THE LOWER BOUND CONJECTURE FOR 3- AND 4-MANIFOLDS

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1. Introduction

For any closed connected d-manifold M let $f(M)$ denote the set of vectors $f(K) = (f_0(K), ..., f_d(K))$, where K ranges over all triangulations of M and $f_k(K)$ denotes the number of k-simplices of K. The principal results of this paper are Theorems 1 through 5 below, which, together with the Dehn-Sommerville equations reviewed in § 2, yield a characterization of $f(M)$ for some of the simpler 3- and 4-manifolds. The results for the 3- and 4-spheres given in Theorems 1 and 5 have immediate and obvious implications for simplicial polytopes, i.e., closed bounded convex polyhedra all of whose proper faces are simplices. In particular they provide a strong affirmative resolution in dimensions 4 and 5 of the so-called lower bound conjecture for simplicial polytopes. For a discussion of this conjecture, which in dimension 4 goes back at least to a paper by Brückner in 1909, and some limited results in higher dimensions the reader is referred to Section 10.3 of Grünbaum's book on polytopes [2]. Theorem 3, which is concerned with triangulations of projective 3-space, also has an immediate implication for a special subclass of the centrally symmetric simplicial polytopes. This result is stated as Theorem 6.

Some special classes $\mathcal{H}^d(n)$, $d \geqslant 1$, $n \geqslant 0$, of abstract simplicial complexes figure in the statement and proof of these theorems. For $d \geqslant 2$ each class $\mathcal{H}^d(n)$ consists of certain especially simple triangulations of a class of closed d-manifolds which might be described as d-spheres with n orientable or nonorientable handles. The classes $\mathcal{H}^d(n)$ may be defined inductively as follows:

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Definition of $\mathcal{H}^d(n)$. (a) The boundary complex of any abstract $(d+1)$-simplex is a member of $\mathcal{H}^d(0)$. (b) If K is in $\mathcal{H}^d(0)$ and σ is any d-simplex of K, then K' is in $\mathcal{H}^d(0)$, where K' is any complex obtained from K by deleting σ and adding the join of the boundary complex Bd σ and a new vertex distinct from the vertices of K. (c) If K is in $\mathcal{H}^d(n)$, then K' is in $\mathcal{H}^d(n+1)$ if there exist d-simplices σ_1 and σ_2 in K with no common vertices and a dimension-preserving simplicial map ϕ from $K-\{\sigma_1\}-\{\sigma_2\}$ onto K' which identifies Bd σ_1 with Bd σ_2 but otherwise is one-to-one.

THEOREM 1. *There exists a triangulation K of the 3-sphere S^3 with f_0 vertices and f_1 edges if and only if $f_0 \geq 5$ and*

$$4f_0 - 10 \leq f_1 \leq f_0(f_0-1)/2.$$

Moreover K is a triangulation of S^3 satisfying $f_1(K) = 4f_0(K) - 10$ if and only if $K \in \mathcal{H}^3(0)$.

THEOREM 2. *Let M be either the orientable 3-handle $H^3_+ = S^2 \times S^1$ or the nonorientable 3-handle H^3_- obtained from $S^2 \times [0, 1]$ by an antipodal identification of $S^2 \times 0$ and $S^2 \times 1$. There exists a triangulation K of M with f_0 vertices and f_1 edges if and only if $f_0 \geq 9$ and*

$$4f_0 \leq f_1 \leq f_0(f_0-1)/2,$$

except that $(f_0, f_1) = (9, 36)$ is impossible if $M = H^3_+$. Moreover, K is a triangulation of H^3_+ or H^3_- satisfying $f_1(K) = 4f_0(K)$ if and only if $K \in \mathcal{H}^3(1)$.

THEOREM 3. *There exists a triangulation K of projective 3-space P^3 with f_0 vertices and f_1 edges if and only if $f_0 \geq 11$ and*

$$4f_0 + 7 \leq f_1 \leq f_0(f_0-1)/2.$$

Moreover K is a triangulation of P^3 satisfying $f_1(K) = 4f_0(K) + 7$ if and only if K can be obtained from K_0 by a sequence of central retriangulations of 3-simplices, where K_0 is the triangulation of P^3 with 11 vertices and 51 edges described in § 8.

THEOREM 4. *Suppose M is any closed connected 3-manifold distinct from S^3, H^3_+, H^3_-, and P^3. Then there exists an integer $\gamma(M) > 7$ such that*

$$f_1(K) \geq 4f_0(K) + \gamma(M)$$

for any triangulation K of M. Conversely, there exists $\gamma^*(M) \geq \gamma(M)$ such that for every (f_0, f_1) satisfying $f_0 \geq 0$ and

$$4f_0 + \gamma^*(M) \leq f_1 \leq f_0(f_0-1)/2$$

there is a triangulation of M with f_0 vertices and f_1 edges.

THEOREM 5. *If K is a connected complex in which the link of each vertex is a triangulation of a closed connected 3-manifold, in particular if K is a triangulation of a closed connected 4-manifold, then*

$$f_1(K) \geq 5f_0(K) - \tfrac{15}{2}\chi(|K|),$$

where $\chi(|K|) = \chi(K)$ *is the Euler characteristic of* K. *Moreover, equality holds if and only if* $K \in \mathcal{H}^4(1 - \frac{1}{2}\chi(K))$.

THEOREM 6. *Let* $\mathcal{D}_p^d$ *denote the class of centrally symmetric simplicial d-polytopes such that no centrally symmetric pair of vertices can be connected by a path consisting of fewer than three edges of the polytope. If* $P \in \mathcal{D}_p^4$ *then*

$$f_1(P) \geq 4f_0(P) + 14.$$

Moreover, $f_1(P) = 4f_0(P) + 14$ *if and only if* P *can be obtained by successively adding pyramids in centrally symmetric pairs to the faces of some member of* $\mathcal{D}_p^4$ *combinatorially equivalent to the polytope* P_0 *with 22 vertices and 102 edges described in § 8.*

For any closed connected triangulable d-manifold M^d let $\gamma(M^d)$ denote the infimum of $f_1(K) - (d+1)f_0(K)$ as K ranges over the triangulations of M^d. It may be conjectured that $\gamma(M^d) \geq \gamma(S^d) = -\binom{d+2}{2}$ for $d \geq 2$, with equality only if M^d is the d-sphere S^d. When $d = 2$ this follows directly from the well-known fact that $\chi(M^2) \leq 2$, with equality only if M^2 is S^2, and the Dehn-Sommerville equation (2.1)

$$f_1(K) = 3f_0(K) - 3\chi(K), \quad |K| \approx M^2.$$

Theorems 1 through 4 show that the conjecture is true for $d = 3$, i.e.,

$$f_1(K) \geq 4f_0(K) - 10 \tag{1.1}$$

for any triangulation K of any M^3, with equality only if M^3 is S^3. Note that the conjecture for $d = 4$ is not inconsistant with Theorem 5 despite the fact that every integer can be realized as the Euler characteristic of some M^4.

Theorems 1 through 4 are stated in a way which emphasizes the characterizations of the sets $f(M)$ separately for the four 3-manifolds S^3, H^3_+, H^3_-, and P^3. The proofs are organized along quite different lines, and certain portions of them may be read independently as indicated below. The section headings are:

1. *Introduction.*
2. *Review of manifolds.*
3. *Surgery on 3-manifolds.*
4. $\mathcal{H}^d(n)$ *and simple d-trees.*
5. *Proof of the lower bounds.*

In § 2 a brief review of the properties of triangulated manifolds is given, and it is shown how the Dehn-Sommerville equations for polytopes can be generalized so as to hold for arbitrary triangulated manifolds. The Dehn-Sommerville equations show that for any 3- or 4-manifold M the numbers $f_0(K)$ and $f_1(K)$ determine the remaining components of the vector $f(K)$ as K ranges over the triangulations of M. Thus Theorems 1 through 3 will in fact characterize $f(M)$ for the four cases S^3, H^3_+, H^3_-, and P^3.

A number of surgical operations on triangulated manifolds are employed in the proofs. One of these operations is of sufficient importance to be sketched here. Suppose K is a triangulation of a 3-manifold M, and suppose K contains the boundary complex of a 3-simplex σ, but not σ itself. Then it is intuitively plausible that K can be cut along Bd σ, opened up, and patched with two 3-simplices to form a new complex K'', and either K'' is the disjoint union of two triangulated 3-manifolds or it is a triangulation of the 3-manifold obtained from M by removing an orientable or nonorientable handle. The details necessary to substantiate this intuitive picture are given in § 3.

In § 4 the cutting and patching operation of § 3 is extended to higher dimensions for a restricted class of complexes containing the classes $\mathcal{H}^d(n)$. The results of this section are required in the proof of the second part of Theorem 5. § 4 also introduces the notion of simple d-trees used in § 7 and § 9.

The important inequality (1.1) noted above is derived in the first half of § 5. A trivial observation establishes

$$f_1(K) - \alpha f_0(K) \geqslant 10 - 5\alpha \tag{1.2}$$

for $\alpha = 2$. Using this result it is then shown that (1.2) holds for $\alpha = 42/13$. Finally (1.2) is proved for $\alpha = 4$, which is just (1.1). The proof is sufficiently involved that special classes $\mathcal{R}(\alpha)$, $\alpha \leqslant 4$, of triangulated 3-manifolds are introduced. The derivation of the properties of the members of these classes is somewhat tedious and is deferred until § 10.

In the second half of § 5 it is shown that any triangulated 3-manifold K which minimizes $f_1(K) - 4f_0(K)$ among all triangulations of $|K|$ can be obtained from members of the class $\mathcal{R}(4)$ and boundary complexes of 4-simplices by the reverse of the cutting and patching operation described above. The end result of § 11 is that K_0 is the only member K of $\mathcal{R}(4)$

such that $f_1(K) \leqslant 4f_0(K)+7$. The lower bounds in Theorems 1 through 4, as well as the characterization of the triangulations which achieve them in Theorems 1 through 3, follow immediately. If only the lower bounds in Theorems 1 and 2, the characterization of the triangulations which achieve them, and a corresponding weaker version of the first part of Theorem 4 are desired, only the material in § 3, § 5, § 10, and a small part of § 9 is required.

In § 7 the interest in lower bounds for $f_1(K)$ given $f_0(K)$ is replaced by an interest in upper bounds. It is shown that every triangulable 3-manifold can be triangulated so that the closed star of some edge contains all the vertices and every pair of vertices is connected by an edge. This result is then used to prove the second part of Theorem 4.

In § 9 the results of § 7 are combined with explicit triangulations of S^3, H^3_+, H^3_-, and variants of K_0 to demonstrate the existence of all triangulations required in Theorems 1 through 3.

2. Review of manifolds

The material in this section is intended primarily for the reader unfamiliar with certain more or less standard results on manifolds. Throughout this and subsequent sections, unless otherwise indicated, *complex* will mean an unoriented closed finite abstract simplicial complex. Generally, terminology and notation will follow [1] or [4].

Unless otherwise indicated, *d-manifold* will mean a closed connected topological d-manifold, that is, a compact connected metric space, every point of which possesses a neighborhood homeomorphic to a d-cell, i.e., homeomorphic to E^d. It is known that any d-manifold can be triangulated if $d \leqslant 3$ [6].

A simplicial complex K is called an homology d-manifold if it is connected, d-dimensional, and for every k-simplex σ, $0 \leqslant k < d$, the complex $\operatorname{BdSt}\sigma$ has the same homology groups as a $(d-1)$-sphere. Any triangulation of a d-manifold is an homology d-manifold, and if K is an homology d-manifold, then any other triangulation of $|K|$ is an homology d-manifold [1; Art. 7–4].

If K is an homology d-manifold, then the link of any vertex v is an homology manifold with the groups of a $(d-1)$-sphere. (A proof of this for the first barycentric subdivision K' is given in [1; Art. 7–7], but obviously $|\operatorname{Lk}(v; K')| \approx |\operatorname{Lk}(v; K)|$.) By induction if K is an homology d-manifold then K is a connected, d-dimensional complex in which $\operatorname{Lk}\sigma$ has the groups of a $(d-k-1)$-sphere for every k-simplex σ of K, $0 \leqslant k < d$. The converse holds also since $\operatorname{BdSt}\sigma$ is the join of $\operatorname{Bd}\sigma$ and $\operatorname{Lk}\sigma$ and therefore has the proper groups [5; p. 111].

Klee [3] calls a simplicial complex K an Euler d-manifold if it is d-dimensional and for every k-simplex σ, $0 \leqslant k < d$, the complex $\operatorname{Lk}\sigma$ has the same Euler characteristic as a

$(d-k-1)$-sphere, that is, $\chi(\mathrm{Lk}\,\sigma)=\sum_{i\geqslant 0}(-1)^i f_i(\mathrm{Lk}\,\sigma)=1-(-1)^{d-k}$. Let us say that an Euler d-manifold K is *locally connected* if Lk σ is connected for every k-simplex σ of K, $0\leqslant k\leqslant d-2$. The Euler–Poincaré formula shows that any homology d-manifold, and hence any triangulation of a topological d-manifold, is necessarily an Euler d-manifold. Clearly any homology d-manifold is locally connected. Conversely, a connected, locally connected Euler d-manifold is a triangulation of a topological d-manifold if $d\leqslant 3$. This is easily seen if $d\leqslant 2$, and for $d=3$ it follows from the well-known fact that the only 2-manifold with Euler characteristic 2 is the 2-sphere. Thus the link of a k-simplex in a locally connected Euler d-manifold, or any triangulated topological d-manifold, is a triangulation of a $(d-k-1)$-sphere if $d-k\leqslant 3$ and a triangulation of a topological $(d-k-1)$-manifold if $d-k=4$. But it is not even known whether the link of a vertex of a triangulated topological 4-manifold is necessarily a 3-sphere.

If K is any Euler d-manifold, the numbers $f_i(K)$ satisfy a set of linear equations which generalize the so-called Dehn-Sommerville equations for simplicial $(d+1)$-polytopes. For d equal to 2, 3, and 4 the generalized equations are equivalent to

$$\left.\begin{aligned} f_1(K)&=3f_0(K)-3\chi(K)\\ f_2(K)&=2f_0(K)-2\chi(K)\end{aligned}\right\}\ (d=2), \tag{2.1}$$

$$\left.\begin{aligned} f_2(K)&=2f_1(K)-2f_0(K)\\ f_3(K)&=f_1(K)-f_0(K)\end{aligned}\right\}\ (d=3),$$

and

$$\left.\begin{aligned} f_2(K)&=4f_1(K)-10f_0(K)+10\chi(K)\\ f_3(K)&=5f_1(K)-15f_0(K)+15\chi(K)\\ f_4(K)&=2f_1(K)-6f_0(K)+6\chi(K).\end{aligned}\right\}\ (d=4), \tag{2.2}$$

These equations can be derived directly from the corresponding equations for polytopes given in Table 3 on page 425 of [2] by multiplying the constant terms by $\frac{1}{2}\chi(K)$, that is, by replacing the standard convention $f_{-1}(K)=1$ by $f_{-1}(K)=\frac{1}{2}\chi(K)$. That this is valid may be seen from Theorem 9.2.5 of [2] and the two paragraphs which follow it.

The Dehn-Sommerville equations for triangulated manifolds may also be obtained with somewhat less dependence on the properties of homology manifolds using results in an earlier but little-known paper [7] by Vaccaro.

3. Surgery on 3-manifolds

Suppose K is an abstract simplicial complex of dimension $d\geqslant 2$, σ and τ are d-simplices of K without vertices in common, and η is a map of the vertices of σ onto the vertices of τ.

Then the triple $T = (\sigma, \tau, \eta)$ determines (within isomorphism) a complex K/T and a simplicial map ϕ taking $K - \{\sigma\} - \{\tau\}$ onto K/T, where ϕ identifies each vertex v of σ with the vertex $\eta(v)$ of τ but acts one-to-one on the remaining vertices of K. As we wish to apply this operation to triangulated manifolds we will clearly want to impose further restrictions on the triple T. Let us say that T is *regular* if σ and τ have no vertices in common, ϕ preserves the dimension of all simplices of $K - \{\sigma\} - \{\tau\}$, and the only simplices of $K - \{\sigma\} - \{\tau\}$ identified by ϕ are pairs of simplices in Bd σ and Bd τ which correspond under the isomorphism defined by η. It can be shown that the triple T is regular if and only if no vertex v of σ can be joined to the vertex $\eta(v)$ of τ by a path consisting of vertices and fewer than three edges of K. With obvious reference to the case of triangulated manifolds we may say that K/T is obtained from K by formation of a handle if T is regular and the same connected component of K contains both σ and τ. If σ and τ are contained in different components of K, then T is clearly regular and we may say that K/T is obtained from these components by manifold addition.

(3.1) LEMMA. *Suppose K is a triangulated 3-manifold and K' is obtained from K by handle formation (or K is the disjoint union of two triangulated 3-manifolds K_1 and K_2 and K' is obtained by manifold addition). Then $|K'|$ is a 3-manifold depending only on $|K|$ and possibly the relative orientability properties of σ, τ, and η with respect to $|K|$.*

Proof. Consider manifold addition; the arguments for handle formation are similar. It is known that there is a closed neighborhood of $|\text{Bd}\,\sigma|$ in $|K_1 - \{\sigma\}|$ homeomorphic to $S^2 \times [0, 1]$. Thus K' is a 3-manifold. Thus also $|K_1 - \{\sigma\}|$ is homeomorphic to $|K_1' - \{\sigma'\}|$, where σ' is a 3-simplex of a subdivision K_1' of K_1 such that the closure of σ' is in the interior of σ. Suppose σ'' is any other 3-simplex of K_1' whose closure is contained in the interior of a 3-simplex of K_1. Then repeated applications of piecewise linear homeomorphisms, each of which is fixed outside some pair of adjacent 3-simplices of K_1, will carry $|K_1' - \{\sigma'\}|$ onto $|K_1' - \{\sigma''\}|$. Moreover, any two pairs of vertices of σ' and σ'' may be matched up. Similar remarks apply to τ and K_2. Thus $|K'|$ depends only on K_1, K_2, and possibly the orientability properties of σ, τ, and η, but not the choice of σ and τ. That $|K'|$ does not depend on the particular triangulations K_1 of $|K_1|$ and K_2 of $|K_2|$ follows from the Hauptvermutung for 3-manifolds [6], which asserts the existence of isomorphic subdivisions of K_1 and any other triangulation of $|K_1|$.

Clearly the orientation of the identifications is topologically significant when handle formation is applied to a triangulation of an orientable manifold. Whether it is significant when the manifold sum of two triangulated orientable 3-manifolds is formed is apparently an open question.

In certain cases it may be possible to perform the reverse of a manifold addition or handle formation on a complex. Suppose K is a d-dimensional complex, $\mathrm{Bd}\,\sigma$ is a subcomplex of K, but the d-simplex σ itself is not a member of K. If there exists a complex K'' such that $K = K''/T$ for some regular triple $T = (\sigma_1, \sigma_2, \eta)$, we shall say that K'' is obtained from K by cutting at $\mathrm{Bd}\,\sigma$ and patching with σ_1 and σ_2. Note that if K is connected and some K'' exists then K'' will have at most two components. In general, even if some K'' exists, it may not be uniquely determined by K and $\mathrm{Bd}\,\sigma$. However:

(3.2) LEMMA. *Suppose K is a triangulation of a 3-manifold, σ is a 3-simplex such that* $\mathrm{Bd}\,\sigma$ *is a subcomplex of K, but σ itself is not in K. Then K can be cut at* $\mathrm{Bd}\,\sigma$ *and patched with two* 3-*simplices to form a complex K''. Moreover K'' is uniquely determined within isomorphism by K and* $\mathrm{Bd}\,\sigma$, *and* $|K''|$ *is a (not necessarily connected) 3-manifold with at most two components.*

Proof. For each vertex v of σ, $\mathrm{Lk}\,(v;\ \mathrm{Bd}\,\sigma)$ is a 1-sphere contained in the 2-sphere $\mathrm{Lk}\,(v;\ K)$ and divides it into two closed disks. Equivalently $\mathrm{St}\,(v;\ \mathrm{Bd}\,\sigma)$ divides the open 3-cell $\mathrm{St}\,(v;\ K)$. It follows that there exists a complex K', a closed subcomplex M of K', and a simplicial map ϕ of K' onto K which takes M two-to-one onto $\mathrm{Bd}\,\sigma$ and $K' - M$ one-to-one onto $K - \mathrm{Bd}\,\sigma$. At each of the 6 vertices v' of M, $\mathrm{Lk}\,(v';\ M)$ is the boundary complex of a 2-simplex. It follows that M must be the disjoint union of the boundary complexes of two 3-simplices σ_1 and σ_2. It is now easy to see that $K'' = K' \cup \{\sigma_1, \sigma_2\}$ is a triangulation of a (not necessarily connected) 3-manifold. The rest of the lemma, including uniqueness, is clear.

The first part of Lemma (3.2) can be extended to manifolds in higher dimensions through the use of the Alexander duality theorem to show that $\mathrm{Lk}\,(v;\ \mathrm{Bd}\,\sigma)$ divides $\mathrm{Lk}\,(v;\ K)$, which is an homology manifold with the groups of a sphere. An attempt to extend the second part of the lemma to successively higher dimensions will meet quickly with a number of standard unsolved problems. An analogue of Lemma (3.2) for a special class of triangulated manifolds in higher dimensions is given at the end of § 4.

The following lemma is easily established. Actually the condition is necessary in any dimension for K' to be a pseudomanifold with connected links. Only the sufficiency in dimension 3 will be needed.

(3.3) LEMMA. *Suppose K is a triangulation of a 3-manifold and K' is the simplicial complex obtained from K by identifying two vertices u and v. Then a necessary and sufficient* *condition that* $|K'|$ *be a 3-manifold homeomorphic with* $|K|$ *is that* (u, v) *be an edge of* K *and* $\mathrm{Lk}\, u \cap \mathrm{Lk}\, v = \mathrm{Lk}\,(u, v)$ *in* K.

Suppose B is a subcomplex of a triangulated d-manifold K such that $|B|$ is a closed d-cell. Let $K' = (K - B) \cup vS$ where S is the boundary complex of B and vS is the closed joint of S and a new vertex v. We shall say that K' is obtained from K by a *central re-triangulation of* B *with center* v. In the special case that $B = \mathrm{ClSt}\,\sigma$ for some simplex σ of K, $S = \mathrm{BdSt}\,\sigma$ and the complex K' can be realized as a subdivision of K (sometimes called an elementary subdivision with respect to σ).

4. $\mathcal{H}^d(n)$ and simple d-trees

A d-dimensional complex T, $d \geq 1$, will be called a *simple d-tree* if it is the closure of its d-simplices, $\sigma_1, \ldots, \sigma_t$, and these d-simplices can be ordered in such a way that

$$\mathrm{Cl}\,\sigma_j \cap \left\{ \bigcup_{i=1}^{j-1} \mathrm{Cl}\,\sigma_i \right\} = \mathrm{Cl}\,\tau_j$$

for some $(d-1)$-face τ_j of σ_j, $j \geq 2$, and the τ_j are all distinct. Any ordering $\sigma_1, \ldots, \sigma_t$ for which the above holds and the related ordering $v_1, \ldots, v_{t+d}$ of vertices of T, where v_{i+d} is the vertex of σ_i not in $\mathrm{Cl}\,\tau_i$, will be called a *natural ordering*. Clearly any simple d-tree T is a triangulation of a closed d-cell whose interior consists exactly of the σ_i and τ_i. The remainder of T, denoted $\mathrm{Bd}\, T$, is a triangulation of S^{d-1}.

(4.1) PROPOSITION. $K \in \mathcal{H}^d(0)$ *if and only if* $K = \mathrm{Bd}\, T$ *for some simple* $(d+1)$*-tree* T. *Moreover* K *uniquely determines* T *if* $d \geq 2$.

Proof. The first part is immediate from the definitions of $\mathcal{H}^d(0)$ and simple $(d+1)$-trees. Suppose then that $d \geq 2$ and $K = \mathrm{Bd}\, T_1 = \mathrm{Bd}\, T_2$ for two simple $(d+1)$-trees T_1 and T_2. Let $v_1, \ldots, v_{d+t}$ and $\sigma_1, \ldots, \sigma_t$ be the vertices and $(d+1)$-simplices of T_1 in some natural order. From the fact that the interior simplicies of T_2 can only be d- or $(d+1)$-simplices, the assumption $d \geq 2$, and the simple form of $\mathrm{ClSt}\,(v_{t+d}; K)$ it follows that σ_t must be a simplex of T_2 as well as T_1. Now σ_t may not be the last $(d+1)$-simplex in a given natural ordering of the $(d+1)$-simplices of T_2, but clearly $T_2 - \mathrm{St}\,(v_{t+d}; T_2)$ is a simple $(d+1)$-tree, as is $T_1 - \mathrm{St}\,(v_{t+d}; T_2)$, and both have the same boundary. By induction it follows that $T_1 = T_2$.

(4.2) LEMMA. *Suppose* K_1 *and* K_2 *are members of* $\mathcal{H}^d(0)$ *and* K *is formed from them by manifold addition. Then* K *is also a member of* $\mathcal{H}^d(0)$. *Conversely, suppose* $K \in \mathcal{H}^d(0)$ *and* $\mathrm{Bd}\,\sigma$ *is a subcomplex of* K, *where* σ *is a d-simplex not in* K. *Then* K *can be cut at* $\mathrm{Bd}\,\sigma$ *and patched with d-simplices to form a unique complex* K'', *which is the union of two disjoint members of* $\mathcal{H}^d(0)$.

Proof. The first part of the lemma is an easy consequence of (4.1). For the second part of the lemma we may suppose $d \geq 2$, since the result for $d=1$ is obvious. Let $K = \mathrm{Bd}\ T_t$, where T_t is a simple $(d+1)$-tree with $(d+1)$-simplices $\sigma_1, \ldots, \sigma_t$ in natural order, and let T_r, $1 \leq r \leq t$, denote the subtree of T_t containing $\sigma_1, \ldots, \sigma_r$. It can be seen that the vertex v_{t+d} cannot be a vertex of σ since $\mathrm{Bd}\,\sigma \subset K$ and $d \geq 2$. Hence $\mathrm{Bd}\,\sigma$ is contained in T_{t-1}. This reasoning may be repeated up to the point that $\sigma \notin \mathrm{Bd}\ T_s$ but $\sigma \in \mathrm{Bd}\ T_{s-1}$, $2 \leq s \leq t$. Then σ must be the interior d-simplex τ_s of T_t. The desired conclusion is immediate.

(4.3) Lemma. *Let $\mathcal{K}^d$, $d \geq 2$, denote the class of triangulations K of (not necessarily connected) d-manifolds such that* Lk $(v; K) \in \mathcal{H}^{d-1}(0)$ *for every vertex v of K. If K' is obtained from members of $\mathcal{K}^d$ by handle formation and manifold addition, then $K' \in \mathcal{K}^d$. Conversely, suppose K is a connected member of $\mathcal{K}^d$, $d \geq 3$, and* Bd σ *is a subcomplex of K, where σ is a d-simplex not in K. Then K can be cut at* Bd σ *and patched with two d-simplices to form a member K'' of $\mathcal{K}^d$.*

Proof. Suppose $K \in \mathcal{K}^d$, $T = (\sigma, \tau, \eta)$ is a regular triple on K, and ϕ is the simplicial map of K onto $K' = K/T$ as defined in § 3. From the characterization of regular triples in terms of paths it follows that Lk $(v; K)$ and Lk $(\eta(v); K)$ are disjoint for any vertex v of σ. Further it can be seen that Lk $(\phi(v); K/T)$ is the result of manifold addition applied to Lk $(v; K)$ and Lk $(\eta(v); K)$. Thus the first part of the lemma follows from the first part of (4.2). Now suppose the conditions in the second part of the lemma hold. For each vertex v_i of σ let τ_i be the $(d-1)$-face of σ opposite v_i. Then $\mathrm{Bd}\,\tau_i = \mathrm{Lk}\,(v_i; \mathrm{Bd}\,\sigma) \subset \mathrm{Lk}\,(v_i; K)$, but $\tau_i \notin \mathrm{Lk}\,(v_i; K)$ since $\sigma \notin K$. Thus by the second part of (4.2), Bd τ_i divides Lk $(v_i; K) \in \mathcal{H}^{d-1}(0)$ into two complexes L_i and L_i' such that

$$L_i \cap L_i' = \mathrm{Bd}\ \tau_i$$

$$L_i \cup \{\tau_i\} \in \mathcal{H}^{d-1}(0)$$

$$L_i' \cup \{\tau_i\} \in \mathcal{H}^{d-1}(0).$$

By the same reasoning used in the proof of (3.2) there exists a complex K'', two d-simplices σ_1 and σ_2 in K'', and a simplicial map ϕ taking $M = \mathrm{Bd}\,\sigma_1 \cup \mathrm{Bd}\,\sigma_2$ two-to-one onto Bd σ and $K'' - M$ one-to-one onto $K - \mathrm{Bd}\,\sigma$. If v' is a vertex of σ_1 and $v_i = \phi(v')$, then Lk $(v'; K'')$ is isomorphic via ϕ to one of the complexes $L_i \cup \{\tau_i\}$ or $L_i' \cup \{\tau_i\}$. Thus $K'' \in \mathcal{K}^d$.

The case $d=4$ of this lemma will be used in the proof of Theorem 5. The first part of the lemma also shows that the member of $\mathcal{H}^d(n)$ are indeed triangulations of d-manifolds, as indicated in the introduction. Any member of $\mathcal{H}^d(n)$, $d \geq 2$, is clearly a connected complex obtained from a collection of boundary complexes of $(d+1)$-simplices by manifold additions and handle formations. The following easy result may be used to show that the converse is also true:

(4.4) PROPOSITION. *Suppose $d \geqslant 2$, $K_1 \in \mathcal{H}^d(n_1)$, $K_2 \in \mathcal{H}^d(n_2)$, and K is formed from K_1 and K_2 by manifold addition. Then $K \in \mathcal{H}^d(n_1+n_2)$.*

5. Proof of the lower bounds

The principal results of this section are Lemmas (5.4) and (5.6). At the end of the section these lemmas are used in connection with the final result from § 11 to establish the lower bound inequalities in Theorems 1 through 4 and the characterizations of the triangulations which achieve them. The medium of communication between this section and the results in § 10 and § 11 are the classes $\mathcal{R}(\alpha)$ defined immediately below. For convenience let $\mathcal{M}^3$ denote the class of triangulations of closed connected 3-manifolds, and for any complex K and any α define $g_\alpha(K) = f_1(K) - \alpha f_0(K)$. Note that $\alpha < \beta$ implies $\mathcal{R}(\alpha) \supseteq \mathcal{R}(\beta)$.

(5.1) *Definition of* $\mathcal{R}(\alpha)$. For any $\alpha \leqslant 4$ the class $\mathcal{R}(\alpha)$ consists of all simplicial complexes K such that $|K|$ is a closed connected 3-manifold and

R1(α): If K' is any simplicial complex such that $|K'| \approx |K|$, then either $g_4(K') \geqslant g_4(K)$ or $g_\alpha(K') > g_\alpha(K)$.

R2: If K contains the boundary complex of a 3-simplex as a subcomplex, then K contains the 3-simplex as well.

R3: K is not the boundary complex of a 4-simplex.

(5.2) LEMMA. *Suppose $2 \leqslant \beta$, $3/2 < \alpha < \beta < \gamma \leqslant 4$, and*

$$K \in \mathcal{M}^3 \Rightarrow g_\beta(K) \geqslant 10 - 5\beta. \tag{5.3}$$

Then either (5.3) *holds with γ in place of β, or there exists $K^* \in \mathcal{R}(\alpha)$ such that $g_\gamma(K^*) < 10 - 5\gamma$.*

Proof. Consider Fig. 1. The point $p = (5, 10)$ represents $f(K) = (f_0(K), f_1(K))$ for the important special case that K is the boundary complex of a 4-simplex. For each α the equation $g_\alpha(K) = 10 - 5\alpha$ determines the line through p with slope α. In particular $g_4(K) \geqslant -10$ is the crucial inequality of Lemma 5.4 following and Theorem 1. Now suppose the hypotheses of the present lemma hold but (5.3) does not hold for γ. Then there exists $K' \in \mathcal{M}^3$ with $f(K')$ to the right of the line psq ... and on or above $ptru$. Let $\mathcal{K}$ be the collection of complexes $K \in \mathcal{M}^3$ such that $f(K)$ lies in the triangular region pqr but not on pq. Choose $K^* \in \mathcal{K}$ so as to minimize $g_\alpha(K^*)$, and if there is a tie choose K^* as far to the right as possible on the line st. Since the region pqr is bounded, the existence of K^* is assured. Let S be the portion of the (f_0, f_1)-plane to the right of the f_1-axis and on or above the broken line $opstru$. It follows from (5.3) and the definition of K^* that $K \in \mathcal{M}^3$ implies $f(K) \in S$. It is also clear that K^*

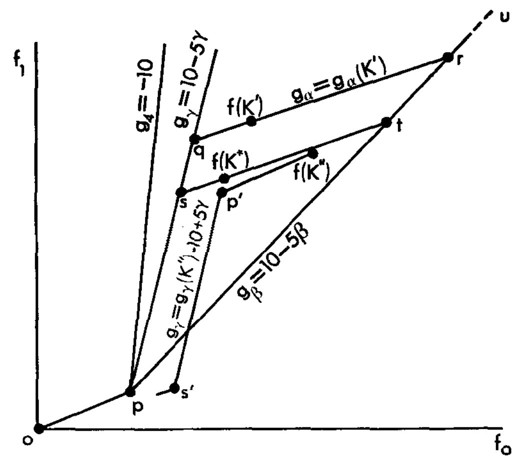

Fig. 1

satisfies conditions $\mathbf{R1}(\alpha)$ and **R3** in Definition (5.1). Suppose K^* does not satisfy condition **R2**, i.e., $\operatorname{Bd}\sigma \subset K^*$, $\sigma \notin K^*$ for some 3-simplex σ. Then by (3.2), K^* can be cut at $\operatorname{Bd}\sigma$ and patched to form a complex K'' which is the union of at most two disjoint members of $\mathcal{M}^3$. A simple count shows $f(K'')=f(K^*)+(4, 6)$. Since $\alpha > 6/4$, $f(K'')$ lies outside S, and hence K'' cannot be connected. Also, the line $g_\gamma = g_\gamma(K'')-10+5\gamma$ passes through $p'=f(K'')-(5, 10)$ and through or below the point $f(K^*)=p'+(1,4)$. From an examination of the symmetric pairs $(0, f(K''))$, (p, p'), (s, s') it is apparent that $f(K'')$ cannot be the sum of two points in S, i.e., K'' does not have two components either. The contradiction establishes that K^* satisfies **R2** and hence $K^* \in \mathcal{R}(\alpha)$, which completes the proof of the lemma.

(5.4) LEMMA. *If K is a triangulation of a closed connected 3-manifold, then* $f_1(K) \geqslant 4f_0(K)-10$.

Proof. Since every vertex of a member of $\mathcal{M}^3$ is the end point of at least 4 edges, obviously

$$K \in \mathcal{M}^3 \Rightarrow g_2(K) \geqslant 0.$$

From an application of Lemma (5.2) with $\alpha=1.9$, $\beta=2$, and $\gamma=42/13$ it follows that either

$$K \in \mathcal{M}^3 \Rightarrow g_{42/13}(K) \geqslant 10-\frac{5\cdot 42}{13} \tag{5.5}$$

or there exists $K^* \in \mathcal{R}(1.9)$ such that $g_{42/13}(K^*) < 10-5\cdot 42/13 < 0$. However, by Lemma (10.3), $K^* \in \mathcal{R}(1.9)$ implies $g_{42/13}(K^*) \geqslant 0$. Hence (5.5) holds. A second application of Lemma (5.2) with $\alpha=3$, $\beta=42/13$, and $\gamma=4$ shows that either (5.4) holds or there exists $K^{**} \in \mathcal{R}(3)$ such that $g_4(K^{**}) < -10$. But Lemma (10.8) will show that $K^{**} \in \mathcal{R}(3)$ implies $g_4(K^{**}) > 0$.

(5.6) LEMMA. *Suppose M is any closed connected 3-manifold. If K is any triangulation of M which minimizes $f_1(K)-4f_0(K)$, then K can be formed from members of $\mathcal{R}(4)$ and the boundary complexes of 4-simplices by manifold addition and formation of handles.*

Proof. Since $g_2(K)\geqslant 0$ for any $K\in\mathcal{M}^3$, it suffices to prove the following inductive proposition: If $K\in\mathcal{M}^3$, K satisfies **R1**(4), but K is not the boundary complex of a 4-simplex or a member of $\mathcal{R}(4)$, then K can be obtained by manifold addition and handle formation from members K_i of $\mathcal{M}^3$ satisfying **R1**(4) and $g_2(K_i)<g_2(K)$. Consider any K satisfying the hypotheses of this proposition. From the definition of $\mathcal{R}(4)$ it is immediate that K does not satisfy **R2**, i.e., $\mathrm{Bd}\,\sigma\subset K$, $\sigma\notin K$, for some 3-simplex σ. Form K'' from K by cutting at $\mathrm{Bd}\,\sigma$ and patching with 3-simplices as in (3.2). Consider first the case that K'' is the disjoint union of two members K_1 and K_2 of $\mathcal{M}^3$. A count of vertices and edges altered in forming K'' shows $g_2(K_1)+g_2(K_2)=g_2(K)-2$. This, combined with $g_2(K_2)\geqslant 0$, yields $g_2(K_1)<g_2(K)$. Suppose K_1 does not satisfy **R1**(4), i.e., $|K_1'|\approx|K_1|$ and $g_4(K_1')<g_4(K_1)$. Then by (3.1) $|K'|\approx|K|$, where K' is some manifold sum of K_1' and K_2, and clearly $g_4(K_1')<g_4(K_1)$ implies $g_4(K')<g_4(K)$. Since this contradicts the assumption that K satisfies **R1**(4), K_1 must satisfy **R1**(4). By symmetry $g_2(K_2)<g_2(K)$ and K_2 satisfies **R1**(4). Since K is the manifold sum of K_1 and K_2, this completes the proof of the inductive proposition in the disconnected case. Consider next the case that K'' is connected. We have immediately that $g_2(K'')=g_2(K)-2$. Suppose K'' does not satisfy **R1**(4), i.e., $|K'''|\approx|K''|$, $g_4(K''')<g_4(K'')$. Now it may not be possible to form a handle directly on K'''. But there do exist triangulations K_1 and K_2 of H^3_+ and H^3_- respectively belonging to $\mathcal{H}^3(1)$ with $f(K_1)=(10, 40)$ and $f(K_2)=(9, 36)$ (see § 9). By Lemma (3.1) the manifold sum K' of K''' and one of the complexes K_1 or K_2 will be a triangulation of $|K|$. From $g_4(K''')<g_4(K'')$ and $g_4(K_1)=g_4(K_2)=0$ it follows that $g_4(K')<g_4(K)$, contradicting the assumption that K satisfies **R1**(4). Thus $g_2(K'')<g_2(K)$ and K'' satisfies **R1**(4). Since K is obtained directly from K'' by formation of a handle, the proof of the inductive proposition and the lemma is complete.

The following lemma is quoted from the end of § 11.

(11.13) LEMMA. *If $K\in\mathcal{R}(3)$ and $f_1(K)\leqslant 4f_0(K)+7$, then K is isomorphic to the complex K_0 defined in § 8.*

Lemma (5.4) yields the lower bound inequality in Theorem 1 directly. It also yields a nontrivial part of Theorem 4, namely, for every closed connected 3-manifold M there exists a minimum value, $\gamma(M)$, of $f_1(K)-4f_0(K)$ for all triangulations K of M. Lemmas (5.6) and (11.13) complete the proof of the lower bounds and characterizations in Theorems 1 through 4 as follows: Suppose M is any 3-manifold for which $\gamma(M)\leqslant 7$. Suppose further that $|K| \approx M$ and $f_1(K) - 4f_0(K) = \gamma(M)$. By (5.6) and (11.13) K is formed from (say) p copies of K_0 and s copies of the boundary complex of a 4-simplex by $p+s-1$ manifold additions and h handle formations. (Recall that $\mathcal{R}(4) \subseteq \mathcal{R}(3)$.) A count of faces shows

$$f_1(K) - 4f_0(K) = 7p - 10s + 10(p+s-1) + 10h = 17p + 10h - 10.$$

Since $\gamma(M) \leqslant 7$, the only possibilities are $h = p = 0$, i.e., $M = S^3$; $h = 1$, $p = 0$, i.e., $M = H^3_+$ or $M = H^3_-$; and $h = 0$, $p = 1$, i.e., $M = P^3$.

6. Proof of Theorem 5

Let $\mathcal{E}^4$ denote the class of connected complexes such that the link of every vertex is a triangulation of a closed connected 3-manifold, and suppose $K \in \mathcal{E}^4$. (Equivalently $\mathcal{E}^4$ is the class of connected, locally connected Euler 4-manifolds as defined in § 2.) By (5.4)

$$f_1(\mathrm{Lk}\ v_i) \geqslant 4f_0(\mathrm{Lk}\ v_i) - 10 \tag{6.1}$$

holds for each vertex v_i of K. Summation over all vertices of K yields

$$3f_2(K) \geqslant 8f_1(K) - 10f_0(K),$$

and an application of the Dehn-Sommerville equation (2.2) for $f_2(K)$ yields

$$f_1(K) \geqslant 5f_0(K) - \tfrac{15}{2}\chi(K), \tag{6.2}$$

which establishes the first part of Theorem 5.

Conversely, suppose equality holds in (6.2) for $K \in \mathcal{E}^4$. Then (6.1) holds with equality for each v_i and by Theorems 1 through 4

$$\mathrm{Lk}\ (v_i;\ K) \in \mathcal{H}^3(0) \tag{6.3}$$

for each vertex v_i of K. Hence in particular $|K|$ is a 4-manifold.

It remains to be shown that

$$K \in \mathcal{H}^4(1 - \tfrac{1}{2}\chi(K)). \tag{6.4}$$

Since every vertex of a member L of $\mathcal{E}^4$ is incident on at least 5 edges,

$$L \in \mathcal{E}^4 \Rightarrow g_{2.5}(L) \geqslant 0. \tag{6.5}$$

(Recall the definition $g_\alpha(L) = f_1(L) - \alpha f_0(L)$.) Thus it suffices in proving (6.4) to assume the inductive hypothesis that K' satisfies (6.4) whenever $K' \in \mathcal{E}^4$, K' satisfies (6.2) with equality, and $g_{2.5}(K') < g_{2.5}(K)$.

Consider any vertex v of K. By (6.3) and (4.1) $\mathrm{Lk}\ v = \mathrm{Bd}\ T$ for some simple 4-tree T. Let $\sigma_1, \ldots, \sigma_s$ and $\tau_2, \ldots, \tau_s$ be respectively the 4-simplices and internal 3-simplices of T as described in § 4. One of three cases must hold.

Case I: $s \geqslant 2$ and none of $\sigma_1, \ldots, \sigma_s$ or $\tau_2, \ldots, \tau_s$ are members of K. Then $K' = (K - \mathrm{St}\, v) \cup T$ is an alternate triangulation of $|K|$ with $f_0(K') = f_0(K) - 1$ and $f_1(K') = f_1(K) - (s+4) < f_1(K) - 5$, which contradicts (6.2) for K'.

Case II: $s \geqslant 2$ and some τ_i is a member of K. On one hand $v\tau_i \notin K$, since $\tau_i \notin \mathrm{Bd}\, T = \mathrm{Lk}\, v$, and on the other $\mathrm{Bd}\,(v\tau_i) = \tau_i \cup v(\mathrm{Bd}\,\tau_i) \subset K$. Thus, in view of (6.3), K satisfies the hypotheses in the second part of Lemma (4.3) with $v\tau_i$ in place of σ. Let K'' be the complex given by (4.3). Then K'' satisfies (6.3), K'' satisfies (6.1) with equality, and hence each component of K'' satisfies (6.2) with equality. Now suppose K'' consists of two components, K_1 and K_2. A count of faces altered in converting K into K'' shows that $g_{2.5}(K'') = g_{2.5}(K_1) + g_{2.5}(K_2) < g_{2.5}(K)$. Thus, in view of (6.5), $g_{2.5}(K_1)$ and $g_{2.5}(K_2)$ are both less than $g_{2.5}(K)$, and hence by the inductive hypothesis $K_1 \in \mathcal{H}^4(1 - \frac{1}{2}\chi(K_1))$ and $K_2 \in \mathcal{H}^4(1 - \frac{1}{2}\chi(K_2))$. A count of altered faces also shows $\chi(K'') = \chi(K_1) = \chi(K_2) = \chi(K) + 2$, and hence K, which is the manifold sum of K_1 and K_2, is a member of $\mathcal{H}^4(2 - \frac{1}{2}\chi(K_1) - \frac{1}{2}\chi(K_2)) = \mathcal{H}^4(1 - \frac{1}{2}\chi(K))$, i.e., K satisfies (6.4). If K'' is connected, a similar argument applies.

Case III: T contains only one 4-simplex σ_1. If $\sigma_1 \in K$, then K is just the boundary complex of a 5-simplex and (6.4) holds trivially. If $\sigma_1 \notin K$, then (4.3) applies to K with σ_1 in place of σ, and the computations at the end of Case II above apply.

7. Neighborly triangulations

In this section it will be shown that every 3-manifold admits a *neighborly* triangulation, that is, one in which every pair of vertices is connected by an edge. Additional observations in (7.3) will complete a proof of the second part of Theorem 4.

(7.1) Lemma. *Every 3-manifold M admits a triangulation in which the closed star of some edge contains all vertices of the triangulation.*

Proof. Let K be any triangulation of M. It is an immediate consequence of the strong connectivity of K that there exists a simple 3-tree T and a simplicial map ϕ of T into K such that ϕ takes the 3-simplices of T into distinct 3-simplices of K and such that ϕT spans the vertices of K. Let $\sigma_1, \ldots, \sigma_t$ be the 3-simplices and $v_1, \ldots, v_{t+3}$ be the vertices of T in some natural order. If $\phi^{-1}\phi v_i$ is always just v_i, then ϕT is a spanning simple 3-tree in K. Otherwise, let v_{s+3} be the last vertex v_i for which $\phi^{-1}\phi v_i$ consists of at least two vertices. It is easily seen that the vertices of the disk $D = \mathrm{ClSt}\,(v_{s+3}; \mathrm{Bd}\, T)$ consist of the four vertices of σ_s and certain of the vertices $v_{s+4}, \ldots, v_{t+3}$. It follows from the choice of v_{s+3} and the dimension preserving properties of ϕ that ϕ is an isomorphism of D into the 3-cell $B = \mathrm{ClSt}\,(\phi v_{s+3}; K)$. In fact the disk ϕD divides B into two closed 3-cells B_1 and $B_2 = \phi\, \mathrm{ClSt}\,(v_{s+3}; T)$. Form K_1 from K by a central retriangulation of B_2 with center v^*. Define

$\phi_1(v_{s+3}) = v^*$, $\phi_1(v_i) = \phi(v_i)$, $i \neq s+3$. It can be checked that ϕ_1 is a simplicial map of T into K_1, ϕ_1 takes the 3-simplices of T into distinct 3-simplices of K_1, $\phi_1 T$ spans the vertices of K_1, and $\phi_1^{-1}\phi_1 v_i = v_i$ for $s+3 \leqslant i \leqslant t+3$. By induction there exists a triangulation K_2 of M and a simple 3-tree T_2 (isomorphic to T) in K_2 spanning the vertices of K_2.

Form K_3 from K_2 by a central retriangulation of T_2 with center w_0. Now observe that ClSt $(w_0; K_3)$ is strongly connected and spans the vertices of K_3. Hence the arguments of the previous paragraph can be applied to K_3 so as to yield a triangulation K_4 of M and a simple 3-tree T_4 in K_4 such that T_4 spans the vertices of K_4 and such that the 3-simplices of T_4 have a common vertex w_0. Finally form K_5 from K_4 by a central retriangulation of T_4 with center w_1. Then ClSt $((w_0, w_1); K_5)$ spans the vertices of K_5 and $|K_5| = M$.

(7.2) LEMMA. *Every 3-manifold M admits a neighborly triangulation in which the closed star of some edge contains all vertices of the triangulation.*

Proof. Let K be a triangulation of M such that ClSt $((w_0, w_1); K)$ spans the vertices of K, and suppose (x, y) is one of $k > 0$ pairs of vertices of K not connected by an edge of K. Clearly x and y are vertices of $L = \mathrm{Lk}((w_0, w_1); K)$ and are a distance $s \geqslant 2$ apart in L, i.e., the vertices of L in natural order are $x, v_2, \ldots, v_s, y, v_{s+2}, \ldots, v_t$. Let R_1 be the path in L from x to y, let R_2 be the path in L from v_s to v_t, let $S = w_0 R_1 \cup w_1 R_2$, and let $T = w_0 w_1 (R_1 \cup R_2)$. Note that T is a simple 3-tree spanning the vertices of K and S is a simple 2-tree in Bd T spanning the vertices of T. Form K_1 from K by a central retriangulation of T with center w_2. Then $w_2 S$ is a simple 3-tree in K_1 spanning the vertices of K_1. Form K_2 from K_1 by a central retriangulation of $w_2 S$ with center w_3. It can be checked that ClSt $((w_2, w_3); K_2)$ spans the vertices of K_2, there are exactly k pairs of vertices in K_2 including (x, y) which are not joined by an edge of K_2, and the vertices of $L_2 = \mathrm{Lk}\,((w_2, w_3); K_2)$ in natural order are $x, z, y, u_4, \ldots, u_{t+2}$, where $z = w_0$. Form K_3 from K_2 by removing (w_2, w_3, x, z), (w_2, w_3, z, y) and the common face (w_2, w_3, z) and then adding (x, y, w_2, z), (x, y, z, w_3), (x, y, w_3, w_2), and their common faces including (x, y). In K_3 let R_3 be the union of $\mathrm{Cl}(x, y)$ and the path from y to u_{t+2} in L_2, let $S_3 = w_2 R_3 \cup \mathrm{Cl}\,(x, y, z) \cup \mathrm{Cl}\,(y, z, w_3)$, and let $T_3 = w_3 S_3$. Then T_3 is a simple 3-tree spanning the vertices of K_3 and S_3 is a simple 2-tree in Bd T_3 spanning the vertices of T_3. Form K_4 from K_3 by a central retriangulation of T_3 with center w_4 and form K_5 from K_4 by a central retriangulation of $w_4 S_3$ with center w_5. It can be checked that ClSt $((w_4, w_5); K_5)$ spans the vertices of K_5 and there are only $k-1$ pairs of vertices in K_5 which are not joined by an edge of K_5. The lemma follows by induction.

(7.3) LEMMA. *Suppose the 3-manifold M admits a neighborly triangulation K containing a spanning simple 3-tree T whose 3-simplices have a vertex u in common. Then for every f_0 and f_1 satisfying*

$$4f_0+\gamma^*\leqslant f_1\leqslant\binom{f_0}{2},\quad f_0\geqslant f_0(K), \tag{7.4}$$

where γ^* *is defined by*

$$4f_0(K)+\gamma^*=f_1(K)=\binom{f_0(K)}{2},$$

there exists a triangulation of M *with* f_0 *vertices and* f_1 *edges.*

Proof. The proof is a natural adaptation of certain arguments for convex 4-polytopes [2; Th. 10.4.2]. Form K_1 from $K_0=K$ by a central retriangulation of $T_0=T$ with center w. Then K_1 is neighborly, ClSt $((u, w); K_1)$ contains all vertices of K_1, and it is immediate that K_1 contains a spanning simple 3-tree T_1 whose 3-simplices have the edge (u, w) in common. Thus, for any $k\geqslant 0$, M admits a neighborly triangulation K_k containing a simple 3-tree T_k spanning the $f_0(K)+k$ vertices of K_k. For any j, $1<j<f_0(K)+k-3$, form $K_{k,j}$ from K_k by a central retriangulation of a subtree of T_k composed of j of the $f_0(K)+k-3$ 3-simplices of T_k. The triangulations K_k and $K_{k,j}$ realize all points (f_0, f_1) satisfying (7.4) which minimize f_0 for fixed values of $f_1-4f_0\geqslant f_1(K)-4f_0(K)$. The remaining points are realized by repeated central retriangulations of 3-simplices applied to these triangulations.

8. Definition of P_0 and K_0

Consider the following 22 points on the unit sphere S^3 in E^4:

$$\begin{array}{ll}
(\ 0,\ \ 0,\ \ 0,\ \pm 1) & \\
(\pm\beta,\ \ 0,\ \ 0,\ \pm\beta) & \\
(\ 0,\ \pm\beta,\ \ 0,\ \pm\beta) & \alpha=\sqrt{3}/3 \\
(\ 0,\ \ 0,\ \pm\beta,\ \pm\beta) & \beta=\sqrt{2}/2. \\
(\pm\alpha,\ \pm\alpha,\ \pm\alpha,\ \ 0) &
\end{array}$$

It is trivial that these are exactly the vertices of a centrally symmetric 4-polytope P_0. It can be verified that each of the following inequalities defines a halfspace supporting P_0 in one of 80 simplicial facets. In each case all combinations of $\pm$ and all permutations of x_1, x_2, x_3 are to be considered.

$$\begin{array}{llllll}
\text{(i)} & \pm(\sqrt{2}-1)x_1 & \pm(\sqrt{2}-1)x_2 & \pm(\sqrt{2}-1)x_3 & & \pm x_4\leqslant 1 \\
\text{(ii)} & \pm(\sqrt{3}/3)x_1 & \pm(\sqrt{3}/3)x_2 & \pm(\sqrt{3}/3)x_3 & \pm(\sqrt{2}-\sqrt{3}/3) & x_4\leqslant 1 \\
\text{(iii)} & \pm(\sqrt{3}/2)x_1 & \pm(\sqrt{3}/2)x_2 & +0\,x_3 & \pm(\sqrt{2}-\sqrt{3}/2) & x_4\leqslant 1 \\
\text{(iv)} & \pm(\sqrt{2})x_1 & \pm(\sqrt{3}-\sqrt{2})x_2 & +0\,x_3 & +0 & x_4\leqslant 1.
\end{array}$$

The relative placement of these 80 facets can be visualized with the aid of a complex J_0 constructed as follows (the construction is such that J_0 may be considered a geometric complex): Let H be the hyperplane tangent to S^3 at $e=(0, 0, 0, 1)$ and let J_0 be the complex in H obtained from the boundary complex Bd P_0 by a central projection with center $-e$. The vertices of J_0 fall in four classes—the point $e \in H$; six vertices $(\pm\gamma, 0, 0, 1)$, $(0, \pm\gamma, 0, 1)$, $(0, 0, \pm\gamma, 1)$, $\gamma = 2\beta/(1+\beta) \cong 0.828$ of an octahedron Q; eight vertices $(\pm 2\alpha, \pm 2\alpha, \pm 2\alpha, 1)$, $2\alpha \cong 1.155$, of a cube Q^* containing Q; and six vertices $(\pm\delta, 0, 0, 1)$, $(0, \pm\delta, 0, 1)$, $(0, 0, \pm\delta, 1)$, $\delta = 2\beta/(1-\beta) \cong 4.83$, of a larger octahedron Q' containing Q^*. There is no vertex of J_0 corresponding to the vertex $-e$ of P_0. Each k-dimensional face of Q or Q' corresponds to a $(2-k)$-dimensional face of the dual Q^* on the same side of e. Certain 3-simplices of J_0 as determined by (i)–(iv) above fall into six subclasses:

(i a) 8 simplices of the form $e\sigma$, where σ is a facet of Q.

(ii a) 8 simplices of the form $v\sigma$, where σ is a facet of Q and v is the corresponding vertex of Q^*.

(iii a) 12 simplices of the form $\tau\sigma$, where τ is an edge of Q and σ is the corresponding edge of Q^*.

(iv) 24 simplices of the form $vv'\sigma$, where v and v' are corresponding vertices of Q and Q', and σ is any one of the four edges of the facet of Q^* pierced by the segment (v, v').

(iii b) 12 simplices as in (iii a), but with Q' in place of Q.

(ii b) 8 simplices as in (ii a), but with Q' in place of Q.

An additional eight 3-simplices of Bd P_0 with $-e$ as vertex are projected onto the boundary of Q'. It has already been noted that $f_0(P)=22$ and $f_3(P)=80$. From the Dehn-Sommerville equations it follows that $f_1(P)=102$ and $f_2(P)=160$.

A further consideration of the complex J_0 reveals a significant property of P_0. Specifically, if v is any vertex of P_0, then v and $-v$ cannot be connected by a path of fewer than 3 edges of P_0. Thus the following proposition applies to P_0.

(8.1) PROPOSITION. *Suppose P is a centrally symmetric simplicial d-polytope, and let K be the essentially unique simplicial complex obtained as the image of a simplicial map ϕ which acts on* Bd P *and identifies centrally symmetric pairs of vertices. If no centrally symmetric pair of vertices of P can be connected by a path of fewer than 3 edges of P, then ϕ acts everywhere two-to-one on* Bd P *and K is a triangulation of projective d-space.*

Proof. Clearly no centrally symmetric vertices of P can be connected by a single edge of P. Hence ϕ must preserve the dimension of all simplices. In addition, if no centrally symmetric vertices are connected by a path of length 2, then ϕ acts 1-to-1 on the closed star of any vertex. The rest is obvious.

Let K_0 be the triangulation of projective 3-space obtained from P_0. Note that Theorem 6 now follows as an easy corollary to Theorem 3 using (8.1). K_0 can be visualized in a number of essentially equivalent ways using J_0. For example, let A_0 be the closed subcomplex of J_0 generated by the simplices in the classes (ia), (iia), and (iiia). Then K_0 is obtained from A_0 by identifying opposite corners and edges of the cube Q^* and adding 12 3-simplices corresponding to the members of class (iv) identified in pairs.

9. Existence of triangulations

In this section the proofs of Theorems 1 through 3 will be completed by exhibiting triangulations with the required numbers of faces. The remainder of Theorem 1 follows from the observation that the boundary complex of a 4-simplex satisfies the hypotheses of (7.3) (or for that matter from the observation that all the indicated values of (f_0, f_1) can be realized by convex polytopes).

Now consider Theorem 2, and let T_1 and T_2 be the simple 4-trees whose 4-simplices are listed below.

T_1:	(a, b, c, d, e)	T_2:	(a, b, c, d, e)
	(f, b, c, d, e)		(f, b, c, d, e)
	(f, g, c, d, e)		(f, g, c, d, e)
	(f, g, h, d, e)		(f, g, h, d, e)
	(f, g, h, i, e)		(f, g, h, i, e)
	(f, g, h, i, j)		(f, g, h, i, a')
	(a', g, h, i, j)		(b', g, h, i, a')
	(a', b', h, i, j)		(b', c', h, i, a')
	(a', b', c', i, j)		(b', c', d', i, a')
	(a', b', c', d', j)		

Of course Bd T_1 is a member of $\mathcal{H}^3(0)$, and it can be checked, using the characterization of regular triples in terms of paths, that removing (a, b, c, d) and (a', b', c', d') from Bd T_1 and identifying primed and unprimed vertices produces a legitimate member K_1 of $\mathcal{H}^3(1)$. The complex K_1 can be oriented by applying the standard boundary operator to the chain of oriented 4-simplices of T_1 as listed above but with alternating signs. Thus $|K_1| = H^3_+ = S^2 \times S^1$. The same construction applied to T_2 yields a complex $K_2 \in \mathcal{H}^3(1)$ with $|K_2| = H^3_-$. With a certain amount of labor it can be shown that K_2 is the unique member of $\mathcal{H}^3(1)$ with 9 vertices. Hence the condition $(f_0, f_1) \neq (9, 36)$ in Theorem 2.

Next let S_2 be the simple 2-tree in K_2 defined by the following six 2-simplices:

$$\begin{array}{ll} (b, c, e) & (b, g, i) \\ (b, d, e) & (b, g, h) \\ (b, d, i) & (f, g, h) \end{array}$$

It can be checked that aS_2 is a spanning simple 3-tree in K_2, and obviously its 3-simplices have the vertex a in common. Theorem 2 for H^3_- follows directly from (7.3).

Again consider the complex $K_1 \in \mathcal{H}^3(1)$ defined above. There are exactly five pairs of vertices of K_1 not connected by an edge of K_1, and with each pair we may associate a pair of 3-simplices of K_1 as follows:

$$\begin{array}{ll} (a, f) \leftrightarrow (a, b, d, e), & (f, b, d, e) \\ (b, g) \leftrightarrow (b, c, e, f), & (g, c, e, f) \\ (c, h) \leftrightarrow (c, d, f, g), & (h, d, f, g) \\ (d, i) \leftrightarrow (d, e, g, h), & (i, e, g, h) \\ (e, j) \leftrightarrow (e, f, h, i), & (j, f, h, i) \end{array}$$

An alternate triangulation of $|K_1|$ can be obtained by removing any pair, such as (a, b, d, e) and (f, b, d, e), along with the common face (b, d, e) and replacing it with (a, f, b, d), (a, f, d, e), (a, f, e, b), and the common faces including (a, f). These alterations can be performed in any combination yielding triangulations of H^3_+ with 10 vertices and any number of edges from 40 to 45. Let K' be the neighborly triangulation so produced. The seven 2-simplices

$$\begin{array}{ll} (b, d, f) & (b, h, i) \\ (b, e, f) & (b, h, j) \\ (b, c, e) & (g, h, j) \\ (b, c, i) & \end{array}$$

define a simple 2-tree S_1, and aS_1 is a spanning simple 3-tree in K' whose 3-simplices have the vertex a in common. By (7.3) there exist triangulations of H^3_+ with f_0 vertices and f_1 edges whenever $f_0 > 10$ and $4f_0 + 5 \leqslant f_1 \leqslant f_0(f_0 - 1)/2$. The triangulations of H^3_+ with 10 vertices and 40 to 44 edges constructed above and the triangulations obtained from them by central retriangulations of 3-simplices realize all (f_0, f_1) satisfying $4f_0 \leqslant f_1 < 4f_0 + 5$, $f_0 \geqslant 10$. This completes the proof of Theorem 2 for H^3_+.

Similar reasoning completes the proof of Theorem 3. The only pairs of vertices of the complex K_0 defined in § 8 which are not connected by an edge of K_0 are the pairs (e, q^*), where q^* is one of the four vertices of K_0 corresponding to the eight vertices of the cube Q^*. Alternate triangulations of P^3 with 11 vertices and any number of edges from 51 to 55

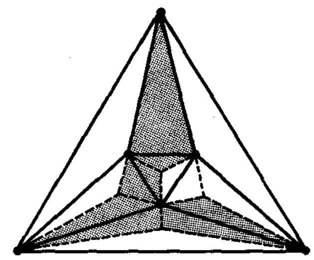

Fig. 2

can be obtained from K_0 by retriangulating the stars of suitably chosen 2-simplices corresponding to faces of the octahedron Q. The heavy lines in Fig. 2 represent Lk $(e; K_0)$. The complete figure represents Lk $(e; K')$, where K' is one of the possible neighborly triangulations. The shaded region determines a simple 2-tree S spanning Lk $(e; K')$. The join eS is a spanning simple 3-tree in K' whose 3-simplices have e as a common vertex.

10. Properties of $\mathcal{R}(\alpha)$

Suppose $K \in \mathcal{R}(\alpha)$, $\alpha \leq 4$, and u is any vertex of K. A convenient way of representing Lk u is shown in Fig. 3, where each diagram is a planar representation of the triangulated disk $D_v u = \text{Lk } u - \text{St } v$ for some vertex v of Lk u. The combinatorial types of Lk u depicted in Fig. 3 are designated 6, 7, 8a, 8b, etc., where the numeral denotes the *valence* $\nu(u) = \nu(u; K) = f_0(\text{Lk } (u; K))$ of u in K. The parenthetical data further designates the valence $\nu(v; \text{Lk } u) = \nu(u, v) = f_0(\text{Lk } (u, v))$ and location of v relative to Lk u. One of the major tasks of this section will be to show that Fig. 3 gives all possible types for $D_v u$ when $\nu(u) \leq 9$ and $K \in \mathcal{R}(3)$.

The disks $D_v u$ are particularly useful in determining the possible combinatorial types and relative orientations of the links of adjacent vertices. If (u, v) is an edge of $K \in \mathcal{R}(\alpha)$, then Lk (u, v) is the common boundary of the two disks $D_v u$ and $D_u v$. But note that this is not sufficient to insure that $D_v u \cup D_u v$ is a 2-sphere. Indeed, Lemma (10.6) will show that $D_v u \cap D_u v$ must include vertices in the interiors of $D_v u$ and $D_u v$ satisfying fairly restrictive conditions if $K \in \mathcal{R}(3)$. As a consequence, it will be shown that there are only a few ways in which a pair of diagrams in Fig. 3 can arise from disks $D_v u$ and $D_u v$ for (u, v) an edge of $K \in \mathcal{R}(3)$.

(10.1) LEMMA. *Suppose* $K \in \mathcal{R}(\alpha)$, $\alpha \leq 4$, *and* (u, v) *is an edge of* K. *Then* $\nu(u, v) \geq 4$, *or equivalently, every vertex of* Lk u *is incident on at least 4 edges of* Lk u.

Proof. If not then (u, v) must be an edge of exactly three 3-simplices of K, say (u, v, a, b), (u, v, b, c), *and* (u, v, c, a). Suppose $(a, b, c) \in K$. Then Bd $(u, a, b, c) \subset K$, and by condition **R2** in the definition (6.1) of $\mathcal{R}(\alpha)$, $(u, a, b, c) \in K$. Similarly $(v, a, b, c) \in K$. Thus all five 3-simplices of Bd (u, v, a, b, c) are in K. But this is possible only if Bd $(u, v, a, b, c) = K$, which contradicts **R3**. Hence $(a, b, c) \notin K$. Now remove the star of (u, v) from K and add (a, b, c), (u, a, b, c), and (v, a, b, c) to form a new complex K'. Since $(a, b, c) \notin K$, it is easy to see that $|K'| \approx |K|$, $f_0(K') = f_0(K)$, and $f_1(K') = f_1(K) - 1$, which contradicts **R1**(α).

(10.2) Lemma. *Suppose* $K \in \mathcal{R}(\alpha)$, $\alpha \leqslant 4$, *and* (u, v) *is an edge of* K. *Then* $W(u, v) = \operatorname{Lk} u \cap \operatorname{Lk} v - \operatorname{Lk}(u, v) = D_v u \cap D_u v - \operatorname{Lk}(u, v)$ *is nonempty.*

Proof. Suppose not. Then necessarily $\operatorname{Lk} u \cap \operatorname{Lk} v = \operatorname{Lk}(u, v)$. Let K' be obtained from K by identifying u and v. By (3.3) $|K'| \approx |K|$, by (10.1) $f_1(K') \leqslant f_1(K) - 5$, and obviously $f_0(K') = f_0(K) - 1$. But this contradicts **R1**(α).

(10.3) Lemma. *If* $K \in \mathcal{R}(\alpha)$, $\alpha \leqslant 4$, *then* $f_1(K) \geqslant 42/13\ f_0(K)$.

Proof. By considering a few cases it can be seen from (10.1) that $\nu(u) \geqslant 6$ for any vertex u of K and that if $\nu(u) = 6$ then Lk u is isomorphic to the boundary complex of an octahedron. Now suppose $(u, v) \in K$ and $\nu(u) = \nu(v) = 6$. Then $D_v u$ and $D_u v$ both have the form shown in Fig. 3-6(4). By (10.2) the interior of $D_v u$ and the interior of $D_u v$ have at least one simplex σ of K in common. An examination of Fig. 3-6(4) shows that this implies $D_v u = D_u v$, whatever the dimension of σ, and hence $K = \operatorname{ClSt} u \cup \operatorname{ClSt} v$. In particular each edge of Lk (u, v) is an edge of only three 3-simplices of K, contrary to (10.1). Thus, if (u, v) is an edge of K and $\nu(u) = 6$, then $\nu(v) \geqslant 7$. Now, for each *ordered* pair of vertices u and v of K such that (u, v) is an edge of K, define a number $\lambda(u, v)$ as follows: If $\nu(u) = 6$ set $\lambda(u, v) = 7/13$ and if $\nu(u) \geqslant 7$ set $\lambda(u, v) = 6/13$. From the preceding argument it is clear that $\lambda(u, v) + \lambda(v, u) \leqslant 1$ for all $(u, v) \in K$. Moreover, by the definition of λ,

$$\Lambda(u) = \sum_{v \in \operatorname{Lk} u} \lambda(u, v) \geqslant \frac{42}{13}.$$

Hence

$$f_1(K) \geqslant \sum_{(u, v) \in K} [\lambda(u, v) + \lambda(v, u)] = \sum_{u \in K} \sum_{v \in \operatorname{Lk} u} \lambda(u, v) \geqslant \sum_{u \in K} \frac{42}{13} = \frac{42}{13} f_0(K).$$

(10.4) Lemma. *Suppose* $K \in \mathcal{R}(3)$, u *is a vertex of* K, *and* Lk u *contains the boundary complex of a 2-simplex* (a, b, c) *as a subcomplex. Then* Lk u *must contain the 2-simplex* (a, b, c) *as well. Hence for any* v, $D_v u$ *cannot contain a* diagonal, *i.e., an interior edge connecting boundary vertices.*

Proof. Suppose (a, b), (b, c) and (c, a) are edges of Lk u but (a, b, c) is not a 2-simplex of Lk u. Then (u, a, b), (u, b, c), and (u, c, a) are 2-simplices of K but (u, a, b, c) is not a 3-simplex of K. It follows immediately from **R2** that (a, b, c) cannot be part of K. Now Bd (a, b, c) divides Lk u into two closed triangulated disks D_1 and D_2. Let K' be the complex

$$K' = [K - \operatorname{St} u] \cup v[D_1 \cup (a, b, c)] \cup w[D_2 \cup (a, b, c)],$$

where v and w are two new vertices. Since $(a, b, c) \notin K$, it is easy to see that $|K'| \approx |K|$. But also $f_0(K') = f_0(K) + 1$ and $f_1(K') = f_1(K) + 3$, which contradicts **R1**(3).

(10.5) LEMMA. *Suppose $K \in \mathcal{R}(3)$, u is a vertex of K of valence at most 9, and v is a vertex of* Lk u. *Then $D_v u$ must have one of the forms shown in Fig. 3.*

Proof. This will follow from (10.1) and (10.4) by an exhaustion of cases. The selection of an efficient scheme for analyzing the cases is left to the reader. However, note that it suffices to obtain only one representation of each type of Lk u, say the one for which $D_v u$ has a maximum number of boundary vertices, and determine any others from it. Also, from the second part of (10.4) it follows that $D_v u$ must have at least one interior vertex and any two interior vertices can be connected by a path consisting of interior vertices and edges of $D_v u$.

(10.6) LEMMA. *Suppose $K \in \mathcal{R}(3)$ and (u, v) is an edge of K. Then $W(u, v) = D_v u \cap D_u v -$* Lk (u, v) *is a nonempty closed subcomplex of K contained in the interiors of $D_v u$ and $D_u v$. In particular there cannot exist any vertex w in $W(u, v)$ and vertex z in* Lk (u, v) *such that (w, z) is an edge in both $D_v u$ and $D_u v$.*

Proof. That $W(u, v)$ is nonempty was shown in (10.2), and clearly $W(u, v)$ is *relatively* closed in the interiors of $D_v u$ and $D_u v$. Consider any simplex σ in $W(u, v)$ and let ω be the smallest face of σ such that $\omega \in W(u, v)$. Then Bd $\omega \subseteq$ Lk (u, v). If ω is a 2-simplex, then Bd ω must be all of Lk (u, v), which contradicts (10.1). If ω is a 1-simplex, then it must be a diagonal of $D_v u$, contrary to (10.4). Thus ω must be a 0-simplex, i.e., every simplex σ in $W(u, v)$ has a vertex in $W(u, v)$. That $W(u, v)$ is a closed complex in the interiors of $D_v u$ and $D_u v$ will now follow if we prove the second part of the lemma. Accordingly suppose (w, z) is an edge of $W(u, v)$ connecting a vertex w of $W(u, v)$ to a vertex z of Lk (u, v). Then $(w, z) \in D_v u$ implies $(u, w, z) \in K$ which implies $(u, w) \in$ Lk z. Similarly, $(w, z) \in D_u v$ implies $(v, w) \in$ Lk z. Further, $z \in$ Lk (u, v) implies $(u, v) \in$ Lk z. Thus (u, v), (v, w), and (u, w) are edges of Lk z, and by (10.4), $(u, v, w) \in$ Lk z. But $(u, v, w) \in K$ implies $w \in$ Lk (u, v), which is impossible.

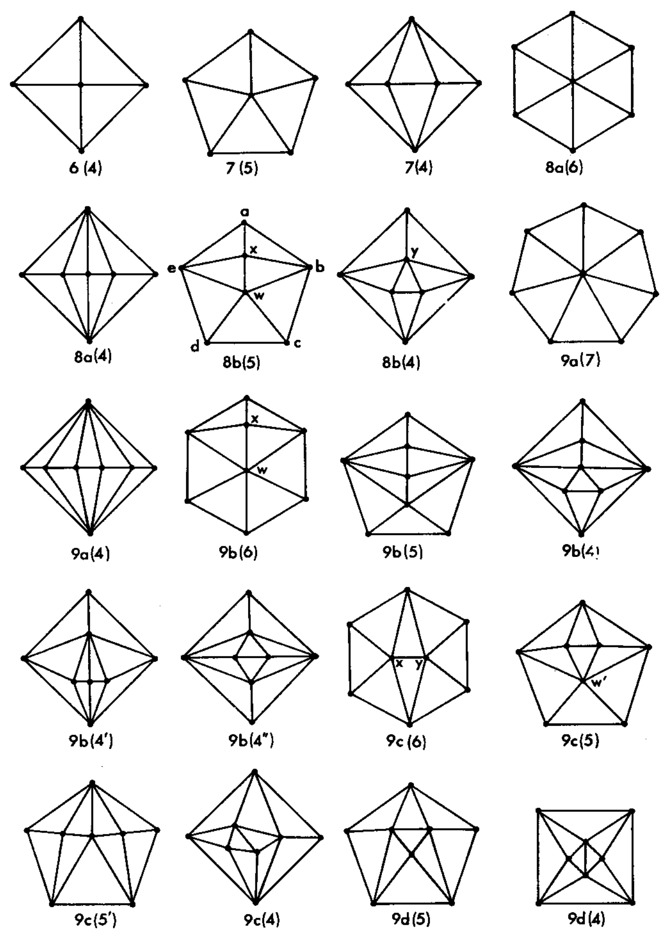

Fig. 3

(10.7) Corollary. *Suppose $K \in \mathcal{R}(3)$ and (u, v) is an edge of K. Then:*

(a) *If $D_v u$ is of type 6(4), 7(5), or 8a(6), then $\nu(v) \geqslant 10$.*

(b) *If $D_v u$ is of type 7(4), 8a(4), or 8b(5), then $\nu(v) \geqslant 9$.*

(c) *If $D_v u$ is of type 8b(4), then $\nu(v) \geqslant 8$.*

Proof. This follows from (10.6) and an examination of Fig. 3. For example, consider (b) and suppose $D_v u$ is of type 7(4), $\nu(v) < 9$. Then $D_u v$ is of type 6(4), 7(4), 8a(4), or 8b(4). Suppose $D_u v$ is of type 8b(4). There are two essentially different ways of identifying the boundaries of Fig. 3-7(4) and Fig. 3-8b(4). However, in both cases it is impossible to identify any vertex in the interior of Fig. 3-7(4) and any vertex in the interior of Fig. 3-8b(4) without violating (10.6).

(10.8) LEMMA. *If* $K \in \mathcal{R}(3)$ *then* $f_1(K) > 4f_0(K)$.

Proof. As in the proof of (10.3), define a function λ as follows:

$$\begin{aligned}\lambda(u, v) &= \tfrac{2}{3} \quad \text{if } \nu(u) = 6,\\ &= \tfrac{3}{4} \quad \text{if } \nu(u) = 7 \text{ and } D_v u \text{ is of type } 7(5),\\ &= \tfrac{1}{2} \quad \text{if } \nu(u) = 7 \text{ and } D_v u \text{ is of type } 7(4),\\ &= \tfrac{1}{2} \quad \text{if } \nu(u) = 8 \text{ or } 9,\\ &= 1 - \lambda(v, u) \quad \text{if } \nu(u) \geqslant 10 \text{ and } \nu(v) \leqslant 9,\\ &= \tfrac{1}{2} \quad \text{otherwise.}\end{aligned}$$

It is easily seen that $\lambda(u, v) + \lambda(v, u) = 1$ for every edge (u, v) of K. This follows from part (a) of (10.7) if $D_v u$ or $D_u v$ is of type 6(4) or 7(4) and is trivial otherwise. Let us show that

$$\Lambda(u) = \sum_{v \in \mathrm{Lk}\, u} \lambda(u, v) \geqslant 4, \quad \text{all} \quad u \in K, \tag{10.9}$$

with strict inequality if $\nu(u) \geqslant 9$. This is easily verified from the definition of λ above provided $\nu(u) \leqslant 9$.

Accordingly, let u be a vertex of K of valence $n \geqslant 10$, let m_6 be the number of vertices of Lk u of valence 6 in K, and let m_7 be the number of vertices v of Lk u such that $D_u v$ is of type 7(5). By the Dehn-Sommerville equation (2.1) there are exactly $2n-4$ 2-simplices in Lk u. Each vertex of Lk u contributing to the count m_6 is incident on 4 of these 2-simplices, each vertex of Lk u contributing to m_7 is incident on 5 of these 2-simplices, and it follows from parts (a) and (b) of (10.7) that no 2-simplex of Lk u is incident on more than one vertex contributing to m_6 or m_7. Thus $\Lambda(u)$ is at least as large as the value $z(n)$ of the linear program

$$\min \; -\frac{1}{6} m_6 - \frac{1}{4} m_7 + \frac{n}{2}$$

$$4 m_6 + 5 m_7 \leqslant 2n - 4$$

$$m_6 \geqslant 0, \;\; m_7 \geqslant 0.$$

It is easily verified that $z(n) \geqslant 4\frac{1}{5}$ if $n \geqslant 10$. This completes the proof of (10.9) and shows, by the argument used in the proof of (10.3), that $f_1(\mathrm{K}) \geqslant 4f_0(K)$, with strict inequality if K contains any vertices of valence 9 or greater. But parts (a) and (b) of (10.7) show that if K has any vertex of valence less than 9 then it also has a vertex of valence at least 9. This completes the proof of the lemma.

11. Further properties of $\mathcal{R}(\alpha)$

(11.1) LEMMA. *Suppose $K \in \mathcal{R}(3)$, (u, v) is an edge of K, and $W(u, v) = \mathrm{Lk}\, u \cap \mathrm{Lk}\, v - \mathrm{Lk}\,(u, v)$ consists of a single point w. Then $\nu(u, w) \geqslant \nu(u, v)$, that is, the valence of w in $D_v u$ is at least as large as the number of boundary vertices of $D_v u$.*

Proof. Let K' be obtained from K by a central retriangulation of $\mathrm{St}\,(u, w)$ with center u^* and let K'' be obtained from K' by identifying u and v. It is easily seen that $W'(u, v) = \mathrm{Lk}'\, u \cap \mathrm{Lk}'\, v - \mathrm{Lk}'\,(u, v)$ is empty and hence $|K| \approx |K'| \approx |K''|$. (Here and elsewhere Lk' will denote a link computed in K', etc.) Moreover $f_0(K'') = f_0(K)$ and $f_1(K'') = f_1(K) + \nu(u, w) - \nu(u, v)$. This will contradict **R1**(3) unless $\nu(u, w) \geqslant \nu(u, v)$.

(11.2) LEMMA. *Suppose $K \in \mathcal{R}(3)$, u is a vertex of K, and v is a vertex of* Lk u *such that* Lk v *is a bipyramid with m base vertices and apexes u and w (that is, $D_u v$ has m boundary vertices and exactly one interior vertex w as in Fig. 3-6(4), -7(5), -8a(6), etc.). Then:*

(a) $W(u, v) = \{w\}$.

(b) $\mathrm{Lk}\,(u, w)$ *is contained entirely in the interior of $D_v u$ and has at least m vertices.*

(c) $\nu(u) \geqslant 2m + 2$.

(d) *For no vertex s of* $\{w\} \cup \mathrm{Lk}\,(u, v)$ *is* Lk s *a bipyramid with apex u.*

Moreover, if $\mathrm{Lk}\,(u, w)$ *has exactly m vertices, then:*

(e) *For each $x \in \mathrm{Lk}\,(u, w)$ there is at least one $y \in \mathrm{Lk}\,(u, w)$ such that (x, y) is an* external diagonal *of* Lk u, *that is, (x, y) is an edge of K but not an edge of* $\mathrm{Lk}\, u \supset \mathrm{Lk}\,(u, w)$.

(f) *For no vertex s of* $\{w\} \cup \mathrm{Lk}\,(u, v) \cup \mathrm{Lk}\,(u, w)$ *is* Lk s *a bipyramid with apex u.*

Proof. Part (a) follows from (10.6) applied to $D_u v$. Part (b) follows from (a), an application of (10.6) to $D_v u$, and from (11.1). Part (c) follows immediately from (b). If $s = w$, then (d) follows trivially from an application of (10.6) to (w, v). If $s \in \mathrm{Lk}\,(u, v)$, then (d) follows from an analysis of the type outlined in (10.7). For example, suppose $D_u v$ is of type 8a(6) and $D_u s$ is of type 9a(7). Then $D_s v$ is of type 8a(4), $D_v s$ is of type 9a(4), and the identification of the boundaries of Fig. 3-8a(4) and Fig. 3-9a(4) is to be performed without a rotation. But it is impossible to identify any interior vertices without violating (10.6). Now consider (e). Suppose $\mathrm{Lk}\,(u, w)$ has exactly m vertices, $x \in \mathrm{Lk}\,(u, w)$, and (e) fails for x, i.e.,

$$y \in \mathrm{Lk}\,(u, w), \quad (x, y) \in K \Rightarrow (x, y) \in \mathrm{Lk}\,(u, w). \tag{11.3}$$

Let K' be obtained from K by a central retriangulation of $\mathrm{St}\,(u, w)$ with center w^* and let K'' be obtained from K' by identifying w^* and x. From (11.3) it follows that

$$y \in \mathrm{Lk}' \, w^*, \quad (x, y) \in K' \Rightarrow (x, y) \in \mathrm{Lk}' \, w^*,$$

and this implies $W'(x, w^*)$ is empty. Hence $|K| \approx |K'| \approx |K''|$. Next let K''' be obtained from K'' by identifying u and v. It is easily seen from $W(u, v) = \{w\}$ that $W''(u, v)$ is empty, and hence $|K'''| \approx |K''| \approx |K|$. A count of altered faces shows $f_0(K''') = f_0(K) - 1$ and $f_1(K''') = f_1(K) - 5$, which contradicts **R1**(3). This proves (e). In view of (d) it suffices in proving (f) to suppose $s \in \mathrm{Lk}\,(u, w)$. Suppose Lk s is a bipyramid with apex u. By (e) there is some $y \in \mathrm{Lk}\,(u, w)$ such that (s, y) is an external diagonal of Lk u. Now $(s, y) \notin \mathrm{Lk}\, u$ implies $y \notin \mathrm{Lk}(s, u)$ and hence $y \in W(s, u)$, i.e., y is the other apex of Lk s. An application of (b) to u and s instead of u and v shows Lk (y, u) and Lk (s, u) must be disjoint. But clearly w is in both. This proves (f) and completes the proof of the lemma.

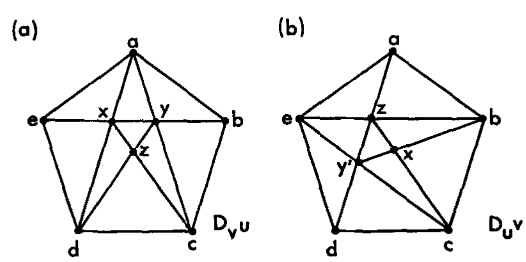

Fig. 4

(11.4) LEMMA. *Suppose $K \in \mathcal{R}(3)$ and (u, v) is an edge of K.*

(a) *If $D_v u$ is of type 9a(7), then $\nu(v) \geqslant 16$.*

(b) *If $D_v u$ is of tyge 9b(6), then $\nu(v) \geqslant 11$.*

(c) *If $D_v u$ is of type 9c(6), then $\nu(v) \geqslant 11$.*

(d) *If $D_v u$ is of type 9c(5), then $\nu(v) \geqslant 10$.*

(e) *If $D_v u$ and $D_u v$ are both of type 9d(5), then either $\nu(w) \geqslant 11$ for some interior vertex w of $D_v u$, or (within symmetries) $W(u, v)$ is exactly the closed edge $[x, z]$ in Fig. 4a and $\nu(x) = \nu(z) = 10$.*

Proof. Part (a) is a special case of (11.2). Suppose $D_v u$ is of type 9b(6) as in (b), and consider vertices w, x of $D_v u$ as indicated in Fig. 3-9b(6). If $w \notin W(u, v)$, then $W(u, v) = \{x\}$, contradicting (11.1). Thus $w \in W(u, v)$. Now w is also an interior vertex of $D_u v$, it must have at least 4 neighbors in $D_u v$, and by (10.6) at most one of these can be a boundary vertex. A count of vertices in $D_u v$ completes the proof of (b). Similar arguments show that both x and y in Fig. 3-9c(6) are in $W(u, v)$ if $D_v u$ is of type 9c(6). The proof of (c) is completed by verifying that it is impossible to construct a satisfactory diagram for $D_u v$ with at most 3 interior vertices, including x and y, which satisfies (10.1), (10.4), and (10.6). We have seen that the vertex x in Fig. 3-9c(6) is in $W(u, v)$ if $D_v u$ is of type 9c(6). Thus $(v, x) \in K$, or equivalently $v \in W(u, x)$, i.e., the vertex labeled w' in Fig. 3-9c(5) is in $W(u', v')$ if $D_{v'}u'$ is of type 9c(5). Part (d) follows easily. Finally, suppose $D_v u$ and $D_u v$ are of type 9d(5) as in (e). By (11.1) either x or y in Fig. 4a is in $W(u, v)$; say x is. By (10.6) $D_u v$ must be oriented as in Fig. 4b with x placed as shown. Now x is incident on (x, u), (x, v), 5 edges of $D_v u$, and 4 edges of $D_u v$. Thus $\nu(x) \geqslant 11 - \varrho$ where ϱ is the number of edges of $W(u, v)$ incident on x. By another application of (10.6), $y \notin W(u, v)$ and hence $\varrho \leqslant 1$. Thus either $\nu(x) \geqslant 11$ or $\nu(x) = 10$ and $W(u, v) = [x, z]$. The rest of (e) follows from the symmetry between $D_v u$ and $D_u v$.

(11.5) LEMMA. *Suppose* $K \in \mathcal{R}(3)$, (u, v) *is an edge of* K, Lk (u, v) *has exactly four vertices,* a, b, c, d, *in cyclic order, and* (a, c) *is not an edge of* K. *Let* K' *be obtained from* K *by removing* St (u, v) *and adding the simplices* (a, c, u, b), (a, c, b, v), (a, c, v, d), (a, c, d, u), *and their common faces including* (a, c). *Then* $|K'| \approx |K|$ *and* $K' \in \mathcal{R}(3)$. (*We shall say that* K' *is obtained from* K *by retriangulating* St (u, v) *using* (a, c).)

Proof. Since (a, c) is not an edge of K, it is easily seen that $|K'| \approx |K|$ and K' satisfies condition **R1**(3) and **R3**. Accordingly, consider **R2** for K' and suppose σ is a 3-simplex, $\sigma \notin K'$, and Bd $\sigma \subset K'$. By condition **R2** for K, $\sigma \notin K$ and Bd $\sigma \subset K$ cannot both hold. Hence either $\sigma \in K$ or Bd $\sigma \not\subset K$. In the first case $\sigma \in K$ and $\sigma \notin K'$ imply $\sigma \in \mathrm{St}\,(u, v)$, whence Bd $\sigma \not\subset K'$, a contradiction. In the second case Bd $\sigma \not\subset K$ and Bd $\sigma \subset K'$ imply (a, c) is an edge of σ, and in fact $\sigma \notin K' \supset \mathrm{St}'(a, c)$ implies $\sigma = (a, c, u, v)$ or $\sigma = (a, c, b, d)$. But $\sigma = (a, c, u, v)$ is not possible because $(u, v) \notin K'$. Hence $\sigma = (a, c, b, d)$, and consequently (a, b, d) and (c, b, d) are members of K' and therefore members of K also. Now (a, b, d), (a, d, u), and (a, u, b) are all in K, and hence by (10.4) $(a, b, d, u) \in K$. Similarly (c, b, d, u), (a, b, d, v), and (c, b, d, v) are in K. Thus (b, d) is an edge of K and in fact $\mathrm{BdSt}\,(b, d) = \mathrm{BdSt}(u, v)$. It follows that $K = \mathrm{ClSt}(b, d) \cup \mathrm{ClSt}(u, v)$. It is readily checked that $(b, d, u, v) \notin K$ and Bd $(b, d, u, v) \subset K$, which contradicts **R2** for K.

(11.6) *Definition.* Let $\mathcal{R}^+$ denote the set of complexes in $\mathcal{R}(3)$ which are minimal with respect to the partial ordering $\geqslant$ defined on $\mathcal{R}(3)$ as follows: For any two members K and K' of $\mathcal{R}(3)$, $K > K'$ if and only if K' can be obtained from K by a sequence of retriangulations of the type described in (11.5) and

$$\begin{aligned} &n_6(K) > n_6(K'), \quad \text{or} \\ &n_6(K) = n_6(K'), \quad n_7(K) < n_7(K'), \quad \text{or} \\ &n_6(K) = n_6(K'), \quad n_7(K) = n_7(K'), \quad n_{8b}(K) > n_{8b}(K'), \end{aligned} \tag{11.7}$$

where $n_6(K)$ denotes the number of vertices of K whose links are of type 6, etc. (Note the direction of the inequalities.)

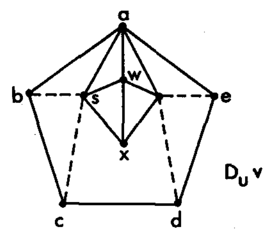

Fig. 5

(11.8) LEMMA. *Suppose $K \in \mathcal{R}^+$, (u, v) is an edge of K, and* Lk u *is of type 8b.*

(a) *If $D_v u$ is of type 8b(5), then both interior vertices of $D_v u$, labeled w and x in Fig. 3-8b(5), are members of $W(u, v)$. Moreover $\nu(v) \geq 11$.*

(b) *If $D_v u$ is of type 8b(4), then the vertex labeled y in Fig. 3-8b(4) is a member of $W(u, v)$. Moreover $\nu(v) \geq 9$.*

Proof. Consider (a), From (11.1) it follows that $w \in W(u, v)$. Suppose $x \notin W(u, v)$. Then $(v, x) \notin K$ and we may form a complex K' from K by retriangulating St (a, u) using (v, x). (See Fig. 3-8b(5).) By parts (b) and (c) of (10.7), $\nu(a) \geq 8$, $\nu(x) \geq 8$, and $\nu(v) \geq 9$. Thus $\nu'(u) = \nu(u) - 1 = 7$, $\nu'(a) = \nu(a) - 1 \geq 7$, $\nu'(x) = \nu(x) + 1 \geq 9$, and $\nu'(v) = \nu(v) + 1 \geq 10$. It follows that the second line of (11.7) holds and $K > K'$. Since this contradicts the assumption $K \in \mathcal{R}^+$, the first statement of (a) is proved. Next suppose the second statement in (a) is false, i.e., $\nu(v) \leq 10$. By the kind of arguments used in proving part (b) of (11.4), $\nu(v) = 10$ and $D_u v$ must take the form indicated in Fig. 5, except possibly for the dashed lines. The dashed lines follow from applications of (10.4) and (10.6). Since $s \notin W(u, v)$, we may retriangulate St (b, v) using (s, u) to form $K' \in \mathcal{R}(3)$. Then $\nu'(u) = \nu(u) + 1 = 9$, $\nu'(v) = \nu(v) - 1 = 9$, $\nu'(b) = \nu(b) - 1$, and $\nu'(s) = \nu(s) + 1$. In order to establish the third line of (11.7) and thus obtain the contradiction $K > K'$, it suffices to show $\nu(b) \geq 10$ and $\nu(s) \geq 8$. We have already established part (a) of the lemma with the weaker inequality $\nu(v) \geq 10$. This result can be applied to $D_b u$ instead of $D_v u$ yielding $\nu(b) \geq 10$. (It is clear from Fig. 3 that $D_b u$ is also of type 8b(5).) Next suppose $\nu(s) \leq 7$. It is obvious from Fig. 5 that $\nu(s, v) \geq 5$, and in fact it is easy to deduce that $\nu(s, v) = 5$ and $D_v s$ is of type 7(5). Part (b) of (11.2) yields the contradiction $\nu(v) \geq 12$. This completes the proof of (a). Finally, suppose $D_v u$ is of type 8b(4) as in (b). Then $D_y u$ is of type 8b(5), $(y, v) \in K$ by (a), and hence $y \in W(u, v)$. The inequality $\nu(v) \geq 9$ follows from arguments of the kind used in part (b) of (11.4).

(11.9) LEMMA. *Suppose $K \in \mathcal{R}^+$. For each ordered pair of vertices $u, v \in K$ such that (u, v) is an edge of K let $\lambda(u, v)$ be defined as follows:*

$\lambda(u, v) = \frac{3}{4}$ *if* $\nu(u) = 6$,

$\lambda(u, v) = 1$ *if* $D_v(u)$ *is of type 7(5).*

$\lambda(u, v) = \frac{3}{4}$ *if* $D_v(u)$ *is of type 8a(6).*

$\lambda(u, v) = \frac{5}{8}$ *if* $D_v(u)$ *is of type 8b(5).*

$\lambda(u, v) = \frac{1}{2}$ *if* $D_v(u)$ *is of type 7(4), 8a(4), 8b(4), or if* $\nu(u) = 9$.

$\lambda(u, v) = 1 - \lambda(v, u)$ *if* $\nu(u) \geqslant 10$ *and* $\nu(v) \leqslant 9$.

$\lambda(u, v) = \frac{1}{2}$ *otherwise.*

Further, for each vertex u of K let $\mu(u) = \sum_{v \in \mathrm{Lk}\, u} \lambda(u, v) - 4\frac{1}{2}$. *Then*

$$\sum_{u \in K} \mu(u) = f_1(K) - 4\tfrac{1}{2} f_0(K) \tag{11.10}$$

and:

(a) $\mu(u) = 0$ *if* $\nu(u) \leqslant 9$.

(b) $\mu(u) \geqslant \frac{1}{4}$ *if* $\nu(u) = 10$.

(c) $\mu(u) \geqslant \frac{1}{2}$ *if* $\nu(u) = 11$ *or* 12.

(d) $\mu(u) > 0$ *if* $\nu(u) \geqslant 13$.

Proof. Part (a) and (11.10) are trivial. Hence consider any vertex u of K, $\nu(u) = n \geqslant 10$. Let m_6, m_7, m_{8a}, and m_{8b} be the number of vertices v of Lk u such that $D_u v$ is of type 6(4), 7(5), 8a(6), and 8b(5) respectively. As in the proof of Lemma (10.8), $\mu(u)$ is at least the value of the program

$$\min -\frac{1}{4} m_6 - \frac{1}{2} m_7 - \frac{1}{4} m_{8a} - \frac{1}{8} m_{8b} + \frac{n-9}{2}$$

$$4 m_6 + 5 m_7 + 6 m_{8a} + 5 m_{8b} \leqslant 2n - 4 \tag{11.11}$$

$$m_6,\ m_7,\ m_{8a},\ m_{8b} \text{ nonnegative integers,}$$

where the derivation of the inequality $4m_6 + 5m_7 + 6m_{8a} + 5m_{8b} \leqslant 2n - 4$ requires part (b) of (11.8) in addition to (10.7).

The value of (11.11), ignoring integer requirements, is $(3n - 41)/10$, which is positive if $n \geqslant 14$. Hence suppose $\nu(u) = n = 13$. If the constraint $m_7 \leqslant 3$ is added to (11.11), then the value of (11.11), again ignoring integer requirements, is 1/16. To complete the proof of (d) it suffices to show $m_7 \leqslant 3$. Suppose $v_1, v_2 \ldots$ are vertices of Lk u contributing to m_7, and denote the unique interior vertex of $D_u v_i$ by w_i. In view of (11.2) part (b), there are two possibilities. Case I: Some w_i, say w_1, has valence 5 in Lk u. Then by part (f) of (11.2) there is only one other vertex of Lk u which can contribute to m_7. Case II: Every w_i has valence 6 in Lk u. In particular w_1 has 6 neighbors, all in the interior of $D_{v_1} u$. By part (d) of (11.2), v_2 (if it exists) is a neighbor of w_1, and by various applications of (10.4) the solid lines in Fig. 6 must be part of $D_{v_1} u$. Now w_2 must also have 6 neighbors and all of them must be in the interior of $D_{v_1} u \subset \mathrm{Lk}\, u$. Hence w_2 can be situated only as shown in Fig. 6

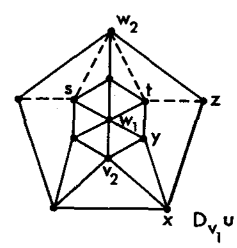

Fig. 6

and the dashed lines of Fig. 6 must be part of $D_{v_1}u$. By applications of (11.2) part (d) to v_1 and v_2, only s and t in Fig. 6 are possible sites for v_3 and v_4 if they exist. But $v_3 = s$ implies $w_3 = x$, which implies $(t, x) \in \mathrm{Lk}\, u$, whereas $v_4 = t$ implies $(y, z) \in \mathrm{Lk}\, u$. This completes the proof of (d).

Next suppose $\nu(u) = n = 12$. By (11.1) part (f), $m_7 \leqslant 1$. If $m_7 = 1$, then again by (11.2) part (f), $m_6 = 0$, and the value of (11.11) subject to these constraints is $\frac{1}{2}$. On the other hand, if $m_7 = 0$ an argument similar to the one used in analyzing the case $n = 13$ above will show $m_6 \leqslant 4$, and hence by (11.11), $\mu(u) \geqslant \frac{1}{2}$. This proves (c) for the case $n = 12$.

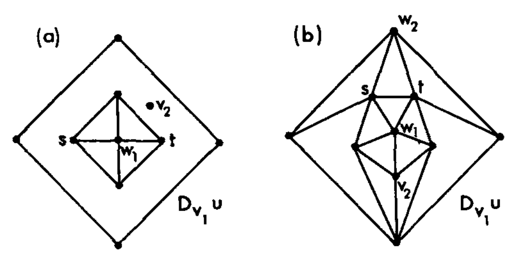

Fig. 7

Now suppose $\nu(u) = n = 11$. By (11.2) part (c), $m_7 = m_{8a} = 0$. Another argument as used for $n = 13$ will show that $m_6 \leqslant 2$ and that if $m_6 = 2$ then D_{v_1} must contain the elements shown in Fig. 7a or Fig. 7b, where v_1, v_2 contribute to m_6 and w_i is the interior vertex of $D_{v_i}u$. If $m_6 = 2$ and Fig. 7b applies, then, by part (a) of (10.7) applied to v_1 and v_2, only s and t can contribute to m_{8b}. Suppose s does. By the same reasoning used in (11.8) part (b), there is a vertex $w \in W(u, s)$ in the interior of $D_s u \subset \mathrm{Lk}\, u$ adjacent to at most one vertex of Lk (s, u). But no such vertex can be found, even in the incomplete diagram Fig. 7b. Thus $m_{8a} = 0$ if Fig. 7b applies. Suppose Fig. 7a applies and s is a vertex of Lk u which contributes to m_{8b}. By part (a) of (10.7) s can only be a vertex of Lk (u, w_1). By part (e) of (11.2), $t \in W(s, u)$. But the fact that t is adjacent in $D_s u$ to three vertices of Lk (s, u) leads to a violation of (10.6). In any case $m_6 = 2$ implies $m_{8b} = 0$ and hence $\mu(u) = \frac{1}{2}$. If $m_6 \leqslant 1$ then (11.11) yields $\mu(u) \geqslant \frac{1}{2}$ directly. This completes the proof of (c).

Finally, if $\nu(u) = n = 10$ then $\mu(u) \geqslant \frac{1}{4}$ follows from (11.2) and (11.8).

(11.12) LEMMA. *If $K \in \mathcal{R}^+$, then $f_1(K) > 4\frac{1}{2}f_0(K)$.*

Proof. The inequality $f_1(K) \geqslant 4\frac{1}{2}f_0(K)$ follows easily from (11.9). Moreover, strict inequality holds unless every vertex of K has valence at most 9, and this is impossible by (10.7) part (a), (11.8) part (a), and (11.4).

(11.13) LEMMA. *If $K \in \mathcal{R}(3)$ and $f_1(K) \leqslant 4f_0(K)+7$, then K is isomorphic to the complex K_0 defined in § 8.*

Proof. Suppose K satisfies the hypotheses of the lemma but $K \not\approx K_0$. Suppose further that the lemma holds if $\mathcal{R}(3)$ is replaced by $\mathcal{R}^+$. Then by the definition of $\mathcal{R}^+$, $K > K_0$, so that K_0 can be obtained from K by a sequence of retriangulations as defined in (11.5). Since these retriangulations are reversible, K can be obtained from K_0 by retriangulations. But an examination of K_0 shows that no such retriangulation is possible. It remains to be shown that the lemma holds assuming $K \in \mathcal{R}^+$.

The four inequalities $f_0(K) > 0$, $f_1(K) \leqslant 4f_0(K)+7$, $f_1(K) > 4\frac{1}{2}f_0(K)$, and

$$f_1(K) \leqslant \binom{f_0(K)}{2}$$

allow only four values for $f(K) = (f_0(K), f_1(K))$, namely, (11,50), (11,51), (12,55), and (13, 59). Obviously then $\nu(u) \leqslant 12$ for any vertex u of K, and consequently $n_{8a}(K) = 0$ by (11.2) part (c), where $n_t(K)$ denotes the number of vertices of K of type (or valence) t. Similarly if $n_7(K) \geqslant 1$ then $n_{12}(K) \geqslant 2$, $f(K) = (13,59)$, and the right side of (11.10) is $\frac{1}{2}$. But if $n_{12}(K) \geqslant 2$ then by (11.9) part (c) the left side of (11.10) is at least 1. Thus $n_7(K) = 0$. A similar argument using (11.8) part (a) shows that $n_{8b}(K) = 0$. Also, if $n_6(K) \geqslant 1$ then from (11.2) part (c) it follows that $n_{10}(K) + n_{11}(K) + n_{12}(K) \geqslant 6$, and from (10.10) and parts (b) and (c) of (11.9) it follows that $n_6(K) = 1$, $n_{10}(K) = 6$, and $n_{11}(K) = n_{12}(K) = 0$.

Consider the vector $\nu(K) = (\nu(u_1), ..., \nu(u_t))$ of valences of the vertices of K arranged in nonincreasing order. It is shown above that $\nu(K)$ is composed of 6's, 9's, 10's, 11's, and 12's only. Using this fact, the equation $\sum_i \nu(u_i) = 2f_1(K)$, and the consequences of $n_6(K) \geqslant 1$ given above, it is a simple matter to deduce that the only possibilities for $\nu(K)$ are:

$$\begin{aligned} f(K) &= (11,51), & \nu(K) &= (10, 10, 10, 10, 10, 10, 9, 9, 9, 9, 6) \\ & & \nu(K) &= (10, 10, 10, 9, ..., 9) \\ f(K) &= (12,55), & \nu(K) &= (11, 9, ..., 9) \\ & & \nu(K) &= (10, 10, 9, ..., 9) \\ f(K) &= (13,59), & \nu(K) &= (10, 9, ..., 9) \\ f(K) &= (11,50), & \nu(K) &= (10, 9, ..., 9). \end{aligned} \tag{11.14}$$

A comparison of (11.4) and (11.14) will show that every vertex of valence 9 must be of

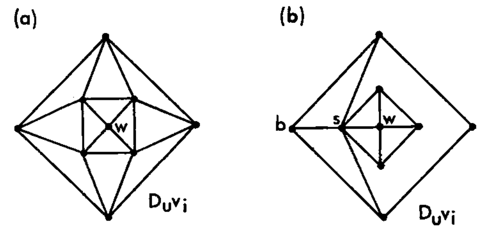

Fig. 8

type 9d. Now consider the second of the six cases in (11.4), and let u be a vertex of valence 9. There are six vertices v of $\mathrm{Lk}\, u$ such that $D_v u$ is of type 9d(5). Choose one such that $\nu(v)=9$. Then (11.4) part (e) applies to u and v, and hence x and z in Fig. 4a have valence 10 in K. If y has valence 9 then (11.4) part (e) also applies to u and y in place of u and v, and hence K contains two more vertices of valence 10 distinct from x and z, which is impossible. If y has valence 10 then y' has valence 9 and a symmetric argument applies. Similar but simpler reasoning will rule out all other cases in (11.14) except the first.

Finally, let u be the vertex of valence 6 in K and let $v_1, \ldots, v_6$ be the vertices of $\mathrm{Lk}\, u$. Of course $\nu(v_i)=10$. We wish to show that $D_u v_i$ must have the form shown in Fig. 8a, where w is the apex of $\mathrm{Lk}\, u$ opposite v_i. From (11.2) part (b) and several applications of (10.4) it can be seen that if Fig. 8a does not hold then Fig. 8b must hold in some orientation. Then $s \notin W(u, v_i)$, and hence the octahedron $\mathrm{St}\,(v_i, b)$ can be retriangulated using (u, s) to form a new complex K'. But $n_6(K') < n_6(K)$ so that $K > K'$, contradicting the assumption that $K \in \mathcal{R}^+$. Let L_0 be the closed subcomplex of the complex J_0 defined in *§ 8* generated by the 3-simplices in the classes (ia), (iia), (iiia), and (iv). From the knowledge of the exact forms of $\mathrm{ClSt}\,(u; K)$ and $\mathrm{ClSt}(v_i; K)$ it can be seen that K contains the image of L_0 under a dimension preserving simplicial map ϕ. But in fact with the knowledge of the sets $W(u, v_i)$ it can be seen that $K=\phi(L_0)$ and $K \approx K_0$.

References

[1]. Cairns, S. S., *Topology*. Ronald Press, New York, 1961.
[2]. Grünbaum, B., *Convex polytopes*. Wiley, New York, 1967.
[3]. Klee, V. M., A combinatorial analogue of Poincaré's duality theorem. *Canad. J. Math.*, 16 (1964), 517–531.
[4]. Lefschetz, S., *Introduction to topology*. Princeton University Press, 1949.
[5]. —— *Topology*. Chelsea, New York, 1956.
[6]. Moise, E. E., Affine structures in 3-manifolds V. The triangulation theorem and Hauptvermutung. *Ann. of Math.*, 56 (1952), 96–114.
[7]. Vaccaro, M., Sulla caratteristica dei complessi simpliciali χ-omogenei. *Ann. Mat. Pura Appl.*, (4) 41 (1956), 1–20.

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