that there exist a neighbourhood U of $\mathbf{0} \in \mathbb{C}^n$ and a constant $c > 0$ such that

$$|\boldsymbol{f}(z)| \geqslant c\,|z|^{\theta} \quad \text{for all} \quad z \in U.$$

Clearly $\mathcal{L}_0(f) = l_0(\nabla f)$.

Lemma 1. *Let f_i for $i = 1, \dots, n$ be a polynomial whose support* $\operatorname{supp} f_i$ *lies in the hyperplane $q_1\alpha_1 + \ldots + q_n\alpha_n = d_i$, where $q_1, \dots, q_n, d_i > 0$ are integers. Suppose that $\boldsymbol{f} = (f_1, \dots, f_n)$ has an isolated zero at $\mathbf{0} \in \mathbb{C}^n$. Then*

$$l_0(\boldsymbol{f}) \leq \frac{\max_{i=1}^n(d_i)}{\min_{i=1}^n(q_i)}.$$

Proof. See [P2], Proposition 2.2. □

Now we can give

Proof of Proposition 1. Let $q_1, \dots, q_n$ and d be positive integers such that $q_i w_i = d$ for $i = 1, \dots, n$. Since f is an isolated singularity we have $\frac{\partial f}{\partial z_i} \neq 0$ for $i = 1, \dots, n$. Obviously $\operatorname{supp}\left(\frac{\partial f}{\partial z_i}\right)$ lies on the hyperplane $q_1\alpha_1 + \ldots + q_n\alpha_n = d - q_i$. Using Lemma 1 we get

$$\mathcal{L}_0(f) = l_0(\nabla f) \leq \frac{\max_{i=1}^n(d - q_i)}{\min_{i=1}^n(q_i)} = \max_{i=1}^n\left(\frac{d}{q_i} - 1\right) = \max_{i=1}^n(w_i - 1).$$

□

Let $f \in \mathbb{C}\{z_1, \dots, z_n\}$ be an isolated singularity and let $l = \sum_{i=1}^n a_i z_i$ be a linear nonzero form. A (*local*) *polar curve of f related to l* is the germ $\Gamma_l(f)$ of the analytic set given by the equations

$$\frac{\partial(f, l)}{\partial(z_i, z_j)} = 0, \qquad 1 \leq i < j \leq n,$$

near the origin. It is easy to check that $\dim \Gamma_l(f) = 1$. In particular $\Gamma_{z_k}(f)$ is given by the equations

$$\frac{\partial f}{\partial z_1} = \cdots = \frac{\partial f}{\partial z_{k-1}} = \frac{\partial f}{\partial z_{k+1}} = \cdots = \frac{\partial f}{\partial z_n} = 0. \tag{3.1}$$

Proposition 2. *Let $f \in \mathbb{C}\{z_1, \dots, z_n\}$ be a weighted homogeneous isolated singularity of type $(w_1, \dots, w_n)$. Suppose that $w_k = \max_{i=1}^n(w_i)$ and $\Gamma_{z_k}(f) \not\subset V(z_k)$. Then*

$$\mathcal{L}_0(f) = \max_{i=1}^n(w_i - 1).$$

Proof. By Proposition 1 we have

$$\mathcal{L}_0(f) \leq w_k - 1.$$

To check that

$$\mathcal{L}_0(f) \geq w_k - 1$$

we choose an open neighbourhood U of $\mathbf{0} \in \mathbb{C}^n$ such that if $\nabla f(\boldsymbol{z}) = 0$, $\boldsymbol{z} \in U$, then $\boldsymbol{z} = \mathbf{0}$. From the assumption $\Gamma_{z_k}(f) \not\subset V(z_k)$ it follows that the system of equations (3.1) has in U a solution $\boldsymbol{a} = (a_1, \dots, a_n)$ such that $a_k \neq 0$. Let $q_1, \dots, q_n$ and d be integers such that $q_i w_i = d$ for $i = 1, \dots, n$. Set

$$\boldsymbol{\varphi}(t) = (a_1 t^{q_1}, \dots, a_n t^{q_n}).$$

Since supp $\left(\frac{\partial f}{\partial z_i}\right)$ lies on the hyperplane $q_1\alpha_1 + \ldots + q_n\alpha_n = d - q_i$ we get

$$\frac{\partial f}{\partial z_i}(\boldsymbol{\varphi}(t)) = t^{d-q_i}\frac{\partial f}{\partial z_i}(\boldsymbol{a}) = 0 \quad \text{for } i \neq k,$$
$$\frac{\partial f}{\partial z_k}(\boldsymbol{\varphi}(t)) = t^{d-q_k}\frac{\partial f}{\partial z_k}(\boldsymbol{a}) \neq 0.$$

Therefore we get

$$\mathcal{L}_0(f) \geq \frac{\operatorname{ord}((\nabla f)\circ\boldsymbol{\varphi}(t))}{\operatorname{ord}\boldsymbol{\varphi}(t)} = \frac{\operatorname{ord}(\frac{\partial f}{\partial z_k}(\boldsymbol{\varphi}(t)))}{\operatorname{ord}\boldsymbol{\varphi}(t)} = \frac{d-q_k}{q_k} = w_k - 1.$$

□

The above propositions give the formula for the Łojasiewicz exponent in a simpler two-dimensional case.

Corollary 4. *Let $f \in \mathbb{C}\{z_1, z_2\}$ be a weighted homogeneous isolated singularity of type (w_1, w_2) at $\mathbf{0} \in \mathbb{C}^2$. Then*

$$\mathcal{L}_0(f) = \max_{i=1}^{2}(w_i - 1).$$

Proof. Assume that $w_1 \leq w_2$. If $V\left(\frac{\partial f}{\partial z_1}\right) \not\subset V(z_2)$, then the corollary follows from Proposition 2. If $V\left(\frac{\partial f}{\partial z_1}\right) \subset V(z_2)$, then $z_2 = A\frac{\partial f}{\partial z_1}$ in $\mathbb{C}\{z_1, z_2\}$. In fact, by the local Hilbert Nullstellensatz $z_2^p = A\frac{\partial f}{\partial z_1}$ in $\mathbb{C}\{z_1, z_2\}$ for some positive integer p. Assume that p is the smallest possible. Then z_2 does not divide A. Since $\mathbb{C}\{z_1, z_2\}$ is a unique factorization domain we get $\frac{\partial f}{\partial z_1} = z_2^p B$, $B(0,0) \neq 0$. Hence there exist $C \in \mathbb{C}\{z_1, z_2\}$ and $g \in \mathbb{C}\{z_2\}$, $g(0) = 0$, such that

$$f(z_1, z_2) = z_2^p C(z_1, z_2) + g(z_2) \quad \text{in } \mathbb{C}\{z_1, z_2\}.$$

If we had $p > 1$, then by condition $\frac{\partial f}{\partial z_2}(0,0) = 0$ we would obtain $g'(0) = 0$. This would imply

$$\frac{\partial f}{\partial z_1}(z_1, 0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial z_2}(z_1, 0) = 0,$$

which contradicts the assumption that f is an isolated singularity. So $p = 1$, i.e. $z_2 = A\frac{\partial f}{\partial z_1}$ in $\mathbb{C}\{z_1, z_2\}$. Hence $\frac{\partial^2 f}{\partial z_1 \partial z_2}(0,0) \neq 0$. This implies that the monomial cz_1z_2 appears with a nonzero coefficient $c \neq 0$ in the Taylor expansion of f. We then get $\frac{1}{w_1} + \frac{1}{w_2} = 1$, which implies $w_1 = w_2 = 2$ (by definition of weighted homogeneous polynomials $w_1, w_2 \geq 2$). Thus f is a homogeneous form of degree 2 and $\mathcal{L}_0(f) = 1 = \max_{i=1}^{2}(w_i - 1)$ by Remark 2. □

Remark 3. It is well known that if $f = f(z_1, z_2)$ defines an isolated curve singularity, then the Milnor number $\mu_0(f)$ and the Łojasiewicz exponent $\mathcal{L}_0(f)$ are topological invariants of the germ $\{f = 0\}$ ([T]). Moreover, if additionally f is weighted homogeneous of type (w_1, w_2), then by [MO]

$$\mu_0(f) = (w_1 - 1)(w_2 - 1),$$

and by Corollary 4

$$\mathcal{L}_0(f) = \max\left((w_1 - 1), (w_2 - 1)\right).$$

Hence the set of weights

$$\{w_1, w_2\} = \left\{ \frac{\mu_0(f)}{\mathcal{L}_0(f)} + 1, \mathcal{L}_0(f) + 1 \right\}$$

is also a topological invariant of the germ $\{f = 0\}$.

4. Proof of Theorem 2

Proof. In the sequel we will use the following notation for any $P \in \mathbb{C}\{z_1, z_2, z_3\}$. Let $P = P_0 + P_1 z_1 + P_2 z_1^2 + \dots$ with $P_i \in \mathbb{C}\{z_2, z_3\}$ for $i = 0, 1, 2, \dots$. Then we put $\widehat{P} = P_1 + P_2 z_1 + \dots$. Thus $P_0 = P(0, z_2, z_3)$ and $P = P_0 + z_1 \widehat{P}$ in $\mathbb{C}\{z_1, z_2, z_3\}$. Note that

$$\left(\frac{\partial P}{\partial z_2} \right)_0 = \frac{\partial P_0}{\partial z_2} \quad \text{and} \quad \left(\frac{\partial P}{\partial z_3} \right)_0 = \frac{\partial P_0}{\partial z_3}.$$

Let us pass to the proof of the theorem. We have to show that there exists a power series $A, B \in \mathbb{C}\{z_1, z_2, z_3\}$ such that

$$z_1 = A \frac{\partial f}{\partial z_2} + B \frac{\partial f}{\partial z_3} \ \text{ in } \ \mathbb{C}\{z_1, z_2, z_3\}.$$

It is easy to check the following three properties:

(1) The system of equations

$$\frac{\partial f_0}{\partial z_2} = \frac{\partial f_0}{\partial z_3} = f_1 = 0$$

has an isolated solution $z_2 = z_3 = 0$ near the origin $\mathbf{0} \in \mathbb{C}^2$ (otherwise, f does not define an isolated singularity).

(2) The analytic set defined by equations

$$\frac{\partial f_0}{\partial z_2} = \frac{\partial f_0}{\partial z_3} = 0$$

near the origin $\mathbf{0} \in \mathbb{C}^2$ is of pure dimension one (since $\Gamma_{z_1}(f)$ is of pure dimension one and lies in $\{z_1 = 0\}$).

(3) For some integer $p > 0$

$$z_1^p = A \frac{\partial f}{\partial z_2} + B \frac{\partial f}{\partial z_3} \ \text{ in } \ \mathbb{C}\{z_1, z_2, z_3\}$$

(by the local Hilbert Nullstellensatz).

Assume that $p > 0$ in (3) is the smallest possible. Hence $A_0 \neq 0$ or $B_0 \neq 0$. Then we have the following fact.

Property 1. $A_0 \not\equiv 0 \pmod{\frac{\partial f_0}{\partial z_3}}$ or $B_0 \not\equiv 0 \pmod{\frac{\partial f_0}{\partial z_2}}$ in $\mathbb{C}\{z_2, z_3\}$.

Proof of Property 1. Suppose that $A_0 \equiv 0 \pmod{\frac{\partial f_0}{\partial z_3}}$; that is, $A_0 = \widetilde{A_0} \frac{\partial f_0}{\partial z_3}$ in $\mathbb{C}\{z_2, z_3\}$. Then

$$\begin{aligned} A = A_0 + z_1 \widehat{A} = \widetilde{A_0} \frac{\partial f_0}{\partial z_3} + z_1 \widehat{A} &= \widetilde{A_0} \left(\frac{\partial f}{\partial z_3} - z_1 \frac{\partial \widehat{f}}{\partial z_3} \right) + z_1 \widehat{A} \\ &= \widetilde{A_0} \frac{\partial f}{\partial z_3} + z_1 C \text{ in } \mathbb{C}\{z_1, z_2, z_3\}. \end{aligned}$$

From (3) we get

$$\begin{aligned} z_1^p &= \left(\widetilde{A_0}\frac{\partial f}{\partial z_3} + z_1 C\right)\frac{\partial f}{\partial z_2} + B\frac{\partial f}{\partial z_3} \\ &= \left(\widetilde{A_0}\frac{\partial f}{\partial z_2} + B\right)\frac{\partial f}{\partial z_3} + z_1 C\frac{\partial f}{\partial z_2}. \end{aligned}$$

By minimality of p we get $\widetilde{A_0}\frac{\partial f}{\partial z_2} + B \not\equiv 0 \pmod{z_1}$, and consequently $\frac{\partial f}{\partial z_3} \equiv 0 \pmod{z_1}$, which implies $\frac{\partial f_0}{\partial z_3} = 0$ in $\mathbb{C}\{z_2, z_3\}$. Similarly the condition $B_0 \equiv 0 \pmod{\frac{\partial f_0}{\partial z_2}}$ implies $\frac{\partial f_0}{\partial z_2} = 0$ in $\mathbb{C}\{z_2, z_3\}$. This proves Property 1. □

From (3) we get

$$A_0\frac{\partial f_0}{\partial z_2} + B_0\frac{\partial f_0}{\partial z_3} = 0 \text{ in } \mathbb{C}\{z_2, z_3\}. \tag{4.1}$$

Suppose to the contrary that $p > 1$. Then differentiating the equality in (3) and putting $z_1 = 0$ we get

$$A_0\frac{\partial f_1}{\partial z_2} + B_0\frac{\partial f_1}{\partial z_3} + A_1\frac{\partial f_0}{\partial z_2} + B_1\frac{\partial f_0}{\partial z_3} = 0 \text{ in } \mathbb{C}\{z_2, z_3\}. \tag{4.2}$$

From (2) it follows that we may write

$$f_0 = g_0 g_1^{k_1} \dots g_r^{k_r} \quad \text{in } \mathbb{C}\{z_2, z_3\},$$

where $k_i \geq 2$ for $i = 1, \dots, r$, $r \geq 1$, g_i are irreducible and g_i does not divide g_j in $\mathbb{C}\{z_2, z_3\}$ for $i \neq j$. Note that

$$\mathrm{GCD}\left(\frac{\partial f_0}{\partial z_2}, \frac{\partial f_0}{\partial z_3}\right) = g_1^{k_1 - 1} \dots g_r^{k_r - 1}. \tag{4.3}$$

Property 2. There exists an $i \in \{1, \dots, r\}$ such that

$$\frac{\partial(g_i, f_1)}{\partial(z_2, z_3)} \equiv 0 \pmod{g_i}.$$

Proof of Property 2. Using Properties (4.1), (4.3) and Property 1 we check that

$$A_0 \not\equiv 0 \left(\mathrm{mod}\, \mathrm{GCD}\left(\frac{\partial f_0}{\partial z_2}, \frac{\partial f_0}{\partial z_3}\right)\right)$$

or

$$B_0 \not\equiv 0 \left(\mathrm{mod}\, \mathrm{GCD}\left(\frac{\partial f_0}{\partial z_2}, \frac{\partial f_0}{\partial z_3}\right)\right).$$

Therefore there is an $i \in \{1, \dots, r\}$ such that

$$A_0 \not\equiv 0 \left(\mathrm{mod}\, g_i^{k_i - 1}\right) \quad \text{or} \quad B_0 \not\equiv 0 \left(\mathrm{mod}\, g_i^{k_i - 1}\right).$$

We may suppose $i = 1$. Write $f_0 = g_1^{k_1}\widehat{g_1}$ in $\mathbb{C}\{z_2, z_3\}$. Obviously $\widehat{g_1} \not\equiv 0 \pmod{g_1}$. Using (4.1) after a simple calculation we get

$$A_0\left(k_1\frac{\partial g_1}{\partial z_2}\widehat{g_1} + g_1\frac{\partial \widehat{g_1}}{\partial z_2}\right) + B_0\left(k_1\frac{\partial g_1}{\partial z_3}\widehat{g_1} + g_1\frac{\partial \widehat{g_1}}{\partial z_3}\right) = 0 \text{ in } \mathbb{C}\{z_2, z_3\}. \tag{4.4}$$

Hence for each integer $m \geq 0$

$$A_0 \equiv 0 \pmod{g_1^m} \quad \text{if and only if } B_0 \equiv 0 \pmod{g_1^m}.$$

Therefore we can write $A_0 = A_0' g_1^{m_1}$ and $B_0 = B_0' g_1^{m_1}$, where $0 \le m_1 < k_1 - 1$ and $A_0' \not\equiv 0 \,(\operatorname{mod} g_1)$, $B_0' \not\equiv 0 \,(\operatorname{mod} g_1)$. From (4.1) and (4.2) we get

$$A_0' \frac{\partial g_1}{\partial z_2} + B_0' \frac{\partial g_1}{\partial z_3} \equiv 0 \,(\operatorname{mod} g_1) \tag{4.5}$$

and

$$A_0' \frac{\partial f_1}{\partial z_2} + B_0' \frac{\partial f_1}{\partial z_3} \equiv 0 \,(\operatorname{mod} g_1)\,. \tag{4.6}$$

Using Cramer's rule to (4.5) and (4.6) we get

$$A_0' \frac{\partial\,(g_1, f_1)}{\partial\,(z_2, z_3)} \equiv 0 \ (\operatorname{mod} g_1),$$

and Property 2 follows since $A_0' \not\equiv 0 \,(\operatorname{mod} g_1)$ and g_1 is irreducible. □

We omit the simple proof of the next property.

Property 3. Let $P, Q \in \mathbb{C}\{x, y\}$ be power series in two variables x, y without constant term. Let P be irreducible and let $\frac{\partial(P,Q)}{\partial(x,y)} \equiv 0 \ (\operatorname{mod} P)$. Then $Q \equiv 0$ $(\operatorname{mod} P)$.

Now we can finish the proof of Theorem 2. The assumption $p > 1$ implies by Properties 2 and 3 that f_1 vanishes on a branch $V(g_i)$ of the curve $V\left(\frac{\partial f_0}{\partial z_2}, \frac{\partial f_0}{\partial z_3}\right)$. This contradicts property (1). Therefore $p = 1$, which ends the proof. □

5. Proof of Theorem 1

Let $f = f(z_1, z_2, z_3)$ be a weighted homogeneous polynomial of type (w_1, w_2, w_3) defining an isolated singularity at the origin $\mathbf{0} \in \mathbb{C}^3$. We may assume that $w_1 = \max(w_1, w_2, w_3)$. If $\Gamma_{z_1}(f) \not\subset V(z_1)$, then $\mathcal{L}_0(f) = w_1 - 1$ by Proposition 2. Suppose then that $\Gamma_{z_1}(f) \subset V(z_1)$. By Theorem 2 there exists a power series $A, B \in \mathbb{C}\{z_1, z_2, z_3\}$ such that $z_1 = A\frac{\partial f}{\partial z_2} + B\frac{\partial f}{\partial z_3}$. Differentiating and putting $z_1 = z_2 = z_3 = 0$ we obtain

$$\frac{\partial^2 f}{\partial z_1 \partial z_2}(\mathbf{0}) \neq 0 \ \text{ or } \ \frac{\partial^2 f}{\partial z_1 \partial z_3}(\mathbf{0}) \neq 0.$$

Thus the support $\operatorname{supp} f$ contains point $(1, 1, 0)$ or $(1, 0, 1)$. Hence $w_1 = w_2 = 2$ or $w_1 = w_3 = 2$. Since $w_1 = \max(w_1, w_2, w_3)$, then $w_1 = w_2 = w_3 = 2$ and f is homogeneous of degree 2. Consequently $\mathcal{L}_0(f) = 1 = w_1 - 1$ by Remark 2, and the theorem is proved.

Remark 4. Let $f = f_0 + f_1 z_1 + f_2 z_1^2 + \ldots$ with $f_i \in \mathbb{C}\{z_2, z_3\}$ for $i = 0, 1, \ldots$ be an isolated surface singularity such that $\Gamma_{z_1}(f) \subset V(z_1)$. From the proofs of Theorems 1 and 2 it follows that f_0 has a multiple factor and $\operatorname{ord} f_1 = 1$. In particular $\operatorname{ord} f = 2$.

6. Appendix

There is another (weaker) definition of a weighted homogeneous polynomial. A polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$ is called a *weak weighted homogeneous polynomial* if there exist n rational positive numbers (*weights*) $(w_1, \ldots, w_n)$ such that f may be written as a sum of monomials $z_1^{\alpha_1} \ldots z_n^{\alpha_n}$ with

$$\frac{\alpha_1}{w_1} + \ldots + \frac{\alpha_n}{w_n} = 1.$$

Observe that we don't assume here that $w_i \geq 2$ for $i = 1, \ldots, n$. The weights are not uniquely determined by the weak weighted homogeneous polynomial. If a weak weighted homogeneous polynomial f of type $(w_1, \ldots, w_n)$ defines an isolated singularity at the origin, then $w_i > 1$ for all $i = 1, \ldots, n$ and

$$\mu_0(f) = \prod_{i=1}^{n}(w_i - 1)$$

([MO], Theorem 1). The class of weak weighted homogeneous polynomials is broader than the class of weighted homogeneous polynomials. However, we can extend our main theorem to this class.

Theorem 3. *Let $f = f(z_1, z_2, z_3)$ be a weak weighted homogeneous polynomial of type (w_1, w_2, w_3) defining an isolated singularity at the origin. Then*

$$\mathcal{L}_0(f) = \min\left(\max_{i=1}^{3}(w_i - 1),\ \prod_{i=1}^{3}(w_i - 1)\right).$$

Note that if $w_i \geq 2$ for all $i = 1, 2, 3$, then $\max_{i=1}^{3}(w_i - 1) \leq \prod_{i=1}^{3}(w_i - 1)$ and we recover Theorem 1.

In the proof we need the following useful lemma:

Lemma 2. *Let $f \in \mathbb{C}\{z_1, \ldots, z_n\}$ define an isolated singularity at the origin. Then*

$$\mathcal{L}_0(f) \leq \mu_0(f)$$

with equality if

$$\operatorname{rk}\left(\frac{\partial^2 f}{\partial z_i \partial z_j}(0)\right) \geq n - 1. \tag{6.1}$$

Proof. It is well known that the monomials $z_1^{\mu}, \ldots, z_n^{\mu}$, $\mu = \mu_0(f)$, belong to the ideal $\left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)$. Whence the inequality $\mathcal{L}_0(f) \leq \mu_0(f)$ follows. If (6.1) holds, then we may assume, by the splitting lemma, that $f = z_1^2 + \ldots + z_{n-1}^2 + z_n^{\mu}$. This obviously implies $\mathcal{L}_0(f) = \mu_0(f)$. □

Remark 5. One can prove that the equality $\mathcal{L}_0(f) = \mu_0(f)$ implies the inequality (6.1)

Proof of Theorem 3. We get $\mathcal{L}_0(f) \leq \mu_0(f) = \prod_{i=1}^{3}(w_i - 1)$ by the Milnor-Orlik formula. On the other hand our proof of Proposition 1 is valid in the case of weak weighted homogeneous isolated singularities, and consequently $\mathcal{L}_0(f) \leq \max_{i=1}^{3}(w_i - 1)$. Summing up we obtain the bound

$$\mathcal{L}_0(f) \leq \min\left(\max_{i=1}^{3}(w_i - 1),\ \prod_{i=1}^{3}(w_i - 1)\right). \tag{6.2}$$

To prove the opposite inequality we suppose, to the contrary, that we have strict "$<$" inequality in (6.2). Then

$$\mathcal{L}_0(f) < \max_{i=1}^{3}(w_i - 1), \tag{6.3}$$

$$\mathcal{L}_0(f) < \prod_{i=1}^{3}(w_i - 1). \tag{6.4}$$

We may assume that $\max_{i=1}^{3}(w_i) = w_1$. Inequality (6.3) implies $V\left(\frac{\partial f}{\partial z_2}, \frac{\partial f}{\partial z_3}\right) \subset V(z_1)$ (cf. the proof of Theorem 1). Using Remark 4 we check that, up to a permutation of variables z_2, z_3,

$$f(z_1, z_2, z_3) = az_3^k + bz_1z_2 + z_1^2 g(z_1, z_3),$$

where $g(z_1, z_3)$ is a polynomial, $ab \neq 0$, and $k \geq 2$. Using Lemma 2 we check that $\mathcal{L}_0(f) = \mu_0(f)$. Since $\mu_0(f) = \prod_{i=1}^{3}(w_i - 1)$ by the Milnor-Orlik formula, then $\mathcal{L}_0(f) = \prod_{i=1}^{3}(w_i - 1)$, which contradicts (6.4). $\square$

References

[A] Abderrahmane, O. M.: *On the Łojasiewicz exponent and Newton polyhedron.* Kodai. Math. J. 28 (2005), 106–110. MR2122194 (2006c:32031)

[CK1] Chadzyński, J. and Krasiński, T.: *The Łojasiewicz exponent of an analytic mapping of two complex variables at an isolated zero.* In: Singularities, Banach Center Publ. 20, PWN, Warszawa, 1988, 139–146. MR1101835 (92e:32018)

[CK2] Chadzyński, J. and Krasiński, T.: *Resultant and the Łojasiewicz exponent.* Ann. Polon. Math. 61 (1995), 95–100. MR1318321 (96a:32061)

[F] Fukui, T.: *Łojasiewicz type inequalities and Newton diagrams.* Proc. Amer. Math. Soc. 112 (1991), 1169–1183. MR1065945 (91j:58021)

[HP] Haraux, A. and Pham, T. S.: *On the Łojasiewicz exponents of quasi-homogeneous functions.* Preprints of the Laboratoire Jacques-Louis Lions 2007, Université Pierre et Marie Curie, No. R07041 (`http://www.ann.jussieu.fr/publications/2007/R07041.pdf`).

[KL] Kuo, T. C. and Lu, Y. C.: *On analytic function germs of two complex variables.* Topology 16 (1977), 299–310. MR0460711 (57:704)

[L-JT] Lejeune-Jalabert, M. and Teissier, B.: *Cloture integrale des idéaux et equisingularité.* École Polytechnique, 1974.

[L] Lenarcik, A.: *On the Łojasiewicz exponent of the gradient of a holomorphic function.* In: Singularities Symposium–Łojasiewicz 70. Banach Center Publ. 44, PWN, Warszawa, 1998, 149–166. MR1677363 (2000h:32039)

[Lt] Lichtin, B.: *Estimation of Łojasiewicz exponents and Newton polygons.* Inventiones Math. 64 (1981), 417–429. MR632982 (83b:32006)

[MO] Milnor, J. and Orlik, P.: *Isolated singularities defined by weighted homogeneous polynomials,* Topology 9 (1970), 385–393. MR0293680 (45:2757)

[OW] Orlik, P. and Wagreich, P.: *Isolated singularities of algebraic surfaces with* $\mathbb{C}^*$*-action.* Ann. of Math. 93 (1971), 205–228. MR0284435 (44:1662)

[P1] Płoski, A.: *Sur l'exposant d'une application analytique I.* Bull. Polish Acad. Sci. Math. 32 (1984), 669–673. MR786190 (86j:32025a)

[P2] Płoski, A.: *Sur l'exposant d'une application analytique II.* Bull. Polish Acad. Sci. Math. 33 (1985), 123–127. MR805025 (86j:32025b)

[S] Saito, K.: *Quasihomogene isolierte Singularitäten von Hyperflächen.* Invent. Math. 14 (1971), 123–142. MR0294699 (45:3767)

[T] Teissier, B.: *Variétés polaires.* Invent. Math. 40 (1977), 267–292. MR0470246 (57:10004)
[Y] Yau, S. S.-T.: *Topological types and multiplicities of isolated quasihomogeneous surface singularities.* Bull. Amer. Math. Soc. (N.S.) 19 (1988), 447–454. MR935021 (92b:32042)

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