THE L^p-INTEGRABILITY OF THE PARTIAL DERIVATIVES OF A QUASICONFORMAL MAPPING

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1. Introduction

Suppose that D is a domain in euclidean n-space R^n, $n \geqslant 2$, and that $f: D \to R^n$ is a homeomorphism into. For each $x \in D$ we set

$$L_f(x) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{|y - x|},$$
$$J_f(x) = \limsup_{r \to 0} \frac{m(f(B(x, r)))}{m(B(x, r))}, \tag{1}$$

where $B(x, r)$ denotes the open n-ball of radius r about x and $m = m_n$ denotes Lebesgue measure in R^n. We call $L_f(x)$ and $J_f(x)$, respectively, the maximum stretching and generalized Jacobian for the homeomorphism f at the point x. These functions are nonnegative and measurable in D, and

$$J_f(x) \leqslant L_f(x)^n \tag{2}$$

for each $x \in D$. Moreover, Lebesgue's theorem implies that

$$\int_E J_f \, dm \leqslant m(f(E)) < \infty \tag{3}$$

for each compact $E \subset D$, and hence that J_f is locally L^1-integrable in D.

Suppose next that the homeomorphism f is K-quasiconformal in D. Then

$$L_f(x)^n \leqslant K J_f(x) \tag{4}$$

a.e. in D, and thus L_f is locally L^n-integrable in D. Bojarski has shown in [1] that a little

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more is true in the case where $n=2$, namely that L_f is locally L^p-integrable in D for $p\in[2, 2+c)$, where c is a positive constant which depends only on K. Bojarski's proof consists of applying the Calderón-Zygmund inequality [2] to the Hilbert transform which relates the complex derivatives of a normalized plane quasiconformal mapping. Unfortunately this elegant two-dimensional argument does not suggest what the situation is when $n>2$.

In the present paper we give a new and quite elementary proof for the Bojarski theorem which is valid for $n\geqslant 2$. More precisely, we show in section 5 that L_f is locally L^p-integrable in D for $p\in[n, n+c)$, where c is a positive constant which depends only on K and n. The argument depends upon an inequality in section 4, relating the L^1- and L^n-means of L_f over small n-cubes, and upon a lemma in section 3, which derives the integrability from this inequality. We conclude in section 6 with a pair of applications.

2. An inequality

We begin with the following inequality for Stieltjes integrals.

LEMMA 1. *Suppose that* $q\in(0,\infty)$ *and* $a\in(1,\infty)$, *that* $h:[1,\infty)\to[0,\infty)$ *is nonincreasing with*

$$\lim_{t\to\infty} h(t)=0, \tag{5}$$

and that

$$-\int_t^\infty s^q\,dh(s)\leqslant a\,t^q h(t) \tag{6}$$

for $t\in[1,\infty)$. *Then*

$$-\int_1^\infty t^p dh(t)\leqslant \frac{q}{aq-(a-1)\,p}\left(-\int_1^\infty t^q dh(t)\right) \tag{7}$$

for $p\in[q, qa/(a-1))$. *This inequality is sharp.*

Proof. Suppose first that there exists a $j\in(1,\infty)$ such that $h(t)=0$ for $t\in[j,\infty)$, and for each $r\in(0,\infty)$ set

$$I(r)=-\int_1^\infty t^r\,dh(t)=-\int_1^j t^r\,dh(t).$$

If $p\in(0,\infty)$, then integration by parts yields

$$I(p)=-\int_1^j t^{p-q}\,t^q\,dh(t)=I(q)+(p-q)\,J,$$

where

$$J=\int_1^j t^{p-q-1}\left(-\int_t^j s^q\,dh(s)\right)dt.$$

Next with (6) and a second integration by parts we obtain

$$J \leqslant a\int_1^j t^{p-1}h(t)\,dt \leqslant -\frac{1}{p}I(q)+\frac{a}{p}I(p),$$

and (7) follows whenever $p\in[q, qa/(a-1))$.

In the general case, (5) implies that

$$j^q h(j) \leqslant -\int_j^\infty t^q dh(t)$$

when $j\in(1,\infty)$. For each such j set

$$h_j(t)=\begin{cases} h(t) \text{ if } t\in[1,j),\\ 0 \text{ if } t\in[j,\infty).\end{cases}$$

Then $h_j\colon[1,\infty)\to[0,\infty)$ is nonincreasing and

$$-\int_t^\infty s^q dh_j(s)\leqslant a t^q h_j(t)$$

for $t\in[1,\infty)$. Hence by what was proved above,

$$-\int_1^j t^p dh(t)\leqslant -\int_1^j t^p dh_j(t)\leqslant \frac{q}{aq-(a-1)p}\left(-\int_1^j t^q dh_j(t)\right)$$
$$\leqslant \frac{q}{aq-(a-1)p}\left(-\int_1^\infty t^q dh(t)\right),$$

and we obtain (7) by letting $j\to\infty$.

The function

$$h(t)=t^{-qa/(a-1)}$$

satisfies the hypotheses of Lemma 1, (7) holds with equality, and hence inequality (7) is sharp.

3. Maximal functions, means, and integrability

Suppose that $q\in(1,\infty)$, that $E\subset R^n$ has finite positive measure, and that $g\colon E\to[0,\infty]$ is L^q-integrable. Then Hölder's inequality implies that the L^1-mean of g over E is dominated by the corresponding L^q-mean of g, with equality if and only if g is a.e. constant, and hence a.e. bounded. We show here that g is L^p-integrable for some $p>q$ if the L^q-mean of g over certain subsets of E do not exceed the corresponding L^1-means of g by more than a fixed factor.

We shall base the proof of this fact on a similar result for maximal functions which may be of independent interest. Suppose that $g\colon R^n\to[0,\infty]$ is locally L^1-integrable. The maximal function $M(g)\colon R^n\to[0,\infty]$ for g is defined by

$$M(g)(x) = \sup \frac{1}{m(B)} \int_B g\, dm$$

for each $x \in R^n$, where the supremum is taken over all n-balls B with center at x. Next if $q \in (1, \infty)$ and g is locally L^q-integrable, then Hölder's inequality implies that

$$M(g)^q \leqslant M(g^q)$$

in R^n.

LEMMA 2. *Suppose that $q, b \in (1, \infty)$, that Q is an n-cube in R^n, that g: $R^n \to [0, \infty]$ is locally L^q-integrable in R^n, and that*

$$M(g^q) \leqslant b\, M(g)^q \tag{8}$$

a.e. in Q. Then g is L^p-integrable in Q with

$$\frac{1}{m(Q)} \int_Q g^p\, dm \leqslant \frac{c}{q+c-p} \left(\frac{1}{m(Q)} \int_Q g^q\, dm \right)^{p/q} \tag{9}$$

for $p \in [q, q+c)$, where c is a positive constant which depends only on q, b and n.

Proof. Inequality (9) is trivial if $g = 0$ a.e. in Q. Hence by replacing g by dg, where d is a suitably chosen constant, we may assume without loss of generality that

$$\int_Q g^q\, dm = m(Q). \tag{10}$$

Next for each $t \in (0, \infty)$ let

$$E(t) = \{x \in Q : g(x) > t\}. \tag{11}$$

We begin by showing that

$$\int_{E(t)} g^q\, dm \leqslant a\, t^{q-1} \int_{E(t)} g\, dm \tag{12}$$

for $t \in [1, \infty)$, where a is a constant which depends only on q, b and n.

Fix $t \in [1, \infty)$ and choose $s \in (t, \infty)$ so that

$$s^q = a_n b \left(\frac{q}{q-1} t \right)^q, \quad a_n = \Omega_n n^{n/2},$$

where $\Omega_n = m(B(0,1))$. Since

$$\frac{1}{m(Q)} \int_Q g^q\, dm \leqslant s^q,$$

we can employ a well known subdivision argument due to Calderón and Zygmund [2] to obtain a disjoint sequence of parallel n-cubes $Q_j \subset Q$ such that

$$s^q < \frac{1}{m(Q_j)} \int_{Q_j} g^q \, dm \leqslant 2^n s^q \tag{13}$$

for all j, and such that $g \leqslant s$ a.e. in $Q \sim G$, where $G = \bigcup_j Q_j$. (See page 418 of [7] or page 18 of [9].) Then $m(E(s) \sim G) = 0$ and with (13) we have

$$\int_{E(s)} g^q \, dm \leqslant \sum_j \int_{Q_j} g^q \, dm \leqslant 2^n s^q m(G). \tag{14}$$

Next if $B = B(x, r)$ where $x \in Q_j$ and $r = \text{dia}\,(Q_j)$, then (13) implies that

$$M(g^q)(x) \geqslant \frac{1}{m(B)} \int_B g^q \, dm > \frac{s^q}{a_n},$$

and with (8) we obtain
$$M(g)(x) > \frac{q}{q-1} t$$

for $x \in F \subset G$, where $m(G \sim F) = 0$.

For each $x \in F$ there exists an n-ball B about x such that

$$\frac{1}{m(B)} \int_B g \, dm \geqslant \frac{q}{q-1} t.$$

Since F is bounded, we can apply a familiar covering theorem to find a disjoint sequence of such balls B_j such that

$$m(G) = m(F) \leqslant 5^n \sum_j m(B_j). \tag{15}$$

(See, for example, page 9 of [9].) For each j,

$$\frac{q}{q-1} t\, m(B_j) \leqslant \int_{B_j} g \, dm \leqslant \int_{B_j \cap E(t)} g \, dm + t\, m(B_j)$$

whence
$$m(B_j) \leqslant \frac{q-1}{t} \int_{B_j \cap E(t)} g \, dm,$$

and combining this inequality with (14) and (15) yields

$$\int_{E(s)} g^q \, dm \leqslant 10^n s^q \frac{q-1}{t} \int_{E(t)} g \, dm. \tag{16}$$

Obviously

$$\int_{E(t)\sim E(s)} g^q\,dm \leqslant s^{q-1}\int_{E(t)} g\,dm,$$

and we obtain (12) with

$$a = 10^n\left(\frac{s}{t}\right)^q (q-1) + \left(\frac{s}{t}\right)^{q-1} < 50^n qb.$$

Now for each $t \in [1, \infty)$ set

$$h(t) = \int_{E(t)} g\,dm.$$

Then $h: [1, \infty) \to [0, \infty)$ is nonincreasing,

$$\lim_{t\to\infty} h(t) = 0,$$

and it is easy to verify that

$$\int_{E(t)} g^r\,dm = -\int_t^\infty s^{r-1}\,dh(s)$$

for all $r, t \in [1, \infty)$. Thus inequality (12) implies that h satisfies the remaining hypothesis (6) of Lemma 1, and we can apply (7) to conclude that

$$\int_{E(1)} g^p\,dm \leqslant \frac{c}{q+c-p}\int_{E(1)} g^q\,dm$$

for $p \in [q, q+c)$, where

$$c = \frac{q-1}{a-1} > \frac{q-1}{50^n qb}.$$

Since $g^p \leqslant g^q$ in $Q \sim E(1)$,

$$\int_Q g^p\,dm \leqslant \frac{c}{q+c-p}\int_Q g^q\,dm$$

for $p \in [q, q+c)$, and this together with (10) yields (9).

LEMMA 3. *Suppose that $q, b \in (1, \infty)$, that Q is an n-cube in R^n, that $g: Q \to [0, \infty]$ is L^q-integrable in Q, and that*

$$\frac{1}{m(Q')}\int_{Q'} g^q\,dm \leqslant b\left(\frac{1}{m(Q')}\int_{Q'} g\,dm\right)^q \tag{17}$$

for each parallel n-cube $Q' \subset Q$. Then g is L^p-integrable in Q with

$$\frac{1}{m(Q)}\int_Q g^p\,dm \leqslant \frac{c}{q+c-p}\left(\frac{1}{m(Q)}\int_Q g^q\,dm\right)^{p/q} \tag{18}$$

for $p \in [q, q+c)$, where c is a positive constant which depends only on q, b and n.

Proof. Assume that (10) holds and define $E(t)$ as in (11). Next for $t\in[1,\infty)$ pick $s\in[1,\infty)$ so that

$$s^q = b\left(\frac{q}{q-1}\,t\right)^q,$$

and choose a disjoint sequence of parallel n-cubes $Q_j\subset Q$ for which (13) and (14) hold. Then (13) and (17) imply that

$$s^q < \frac{1}{m(Q_j)}\int_{Q_j} g^q\,dm \leqslant b\left(\frac{1}{m(Q_j)}\int_{Q_j} g\,dm\right)^q$$

and hence that

$$m(Q_j)\leqslant \frac{q-1}{t}\int_{Q_j\cap E(t)} g\,dm$$

for each j. Combining this inequality with (14) yields (16) with 2^n in place of 10^n, and we obtain (12) with

$$a = 2^n\left(\frac{s}{t}\right)^q (q-1) + \left(\frac{s}{t}\right)^{q-1} < 2^{n+2}qb.$$

This then yields (18) with

$$c = \frac{q-1}{a-1} > \frac{q-1}{2^{n+2}qb}.$$

If $g=0$ in $R^n\sim Q$, then inequality (17) implies that

$$M(g^q)\leqslant dM(g)^q$$

in Q, where d is a constant which depends only on q, b and n. Hence Lemma 3 is a direct consequence of Lemma 2. However, the direct argument sketched above yields a substantially better estimate for the constant c.

4. An inequality for quasiconformal mappings

We show next that for a quasiconformal mapping f, the L^n-mean of L_f over a small n-cube is dominated by a fixed factor times the corresponding L^1-mean of L_f.

LEMMA 4. *Suppose that D is a domain in R^n, that $f\colon D\to R^n$ is a K-quasiconformal mapping, and that Q is an n-cube in D with*

$$\operatorname{dia} f(Q) < \operatorname{dist}(f(Q), \partial f(D)). \tag{19}$$

Then

$$\frac{1}{m(Q)}\int_Q L_f^n\,dm \leqslant b\left(\frac{1}{m(Q)}\int_Q L_f\,dm\right)^n, \tag{20}$$

where b is a constant which depends only on K and n.

Proof. We begin with some notation. We denote by $e_1, \ldots, e_n$ the basis vectors in R^n, and by $\bar{R}^n$ the one point compactification $R^n \cup \{\infty\}$ of R^n. Next for $t \in (0, \infty)$ we let $R_T(t)$ denote the ring with

$$\{x = se_1 : s \in [-1, 0]\}, \quad \{x = se_1 : s \in [t, \infty]\}$$

as its complementary components in $\bar{R}^n$. Then

$$\text{mod } R_T(t) \leqslant \log \lambda^2(t+1), \tag{21}$$

where λ is a constant which depends only on n,

$$\lambda \leqslant 4 \exp\left(\int_1^\infty \left(\left(\frac{s^2+1}{s^2-1}\right)^{\frac{n-2}{n-1}} - 1\right) \frac{ds}{s}\right).$$

(See, for example, [3] or [4].) In particular, it is easy to verify that

$$4 \leqslant \lambda \leqslant 4\left(\frac{e^n}{2}\right)^{\frac{n-2}{n-1}}.$$

By performing preliminary isometries, we may assume that Q is the closed n-cube

$$Q = \{(x_1, \ldots, x_n) : |x_i| \leqslant s, \; i = 1, \ldots, n\}, \quad s \in (0, \infty),$$

and that $f(0) = 0$. Let

$$r = \frac{s}{3^K \lambda^{2K} n^{\frac{1}{2}}},$$

and let R_1 be the ring with

$$C_1 = \{(x_1, \ldots, x_n) : |x_i| \leqslant r, \; i = 1, \ldots, n\}, \quad C_2 = \bar{R}^n \sim \text{int } Q$$

as its complementary components. Since C_1 and C_2 are separated by the spherical annulus

$$R = \{x \in R^n : n^{\frac{1}{2}} r < |x| < s\},$$

we have

$$\text{mod } R_1 \geqslant \text{mod } R = K \log 3\lambda^2. \tag{22}$$

Next let

$$r' = \max_{x \in \partial C_1} |f(x)|, \quad s' = \min_{x \in \partial C_2} |f(x)|, \quad t' = \max_{x \in \partial C_2} |f(x)|,$$

and choose points $x \in \partial C_1$ and $y \in \partial C_2$ such that $|f(x)| = r'$ and $|f(y)| = s'$. The ring $f(R_1)$ then separates $f(x)$ and 0 from $f(y)$ and ∞, and hence

$$\text{mod } f(R_1) \leqslant \text{mod } R_T\left(\frac{|f(y)|}{|f(x)|}\right) = \text{mod } R_T\left(\frac{s'}{r'}\right). \tag{23}$$

(See, for example, [3], [4], or [8].) Thus (21), (22), (23) and the fact that f is K-quasiconformal imply that

$$K \log 3\lambda^2 \leqslant K^{1/(n-1)} \operatorname{mod} f(R_1) \leqslant K \log \lambda^2 \left(\frac{s'}{r'}+1\right)$$

or simply that

$$s' \geqslant 2r'. \tag{24}$$

Let P: $R^n \to R^{n-1}$ denote the projection

$$P(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}),$$

and for each $y \in P(C_1)$ let $\gamma = \gamma(y)$ denote the closed segment joining $y + re_n$ to $y + se_n$. Since f is quasiconformal, there exists a Borel set $E \subset P(C_1)$ such that

$$m_{n-1}(E) = m_{n-1}(P(C_1)) = (2r)^{n-1}$$

and such that f is absolutely continuous on γ whenever $y \in E$. By Fubini's theorem, we can choose a $y \in E$ such that

$$\int_\gamma L_f ds \leqslant \frac{1}{m_{n-1}(E)} \int_Q L_f dm = \frac{1}{(2r)^{n-1}} \int_Q L_f dm. \tag{25}$$

Then since $y + re_n \in \partial C_1$ and $y + se_n \in \partial C_2$,

$$s' - r' \leqslant |f(y+se_n)| - |f(y+re_n)| \leqslant \int_\gamma L_f \, ds,$$

and we obtain

$$s' \leqslant \frac{2}{(2r)^{n-1}} \int_Q L_f \, dm \tag{26}$$

from (24) and (25).

Now suppose that $s' < t'$ and let

$$R_2' = \{x \in R^n: s' < |x| < t'\}.$$

Then (19) implies that $R_2' \subset f(D)$, and hence $R_2 = f^{-1}(R_2')$ is a ring which separates x and 0 from y and ∞, where $x, y \in \partial C_2$. Thus

$$\operatorname{mod} R_2 \leqslant \operatorname{mod} R_T\left(\frac{|y|}{|x|}\right) \leqslant \operatorname{mod} R_T(n^{\frac{1}{2}}),$$

and we obtain

$$\log \frac{t'}{s'} = \operatorname{mod} R_2' \leqslant K^{1/(n-1)} \operatorname{mod} R_2 \leqslant K \log \lambda^2 (n^{\frac{1}{2}} + 1),$$

or simply

$$t' \leqslant a\, s', \quad a = \lambda^{2K}(n^{\frac{1}{2}}+1)^K, \tag{27}$$

from (21) and the fact that f is K-quasiconformal. (See also Lemma 3 in [6].) Since $a > 1$, (27) also holds if $t' = s'$.

Finally $f(Q)$ obviously lies inside the closed ball $\bar{B}(0, t')$. Hence if we combine (3), (4), (26) and (27), we obtain

$$\frac{1}{m(Q)}\int_Q L_f^n\,dm \leqslant K\,\frac{m(f(Q))}{m(Q)} \leqslant K\Omega_n\left(\frac{as'}{2s}\right)^n$$

$$\leqslant K\Omega_n\left(2a\left(\frac{s}{r}\right)^{n-1}\frac{1}{m(Q)}\int_Q L_f\,dm\right)^n$$

$$= b\left(\frac{1}{m(Q)}\int_Q L_f\,dm\right)^n,$$

where

$$b = K\Omega_n(2a)^n(3^K\lambda^{2K}n^{\frac{1}{2}})^{n(n-1)}. \tag{28}$$

This completes the proof of Lemma 4.

5. Main result

We now apply Lemmas 3 and 4 to obtain the following n-dimensional version of Bojarski's theorem.

THEOREM 1. *Suppose that D is a domain in R^n and that $f\colon D \to R^n$ is a K-quasiconformal mapping. Then L_f is locally L^p-integrable in D for $p \in [n, n+c)$, where c is a positive constant which depends only on K and n.*

Proof. Choose an n-cube $Q \subset D$ such that

$$\text{dia}\,(f(Q)) < \text{dist}\,(f(Q), \partial f(D)). \tag{29}$$

Then L_f is L^n-integrable in Q. If $Q' \subset Q$ is an n-cube, then (29) implies that

$$\text{dia}\,(f(Q')) < \text{dist}\,(f(Q'), \partial f(D))$$

and hence, with Lemma 4, that

$$\frac{1}{m(Q')}\int_{Q'} L_f^n\,dm \leqslant b\left(\frac{1}{m(Q')}\int_{Q'} L_f\,dm\right)^n,$$

where b depends only on K and n. Thus by (3), (4) and Lemma 3, L_f is L^p-integrable in Q with

$$\frac{1}{m(Q)}\int_Q L_f^p\,dm \leqslant \frac{c}{n+c-p}\left(K\,\frac{m(f(Q))}{m(Q)}\right)^{p/n} < \infty$$

for $p \in [n, n+c)$, where c is a positive constant which depends only on K and n,

$$c > \frac{n-1}{2^{n+2}nb}. \tag{30}$$

Since each compact $E \subset D$ can be covered by a finite number of n-cubes Q satisfying (29), it follows that L_f is locally L^p-integrable in D for $p \in [n, n+c)$, where c is as above. This completes the proof.

Inequalities (27), (28) and (30) yield an explicit positive lower bound for the constant c in Theorem 1. However, this estimate is undoubtedly far from best possible since we have made no attempt to obtain sharp bounds in Lemmas 3 and 4.

To obtain an upper bound for the constant c in Theorem 1, set

$$f(x) = |x|^{a-1}x, \quad a = K^{1/(1-n)}.$$

Then f: $R^n \to R^n$ is a K-quasiconformal mapping with

$$L_f(x) = |x|^{a-1}.$$

Since L_f is not L^p-integrable near the origin whenever $p(a-1) \leqslant -n$, we see that

$$c \leqslant \frac{n}{K^{1/(n-1)} - 1}.$$

It seems probable that this upper bound for c is sharp.

6. Final remarks

We conclude this paper with two applications of Theorem 1. The first of these sharpens the well known result that a quasiconformal mapping is absolutely continuous with respect to Lebesgue measure.

THEOREM 2. *Suppose that D is a domain in R^n, that f: $D \to R^n$ is a K-quasiconformal mapping, and that c is the constant in Theorem* 1. *For each $a \in (0, c/(n+c))$ and each compact $F \subset D$ there exists a constant b such that*

$$m(f(E)) \leqslant b\, m(E)^a$$

for each measurable $E \subset F$.

Proof. Choose $a \in \left(0, \dfrac{c}{n+c}\right)$ and set

$$q = \frac{1}{1-a} \in \left(1, 1 + \frac{c}{n}\right).$$

Then Theorem 1 and (2) imply that J_f is locally L^q-integrable in D,

$$b = \left(\int_F J_f^q \, dm\right)^{1/q} < \infty,$$

and with Hölder's inequality we obtain

$$m(f(E)) = \int_E J_f \, dm \leqslant b\, m(E)^a$$

for each measurable $E \subset F$.

The second application is concerned with Hausdorff dimension. Suppose that $E \subset R^n$. For $a \in (0, \infty)$ the *Hausdorff a-dimensional outer measure* of E is defined as

$$H_a(E) = \lim_{d \to 0} \left(\inf \sum_j \operatorname{dia}(E_j)^a\right),$$

where the infimum is taken over all countable coverings of E by sets E_j with $\operatorname{dia}(E_j) < d$. The *Hausdorff dimension* of E is then given by

$$H\text{-}\dim E = \inf \{a \colon H_a(E) = 0\}.$$

Obviously $0 \leqslant H\text{-}\dim E \leqslant n$.

The following result describes what happens to the Hausdorff dimension of a set under a quasiconformal mapping. (See Theorems 8 and 12 in [6].)

THEOREM 3. *Suppose that D is a domain in R^n, that $f\colon D \to R^n$ is a K-quasiconformal mapping, and that c is the constant in Theorem* 1. *Then*

$$\frac{c\alpha}{c+n-\alpha} \leqslant H\text{-}\dim f(E) \leqslant \frac{(c+n)\,\alpha}{c+\alpha} \tag{31}$$

for each $E \subset D$ with H-dim $E = \alpha$.

Proof. A simple limiting argument shows we may assume that E is contained in an open set with compact closure $F \subset D$. Next for each $a \in (\alpha, \infty)$ and each $\gamma \in (0, c)$ set

$$b = \frac{(\gamma + n)\, a}{\gamma + a}, \quad q = 1 + \frac{\gamma}{n}.$$

Then $H_a(E) = 0$, J_f is L^q-integrable in F, and we obtain

$$H_b(f(E)) = 0$$

from the proof of Theorem 12 in [6] with 2 replaced by n. Letting $a \to \alpha$ and $\gamma \to c$ then yields the right-hand side of (31). The left-hand side of (31) follows from applying what was proved above to f^{-1}.

Theorem 3 shows that sets of Hausdorff dimension 0 and n are preserved under n-dimensional quasiconformal mappings, thus completing the proof of Conjecture 15 in [6]. Theorem 5 in [6] shows, on the other hand, that no such statement is true for sets of Hausdorff dimension α when $\alpha \in (0, n)$.

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