Bull. Korean Math. Soc. **48** (2011), No. 1, pp. 79–83
DOI 10.4134/BKMS.2011.48.1.079

THE LOWER AUTOCENTRAL SERIES OF ABELIAN GROUPS

Mohammad Reza R. Moghaddam, Foroud Parvaneh, and Mohammad Naghshineh

Abstract. In the present paper we introduce the lower autocentral series of autocommutator subgroups of a given group. Following our previous work on the subject in 2009, it is shown that every finite abelian group is isomorphic with n^{th}-term of the lower autocentral series of some finite abelian group.

1. Introduction

Let $A = \mathrm{Aut}(G)$ denote the group of automorphisms of a given group G. For any element $g \in G$ and $\alpha \in A$ the element $[g, \alpha] = g^{-1}g^{\alpha}$ is an *autocommutator* of g and α. We define the autocommutator of higher weight inductively as follows:

$$[g, \alpha_1, \alpha_2, \ldots, \alpha_i] = [[g, \alpha_1, \alpha_2, \ldots, \alpha_{i-1}], \alpha_i]$$

for all $\alpha_1, \alpha_2, \ldots, \alpha_i \in A$.

So the *autocommutator subgroup* of *weight* $i + 1$ is defined in the following way:

$$K_i(G) = [G, \underbrace{A, \ldots, A}_{i\text{-times}}] = \langle [g, \alpha_1, \alpha_2, \ldots, \alpha_i] \mid g \in G, \alpha_1, \alpha_2, \ldots, \alpha_i \in A \rangle.$$

Clearly $K_i(G)$ is a characteristic subgroup of G for all $i \geq 1$. Therefore, one obtains a descending chain of autocommutator subgroups of G as follows:

$$G \supseteq K_1(G) \supseteq K_2(G) \supseteq \cdots \supseteq K_i(G) \supseteq \cdots,$$

which we may call it the *lower autocentral series* of G. The aim of the present paper is to prove the following main result.

Theorem 1.1. *For any finite abelian group G and every natural number $n \in \mathbb{N}$, there exists a finite abelian group H such that*

$$G \cong K_n(H).$$

Received May 11, 2009.
2010 *Mathematics Subject Classification.* 20D45, 20D25, 20E34.
Key words and phrases. autocommutator subgroup, autocentral series, abelian group.

2. Preliminary results

In order to prove our main result, we need to prove some technical lemmas, which are interested in their own right.

Lemma 2.1. *Let $G = H \times K$ be the direct product of two characteristic subgroups H and K. Then for any natural number n,*

$$K_n(H \times K) = K_n(H) \times K_n(K).$$

Proof. Clearly every element $g \in G$ can be written as $g = hk$, where $h \in H$ and $k \in K$. By Lemma 2.1 of [5],

$$\mathrm{Aut}(G) \cong \mathrm{Aut}(H) \times \mathrm{Aut}(K).$$

Using induction on n, we show that

$$[hk, \alpha_1, \alpha_2, \ldots, \alpha_n] = [h, \alpha_{1|H}, \alpha_{2|H}, \ldots, \alpha_{n|H}][k, \alpha_{1|K}, \alpha_{2|K}, \ldots, \alpha_{n|K}],$$

for all $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathrm{Aut}(G)$. If $n = 1$, then since the subgroups H and K are characteristic in G and $hk = kh$, we have

$$[hk, \alpha_1] = [h, \alpha_1][k, \alpha_1].$$

Now, assume the result holds for $n - 1$, then

$$\begin{aligned}
&[hk, \alpha_1, \alpha_2, \ldots, \alpha_n] \\
&= [[hk, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}], \alpha_n] \\
&= [[h, \alpha_{1|H}, \alpha_{2|H}, \ldots, \alpha_{n-1|H}][k, \alpha_{1|K}, \alpha_{2|K}, \ldots, \alpha_{n-1|K}], \alpha_n] \\
&= [h, \alpha_{1|H}, \alpha_{2|H}, \ldots, \alpha_{n-1|H}, \alpha_{n|H}][k, \alpha_{1|K}, \alpha_{2|K}, \ldots, \alpha_{n-1|K}, \alpha_{n|K}].
\end{aligned}$$

This implies that $K_n(G) \subseteq K_n(H) \times K_n(K)$.

By Lemma 2.1 of [5], any automorphism μ of H, can be extended to an automorphism $\bar{\mu}$ of G. Hence, for all $\mu_1, \ldots, \mu_n \in \mathrm{Aut}(H)$ and $h \in H$

$$[h, \mu_1, \ldots, \mu_n] = [h, \bar{\mu_1}, \ldots, \bar{\mu_n}] \in K_n(G).$$

Therefore $K_n(H) \subseteq K_n(G)$. Similarly, $K_n(K)$ is contained in $K_n(G)$. Thus

$$K_n(G) = K_n(H) \times K_n(K).$$

□

Using the above notation, we have the following

Lemma 2.2. *If G is a finite cyclic group, then for any natural number n,*

$$K_n(G) = G^{2^n}.$$

Proof. Let $G = \langle x \mid x^m = 1\rangle$ be the cyclic group of order m. Clearly $\varphi : x \mapsto x^i$ is an automorphism of G if and only if $(i, m) = 1$. Since G is abelian, it follows that $\alpha : x \mapsto x^{-1}$ is an automorphism. So by an easy induction, for all $g \in G$, if n is even

$$g^{2^n} = [g, \underbrace{\alpha, \ldots, \alpha}_{n\text{-times}}] \in G^{2^n},$$

and if n is odd, then

$$g^{2^n} = [g^{-1}, \underbrace{\alpha, \ldots, \alpha}_{n\text{-times}}] \in G^{2^n},$$

which implies that G^{2^n} is contained in $K_n(G)$.

Conversely, assume $(2^n, m) = 1$ then m is odd number and hence $G = G^{2^n}$, which follows that $K_n(G) \subseteq G^{2^n}$. Now, assume $(2^n, m) \neq 1$, i.e., m is an even number then using an easy induction for all $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathrm{Aut}(G)$, we obtain that

$$[g, \alpha_1, \alpha_2, \ldots, \alpha_n] = g^{(i_1-1)\cdots(i_n-1)} \in G^{2^n},$$

where $\alpha_j(g) = g^{i_j}$, $(i_j, m) = 1$ that is to say $i_j - 1$ is even, for all $1 \leq j \leq n$. This completes the proof. □

Lemma 2.3. *Let G be a finite abelian group of odd order m and Z_2 the cyclic group of order 2. Then $K_n(G)$ and $K_n(G \times Z_2)$ are both isomorphic with G for all natural number n.*

Proof. Clearly, by the assumption $(2^n, m) = 1$ and hence by Lemma 2.2, $K_n(G) = G$.

One notes that G and Z_2 are both characteristic subgroups in the direct product $G \times Z_2$. Therefore, Lemma 2.1 implies that

$$K_n(G \times Z_2) = K_n(G) \times K_n(Z_2).$$

Now, the triviality of $K_n(Z_2)$ gives the result. □

The following proposition is very useful in our further investigations.

Proposition 2.4. *Let G be a cyclic group of order 2^m and H be an abelian 2-group of exponent 2^n with $n < m$. Then*

$$K_n(G \times H) = G^{2^n} \times H^{2^{n-1}}.$$

Proof. Let $G = \langle g \mid g^{2^m} = 1 \rangle$ be the cyclic group of order 2^m. Then we define the automorphisms α_h and α'_h of the group $G \times H$, given by $g^{\alpha_h} = gh, h^{\alpha_h} = h$ and $g^{\alpha'_h} = gh^{-1}, h^{\alpha'_h} = h^{-1}$ for all $h \in H$.

Now, if n is even, then

$$h^{2^{n-1}} = [\, g, \underbrace{\alpha'_h, \ldots, \alpha'_h}_{n\text{-times}} \,],$$

and if n is odd we have

$$h^{2^{n-1}} = [\, g, \alpha_h, \underbrace{\alpha'_h, \ldots, \alpha'_h}_{(n-1)\text{-times}} \,].$$

These imply that $H^{2^{n-1}} \subseteq K_n(G \times H)$ and $G^{2^n} \subseteq K_n(G) \subseteq K_n(G \times H)$. Thus

$$G^{2^n} \times H^{2^{n-1}} \subseteq K_n(G \times H).$$

On the other hand, using the structures of the groups G and H for all $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathrm{Aut}(G \times H)$ and $x \in G \times H$,

$$[x, \alpha_1, \alpha_2, \ldots, \alpha_n] \in G^{2^n} \times H^{2^{n-1}},$$

which implies that $K_n(G \times H) \subseteq G^{2^n} \times H^{2^{n-1}}$ and the proof is complete. □

The following theorem follows from the above proposition, which is interested in its own right.

Theorem 2.5. *For all natural numbers $m \geq n_1 \geq \cdots \geq n_r$ and $n \geq 2$,*

$$K_n(Z_{2^m} \times Z_{2^{n_1}} \times \cdots \times Z_{2^{n_r}}) = Z_{2^{m-n}} \times Z_{2^{n_1-(n-1)}} \times \cdots \times Z_{2^{n_r-(n-1)}}.$$

Proof of Theorem 1.1. Let G be a finite abelian group, which can be written as a product of its Sylow subgroups. Now, if $|G|$ is an odd number, then by Lemma 2.3,

$$G = K_n(G).$$

Assume 2 divides $|G|$ and A is the Sylow 2-subgroup of G, then $G = A \times P_1 \times \cdots \times P_s$, where $P_i^{'}s$ are Sylow p_i-subgroups of G $(1 \leq i \leq r)$. By Lemma 2.1,

$$K_n(G) = K_n(A) \times P_1 \times \cdots \times P_s.$$

As A is an abelian 2-group, we may write A as a direct product of cyclic groups of orders some powers of 2, as follows:

$$A \cong Z_{2^m} \times Z_{2^{n_1}} \times \cdots \times Z_{2^{n_r}},$$

where $m \geq n_1 \geq \cdots \geq n_r$.

Now, we choose the abelian group

$$H = Z_{2^{m+n}} \times Z_{2^{n_1+n-1}} \times \cdots \times Z_{2^{n_r+(n-1)}} \times P_1 \times \cdots \times P_s.$$

It can be easily seen that

$$K_n(H) = G,$$

and hence the claim is proved. □

References

[1] C. Chiš, M. Chiš, and G. Silberberg, *Abelian groups as autocommutator groups*, Arch. Math. (Basel) **90** (2008), no. 6, 490–492.

[2] M. Deaconescu and G. L. Walls, *Cyclic groups as autocommutator groups*, Comm. Algebra **35** (2007), no. 1, 215–219.

[3] P. Hegarty, *The absolute centre of a group*, J. Algebra **169** (1994), no. 3, 929–935.

[4] ———, *Autocommutator subgroups of finite groups*, J. Algebra **190** (1997), no. 2, 556–562.

[5] M. Naghshineh, M. R. R. Moghaddam, and F. Parvaneh, *The third term of the lower autocentral series of abelian groups*, Journal of Mathematical Extension, Vol. 4, No. 1 (2009), 1–6.

[6] D. J. S. Robinson, *A Course in the Theory of Groups*, Graduate Texts in Mathematics, 80. Springer-Verlag, New York-Berlin, 1982.

Mohammad Reza R. Moghaddam
Faculty of Mathematical Sciences
Ferdowsi University of Mashhad
and
Centre of Excellence in Analysis on Algebraic Structures
Mashhad, Iran
E-mail address: rezam@ferdowsi.um.ac.ir

Foroud Parvaneh
Department of Mathematics
Islamic Azad University
Mashhad-Branch, Mashhad, Iran
E-mail address: fparvaneh@iauksh.ac.ir

Mohammad Naghshineh
Department of Mathematics
Islamic Azad University
Mashhad-Branch, Mashhad, Iran
E-mail address: mnaghinehfard@yahoo.com